Polyhedral semantics and the tractable approximation of Łukasiewicz infinitely-valued logic

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Abstract

In this work, we present polyhedral semantics as a means to tractably approximate Łukasiewicz infinitely-valued logic ($Ł_\infty$). As $Ł_\infty$ is an expressive multivalued propositional logic whose decision problem is NP-complete, we show how to to obtain an approximation for this problem providing a family of multivalued logics over the same language as $Ł_\infty$. Each element of the family is associated to a polynomial-time linear program, thus providing a tractable way of deciding each intermediate step. We also investigate properties of the logic system derived from polyhedral semantics and the details of an algorithm for the approximation process.

Keywords: Łukasiewicz logic, approximation logics, linear constraints

1 Introduction

The complexity of logic inference has been a major drawback for its wider adoption and dissemination. Several strategies have been proposed to face this problem, chief among which is the identification of fragments of logic that are tractable, such as the Horn fragment [8], the 2-clause fragment of propositional logic [15] and the EL-family of description logics [1], which are a fragment of ALC-description logics [2]. Note that all those fragments can be seen as subsets of *classical* propositional, modal or first-order logic. In the context of non-classical logics, the fragments of simple Ł-clausal forms and 2-clause (restricted) Ł-clausal forms of Łukasiewicz logics were recently identified to be tractable [3].

Another way in which the tractability of logic was approached consisted of approximating classical logic by a family of parameterized non-classical logics, such that each intermediate logic system is tractable in relation to the size of the set of its parameters. This idea was initially proposed by [24], who proposed two systems, S_3 and S_1, which approximated clausal propositional logic validity and invalidity, respectively. Those systems were followed by studies that approximated propositional logic in general, providing both semantic and proof-theoretic approach, as well as an incremental way to proceed in the approximating path [9–11]. Other approximations of classical propositional logics were proposed with both philosophical and computational motivation [6]. In all approaches mentioned, each approximation step is a different parameterized logic whose decision procedure is polynomial-time over the size of the parameter set; in some cases, the 'approximation step' is

Vol. 00, No. 00,
https://doi.org/10.1093/logcom/exad059

incremental, in the sense that an inference at step $i+1$ could proceed from the point it stopped at step i.

The vast majority of fragments and approximations in the literature have classical logic as their target.[1] But several non-classical logics have reached prominence, chief among which is Łukasiewicz infinitely-valued logic ($Ł_\infty$), one of the aforementioned Łukasiewicz logics. $Ł_\infty$ is a system that has attracted the attention for quite sometime [16], due to its rich semantics in terms of Chang-algebras [4], important properties [17], complex analysis tools [5], important role among the class of fuzzy logics [14] and foundational applications in probability theory [20]. Furthermore, $Ł_\infty$ has been recently applied to the modelling and proving properties about neural networks [21, 22], an application that leads us towards the search for efficient ways of deciding large $Ł_\infty$-theories.

In this work, we introduce polyhedral semantics, a semantic approximation of $Ł_\infty$ that yields a multivalued and interval-valued logic over the same language as $Ł_\infty$. First, we study some properties of the polyhedral semantics as logical equivalences and derivatives of connectives.

Then, as $Ł_\infty$ is an expressive multivalued propositional logic whose decision problem is NP-complete [18], we aim at providing a tractable approximation for this problem, defining a parameterized family of logics based on polyhedral semantics, whose decision procedure is shown to be polynomial-time over the parameter set. The polyhedral semantics is directly coupled with a set of linear programs, and is thus associated with a polynomial-time algorithm.

The rest of the paper is organized as follows. Section 1.1 brings some of the fundamentals of Łukasiewicz infinitely-valued logic together with a known translation from its formulas to mixed integer linear programming (MILP) constraints. Section 2 introduces polyhedral semantics and presents some of its properties. Section 3 presents an approximation procedure for $Ł_\infty$ through a family of logics determined by polyhedral semantics while Section 4 structures the algorithm and proposes a heuristic needed in one of its steps. In Section 5, we draw some conclusions.

Notation. The language of $Ł_\infty$ contains propositional symbols in $\mathbb{P}$, unary negation connective $\neg$ and binary connectives for disjunction $\oplus$, conjunction $\odot$, idempotent disjunction $\vee$ and idempotent conjunction $\wedge$. Propositional symbols are atomic formulas; if φ and ψ are formulas, then so are $\neg\varphi$, $(\varphi \oplus \psi)$, $(\varphi \odot \psi)$, $(\varphi \vee \psi)$ and $(\varphi \wedge \psi)$. We designate the language of $Ł_\infty$ as $\mathcal{L}$. If A is a set, $|A|$ denotes the cardinality of A.

1.1 Background: semantics for $Ł_\infty$

The negation $\neg$ and disjunction $\oplus$ connectives are usually considered primitive, and the remaining connectives are defined in terms of them. For the semantics, consider a propositional symbol assignment $v_{\mathbb{P}}$ that assigns each symbol in $\mathbb{P}$ a value in $[0,1]$. Such assignment may be extended to a valuation $v : \mathcal{L} \to [0,1]$ such that, for $\varphi, \psi \in \mathcal{L}$:

$$v(\varphi \oplus \psi) = \min(1, v(\varphi) + v(\psi)); \tag{1}$$

$$v(\neg\varphi) = 1 - v(\varphi). \tag{2}$$

[1] Notable recent exceptions are [3] and [7].

We denote the set of all valuations by **Val**. From disjunction and negation, we derive the following operators:

$$
\begin{array}{rll}
\text{Conjunction: } & \varphi \odot \psi =_{\text{def}} \neg(\neg\varphi \oplus \neg\psi) & v(\varphi \odot \psi) = \max(0, v(\varphi) + v(\psi) - 1) \\
\text{Maximum: } & \varphi \vee \psi =_{\text{def}} \neg(\neg\varphi \oplus \psi) \oplus \psi & v(\varphi \vee \psi) = \max(v(\varphi), v(\psi)) \\
\text{Minimum: } & \varphi \wedge \psi =_{\text{def}} \neg(\neg\varphi \vee \neg\psi) & v(\varphi \wedge \psi) = \min(v(\varphi), v(\psi)) \\
\text{Implication: } & \varphi \rightarrow \psi =_{\text{def}} \neg\varphi \oplus \psi & v(\varphi \rightarrow \psi) = \min(1, 1 - v(\varphi) + v(\psi)) \\
\text{Bi-implication: } & \varphi \leftrightarrow \psi =_{\text{def}} (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi) & v(\varphi \leftrightarrow \psi) = 1 - |v(\varphi) - v(\psi)|
\end{array}
$$

For any propositional symbol p and valuation v, we have that $v(p \oplus \neg p) = 1$; so we define constants $\mathbf{1} =_{\text{def}} z \oplus \neg z$ and $\mathbf{0} =_{\text{def}} \neg\mathbf{1}$.

Let $\alpha \in [0,1]$; a formula φ is *α-satisfiable* if there exists a $v \in \mathbf{Val}$ such that $v(\varphi) = \alpha$; otherwise, it is *α-unsatisfiable*. Similarly, φ is *α^+-satisfiable* if there exists a v such that $v(\varphi) > \alpha$; otherwise, it is *α^+-unsatisfiable*. A set of formulas Φ is α/α^+-satisfiable if there exists a v such that $v(\varphi) = \alpha/v(\varphi) > \alpha$, for all $\varphi \in \Phi$. Most frequently, one is interested in deciding $1/0^+$-satisfiability. The problem of deciding α/α^+-satisfiability in $Ł_\infty$ for a given formula was shown to be NP-complete by Mundici [18].

Hähnle proposed methods for deciding whether a formula is a *$[0,d]$-tautology* or a *$[c,1]$-tautology*, i.e. whether it has value at most d or at least c, respectively, under any valuation [13]. Based on such methods, let us show how the semantics of a formula can be expressed as a set of MILP restrictions. Suppose y_φ and y_ψ stand for the values of $v(\varphi)$ and $v(\psi)$, respectively; in case they are propositional symbols, such a restriction is just an equality to a constant. For the disjunction, let $y_{\varphi\oplus\psi} = v(\varphi \oplus \psi)$, then by considering an extra symbol $b_{\varphi\oplus\psi}$, the following restrictions imposed on $y_{\varphi\oplus\psi}$ guarantee the same values as (1):

$$
\begin{aligned}
b_{\varphi\oplus\psi} &\in \{0,1\} \\
b_{\varphi\oplus\psi} &\leq y_{\varphi\oplus\psi} \leq 1 \\
y_\varphi + y_\psi - b_{\varphi\oplus\psi} &\leq y_{\varphi\oplus\psi} \leq y_\varphi + y_\psi
\end{aligned}
\tag{3}
$$

When $b_{\varphi\oplus\psi} = 0$, $y_{\varphi\oplus\psi} = y_\varphi + y_\psi \leq 1$, and when $b_{\varphi\oplus\psi} = 1$, $y_{\varphi\oplus\psi} = 1 \leq y_\varphi + y_\psi$. So $y_{\varphi\oplus\psi} = \min(1, y_\varphi + y_\psi)$. Negation remains a simple equality; letting $y_{\neg\varphi} = v(\neg\varphi)$, the following restriction guarantees the same value as (2):

$$
y_{\neg\varphi} = 1 - y_\varphi \tag{4}
$$

In a similar way to the disjunction, the remaining connectives can also be expressed as MILP restrictions:

$$
v(\varphi \odot \psi) = y_{\varphi\odot\psi} \qquad
\begin{aligned}
b_{\varphi\odot\psi} &\in \{0,1\} \\
0 \leq y_{\varphi\odot\psi} &\leq b_{\varphi\odot\psi} \\
y_\varphi + y_\psi - 1 \leq y_{\varphi\odot\psi} &\leq y_\varphi + y_\psi - b_{\varphi\odot\psi}
\end{aligned}
\tag{5}
$$

$$
v(\varphi \vee \psi) = y_{\varphi\vee\psi} \qquad
\begin{aligned}
b_{\varphi\vee\psi} &\in \{0,1\} \\
y_\varphi \leq y_{\varphi\vee\psi} &\leq y_\varphi + b_{\varphi\vee\psi} \\
y_\psi \leq y_{\varphi\vee\psi} &\leq y_\psi + 1 - b_{\varphi\vee\psi}
\end{aligned}
\tag{6}
$$

$$v(\varphi \wedge \psi) = y_{\varphi\wedge\psi} \qquad \begin{aligned} &b_{\varphi\wedge\psi} \in \{0,1\} \\ y_\varphi - b_{\varphi\wedge\psi} &\le y_{\varphi\wedge\psi} \le y_\varphi \\ y_\psi - (1 - b_{\varphi\wedge\psi}) &\le y_{\varphi\wedge\psi} \le y_\psi \end{aligned} \tag{7}$$

$$v(\varphi \to \psi) = y_{\varphi\to\psi} \qquad \begin{aligned} &b_{\varphi\to\psi} \in \{0,1\} \\ b_{\varphi\to\psi} &\le y_{\varphi\to\psi} \le 1 \\ 1 - y_\varphi + y_\psi - b_{\varphi\to\psi} &\le y_{\varphi\to\psi} \le 1 - y_\varphi + y_\psi \end{aligned} \tag{8}$$

$$v(\varphi \leftrightarrow \psi) = y_{\varphi\leftrightarrow\psi} \qquad \begin{aligned} &b_{\varphi\leftrightarrow\psi} \in \{0,1\} \\ 1 - y_\varphi + y_\psi - 2b_{\varphi\leftrightarrow\psi} &\le y_{\varphi\leftrightarrow\psi} \le 1 - y_\varphi + y_\psi \\ -1 + y_\varphi - y_\psi + 2b_{\varphi\leftrightarrow\psi} &\le y_{\varphi\leftrightarrow\psi} \le 1 + y_\varphi - y_\psi \end{aligned} \tag{9}$$

As an example, evaluating formula $p \oplus \neg p$ under valuation v is the same as evaluating $y_{p\oplus\neg p}$ constrained to:

$$\begin{aligned} y_p &= v(p) \\ y_{\neg p} &= 1 - y_p \\ b_{p\oplus\neg p} &\in \{0,1\} \\ b_{p\oplus\neg p} &\le y_{p\oplus\neg p} \le 1 \\ y_p + y_{\neg p} - b_{p\oplus\neg p} &\le y_{p\oplus\neg p} \le y_p + y_{\neg p} \end{aligned}$$

For any value of y_p, we have that $y_p + y_{\neg p} = 1$. Then, for both possibilities $b_{p\oplus\neg p} = 0$ and $b_{p\oplus\neg p} = 1$, we have $y_{p\oplus\neg p} = 1$ by the last two lines in MILP restrictions above. So, $p \oplus \neg p$ is a 1-tautology.

From such semantic expression, one may reduce, for instance, the problem of α-satisfiability to MILP by building a set of MILP restrictions according to the construction of a formula $\varphi \in \mathcal{L}$. For the atoms $p_1, \ldots, p_k$ of φ, add the restrictions $0 \le y_{p_i} \le 1$, for $i = 1, \ldots, k$. Then, for each subformula in the inductive construction of φ, add the restrictions given by Equations (3)–(9). Finally, add the restriction $y_\varphi = \alpha$. If the resulting set of MILP restrictions is feasible, φ is α-satisfiable; otherwise, it is α-unsatisfiable.

Therefore, deciding about the α-satisfiability of formula $p \oplus \neg p$ is equivalent to deciding about the feasibility of the following MILP restrictions:

$$\begin{aligned} 0 \le y_p &\le 1 \\ y_{\neg p} &= 1 - y_p \\ b_{p\oplus\neg p} &\in \{0,1\} \\ b_{p\oplus\neg p} &\le y_{p\oplus\neg p} \le 1 \\ y_p + y_{\neg p} - b_{p\oplus\neg p} &\le y_{p\oplus\neg p} \le y_p + y_{\neg p} \\ y_{p\oplus\neg p} &= \alpha \end{aligned}$$

Note that besides being 1-satisfiable, if $p \oplus \neg p$ is α-satisfiable, then we must have $\alpha = 1$.

2 Polyhedral semantics

Based on the MILP presentation of the connectives, we propose a new form of semantics. In this case, we do not deal with truth-functional valuations, but with interpretations that map a formula $\varphi \in \mathcal{L}$ to sets of linear restrictions containing a distinguished variable y_φ, which takes values in a subinterval of $[0, 1]$. Such linear restrictions describe polyhedra in $\mathbb{R}^n$, for some n.

In order to obtain the new semantics, we first relax the integral conditions in Equations (3)–(9) into continuous intervals. As a result of this relaxation, the binary value b now becomes a continuous value $\beta \in [0, 1]$, which we call *relaxed variable*. Thus, each connective projects a set of linear inequalities describing a polyhedron:

$$J(\varphi \oplus \psi) \qquad \begin{aligned} 0 \leq \beta_{\varphi\oplus\psi} &\leq 1 \\ \beta_{\varphi\oplus\psi} \leq y_{\varphi\oplus\psi} &\leq 1 \\ y_\varphi + y_\psi - \beta_{\varphi\oplus\psi} \leq y_{\varphi\oplus\psi} &\leq y_\varphi + y_\psi \end{aligned} \tag{10}$$

$$J(\neg\varphi) \qquad y_{\neg\varphi} = 1 - y_\varphi \tag{11}$$

$$J(\varphi \odot \psi) \qquad \begin{aligned} 0 \leq \beta_{\varphi\odot\psi} &\leq 1 \\ 0 \leq y_{\varphi\odot\psi} &\leq \beta_{\varphi\odot\psi} \\ y_\varphi + y_\psi - 1 \leq y_{\varphi\odot\psi} &\leq y_\varphi + y_\psi - \beta_{\varphi\odot\psi} \end{aligned} \tag{12}$$

$$J(\varphi \vee \psi) \qquad \begin{aligned} 0 \leq \beta_{\varphi\vee\psi} &\leq 1 \\ y_\varphi \leq y_{\varphi\vee\psi} &\leq y_\varphi + \beta_{\varphi\vee\psi} \\ y_\psi \leq y_{\varphi\vee\psi} &\leq y_\psi + 1 - \beta_{\varphi\vee\psi} \end{aligned} \tag{13}$$

$$J(\varphi \wedge \psi) \qquad \begin{aligned} 0 \leq \beta_{\varphi\wedge\psi} &\leq 1 \\ y_\varphi - \beta_{\varphi\wedge\psi} \leq y_{\varphi\wedge\psi} &\leq y_\varphi \\ y_\psi - (1 - \beta_{\varphi\wedge\psi}) \leq y_{\varphi\wedge\psi} &\leq y_\psi \end{aligned} \tag{14}$$

$$J(\varphi \to \psi) \qquad \begin{aligned} 0 \leq \beta_{\varphi\to\psi} &\leq 1 \\ \beta_{\varphi\to\psi} \leq y_{\varphi\to\psi} &\leq 1 \\ 1 - y_\varphi + y_\psi - \beta_{\varphi\to\psi} \leq y_{\varphi\to\psi} &\leq 1 - y_\varphi + y_\psi \end{aligned} \tag{15}$$

$$J(\varphi \leftrightarrow \psi) \qquad \begin{aligned} 0 \leq \beta_{\varphi\leftrightarrow\psi} &\leq 1 \\ 1 - y_\varphi + y_\psi - 2\beta_{\varphi\leftrightarrow\psi} \leq y_{\varphi\leftrightarrow\psi} &\leq 1 - y_\varphi + y_\psi \\ -1 + y_\varphi - y_\psi + 2\beta_{\varphi\leftrightarrow\psi} \leq y_{\varphi\leftrightarrow\psi} &\leq 1 + y_\varphi - y_\psi \end{aligned} \tag{16}$$

In detail, given a valuation v, a *polyhedral interpretation* L_v is a map from $\mathcal{L}$ to the power set of the set of linear restrictions inductively defined as follows. For each propositional symbol p, we have

$L_v(p) = \{y_p = v(p)\}$. Then, for formulas $\varphi, \psi \in \mathcal{L}$ and $\diamond \in \{\oplus, \odot, \vee, \wedge, \rightarrow, \leftrightarrow\}$:

$$L_v(\neg\varphi) = J(\neg\varphi) \cup L_v(\varphi);$$

$$L_v(\varphi \diamond \psi) = J(\varphi \diamond \psi) \cup L_v(\varphi) \cup L_v(\psi).$$

For a formula $\varphi \in \mathcal{L}$, a polyhedral interpretation $L_v(\varphi)$ determines a polyhedron in $\mathbb{R}^n$, with $n = s+c$, where s is the number of distinct subformulas of φ (including φ itself) and c is the number of connectives $\oplus$, $\odot$, $\vee$, $\wedge$, $\rightarrow$ and $\leftrightarrow$ occurring in φ. From now on, we freely identify a set of linear restrictions $L_v(\varphi)$ with the polyhedron it determines.

Now, given a polyhedral interpretation L, we define an *interval-valuation* I_L, for formulas $\varphi \in \mathcal{L}$, by

$$I_L(\varphi) = \left\{ y_\varphi \;\middle|\; \langle \dots, y_\varphi, \dots \rangle \in L(\varphi) \right\} \subseteq [0, 1].$$

Note that $I_L(\varphi)$ is the projection of $L(\varphi)$ to the y_φ-axis. We define *polyhedral semantics* comprehending the interval-valuations I_L yielded by interpretations L that map formulas in $\mathcal{L}$ to polyhedra. Let us denote by $\text{Ł}_\infty^{\mathcal{P}}$ the the multivalued logic system that has language $\mathcal{L}$ and is evaluated with the polyhedral semantics.

LEMMA 2.1
Let $p, q \in \mathbb{P}$ and $v \in \mathbf{Val}$. Then,

$$I_{L_v}(p \oplus q) = \left[\frac{v(p) + v(q)}{2},\ v(p \oplus q) \right] \text{ and}$$

$$I_{L_v}(p \odot q) = \left[v(p \odot q),\ \frac{v(p) + v(q)}{2} \right].$$

PROOF. For $v \in \mathbf{Val}$, we have

$$L_v(p \oplus q) = \left\{ \begin{array}{rl} y_p & = v(p),\ y_q = v(q), \\ 0 & \leq \beta_{p\oplus q} \leq 1, \\ \beta_{p\oplus q} & \leq y_{p\oplus q} \leq 1, \\ y_p + y_q - \beta_{p\oplus q} & \leq y_{p\oplus q} \leq y_p + y_q \end{array} \right\}.$$

Taking $\beta_{p\oplus q} = \frac{v(p)+v(q)}{2}$, we have $y_{p\oplus q} = \beta_{p\oplus q}$, which is the least possible value for $y_{p\oplus q}$. In fact, suppose $y_{p\oplus q} < \frac{v(p)+v(q)}{2}$; then, we would have $\beta_{p\oplus q} < \frac{v(p)+v(q)}{2}$, that yields $y_{p\oplus q} \geq y_p + y_q - \beta_{p\oplus q} > \frac{v(p)+v(q)}{2}$ and contradicts the original supposition. Also, by the inequalities in $L_v(p \oplus q)$, we have that $y_{p\oplus q} \leq \min\{1, v(p) + v(q)\} = v(p \oplus q)$, which is the maximum possible value for $y_{p\oplus q}$. The computation of interval limits for $I_{L_v}(p \odot q)$ is similar. □

Within such context of polyhedral semantics, we say that a formula $\varphi \in \mathcal{L}$ is $\alpha_{\mathcal{P}}$*-satisfiable* if there exits a polyhedral interpretation L such that $\alpha \in I_L(\varphi)$; otherwise it is $\alpha_{\mathcal{P}}$*-unsatisfiable*. Similarly, φ is $\alpha_{\mathcal{P}}^+$*-satisfiable* if there exists a polyhedral interpretation L such that $\alpha' \in I_L(\varphi)$, for some $\alpha' > \alpha$; otherwise it is $\alpha_{\mathcal{P}}^+$*-unsatisfiable*. Moreover, we say that a set of formulas $\Phi \subseteq \mathcal{L}$ is $\alpha_{\mathcal{P}}$*-satisfiable* if there is a valuation v such that $\alpha \in I_{L_v}(\varphi)$, for all $\varphi \in \Phi$. And Φ is α^+*-satisfiable* if there is a valuation v such that, for each $\varphi \in \Phi$, $\alpha'_\varphi \in I_{L_v}(\varphi)$, for some $\alpha'_\varphi > \alpha$. An immediate consequence of the relaxation in polyhedral semantics is that the decision for $\alpha_{\mathcal{P}}$-satisfiability and $\alpha_{\mathcal{P}}^+$-satisfiability becomes tractable, for instead of MILP problems we now have reductions to linear programming problems, which can be solved in polynomial time [11].

THEOREM 2.2
There are polynomial time decision procedures for the satisfiability problems in $Ł_\infty^{\mathcal{P}}$.

PROOF. Let $\Phi \subseteq \mathcal{L}$. Based on polyhedral interpretations, we inductively build from Φ a set of linear restrictions Λ by means of Equations (10)–(16). However, instead of equalities to constants $y_p = \kappa$, we add to Λ restrictions $0 \leq y_p \leq 1$, for all propositional symbol p occurring in formulas in Φ. Therefore, Φ is $\alpha_{\mathcal{P}}$-satisfiable if, and only if,

$$\Lambda \cup \left\{ y_\varphi = \alpha \;\middle|\; \varphi \in \Phi \right\} \tag{17}$$

is feasible. A polynomial procedure for solving $\alpha_{\mathcal{P}}$-satisfiability follows from the tractability of linear programming feasibility decision. A routine for solving $\alpha^+_{\mathcal{P}}$-satisfiability is achieved by, first, building set Λ as above for $\psi = \bigwedge \Phi$ and, then, maximizing y_ψ subject to Λ. In this way, Φ is α^+-satisfiable if, and only if, the maximum possible value for y_ψ is greater than α. Such routine is also polynomial. □

In the following, we investigate some properties of $Ł_\infty^{\mathcal{P}}$. Let us say that $\varphi, \psi \in \mathcal{L}$ are *equivalent*, denoting by $\varphi \equiv \psi$, if $I_L(\varphi) = I_L(\psi)$, for all polyhedral interpretation L. We show some equivalences from $Ł_\infty$ that are still valid in $Ł_\infty^{\mathcal{P}}$ and some that are not.

THEOREM 2.3
The following logical equivalences are valid in $Ł_\infty^{\mathcal{P}}$:

(i) $\varphi \odot \psi \equiv \neg(\neg\varphi \oplus \neg\psi)$;
(ii) $\varphi \oplus \psi \equiv \psi \oplus \varphi$;
(iii) $\varphi \odot \psi \equiv \psi \odot \varphi$.

PROOF.

(i) For an interpretation L, we have that $I_L(\varphi \odot \psi) \subseteq I_L(\neg(\neg\varphi \oplus \neg\psi))$ because each point in $L(\varphi \odot \psi)$ corresponds to a point in $L(\neg(\neg\varphi \oplus \neg\psi))$ such that $y_{\varphi\odot\psi} = y_{\neg(\neg\varphi\oplus\neg\psi)}$ by maintaining the values assigned to the restriction variables in $L(\varphi)$ and $L(\psi)$, obeying the negation restrictions as (11) and making:

$$\beta_{\neg\varphi\oplus\neg\psi} = 1 - \beta_{\varphi\odot\psi};$$

$$y_{\neg\varphi\oplus\neg\psi} = 1 - y_{\varphi\odot\psi}.$$

The converse set inclusion is analogous.
(ii) By a similar argument to (i) with $\beta_{\varphi\oplus\psi} = \beta_{\psi\oplus\varphi}$ and $y_{\varphi\oplus\psi} = y_{\psi\oplus\varphi}$.
(iii) By a similar argument to (i) with $\beta_{\varphi\odot\psi} = \beta_{\psi\odot\varphi}$ and $y_{\varphi\odot\psi} = y_{\psi\odot\varphi}$. □

THEOREM 2.4
The disjunction and conjunction operators are not associative in $Ł_\infty^{\mathcal{P}}$. That is, in general, $(\varphi \oplus \psi) \oplus \eta$ is not equivalent to $\varphi \oplus (\psi \oplus \eta)$ and $(\varphi \odot \psi) \odot \eta$ is not equivalent to $\varphi \odot (\psi \odot \eta)$.

PROOF. For an interpretation L that assigns to propositional symbols p, q and r the restrictions $y_p = 0.6$, $y_q = 0.5$ and $y_r = 0.4$, respectively, we have that $0.475 \in I_L((p \oplus q) \oplus r)$. However, the minimum value in $I_L(p \oplus (q \oplus r))$ is 0.525. On the other hand, we have that $0.525 \in I_L(p \odot (q \odot r))$, but the maximum value in $I_L((p \odot q) \odot r)$ is 0.475. □

A *t-norm* is a function $T : [0,1] \times [0,1] \to [0,1]$ that satisfies the properties of commutativity (i.e., $T(a,b) = T(b,a)$), associativity (i.e., $T(a, T(b,c)) = T(T(a,b),c)$), monotonicity (i.e., $T(a,b) \leq$

$T(c,d)$, if $a \le c$ and $b \le d$) and the value 1 acts as an identity (i.e., $T(a,1) = a$). An *s-norm* is a function $S : [0,1] \times [0,1] \to [0,1]$ that satisfies the properties of commutativity, associativity, monotonicity and the value 0 acts as an identity. In $Ł_\infty$, the semantics of the conjunction operator is a t-norm and the semantics of the disjunction operator is an s-norm.

In order to investigate properties of t-norm and s-norm in relation to the operators of $Ł_\infty^{\mathcal{P}}$, such properties should be adapted to fit polyhedral semantics. For instance, we might say that disjunction in $Ł_\infty^{\mathcal{P}}$ is commutative because, by the equivalence (ii) in Theorem 2.3, we have that $I_{L_v}(p \oplus q) = I_{L_v}(q \oplus p)$, for every propositional symbols $p, q \in \mathbb{P}$ and valuation $v \in \mathbf{Val}$. An analogous version of commutativity holds for conjunction in $Ł_\infty^{\mathcal{P}}$ also by Theorem 2.3. On the other hand, by Theorem 2.4, equivalences standing for associativity of conjunction and disjunction in $Ł_\infty^{\mathcal{P}}$ do not hold. The following results have versions of monotonicity and identity element properties for conjunction and disjunction in $Ł_\infty^{\mathcal{P}}$.

THEOREM 2.5
Let $p_1, p_2, q_1, q_2 \in \mathbb{P}$ and $v \in \mathbf{Val}$ be such that $v(p_1) \le v(p_2)$ and $v(q_1) \le v(q_2)$. If $I_{L_v}(p_1 \oplus q_1) = [l, r]$ and $I_{L_v}(p_2 \oplus q_2) = [L, R]$, then $l \le L$ and $r \le R$. For the conjunction operator ($\odot$), the result is analogous.

PROOF. Immediately from Lemma 2.1. □

THEOREM 2.6
Let $p \in \mathbb{P}$ and $v \in \mathbf{Val}$. Then,

- $v(p) \in I_{L_v}(p \oplus \mathbf{0})$; and
- $v(p) \in I_{L_v}(p \odot \mathbf{1})$.

PROOF. Also immediately from Lemma 2.1. □

We finish this section by investigating differential properties of some connectives according to the polyhedral semantics.

Let $\mathcal{CI}$ denote the set of closed subintervals of [0, 1]. Then, fixing propositional symbols p and q and letting a valuation v be such that $v(p) = x$ and $v(q) = y$, we define functions $f_\oplus, f_\odot : [0,1]^2 \to \mathcal{CI}$ by:

$$f_\oplus(x,y) = I_{L_v}(p \oplus q) \qquad \text{and} \qquad f_\odot(x,y) = I_{L_v}(p \odot q).$$

For $\diamond \in \{\oplus, \odot\}$, we denote by $\min f_\diamond(x,y)$ and $\max f_\diamond(x,y)$ the left and right extremes of the interval $f_\diamond(x,y)$, respectively.

THEOREM 2.7
The following differential properties for disjunction and conjunction in $Ł_\infty^{\mathcal{P}}$ hold:

$$\frac{\partial \min f_\oplus}{\partial x}(x,y) = \frac{\partial \min f_\oplus}{\partial y}(x,y) = \frac{1}{2}$$

$$\frac{\partial \max f_\oplus}{\partial x}(x,y) = \frac{\partial \max f_\oplus}{\partial y}(x,y) = \begin{cases} 1, & x+y<1 \\ 0, & x+y>1 \end{cases}$$

$$\frac{\partial \min f_\odot}{\partial x}(x,y) = \frac{\partial \min f_\odot}{\partial y}(x,y) = \begin{cases} 0, & x+y<1 \\ 1, & x+y>1 \end{cases}$$

$$\frac{\partial \max f_\odot}{\partial x}(x,y) = \frac{\partial \max f_\odot}{\partial y}(x,y) = \frac{1}{2}$$

PROOF. The derivatives directly follow from Lemma 2.1. □

3 Approximating Ł$_\infty$-inference using polyhedral semantics

The idea of *approximate entailment* has been proposed and developed both as a means of modeling the reasoning of an agent with limited resources and as a means to convey tractable reasoning to intractable systems [11]. Even if all approximate reasoning systems consist of non-classical logics, the vast majority of approximate reasoning systems have classical propositional logic as its target.

Here we propose to use a parameterised polyhedral semantics as a system that approximates non-classical multivalued logic Ł$_\infty$. The approximation occurs by forcing a few (but not necessarily all) of the the relaxed variables β to take binary values in $\{0, 1\}$ instead of continuous values in $[0, 1]$.

Suppose we are trying to verify whether a set of formulas Φ is α-unsatisfiable, i.e. we are going to show that there is no valuation v that attributes $v(\varphi) = \alpha$, for every $\varphi \in \Phi$. One possible solution is to transform Φ into a set of restrictions based on Equations (3)–(9) as in Section 1.1 and solve them with MILP programming techniques; the problem is satisfiable iff there is a solution for the set of mixed integer and continuous linear constraints. As this problem is NP-complete, we are going to try to approximate it in a tractable way.

The idea is to obtain a sequence of decision steps starting with a single set of linear restrictions obtained from Φ based on Equations (10)–(16), i.e. the set (17) in the proof of Theorem 2.2. At each step, a set of sets of tractable linear inequalities are generated such that $\alpha_{\mathcal{P}}$-unsatisfiability is obtained if all the sets are unfeasible. In that case, we are guaranteed that the formula is also α-unsatisfiable. Otherwise, a further approximation step is required. The approximation either stops when $\alpha_{\mathcal{P}}$-unsatisfiability is obtained or until an exponential number of sets of linear inequalities is generated, corresponding to the initial MILP given by transforming Φ using Equations (3)–(9) (Section 1.1), which is the only point in which it is possible to state α-satisfiability.

Let us exemplify this process with the α-unsatisfiability decision of the set of formulas $\Phi = \{p \odot \neg p, q \oplus r\}$. Let $y_p, y_q, y_r \in [0, 1]$ stand for the valuations of p, q, r, respectively; the translation of Φ into a set of (relaxed) inequalities is as follows:

$$
\begin{aligned}
0 \le y_p, y_q, y_r &\le 1 \\
y_{\neg p} &= 1 - y_p \\
0 \le y_{p\odot\neg p} &\le \beta_{p\odot\neg p} \le 1 \\
y_p + y_{\neg p} - 1 \le y_{p\odot\neg p} &\le y_p + y_{\neg p} - \beta_{p\odot\neg p} \\
0 \le \beta_{q\oplus r} &\le y_{q\oplus r} \le 1 \\
y_q + y_r - \beta_{q\oplus r} \le y_{q\oplus r} &\le y_q + y_r \\
y_{p\odot\neg p} = y_{q\oplus r} &= \alpha
\end{aligned}
\tag{18}
$$

Let R be a set of relaxed variables; initially, $R_0 = \emptyset$. If $\alpha = 1$, (18) is already unfeasible, so Φ is 1-unsatisfiable. However, for $\alpha = \frac{1}{4}$, (18) is feasible, so nothing can be stated about $\frac{1}{4}$-satisfiability of Φ (although it is $\frac{1}{4}_{\mathcal{P}}$-satisfiable).

For the second step of the approximation, we make $R_1 = \{\beta_{p\odot\neg p}\} \supset R_0$. This generates two sets of linear inequalities that expand (18), one for $\beta_{p\odot\neg p} = 0$ and one for $\beta_{p\odot\neg p} = 1$:

$$(18) \cup \{\beta_{p\odot\neg p} = 0\} \qquad\qquad (18) \cup \{\beta_{p\odot\neg p} = 1\}$$

As both those systems of inequalities are unfeasible for $\alpha = \frac{1}{4}$, it follows that $\frac{1}{4}$-unsatisfiability of Φ hold. However, had we chosen instead $R'_1 = \{\beta_{q\oplus r}\} \supset R_0$, system (18) $\cup \{\beta_{q\oplus r} = 1\}$ would be unfeasible, but (18) $\cup \{\beta_{q\oplus r} = 0\}$ would be feasible, and nothing would be implied about $\frac{1}{4}$-unsatisfiability. So we would need to proceed with the approximation to the next step:

$$R_2 = \{\beta_{p\odot\neg p}, \beta_{q\oplus r}\} \supset R'_1 \supset R_0,$$

at which point the $\frac{1}{4}$-satisfiability would be decided negatively. In fact, for the set of formulas Φ, $R_2 = R_{\text{end}}$ as it contains all the possible relaxed variables, in which case we consider all the $2^{|R_{\text{end}}|}$ $\{0,1\}$-combinations for those variables, effectively operating in $Ł_\infty$.

A totally analogous example could be made for α^+-satisfiability by suppressing the final restriction in (18) and adding:

$$0 \leq \beta_{(p\odot\neg p)\wedge(q\oplus r)} \leq 1$$

$$y_{p\odot\neg p} - \beta_{(p\odot\neg p)\wedge(q\oplus r)} \leq y_{(p\odot\neg p)\wedge(q\oplus r)} \leq y_{p\odot\neg p}$$

$$y_{q\oplus r} - (1 - \beta_{(p\odot\neg p)\wedge(q\oplus r)}) \leq y_{(p\odot\neg p)\wedge(q\oplus r)} \leq y_{q\oplus r}$$

This is the construction of Λ in Theorem 2.2 for $\bigwedge \Phi$. Then, a step of the approximation consists of maximizing $y_{(p\odot\neg p)\wedge(q\oplus r)}$ subject to each of the sets of linear constraints; if all of the maximization problems in the step are either unfeasible or have maximum at most α, set Φ is α^+-unsatisfiable.

In general, an approximation process to decide α/α^+-satisfiability for a set Φ of formulas would be parameterized by a sequence of sets of relaxed variables

$$\emptyset = R_0 \subset R_1 \subset \cdots \subset R_i \subset \cdots \subset R_{\text{end}};$$

at each step i, one has to consider 2^i relaxed linear systems. If all those systems are unfeasible/have maximum at most α, then Φ is α/α^+-unsatisfiable and the decision process terminates; otherwise, nothing can be inferred about Φ in view of $Ł_\infty$. If no unsatisfiability decision is reached, the process proceeds until R_{end} is reached and a final decision is made. This process is called a *logical approximation from below* by Finger & Wassermann [10].

LEMMA 3.1
(Soundness of $Ł_\infty$ -approximation).
Suppose we are at step i in a logical approximation from below for α-satisfiability of a set of formulas Φ, such that all the 2^i relaxed linear systems are unfeasible. Then, Φ is α-unsatisfiable. Similarly for α^+-unsatisfiability.

PROOF. If Φ were α-satisfiable, then some of the sets of constraints in step i would be feasible in such a way that its relaxed variables solutions were in $\{0,1\}$. As an unfeasible set of constraints remains unfeasible if extra restrictions are added, Φ is necessarily α-unsatisfiable in case all 2^i linear systems in step i are unfeasible. For α^+-unsatisfiability, note that the solution of a linear maximization problem may only decrease in the case of adding extra restrictions. Thus, if all such problems in step i either have maximum at most α or are unfeasible, Φ must be α^+-unsatisfiable. □

Let $Ł^{\mathcal{P}}_\infty(\Psi)$, where $\Psi \subseteq \mathcal{L}$, be a multivalued logic system with language $\mathcal{L}$ that is evaluated by an *extended polyhedral semantics* such that interpretations $L(\psi)$ also contain restriction $\beta_\psi \in \{0,1\}$, for every $\psi \in \Psi$. It is immediate that for each R_i in a sequence of relaxed variables there exists a corresponding $Ł_\infty^{\mathcal{P}}(\Psi_i)$, where

$$\emptyset = \Psi_0 \subset \Psi_1 \subset \cdots \subset \Psi_i \subset \cdots \subset \Psi_{\text{end}}.$$

Note that $Ł_\infty^{\mathcal{P}}(\Psi_0) = Ł_\infty^{\mathcal{P}}(\emptyset) = Ł_\infty^{\mathcal{P}}$. We define $\alpha_\Psi/\alpha_\Psi^+$*-satisfiability* just as $\alpha_{\mathcal{P}}/\alpha_{\mathcal{P}}^+$-satisfiability except that the extended polyhedral interpretations of $Ł_\infty^{\mathcal{P}}(\Psi)$ are considered instead of the original polyhedral interpretations.

THEOREM 3.2

Let Φ be a set of formulas such that the number of subformulas in Φ is n. Suppose approximate logic $Ł_\infty^{\mathcal{P}}(\Psi)$ is such that $|\Psi| = O(\log n)$. Then, the decision of $\alpha_\Psi/\alpha_\Psi^+$-satisfiability of Φ in $Ł_\infty^{\mathcal{P}}(\Psi)$ has complexity time polynomial in n.

PROOF. The decision of $\alpha_\Psi/\alpha_\Psi^+$-satisfiability may be done by the method described in this section, but only the relaxed variables β_ψ, for $\psi \in \Psi$, may be chosen to be fixed. As $|\Psi| = O(\log n)$, there are $2^{O(\log n)}$ sets of constraints in the last step of the approximation, each of which may be solved in polynomial time. Then, the whole decision may be achieved in polynomial time. □

So, in an approximation process for $Ł_\infty$ grounded on $Ł_\infty^{\mathcal{P}}(\Psi)$, the decision is tractable while $|\Psi| = O(\log n)$. However, only an undecidable verdict is transferred to $Ł_\infty$. Also note that each approximation step can be *incremental*, in the sense that if for some attribution of $0/1$ values to elements of R_i leads to an unfeasible system, there is no need to explore those configurations at subsequent $R_j, j > i$, as those will also be unsatisfiable. Thus, the closed paths in R_i remain closed in R_j.

Conversely, for fast α/α^+-satisfiability, an appropriate $\{0, 1\}$-valuation can be guessed for the relaxed variables in R_{end} such that, in polynomial time, the corresponding set of constraints can be shown to be feasible. A *logical approximation process from above* for $Ł_\infty$ can be obtained by a generate-and-test of valuations for relaxed values; a single feasible system would imply Φ to be α/α^+-satisfiable in $Ł_\infty$.

4 Fixing relaxed variables

The approximation satisfiability algorithm devised in last section is structured as the procedure APPROX-$Ł_\infty$ in Algorithm 1. Besides a set of formulas $\Phi \subseteq \mathcal{L}$ and a number $\alpha \in [0, 1] \cap \mathbb{Q}$, indicating that APPROX-$Ł_\infty$ is to decide about the $\alpha_{\mathcal{P}}$-satisfiability of Φ, the procedure also takes a natural number N as input. Such N must be at most the maximum number of subformulas of a formula in Φ and it indicates how far the procedure may go in approximating its decision, i.e. how many relaxed variables may be fixed.

A negative decision of $\alpha_{\mathcal{P}}$-unsatisfiability by Algorithm 1 is also a decision of α-unsatisfiability. Moreover, a decision of $\alpha_{\mathcal{P}}$-satisfiability is particularly a decision of $\alpha_{\Psi_{R_N}}$-satisfiability in $Ł_\infty(\Psi_{R_N})$, where

$$\Psi_{R_N} =_{\text{def}} \left\{ \psi \in \mathcal{L} \;\middle|\; \beta_\psi \in R_N \right\};$$

R_N is the set of fixed relaxed variables in the last iteration of the approximation. Note that if N is the maximum number of subformulas of a formula in Φ, procedure APPROX-$Ł_\infty$ returns an exact decision on the α-satisfiability of Φ.

ALGORITHM 1 APPROX-$Ł_\infty$

Input: A set of formulas $\Phi \subseteq \mathcal{L}$, a number $\alpha \in [0,1] \cap \mathbb{Q}$ and a number $N \in \mathbb{N}$.
Output: **No**, if Φ is α-unsatisfiable. Or **Yes**, if Φ is $\alpha_{\mathcal{P}}$-satisfiable.

1: $\Gamma^{(0)} := \big\{ \Lambda \cup \{y_\varphi = \alpha \mid \varphi \in \Phi\} \big\};$ ▷ equation (17) computed for Φ
2: $R_0 := \emptyset;$
3: **for** $i = 0, \dots, N$ **do**
4:     **if** all sets in $\Gamma^{(i)}$ are unfeasible **then**
5:         **return No**; ▷ Φ is α-unsatisfiable
6:     **else**
7:         Choose a relaxed variable $\beta \notin R_i$; ▷ nondeterministic step
8:         $\Gamma^{(i+1)} := \emptyset;$
9:         $R_{i+1} := R_i \cup \{\beta\};$
10:         **for all** $\Gamma \in \Gamma^{(i)}$ **do**
11:             $\Gamma^{(i+1)} := \Gamma^{(i+1)} \cup \big\{ \Gamma \cup \{\beta = 0\},\ \Gamma \cup \{\beta = 1\} \big\};$
12:         **end for**
13:     **end if**
14: **end for**
15: **return Yes**; ▷ Φ is $\alpha_{\mathcal{P}}$-satisfiable

Algorithm 1 is almost completely determined except for a nondeterministic step in line 7. As we saw in last section, in the example where $\Phi = \{p \odot \neg p, q \oplus r\}$, the choice of a relaxed variable in each iteration of the algorithm may interfere with how quickly it converges to a negative decision (α-unsatisfiability). Let us resume the discussion about that example in order to propose a heuristic for choosing relaxed variables.

In the first iteration of APPROX-$Ł_\infty$ taking as input $\Phi = \{p \odot \neg p, q \oplus r\}$ and $\alpha = \frac{1}{4}$, the only element in $\Gamma^{(0)} = \{\Gamma_1^{(0)}\}$ comprehends the restrictions in (17), which are jointly feasible. Then, in order to determine $\Gamma^{(1)}$, the algorithm proceeds by choosing some relaxed variable to fix. An intuition about such choice may be drawn from the following linear programming problems:

$$\begin{array}{ll} \min & \beta_{p\odot\neg p} + \beta_{q\oplus r} \\ \text{s.t.} & \Gamma_1^{(0)} \end{array} \qquad\qquad \begin{array}{ll} \max & \beta_{p\odot\neg p} + \beta_{q\oplus r} \\ \text{s.t.} & \Gamma_1^{(0)} \end{array}$$

The minimum value of the sum of relaxed variables subject to $\Gamma_1^{(0)}$ is 0.25, which may be achieved by $\beta_{p\odot\neg p} = 0.25$ and $\beta_{q\oplus r} = 0$. And the maximum value is 1, which may be achieved by $\beta_{p\odot\neg p} = 0.75$ and $\beta_{q\oplus r} = 0.25$.

We may interpret the values for $\beta_{p\odot\neg p}$ in both problems as an indication of the impossibility of this variable reaching 0 or 1. In this way, fixing $\beta_{p\odot\neg p}$ will perhaps quickly lead to $\frac{1}{4}$-unsatisfiability, which we already know to be the case. Indeed, if we set $R_1 = \{\beta_{p\odot\neg p}\}$, both sets of restrictions in $\Gamma^{(1)}$ are unfeasible.

Had we chosen $\beta_{q\oplus r}$ to fix and set $R_1' = \{\beta_{q\oplus r}\}$, the set of restrictions $\Gamma_1^{(1)'} = \Gamma^{(0)} \cup \{\beta_{q\oplus r} = 0\}$ in $\Gamma^{(1)'}$ would be feasible and one more iteration would be needed. Although $\beta_{p\odot\neg p}$ would be the only option left among the yet unfixed relaxed variables, let us examine the following linear programming problems:

$$\begin{array}{ll} \min & \beta_{p\odot\neg p} \\ \text{s.t.} & \Gamma_1^{(1)'} \end{array} \qquad\qquad \begin{array}{ll} \max & \beta_{p\odot\neg p} \\ \text{s.t.} & \Gamma_1^{(1)'} \end{array}$$

In these problems, the minimum value of $\beta_{p\odot\neg p}$ is 0.25 and the maximum is 0.75, pointing to the impossibility of $\beta_{p\odot\neg p}$ reaching 0 or 1. Both problems generated by fixing $\beta_{p\odot\neg p}$ are indeed unfeasible, leading to the $\frac{1}{4}$-unsatisfiability of Φ.

In view of the above exposure, we now delineate a proposal of heuristic for fixing relaxed variables. In the i-th iteration of APPROX-Ł$_\infty$, if any of the sets of constraints in $\Gamma^{(i)} = \{\Gamma_1^{(i)}, \ldots, \Gamma_{2^i}^{(i)}\}$ is feasible, the algorithm will choose some of the yet unfixed relaxed variables to be fixed next.

Let $S_1, \ldots, S_k \in \Gamma^{(i)}$ be the k feasible sets of constraints, $k \leq 2^i$ and B_{unf} be the set of yet unfixed relaxed variables. The heuristic consists, first, in solving the $2k$ linear programming problems, for $j = 1, \ldots, k$, in the following:

$$\begin{array}{ll} \min & \sum B_{\text{unf}} \\ \text{s.t.} & S_j \end{array} \qquad\qquad \begin{array}{ll} \max & \sum B_{\text{unf}} \\ \text{s.t.} & S_j \end{array} \tag{19}$$

For each $\beta \in B_{\text{unf}}$, there are k values corresponding to the solutions of the minimization problems, which are, then, combined in a single value $\underline{\beta} \in [0, 1]$. Such combination might be, for instance, the average of the k values or the minimum among them. Analogously, $\overline{\beta}$ is a combination of the k values corresponding the the solutions of the maximization problems (such as their average or maximum). Remember that linear programming problems may be solved in polynomial time [12].

Finally, in an attempt to choose the relaxed variable with the strongest indication of impossibility to reach 0 or 1, the heuristic choice follows the following rule:

$$\beta_{\text{choice}} := \underset{\beta \in B_{\text{unf}}}{\arg\max} \Big\{ \min\{\underline{\beta}, \overline{\beta}\} \Big\}. \tag{20}$$

A procedure for dealing with $\alpha_{\mathcal{P}}^{+}$-satisfiability may be easily adapted from Algorithm 1. In such adaptation, the condition on line 4 is replaced by 'all linear programs for maximizing $y_{\bigwedge \Phi}$ subject to each set of restrictions in $\Gamma^{(i)}$ are either unfeasible or have value at most α'. The same heuristic just described for fixing relaxed variables in the $\alpha_{\mathcal{P}}$-satisfiability case may be used in such a $\alpha_{\mathcal{P}}^{+}$-satisfiability procedure. In addition to maximizing $y_{\bigwedge \Phi}$ subject to a set of constraints S_j, the linear programs in (19) are also solved and Equation (20) is used for choosing a relaxed variable to fix.

5 Conclusion

In this work, we introduced the polyhedral semantics to formulas of Łukasiewicz infinitely-valued logic through relaxations of variables in the widely known codification of the traditional semantics into sets of MILP-constraints. By means of the polyhedral semantics, we established the logic system Ł$_\infty^{\mathcal{P}}$, where compound formulas are interval-valued from valuation assignments to their propositional symbols.

Then, we presented properties of Ł$_\infty^{\mathcal{P}}$ as the tractability of its satisfiability problems (Theorem 2.2). We established some logical equivalences in Ł$_\infty^{\mathcal{P}}$, such as the ones that state the commutativity property for disjunction and conjunction (Theorem 2.3), and showed that associativity does not hold for these operators (Theorem 2.4). In this way, although disjunction and conjunction in Ł$_\infty^{\mathcal{P}}$ have some of the properties of s-norms and t-norms, they lack associativity. Nevertheless, we showed that such connectives comply with notions of monotonicity and identity element, which are suitable for polyhedral semantics (Theorems 2.5 and 2.6). We also established differential properties for disjunction and conjunction in Ł$_\infty^{\mathcal{P}}$ (Theorem 2.7).

Based on polyhedral semantics, we introduced an approximation procedure for satisfiability in $Ł_\infty$, where the decision in $Ł_\infty^{\mathcal{P}}$ is the loosest approximation and better approximations are iteratively achieved in a sequence of polyhedral semantics-based systems as relaxed variables are fixed. A maximum number of iterations in order that approximation satisfiability is tractable was devised (Theorem 3.2). Finally, we proposed a heuristic for choosing relaxed variables to fix in such approximation procedure.

For the future, we intend to implement the approximation procedure for satisfiability in $Ł_\infty$ and empirically analyze its performance. This investigation might ground a comparison among the heuristic for choosing relaxed variables proposed in this work and other possible approaches for such choice. Also, a deepest investigation about logic systems based on polyhedral semantics might be carried out.

Moreover, an investigation about the use of the approximation algorithm for satisfiability in $Ł_\infty$ as part of approximation procedures for other problems might also be pursued. For instance, in an approximation procedure for the Boolean maximum satisfiability problem (MaxSAT), which may be reduced both to 0^+-satisfiability [19] and 1-satisfiability in $Ł_\infty$ [23].

Acknowledgements

This work was carried out at the Center for Artificial Intelligence (C4AI-USP), with support by the São Paulo Research Foundation (FAPESP) [grant #2019/07665-4] and by the IBM Corporation. This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior–Brasil (CAPES) [Finance Code 001]; the São Paulo Research Foundation (FAPESP) [grants #2015/21880-4 and #2014/12236-1 to M.F., #2021/03117-2 to S.P.]; and the National Council for Scientific and Technological Development (CNPq) [grant PQ 303609/2018-4 to M.F.].

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Received 4 July 2023