Outer Kernel Theorem for Co-orbit Spaces of Localised Frames

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***Abstract*—We prove a so-called outer kernel theorem for bounded linear operators between co-orbit spaces generated from a localised frame Ψ. In particular, we show that there is a bijective correspondence between the bounded linear operators mapping the co-orbit space of test functions $H^1_w(\Psi)$ to the co-orbit space of distributions $H^\infty_{1/w}(\Psi)$ and their kernels in $H^{\infty,\otimes}_{1/w}(\Psi)$. The proof of the theorem relies on general properties of localised frames, tensor products and Galerkin's method for matrix represention of operators.**

I. Introduction

Kernel theorems have a long history dating back to Laurent Schwartz' characterization of operators from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$. In principle, kernel theorems state that 'any' operator O can be 'nicely' represented. In concrete function spaces, this means that it can be written as an integral operator with kernel K - in reminiscence of the matrix representation in finite dimensional spaces. In a more formal way, this principle may be written as

$$\langle Of, g\rangle = \langle K, f \otimes g\rangle, \tag{1}$$

with the definition of the brackets depending on the particular spaces involved. Over the years, kernel theorems were established for various function spaces, see, for example, [14], [7], or [3]. See also [5], [11] for further variations of the topic.

In this article, we prove the so-called outer kernel theorem (Theorem 4.1) for the case that the involved function spaces are co-orbit spaces generated by a localised frame Ψ. Leaving the technical details for later (see Sections II and III), co-orbit spaces [9], [10] are defined as those functions whose frame coefficients are contained in certain function or sequence spaces. Typical examples are modulation [8], or Besov spaces [15]. Classically, co-orbit spaces are generated by integrable representations of locally compact groups [9], [10]. In this article we consider co-orbit spaces generated by (intrinsically) localised frames, i.e. those frames whose Gramian matrices $(\langle \psi_i, \psi_j\rangle)_{i,j}$ are 'nice' [12]. One should note here that many of the classical co-orbit spaces can also be understood in terms of localised frames. Kernel theorems in the classical setting that are of similar nature than Theorem 4.1 were established in [3]. This shows that the essential ingredient to prove kernel theorems for co-orbit spaces is in fact the localisation of the frame and not the group structure.

Our proof of Theorem 4.1 relies on Galerkin's method to represent an operator O by the matrix $(\langle O\psi_i, \psi_j\rangle)_{i,j}$. There is every reason to specifically consider localised frames $\Psi = \{\psi_i\}_{i\in I}$ in the Galerkin matrix [2]. For one, localised frames allow atomic decompositions of the elements of their co-orbit spaces. For another, the Gram matrix of any localised frame is, by definition, well-behaved. Using Galerkin's method with localised frames therefore gives rise to matrices representing bounded operators mapping co-orbit spaces on one another [2]. This can be very useful for solving corresponding operator equations numerically [6].

As we shall see, the kernel of any bounded operator mapping one co-orbit space on the other can be constructed from the matrix that represents the operator. If the operator shows certain regularity, for example, if it maps distributions to test-functions, then we are guaranteed to have a well-behaved Galerkin matrix, see [2, Proposition 6]. The interpretation of this result in terms of the kernels is part of ongoing work of ours.

II. Background information and notation

In this section we provide the bare minimum of information necessary to follow the arguments of our proof in Section IV. More information on (localised) frames and co-orbit spaces can be found in [4], [12], [2], [7], [9], and [10].

We will write $f \asymp g$ if there exist constants $c, C > 0$ such that $cf \leqslant g \leqslant Cf$. A countable set $\Psi := \{\psi_i\}_{i\in I} \subset H$ is called a *frame* for the Hilbert space H if there exist constants $A_\Psi, B_\Psi > 0$ such that

$$A_\Psi \|f\|_H^2 \leqslant \sum_{i\in I} |\langle f, \psi_i\rangle|^2 \leqslant B_\Psi \|f\|_H^2, \tag{2}$$

for every $f \in H$. The *analysis operator* and *synthesis operator* are defined by

$$C_\Psi : H \to \ell^2(I), \quad f \mapsto \{\langle f, \psi_i \rangle\}_{i\in I},$$

and

$$D_\Psi : \ell^2(I) \to H, \quad \{c_i\}_{i\in I} \mapsto \sum_{i\in I} c_i\psi_i$$

respectively. Their combination $S_\Psi = D_\Psi C_\Psi$ is called the *frame operator*, and is always positive and self-adjoint. Moreover, S_Ψ is invertible if Ψ is a frame. The *canonical dual frame* $\widetilde{\Psi} := \{\widetilde{\psi_i}\}_{i\in I} = \{S_\Psi^{-1}\psi_i\}_{i\in I}$ allows the representation

$$f = \sum_{i\in I} \langle f, \psi_i \rangle \tilde{\psi}_i = \sum_{i\in I} \left\langle f, \tilde{\psi}_i \right\rangle \psi_i, \quad f \in H.$$

To introduce the concept of localised frames we first need the notion of a spectral matrix algebra.

Definition 2.1: An involutive Banach algebra $\mathcal{A}$ of infinite matrices equipped with the norm $\|\cdot\|_\mathcal{A}$ is called spectral matrix algebra *if*

(i) *any $A \in \mathcal{A}$ defines a bounded operator on $\ell^2(I)$ or, in shorthand, $\mathcal{A} \subset B(\ell^2(I))$;*
(ii) *$\mathcal{A}$ is inverse-closed in $B(\ell^2(I))$, that is if $A \in \mathcal{A}$ is invertible on $B(\ell^2(I))$, then $A^{-1} \in \mathcal{A}$; and*
(iii) *$\mathcal{A}$ is solid, that is $A \in \mathcal{A}$ and $|b_{i,j}| \leqslant |a_{i,j}|$ for any $i, j \in I$ implies that $B \in \mathcal{A}$ and $\|B\|_\mathcal{A} \leqslant \|A\|_\mathcal{A}$.*

*A weight sequence $w := \{w_i\}_{i\in I} \subset \mathbb{R}^+$ is called $\mathcal{A}$-*admissible *if every $A \in \mathcal{A}$ defines a bounded operator on $\ell^p_w(I)$ for every $1 \leqslant p \leqslant \infty$.*

Throughout the rest of this article, we will always assume that $\mathcal{A}$ is a spectral matrix algebra and that w is an $\mathcal{A}$-admissible weight.

Examples of spectral matrix algebras are given by Jaffard class, Sjöstrand class, and by Schur-type conditions, see [12].

*Definition 2.2: A frame $\Psi = \{\psi_i\}_{i\in I} \subset H$ is said to be $\mathcal{A}$-*localised *if its Gram matrix $G_\Psi := (\langle \psi_i, \psi_j \rangle)_{(i,j)\in I^2}$ belongs to the spectral matrix algebra $\mathcal{A}$.*

Let $\Psi := \{\psi_i\}_{i\in I} \subset H$ be an $\mathcal{A}$-localised frame, and $\widetilde{\Psi} := \{\widetilde{\psi_i}\}_{i\in I} \subset H$ its canonical dual. Moreover, let

$$H_{00} := \left\{ \sum_{i\in I} c_i \psi_i : \ \{c_i\}_{i\in I} \in c_{00} \right\},$$

where c_{00} denotes the space of all infinite sequences with finitely many non-zero terms. For $1 \leqslant p < \infty$, the *co-orbit space* $H^p_w(\Psi)$ is defined as the norm completion of H_{00} with respect to the norm

$$\|f\|^p_{H^p_w} := \left\|C_{\widetilde{\Psi}} f\right\|^p_{\ell^p_w(I)} = \sum_{i\in\mathcal{I}} \left|(C_{\widetilde{\Psi}} f)_i\right|^p w_i^p,$$

while $H^\infty_w(\Psi)$ is defined as a certain weak-$*$ completion of H_{00} with respect to the metric $\|C_{\widetilde{\Psi}} f\|_{\ell^\infty_w}$. For the details of the construction of $H^\infty_{1/w}(\Psi)$ we refer to [2, Section 3.1].

Proposition 2.3: Let Ψ be an $\mathcal{A}$-localised frame, and $1 \leqslant p \leqslant \infty$. Then

(i) *the co-orbit space $H^p_w(\Psi)$ is a Banach space;*
(ii) *the synthesis operator $D_\Psi : \ell^p_w(I) \to H^p_w(\Psi)$ is continuous;*
(iii) *$H^2_1(\Psi) = H$;*
(iv) *if $1 \leqslant p < \infty$ and q satisfies $1/p + 1/q = 1$, then $(H^p_w(\Psi))^* = H^q_{1/w}(\Psi)$ where the duality pairing is given by*

$$\langle f, g \rangle_{H^p_w, H^q_{1/w}} = \langle C_{\widetilde{\Psi}} f, C_{\widetilde{\Psi}} g \rangle_{\ell^p_w, \ell^q_{1/w}}.$$

Throughout the rest of this paper we will omit the subscripts whenever it does not undermine clarity of the arguments;
(v) *the canonical dual frame $\widetilde{\Psi}$ is also $\mathcal{A}$-localised, and $H^p_w(\Psi) = H^p_w(\widetilde{\Psi})$ with their norms being equivalent;*
(vi) *the Gram matrix G_Ψ defines a bounded operator on $\ell^p_w(I)$.*

The next result can be found in [12].

Proposition 2.4: For any $f \in H^1_w(\Psi)$ there is a sequence $c := \{c_i\}_{i\in I} \in \ell^1_w(I)$ such that

$$f = \sum_{i\in I} c_i \psi_i, \tag{3}$$

where the series converges absolutely and

$$\|c\|_{\ell^1_w} \asymp \|f\|_{H^1_w(\Psi)}.$$

III. Co-orbit spaces induced by tensor product frames

Let $f_1, f_2 \in H$. We define the *simple tensor* $f_1 \otimes f_2$ as the rank one operator

$$(f_1 \otimes f_2)(f) := \langle f, f_1 \rangle f_2, \quad f \in H. \tag{4}$$

The tensor product $H \otimes H$ is defined as the completion of the linear span of all simple tensors with respect to the metric induced by the inner product

$$\langle f_1 \otimes f_2, g_1 \otimes g_2 \rangle = \overline{\langle f_1, g_1 \rangle} \langle f_2, g_2 \rangle. \tag{5}$$

Observe that for $O \in H \otimes H$, and $f_1, f_2 \in H$ it holds

$$\langle O, f_1 \otimes f_2 \rangle = \langle O f_1, f_2 \rangle.$$

We note here that if Ψ is a frame for H, then

$$\Psi \otimes \Psi := \{\psi_i \otimes \psi_j\}_{(i,j)\in I^2}$$

is a frame for $H \otimes H$, see, for example, [1]. This now allows us to define co-orbit spaces generated by $\Psi \otimes \Psi$.

Definition 3.1: Let $1 \leqslant p < \infty$, and Ψ be an $\mathcal{A}$-localised frame. The co-orbit space $H^{p,\otimes}_w(\Psi)$ associated with $\Psi \otimes \Psi$ is defined as the norm-completion of

$$H^\otimes_{00} = \left\{ \sum_{i,j\in I} c_{i,j}\, \psi_i \otimes \psi_j : \ \{c_{i,j}\}_{(i,j)\in I^2} \in c_{00} \right\},$$

with respect to

$$\|F\|_{H_w^{p,\otimes}(\Psi)} := \|C_{\widetilde{\Psi\otimes\Psi}} F\|_{\ell^p_{w\otimes w}},$$

while $H_w^{\infty,\otimes}(\Psi)$ is defined as the weak-$*$ completion of $H_{00}^{\otimes}$ with respect to the metric $\|C_{\widetilde{\Psi\otimes\Psi}} F\|_{\ell^\infty_{w\otimes w}}$. The construction works the same as in the setting described in Section II

Observe that, while these spaces are well-defined, it can lead to pathological examples [13, Example 2.18]. It can, e.g., yield empty spaces. Localization guarantees that those spaces are well-behaved. Also note that, in order for the co-orbit spaces to have the properties of Propositions 2.3 and 2.4, $\Psi \otimes \Psi$ would have to be localised. This seems to be a non-trivial problem though, and is the topic of current research. Therefore, we have to derive some results "by hand", without the advantages of the localization property.

As before, we may introduce a duality pairing via

$$\langle F, G\rangle_{H_w^{p,\otimes}, H_{1/w}^{q,\otimes}} = \langle C_{\widetilde{\Psi\otimes\Psi}} F, C_{\widetilde{\Psi\otimes\Psi}} G\rangle_{\ell^p_{w\otimes w}, \ell^q_{1/(w\otimes w)}},$$

and note that one can show

$$(H_w^{p,\otimes}(\Psi))^* = H_{1/w}^{q,\otimes}(\Psi), 1 \leq p < \infty. \tag{6}$$

Let us shortly state two auxiliary results.

Proposition 3.2: Let Ψ be an $\mathcal{A}$-localised frame. Then for any $F \in H_w^{1,\otimes}(\Psi)$ there exists a sequence $c = \{c_{i,j}\}_{(i,j)\in I^2} \in \ell^1_{w\otimes w}(I^2)$ such that

$$F = \sum_{i\in I}\sum_{j\in I} c_{i,j}\, \psi_i \otimes \psi_j = D_{\Psi\otimes\Psi} c \tag{7}$$

converges absolutely and $\|c\|_{\ell^1_{w\otimes w}} \asymp \|F\|_{H_w^{1,\otimes}(\Psi)}$.

The proof of this Proposition follows closely the arguments of the proof of [12, Proposition 2.4] and that of [2, Lemma 6]; and will be provided elsewhere.

The next lemma is a version of the so-called *correspondence principle* in the setting of the co-orbit spaces $H_w^{p,\otimes}(\Psi)$ generated by tensor products of frames. Its proof is similar to the corresponding result for $H_w^p(\Psi)$ and will therefore be omitted.

Lemma 3.3: Let Ψ be an $\mathcal{A}$-localised frame. For $M \in \ell^\infty_{w\otimes w}(I^2)$ there exists $F \in H_w^{\infty,\otimes}(\Psi)$ such that

$$M = C_{\widetilde{\Psi\otimes\Psi}} F \tag{8}$$

if and only if

$$M = G_{\widetilde{\Psi\otimes\Psi},\,\Psi\otimes\Psi}\, M = C_{\widetilde{\Psi\otimes\Psi}} D_{\Psi\otimes\Psi}\, M. \tag{9}$$

The statement also applies to $M \in \ell^\infty_{1/(w\otimes w)}\left(I^2\right)$ and $F \in H_{1/w}^{\infty,\otimes}(\Psi)$.

IV. Outer kernel theorem

With all notation and auxiliary results in place, we are now ready to state and prove our main result.

Theorem 4.1: Let $\Psi \subset H$ be an $\mathcal{A}$-localised frame. Then, to any $K \in H_{1/w}^{\infty,\otimes}(\Psi)$ corresponds a unique bounded linear $O : H_w^1(\Psi) \to H_{1/w}^\infty(\Psi)$ via

$$\langle O f_1, f_2\rangle = \langle K, f_1 \otimes f_2\rangle, \quad f_1,\, f_2 \in H_w^1 \tag{10}$$

and

$$\|K\|_{H_{1/w}^{\infty,\otimes}(\Psi)} \asymp \|O\|_{H_w^1(\Psi)\to H_{1/w}^\infty(\Psi)}\,. \tag{11}$$

Conversely, to any bounded operator $O : H_w^1(\Psi) \to H_{1/w}^\infty(\Psi)$ corresponds a unique $K \in H_{1/w}^{\infty,\otimes}(\Psi)$ that satisfies (10).

Remark 4.2: Note that for simplicity we restrict ourselves to the case when both co-orbit spaces are generated by the same localized frame and the same weight. The statement of Theorem 4.1 can however be easily adjusted for $O : H_{w_1}^1(\Psi_1) \to H_{1/w_2}^\infty(\Psi_2)$.

Proof of Theorem 4.1. First of all, we note that the scalar product on the right hand side of (10) is a duality relation by definition and (6). The fact that $C_{\widetilde{\Psi\otimes\Psi}}(f_1 \otimes f_2) = \overline{C_{\tilde\Psi} f_1}\, C_{\tilde\Psi} f_2$, implies by (6) that

$$|\langle K, f_1 \otimes f_2\rangle| \leqslant \|K\|_{H_{1/w}^{\infty,\otimes}(\Psi)}\, \|f_1\|_{H_w^1(\Psi)} \|f_2\|_{H_w^1(\Psi)}.$$

Secondly, $(f_1, f_2) \mapsto \langle K, f_1 \otimes f_2\rangle$ is a bounded sesquilinear form on $H_w^1(\Psi) \times H_w^1(\Psi)$ and therefore, for any fixed $f_1 \in H_w^1(\Psi)$, the map $f_2 \mapsto \langle K, f_1 \otimes f_2\rangle$ is a bounded antilinear functional on $H_w^1(\Psi)$, which we call $O f_1$. The mapping $f_1 \mapsto O f_1$ is linear and so (10) defines a linear operator O mapping $H_w^1(\Psi)$ on $H_{1/w}^\infty(\Psi)$. This operator is bounded. Indeed,

$$\begin{aligned}\left|\langle O f_1, f_2\rangle\right| &= |\langle K, f_1 \otimes f_2\rangle| \\ &\leqslant \|K\|_{H_{1/w}^{\infty,\otimes}(\Psi)}\, \|f_2\|_{H_w^1(\Psi)}\, \|f_1\|_{H_w^1(\Psi)}\,,\end{aligned}$$

and consequently

$$\|O f_1\|_{H_{1/w}^\infty(\Psi)} \leqslant \|K\|_{H_{1/w}^{\infty,\otimes}(\Psi)}\, \|f_1\|_{H_w^1(\Psi)}\,,$$

as well as

$$\|O\|_{H_w^1(\Psi)\to H_{1/w}^\infty(\Psi)} \leqslant \|K\|_{H_{1/w}^{\infty,\otimes}(\Psi)}\,. \tag{12}$$

This also proves that the map $K \mapsto O$ is bounded.

For the first direction, it is left to show that this mapping is injective. So let us assume the contrary, i.e. that the bounded linear operator O mapping $H_w^1(\Psi)$ on $H_{1/w}^\infty(\Psi)$ has two distinct kernels K_1 and $K_2 \in H_{1/w}^{\infty,\otimes}(\Psi)$ both satisfying (10). Then, in particular,

$$\left\langle K_1, \psi_i \otimes \psi_j\right\rangle = \left\langle O\psi_i, \psi_j\right\rangle = \left\langle K_2, \psi_i \otimes \psi_j\right\rangle,$$

for any ψ_i and $\psi_j \in \Psi$. By Proposition 3.2, any $F \in H_w^{1,\otimes}(\Psi)$ can be expressed by an absolutely convergent series as in (7). Hence we infer that

$$\begin{aligned}\langle K_1, F\rangle &= \Big\langle K_1\,, \sum_{i\in I}\sum_{j\in I} c_{i,j}\,\psi_i\otimes\psi_j\Big\rangle\\ &= \sum_{i\in I}\sum_{j\in I} c_{i,j}\,\langle K_1, \psi_i\otimes\psi_j\rangle\\ &= \sum_{i\in I}\sum_{j\in I} c_{i,j}\langle K_2, \psi_i\otimes\psi_j\rangle\\ &= \Big\langle K_2, \sum_{i\in I}\sum_{j\in I} c_{i,j}\,\psi_i\otimes\psi_j\Big\rangle = \langle K_2, F\rangle,\end{aligned}$$

and so $K_1 = K_2$, by (6).

To prove the second statement, let us assume that the operator $O : H_w^1(\Psi) \to H_{1/w}^\infty(\Psi)$ is bounded and consider its Galerkin matrix M defined by

$$M_{i,j} := \langle O\widetilde{\psi}_i, \widetilde{\psi}_j\rangle, \text{ for } (i,j) \in I^2. \qquad (13)$$

By [2, Prop. 6 (23)] $M \in \ell^\infty_{1/(w\otimes w)}(I^2)$. We will derive a concrete bound. By Proposition 2.3, the Gram matrix $G_{\widetilde{\Psi}}$ is a bounded operator on $\ell_w^1(I)$ as Ψ is $\mathcal{A}$-localised and w is $\mathcal{A}$-admissible. Therefore

$$\begin{aligned}\|\widetilde{\psi}_i\|_{H_w^1(\Psi)} &= \sum_{k\in I}\big|\langle\widetilde{\psi}_i, \widetilde{\psi}_k\rangle\big|\, w_k = \sum_{k\in I}\big|(G_{\widetilde{\Psi}})_{k,i}\big|\, w_k\\ &= \sum_{k\in I}\big|(G_{\widetilde{\Psi}} e_i)_k\big|\, w_k = \|G_{\widetilde{\Psi}} e_i\|_{\ell_w^1} \lesssim \|e_i\|_{\ell_w^1} = w_i.\end{aligned}$$

where e_i denotes the i-th vector of the standard basis of $\ell_w^1(I)$. Combining this estimate with (13) shows

$$\begin{aligned}|M_{i,j}| &\leqslant \|O\|_{H_w^1(\Psi)\to H_{1/w}^\infty(\Psi)}\|\widetilde{\psi}_i\|_{H_w^1}\,\|\widetilde{\psi}_j\|_{H_w^1}\\ &\lesssim \|O\|_{H_w^1(\Psi)\to H_{1/w}^\infty(\Psi)}\;\omega_i\,\omega_j. \qquad (14)\end{aligned}$$

In what follows, we show that M satisfies (9), which, according to Lemma 3.3, implies that there is a unique $K \in H_{1/w}^{\infty,\otimes}(\Psi)$ that satisfies $C_{\widetilde{\Psi\otimes\Psi}} K = M$. Indeed, for any $(i,j) \in I^2$,

$$\begin{aligned}&\Big(C_{\widetilde{\Psi\otimes\Psi}} D_{\Psi\otimes\Psi} M\Big)_{r,s} =\\ &= \sum_{i\in I}\sum_{j\in I}\langle O\widetilde{\psi}_i, \widetilde{\psi}_j\rangle\,\langle\psi_i\otimes\psi_j, \widetilde{\psi}_r\otimes\widetilde{\psi}_s\rangle\\ &= \sum_{i\in I}\sum_{j\in I}\langle O\widetilde{\psi}_i, \widetilde{\psi}_j\rangle\,\overline{\langle\psi_i, \widetilde{\psi}_r\rangle}\,\langle\psi_j, \widetilde{\psi}_s\rangle\\ &= \sum_{i\in I}\langle O\widetilde{\psi}_i, \widetilde{\psi}_s\rangle\,\overline{\langle\psi_i, \widetilde{\psi}_r\rangle} \qquad (15)\\ &= \sum_{i\in I}\langle\widetilde{\psi}_i, O'\widetilde{\psi}_s\rangle\,\langle\widetilde{\psi}_r, \psi_i\rangle\\ &= \langle\widetilde{\psi}_r, O'\widetilde{\psi}_s\rangle = \langle O\widetilde{\psi}_r, \widetilde{\psi}_s\rangle = M_{r,s},\end{aligned}$$

where $O' : H_w^1(\Psi) \to H_{1/w}^\infty(\Psi)$ is the unique operator that satitsfies $\langle Of, g\rangle = \langle f, O'g\rangle$ for every $f, g \in H_w^1(\Psi)$. Note that (15) is independent of the order chosen because

$$\big\{\langle O\widetilde{\psi}_i, \widetilde{\psi}_j\rangle\,\langle\widetilde{\psi}_r, \psi_i\rangle\,\langle\psi_j, \widetilde{\psi}_s\rangle\big\}_{(i,j)\in I^2}$$

is in $\ell^1(I^2)$ for any given $(r,s) \in I^2$ as

$$\big\{\langle O\widetilde{\psi}_i, \widetilde{\psi}_j\rangle\big\}_{(i,j)\in I^2} \in \ell^\infty_{1/(w\otimes w)}(I^2),$$

and $\big\{\langle\widetilde{\psi}_r, \psi_i\rangle\,\langle\psi_j, \widetilde{\psi}_s\rangle\big\}_{(i,j)\in I^2} \in \ell^1_{w\otimes w}(I^2)$, by Proposition 2.3 and Definition 2.1. Moreover, (14) implies that

$$\begin{aligned}\|K\|_{H_{1/w}^{\infty,\otimes}(\Psi)} &= \|C_{\widetilde{\Psi\otimes\Psi}} K\|_{\ell^\infty_{1/(w\otimes w)}} = \|M\|_{\ell^\infty_{1/(w\otimes w)}}\\ &\lesssim \|O\|_{H_w^1(\Psi)\to H_{1/w}^\infty(\Psi)}. \qquad (16)\end{aligned}$$

According to the first direction, to this unique $K \in H_{1/w}^{\infty,\otimes}(\Psi)$ corresponds a bounded linear operator mapping $H_w^1(\Psi)$ on $H_{1/w}^\infty(\Psi)$ via (10). This operator is the bounded linear operator O. Indeed let us assume that to the kernel $K \in H_{1/w}^{\infty,\otimes}(\Psi)$ corresponds via (10) a linear operator $\widehat{O}$ mapping $H_w^1(\Psi)$ on $H_{1/w}^\infty(\Psi)$ that is different from O. By (13) and $C_{\Psi\otimes\Psi} K = M$ we get

$$\langle\widehat{O}\widetilde{\psi}_i, \widetilde{\psi}_j\rangle = \langle K, \widetilde{\psi}_i\otimes\widetilde{\psi}_j\rangle = \langle O\widetilde{\psi}_i, \widetilde{\psi}_j\rangle,$$

for any $\psi_i\otimes\psi_j \in H_w^{1,\otimes}(\Psi)$. From this and the fact that, according to Proposition 2.4, any $f_1,\ f_2 \in H_w^1$ can be expressed by an absolutely convergent series as in (3) we infer that

$$\begin{aligned}\langle\widehat{O}f_1, f_2\rangle &= \Big\langle\widehat{O}\sum_{i\in I} c_{1,i}\,\widetilde{\psi}_i, \sum_{j\in I} c_{2,j}\,\widetilde{\psi}_j\Big\rangle\\ &= \sum_{i\in I}\sum_{j\in I} c_{1,i}\,c_{2,j}\,\langle\widehat{O}\widetilde{\psi}_i, \widetilde{\psi}_j\rangle\\ &= \sum_{i\in I}\sum_{j\in J} c_{1,i}\,c_{2,j}\,\langle O\widetilde{\psi}_i, \widetilde{\psi}_j\rangle\\ &= \Big\langle O\sum_{i\in I} c_{1,i}\,\widetilde{\psi}_i, \sum_{j\in I} c_{2,j}\,\widetilde{\psi}_j\Big\rangle = \langle Of_1, f_2\rangle\,.\end{aligned}$$

Thus $\widehat{O}f_1 = Of_1$ for any $f_1 \in H_w^1(\Psi)$ or, in other words, $\widehat{O} = O$. Therefore, the function $O \mapsto K$ that maps the Banach space $B\big(H_w^1(\Psi), H_{1/w}^\infty(\Psi)\big)$ on the Banach space $H_{1/w}^{\infty,\otimes}(\Psi)$ is injective and bounded by (16). Combining (12) and (16) finally shows (11) which concludes the proof. □

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