# Clustering-Based Edge Augmentation for Minimizing the Kirchhoff Index

**Prasanth Yalamanchili**
University of Utah

**Aditya Bhaskara**
University of Utah

## Abstract

The Kirchhoff index ($\mathcal{K}_G$), defined as the sum of effective resistances over all pairs of nodes in a connected undirected graph $G$, is a fundamental metric for real-world networks. It corresponds to average power consumption in electrical circuits, average commute time of random walks, and more relevantly to optimization, is equal to $\mathrm{Tr}(\mathcal{L}^\dagger)$, where $\mathcal{L}$ is the graph Laplacian. In this paper, we study the problem of augmenting a given graph by adding $k$ edges to minimize the Kirchhoff index. The problem was introduced in a work of Ghosh, Boyd, and Saberi (2008), and is known to be NP-hard; the state-of-the-art algorithms mostly employ greedy heuristics and have very weak guarantees. We design novel algorithms and show bi-criteria approximation guarantees, i.e., the algorithm adds $c \cdot k$ edges and obtains an $\alpha$ factor approximation to the optimum objective value with $k$ edges. Specifically, an algorithm based on $k$-median clustering with penalties achieves $c = 2$ and $\alpha = O(k)$. By using known submodularity ideas, we extend this to achieve $c = O(\log k)$ and $\alpha = (4 + \epsilon)$. The problem corresponds to an augmentation version of the classic A-optimal experimental design problem in statistics. We also prove strong integrality gaps for the natural convex relaxation and demonstrate the performance of our algorithm on real and synthetic graphs.

## 1 INTRODUCTION

In several graph theoretic applications, ensuring *good connectivity* is a crucial and desirable property. Connectivity can be defined in several ways; notable combinatorial notions include the diameter and conductance. Likewise, spectral notions such as mixing times and Laplacian eigenvalue lower bounds have been used in several applications. One useful way to measure connectivity between two vertices $u, v$ is via the *effective resistance* between them. While the notion originated in electrical circuits, the effective resistance is closely related to random walks (specifically the commute time), and also turns out to be a robust notion of distance between vertices [Doyle and Snell, 1984]. Further, the effective resistance is closely related to random spanning trees [Madry et al., 2014], and more interestingly, to the notion of statistical leverage scores [Spielman and Srivastava, 2008, Jambulapati et al., 2025].

The Kirchhoff Index of a graph is the sum of all pairwise effective resistances. It has found several applications as a measure of the "average connectivity" of a graph. Our goal in the paper is to study the Kirchhoff Index through a network-design perspective. Specifically, we consider the following *optimal augmentation* problem: given a graph $G$ (undirected, unweighted, connected) and a budget $k$, which $k$ edges do we add to $G$ so as to minimize the Kirchhoff Index?

This problem was first studied by [Ghosh et al., 2008], who gave a convex relaxation and developed heuristic algorithms. [Predari et al., 2023, Zhou et al., 2025, Achterberg and Kooij, 2025, Summers et al., 2015, Yang et al., 2019] studied the natural greedy approach of adding edges iteratively to optimize the objective. These works use the fact that the Kirchoff Index is exactly $n \cdot \mathrm{Tr}(\mathcal{L}_G^\dagger)$, where $\mathcal{L}_G$ is the Laplacian of $G$. However, we are not aware of any algorithms that come with good theoretical guarantees on the solution they produce. (The latter works above have approximation bounds, but in general, they lose factors of $n$ either in the approximation ratio or in the number of edges added.) On the hardness side, the best known result is an NP-hardness for (exactly) solving the problem [Kooij and Achterberg, 2023]. Our goal is to design new algorithms and show

Proceedings of the 29th International Conference on Artificial Intelligence and Statistics (AISTATS) 2026, Tangier, Morocco. PMLR: Volume 300. 

effective approximation guarantees.

**Related work and applications.** As we will see, the Kirchhoff Index is equal to (up to a constant), $\mathrm{Tr}(\mathcal{L}_G^{\dagger})$. Thus, the question of adding $k$ edges to improve this quantity turns out to be an "augmentation version" of the classic A-optimal experiment design problem in statistics (e.g., [Pukelsheim, 2006, Fedorov, 2013, Allen-Zhu et al., 2017]). Indeed, the original formulation of Ghosh, Boyd, and Saberi [Ghosh et al., 2008] exploit this connection in formulating their convex program. In the case $k > n$, the convex programming approach has been shown to yield efficient multiplicative approximations [Madan et al., 2019, Lau and Zhou, 2022, Nikolov et al., 2022]. However, the case $k < n$ is more challenging (but also more relevant in a graph augmentation setting). Recent works such as [De la Calle-Arroyo et al., 2023] studied this setting, but do not obtain guarantees for our setting.

Improving graph connectivity by adding edges is a classic problem in network design. For combinatorial notions such as diameter reduction, works of [Demaine and Zadimoghaddam, 2010, Bilò et al., 2012] obtained nearly optimal algorithms using $k$-center clustering methods. For the average shortest path [Meyerson and Tagiku, 2009] came up with algorithms using $k$-median with penalty clustering methods, and for centrality-based measures [Crescenzi et al., 2015, Shan et al., 2018] proved that greedy algorithms give very good approximations. Similarly, for the edge augmentations in probabilistic graphs [Bhaskara et al., 2025], proposed algorithms for improving the probabilistic diameter using clustering methods.

For augmenting a graph to improve spectral metrics like $\lambda_2(\mathcal{L}(G))$, [Kolla et al., 2010] studied algorithms based on the notion of *ultra-sparsifiers*. Our techniques are inspired by several of these results, as well as tools from the analysis of electrical networks such as Thompson's principle[Doyle and Snell, 1984].

## 1.1 Our Results and Techniques

We will develop efficient algorithms with approximation guarantees for the edge augmentation problem defined above. First, we ask: by adding $k$ (or $O(k)$) edges, what is the best possible approximation that one can obtain for the Kirchhoff index objective? We will then ask if one can obtain an $O(1)$ (constant factor) approximation to the objective by adding only $c \cdot k$ edges, for some small parameter $c$. We state these results below.

**Theorem 1.** *Let $G = (V, E)$ be a connected, unweighted, undirected graph, and let $\mathcal{K}_{G^*}$ be the minimum Kirchhoff Index of a graph of the form $G \cup F$, where $F$ is a set of $k$ new edges and $G \cup F$ is the graph formed by augmenting the edges in $F$ to graph $G$. There is an efficient algorithm that constructs a set $F'$ of at most $2k$ new edges, such that $\mathcal{K}_{G\cup F'} \leq O(k)\cdot\mathcal{K}_{G^*}$.*

The algorithm is based on finding an approximate solution to a variant of the $k$-median problem (known as $k$-median with penalties), on the effective resistance metric defined on the vertices. Given an algorithm for this variant of $k$-median that obtains an $O(1)$ approximation to the objective in $T_{\text{median}}(n)$ running time, our algorithm has expected running time $O\left(n^2 + T_{\text{median}}\right)$.

An outline of the analysis is as follows: we first note that for a random vertex $s$ in the optimal augmentation (call it $F^*$), the sum of effective resistances from $s$ to the remaining vertices is $O(\mathcal{K}_{G^*})/n$. Then we show, using a property of the effective resistance known as the Thompson's principle, that if $T$ is a set of centers that approximately optimize a variant of the $2k$-median objective, then every vertex $u \in V$ is "close enough" either to one of the vertices in $T$, or to $s$ itself. This yields the desired approximation guarantee.

Next, we show how to use Theorem 1 in conjunction with a known submodularity argument from [Shan et al., 2018] to obtain the following result:

**Theorem 2.** *Let $G = (V, E)$ be a connected unweighted graph, and let $\mathcal{K}_{G^*}$ be as defined above. There is an efficient algorithm that constructs a set $F'$ of $O(k\log(k/\delta))$ edges, such that $\mathcal{K}_{G\cup F'} \leq (4+\delta)\cdot\mathcal{K}_{G^*}$, for any $\delta > 0$.*

It is an interesting open problem whether we can achieve an approximation ratio $(1 + \delta)$. We contrast our result with the observation by [Achterberg and Kooij, 2025] who showed that the objective $h(F) := \mathcal{K}_{G\cup F}$ viewed as a set function in $F$ is not submodular. Indeed, they showed that the submodularity ratio, defined as

$$\inf_{S\subseteq R\subset V} \frac{h(S\cup\{e\}) - h(S)}{h(R\cup\{e\}) - h(R)},$$

can be as small as $1/n$. (Note that the ratio would be $\geq 1$ if $h$ were submodular.) This implies that one cannot analyze the greedy edge addition heuristic using techniques for maximizing approximately submodular function (such as [Sviridenko et al., 2017]), unless we add an $n$ factor more edges. Our result bypasses this limitation by instead focusing on edges out of a single "source" $s$. Surprisingly, in this case, the "local" objective turns out to be submodular, as was shown in [Shan et al., 2018]. One technical caveat in our result is that we allow the algorithm to add parallel edges (multiple edges between the same pair $u, v$).[1]

[1]In practice, as long as there is a vertex $w$ that is not

Finally, we consider the natural convex relaxation for the problem, as proposed in [Ghosh et al., 2008]. The relaxation is based on the fact that $\mathrm{Tr}(X^{-1})$ is convex over the positive definite cone (e.g., [Boyd and Vandenberghe, 2004]).

$$\min n \cdot \mathrm{Tr}\left(\mathcal{L}_G + \sum_{\{i,j\}\notin E} w_{ij}(e_i - e_j)(e_i - e_j)^T\right)^{\dagger}$$
$$\text{subject to} \quad \sum_{\{i,j\}} w_{ij} \le k,$$
$$0 \le w_{ij} \le 1. \tag{1}$$

We exhibit an integrality gap for this convex program, where the optimal integral solution has an objective value that is $\Omega\left(\frac{1}{c}\cdot\sqrt{\frac{n}{k}}\right)$ times the optimal objective value of (1), even if the integral soluion is allowed to contain $ck$ edges. This suggests that one cannot match our algorithmic results using the direct relaxation. We refer to Section 5 for details.

In Section 6, we also perform experiments on both real and synthetic datasets to empirically compare the performance of our algorithms with greedy edge addition. We also propose a new heuristic based on our $k$-median algorithm and show its empirical behavior.

## 2 PRELIMINARIES

Throughout the paper, $G = (V, E)$ will denote an unweighted undirected graph. We will also write $n = |V|$. For any pair of vertices $u, v \in V$, the *effective resistance* $R_G(u, v)$ is defined as the potential difference induced between $u$ and $v$ when one unit of current is injected at $u$ and extracted at $v$, assuming each edge acts as a unit resistor. It is well-known that

$$R_G(u,v) = (e_u - e_v)^\top \mathcal{L}_G^{\dagger}(e_u - e_v), \tag{2}$$

where $\mathcal{L}_G$ is the Laplacian matrix of $G$ and $\mathcal{L}_G^{\dagger}$ its Moore–Penrose pseudoinverse and $e_j$ denotes the $j$-th standard basis in $\mathbb{R}^n$. The effective resistance forms a metric on the vertex set $V$ and is monotonically decreasing with respect to adding edges [Klein and Randic, 1993, Doyle and Snell, 1984]. The *Kirchhoff index* $\mathcal{K}_G$ of graph $G$ is defined as the sum of all pairwise effective resistances. In terms of the Laplacian spectrum, we have

$$\mathcal{K}_G = \sum_{u<v} R_G(u,v) = n \cdot \mathrm{Tr}\left(\mathcal{L}_G^{\dagger}\right) = n\left(\sum_{i=2}^{n} \frac{1}{\lambda_i}\right),$$

where $\lambda_i$ is the $i$-th smallest eigenvalue of $\mathcal{L}_G$. Also, for a fixed $s \in V$, we write $R_G(s) = \sum_{v\in V} R_G(s, v)$.

**Thompson's Principle.** In our analysis, we use the classic variational formulation of the effective resistance, which is as a flow that minimizes the total *energy* over the edges. Formally,

**Lemma 3** ([Doyle and Snell, 1984]). *Let $G = (V, E)$ be a connected graph and let $i, j$ be two distinct vertices. Let $\mathcal{F}_{ij}$ denote the set of all unit flows from $i$ to $j$ in $G$. The effective resistance $R_G(i, j)$ between vertices $i$ and $j$ is given by:*

$$R_G(i,j) = \min_{f\in\mathcal{F}_{ij}} \mathcal{E}(f)$$

*where the energy $\mathcal{E}(f)$ of a flow $f \in \mathbb{R}^{|E|}$ is defined as $\mathcal{E}(f) := \sum_{e\in E} f_e^2$, where $f_e$ denotes the amount of flow passing through edge $e$ in the flow $f$ (i.e the $e^{th}$ coordinate of vector $f$).*

**Metric $k$-Median with Penalties.** Our algorithms use a variant of the classic $k$-median problem, in which the algorithm can "deny service" to any client at a specified penalty cost. Formally,

**Definition 4.** *($k$-median with penalties; KMP) Let $C$ be a set of clients and $F$ be a set of potential facility locations, both in a metric space with distance function $d$ Additionally, for each $j \in C$, we are given a penalty $p_j$ for not being served. The goal is to open at most $k$ facilities, i.e., select $S \subseteq F$ with $|S| \le k$, and for each client $j$, either assign $j$ to its nearest open facility in $S$ (incurring cost $d(j, S)$), or refuse service and pay penalty $p_j$. The goal is to minimize the total cost,*

$$\sum_{j\in C} min(p_j, d(j, S)).$$

The work of [Meyerson and Tagiku, 2009] gave a constant factor approximation based on local search:

**Theorem 5** (Approximation for $k$-Median with Penalties). *There exists a polynomial-time algorithm that achieves a factor 5-approximation for the metric $k$-median with penalties problem.*

## 3 APPROXIMATION USING CLUSTERING

First, we present our algorithm that adds at most $2k$ edges and gives an $O(k)$ approximation to the Kirchoff index objective. This establishes Theorem 1.

**Algorithm Outline.** Our algorithm has a simple structure: first we choose a random $s \in V$ as the "center". We then set up an instance of $2k$-median with penalties (KMP) with a carefully chosen metric, and

in $\Gamma(u) \cup \Gamma(v)$, we can mimic a parallel edge $uv$ by adding the two edges $uw$ and $wv$. This increases the edge budget by a factor of 2 and looses approximation by a factor of 2.

solve it using the algorithm from Theorem 5. Let $P$ be the set of facilities (vertices) that are output. Our algorithm then outputs the edges $\{s, p\}$ for all $p \in P$ (not already present).

To make the outline above formal (and for the guarantee to hold with high probability), we need to take $O(\log n)$ random samples $s$ — see Algorithm 1.

**Algorithm 1** Edge Augmentation

```
Input: Graph G = (V, E), parameters k, δ
S ← random sample of O(log n / δ) vertices
for each vertex s ∈ S do
    P = Approx solution to KMP(G, s, 2k)
    Let F_s be the edges {{s, p} : p ∈ P}
    Compute Kirchhoff index K_{G∪F_s}
end for
Let s* = arg min_{s∈S} K_{G∪F_s}
Output: edges F_{s*}
```

In what follows, we will write $F_{\text{opt}}$ to denote the optimal $k$-edge augmentation. For convenience, we also write $G' = G \cup F_{\text{opt}}$. The first lemma is the following:

**Lemma 6** (Existence of Good Centers)**.** *Let $G = (V, E)$, and let $G'$, $F_{opt}$ be defined as before. Let $\mathcal{S}$ be a set of $O(\frac{\log n}{\delta})$ vertices chosen at random from $V$. Then with probability at least $1 - \frac{1}{n}$,*

$$\min_{s \in \mathcal{S}} R_{G'}(s) \leq \frac{(2+\delta)}{n} \cdot \mathcal{K}_{G'}$$

The proof is a simple argument based on Markov's inequality, and is deferred to Appendix A.1. The next lemma, which is crucial to our analysis, relates the effective resistance in the optimally augmented graph to that of the original graph, and helps us reduce the problem to an instance of $2k$-median with penalties, as we will see.

**Lemma 7.** *Let $G = (V, E)$, $G'$, $F_{\text{opt}}$ be as defined above, and let $s$ be any fixed vertex. Then there exists a set $C \subseteq V$ of size at most $2k$, such that for all $v \in V$, one of the following holds:*

1. *$R_G(C, v) \leq (k+1) \cdot R_{G'}(s, v) - 1$*
2. *$R_G(s, v) \leq (k+1) \cdot R_{G'}(s, v)$*

*Where $R_G(C, v) = min_{u \in C} R_G(u, v)$*

*Proof.* We will show that the Lemma holds for $C$ being the set of endpoints of the $k$ edges in $F_{\text{opt}}$. (Thus we clearly have $|C| \leq 2k$.)

Fix any vertex $v$, and consider a unit flow $f^* \in \mathbb{R}^{|E'|}$ (where $E' = E \cup F_{opt}$) from $s$ to $v$ achieving $R_{G'}(s, v) = \sum_e (f_e^*)^2$, as guaranteed by Lemma 3. Since any flow can be decomposed in paths, there exist a collection of simple $s - v$ paths $\{P_1, P_2, \ldots, P_m\}$, each carrying flow $f_{P_i} > 0$, such that

$$\sum_{i=1}^{m} f_{P_i} = 1 \quad \text{and} \quad f^* = \sum_{i=1}^{m} f_{P_i} P_i.$$

Where $P_i \in \mathbb{R}^{|E'|}$ is a binary vector with each entry representing whether that edge is present in the path or not.

Each of these paths uses a (possibly empty) subset of the edges from $F_{\text{opt}}$. We can thus partition the paths into $(k+1)$ equivalence classes $\{Q_j\}_{j=1}^{k+1}$, based on the *last* edge from $F_{\text{opt}}$ used in the path. These yield $k$ classes; the $(k+1)$th class consists of paths that do not use any edges in $F_{\text{opt}}$. Additionally, note that for any edge $\{u, v\}$, all the paths $P_i$ that use the edge use the same "orientation" (either they all send flow from $u$ to $v$ or vice versa).

Now, let $F_j = \sum_{P_i \in Q_j} f_{P_i}$ denote the total flow in class $Q_j$. For each $j$ with $F_j > 0$, define the normalized flow

$$f^{(j)} = \frac{1}{F_j} \sum_{P_i \in Q_j} f_{P_i} P_i$$

which is a unit flow from $s$ to $v$ using paths in $Q_j$.

For each edge $e$, let $f_e^*$ denote the flow passing through edge $e$ in unit flow $f^*$, and $f_e^*(j)$ be the flow passing through edge $e$ by using paths in class $Q_j$ (so $f_e^* = \sum_{j=1}^{k+1} f_e^*(j)$). Now consider the energy (sum of squared flows) of each unit flow $f^{(j)}$:

$$\mathcal{E}(f^{(j)}) = \sum_e \frac{f_e^*(j)^2}{F_j^2}.$$

By using the inequality $\forall a_i, b_i \geq 0$, $\min_i(\frac{a_i}{b_i}) \leq \frac{\sum_i a_i}{\sum_i b_i}$. We are interested in bounding $\min_j \mathcal{E}(f_j)$

$$\begin{aligned}
\min_j \mathcal{E}(f^{(j)}) &= \min_j \left( \frac{\sum_e f_e^*(j)^2}{F_j^2} \right) \\
&\leq \frac{\sum_{j=1}^{k+1} \sum_e f_e^*(j)^2}{\sum_{j=1}^{k+1} F_j^2} \\
&\leq \frac{\sum_e \sum_{j=1}^{k+1} f_e^*(j)^2}{\sum_{j=1}^{k+1} F_j^2}
\end{aligned}$$

Since $f_e^*(j) \geq 0$ implies that $\sum_{j=1}^{k+1} f_e^*(j)^2 \leq \left(\sum_{j=1}^{k+1} f_e^*(j)\right)^2 \leq f_e^2$ and also $\sum_{j=1}^{k+1} F_j = 1$ implies that from Cauchy–Schwarz inequality $\sum_{j=1}^{k+1} F_j^2 \geq \frac{1}{k+1}$.

$$\min_j \mathcal{E}(f^{(j)}) \leq \frac{\sum_e (f_e^*)^2}{\frac{1}{k+1}} \leq (k+1) \cdot R_{G'}(s, v). \quad (3)$$

Let $j$ be the index at which the minimum is attained above. The remainder of the proof goes by showing that if $j \leq k$, then for one of the edge endpoints $x$, $R_G(x, v)$ is small (and since $x \in C$, $R_G(C, v)$ is small), and if $j = (k+1)$, then $R_G(s, v)$ is already small enough. (These correspond to the two cases in the Lemma statement.) The details of this argument are deferred to Appendix A.2. □

Next, we show how to construct the instance of $2k$-median with penalties for finding a $C'$ that (roughly) satisfies the conditions in Lemma 7 (step 4 of Algorithm 1). We start by defining an appropriate metric space. Given $G = (V, E)$, we define

$$d(u, v) := \begin{cases} 1 + R_G(u, v) & \text{if } u \neq v, \\ 0 & \text{if } u = v, \end{cases}$$

where $R_G(u, v)$ is the effective resistance between vertices $u, v$ in $G$.

**Observation 8.** *$d : V \times V \mapsto \mathbb{R}_{\geq 0}$ defined above is a metric.*

The proof is straightforward and is deferred to Appendix A.3. Next, we show how to construct an instance of clustering with penalties.

**Definition 9** ($\text{KMP}(G, s, \ell)$)**.** *Given $G$, a vertex $s \in V$, and an integer $\ell$, $\text{KMP}(G, s, \ell)$ is an instance of the $k$-median with penalties problem (Definition 4), with $F = V$, $C = V \setminus \{s\}$, metric $d$ as in Observation 8, and penalty $p_j := R_G(s, j)$.*

*Thus the goal is to return a subset $T \subseteq V$ of size $\leq \ell$ minimizing the objective*

$$\sum_{j \in V \setminus s} \min(p_j, d(j, T)).$$

The following lemma shows a bound on $\text{KMP}(G, s, 2k)$ in terms of $\mathcal{K}_{G^*}$.

**Lemma 10.** *Let $F_{\text{opt}}$ be the optimal augmentation to $G$ as before, and let $s$ be any vertex. Then the optimal objective value $\Lambda_{kmp}$ for the problem $\text{KMP}(G, s, 2k)$ satisfies*

$$\Lambda_{kmp} \leq (k+1) \cdot R_{G'}(s).$$

The proof is fairly straightforward using the definition of $d$, and is deferred to Appendix A.4. The next lemma is crucial to the analysis, and relates an approximate solution to KMP with the Kirchoff index objective, restricted to $s$.

**Lemma 11.** *Suppose $C$ be a set of centers returned by a $\gamma$ factor approximation algorithm for $\text{KMP}(G, s, 2k)$, and let $\Lambda_{kmp}$ be the optimum objective value. Let $F$ be the set of all edges of the form $\{s, c\}$, where $c \in C$ that are not already present in $G$. Then we have*

$$R_{G \cup F}(s) \leq \gamma \cdot \Lambda_{kmp} + 2k.$$

The proof requires a careful use of the properties of effective resistances, as well as the setup of the KMP problem. It is deferred to Appendix A.5. One seeming issue with the bound above is the additive factor of $2k$. It turns out that since $R_G(s)$ cannot be too small $< 1$ (even for a complete graph!), this factor can be absorbed into the approximation ratio. Then, we need to argue that one can bound $\mathcal{K}_{G \cup F}$ in terms of $R_{G \cup F}(s)$ for some $s$. This follows from the triangle inequality for effective resistances. Putting all these together, we obtain our main result (a formal version of Theorem 1).

**Theorem 12.** *Let $G = (V, E)$ be a connected, unweighted, undirected graph and let $F_{\text{opt}}$ be an optimal set of $k$ edges whose addition minimizes the Kirchhoff index of $G' = (V, E \cup F_{\text{opt}})$. Then Algorithm 1 runs in polynomial-time, adds at most $2k$ edges, and yields a graph with Kirchhoff index at most $(14k + 10 + \epsilon)\mathcal{K}_{G'}$.*

**Running time of Algorithm 1.** For each $s \in \mathcal{S}$, it takes $T_{\text{median}}$ time to solve the $2k$-median problem approximately and $\widetilde{O}(n^2)$ time for finding the Laplacian inverse using the recent techniques of [Jambulapati and Sidford, 2025]. This in turn gives us the effective resistances between all pairs of vertices in $\widetilde{O}(n^2)$ time. Together, this yields a running time of $\widetilde{O}(n^2 + T_{\text{median}})$.[2]

*Proof of Theorem 12.* From Lemma 6, we know that with probability $\geq 1 - \frac{1}{n}$, for some $s \in \mathcal{S}$, $R_{G'}(s) \leq \frac{2+\delta}{n} \cdot \mathcal{K}_{G'}$. Let $P$ be the set of vertices that are output by the approximation algorithm from Theorem 5 on the instance $\text{KMP}(G, s, 2k)$ we constructed for this $s$. As in the Algorithm, let $F_s$ be the edges of the form $\{s, p\}$, for $p \in P$. For convenience, write $G_s := (V, E \cup F_s)$. Then from Lemma 11,

$$R_{G_s}(s) \leq 5\Lambda_{kmp} + 2k,$$

since we have $\gamma = 5$ from Theorem 5. Now from Lemma 10, we have $\Lambda_{kmp} \leq (k+1) \cdot R_{G'}(s)$, and thus

$$R_{G_s}(s) \leq 5(k+1)R_{G'}(s) + 2k.$$

Next, we observe that for $G'$ (and in fact *any* simple graph), $R_{G'}(s) \geq 1$. This is because the value

[2] A simple implementation of the Local Search algorithm of [Meyerson and Tagiku, 2009] takes time $O(nk)$ time per iteration. If we ensure that the objective improves by a $(1 + \frac{\Omega(1)}{k})$ factor each iteration, the number of iterations is only $\sim k \log n$, so this term is lower order for small $k$.

$\sum_{v \in V} R(s,v) \geq 1$ for the complete graph, and effective resistance values are monotone. Thus, we can write

$$R_{G_s}(s) \leq 5(k+1)R_{G'}(s) + (2k)R_{G'}(s) \qquad (4)$$

Next, we note that by triangle inequality $R_{G_s}(u,v) \leq R_{G_s}(s,u) + R_{G_s}(s,v)$. Thus, since $R(u,u) = 0$,

$$\begin{aligned}\mathcal{K}_{G_s} &= \sum_{u<v} R(u,v) = \frac{1}{2}\sum_u \sum_v R(u,v) \\ &\leq \frac{1}{2}\sum_u \sum_v R(s,u) + R(s,v) \\ &= n \cdot R(s), \qquad (5)\end{aligned}$$

where $R()$ refers to $R_{G_s}()$ for convenience. Hence applying the inequality in (4), we get

$$\mathcal{K}_{G_s} \leq n \cdot (5(k+1) + 2k)R_{G'}(s)$$

Since $s$ is a "good center" from Lemma 6, $R_{G'}(s) \leq \frac{2+\delta}{n}\mathcal{K}_{G'}$. Using this we can say that

$$\mathcal{K}_{G_s} \leq (2+\delta) \cdot (7k+5)\mathcal{K}_{G'}$$

With small enough $\delta \in (0,1)$ that gives small enough $\epsilon \in (0,1)$, we can write

$$\mathcal{K}_{G_s} \leq (14k + 10 + \epsilon)\mathcal{K}_{G'}$$

□

## 4 CONSTANT FACTOR APPROXIMATION

We will now use techniques from Section 3, together with known submodularity results on the principal minors of a graph Laplacian to obtain a factor $(4+\epsilon)$ approximation to the Kirchhoff Index objective by adding $O(k \log k)$ edges. This establishes Theorem 2. As mentioned earlier, in this section, we allow the algorithm to add parallel edges; thus for a set of edges $F$, we will denote the (multi-)graph obtained by adding a set $F$ of edges as $G + F := G(V, E+F)$, where $E+F$ adds edges by multiplicity.

As before, we will let $F_{\text{opt}}$ denote the optimal $k$-subset of edges to add to $G$ to minimize the $\mathcal{K}_G$ objective, and define $G' = G + F_{\text{opt}}$.

**Algorithm Outline.** As before, we first sample a random subset of *center* vertices $\mathcal{S}$, and try to add edges out of $s \in \mathcal{S}$. In this case, we simply add edges greedily to optimize the objective $R_G(s)$ (i.e., the sum of the effective resistances between $s$ and the other vertices in the graph). Using known submodularity results, we can argue that as long as the graph we begin with has Kirchhoff Index at most $\Delta$ times $\mathcal{K}_{G'}$, the algorithm achieves the desired approximation ratio in $k \log \Delta$ iterations. To obtain the initial guarantee, we simply use Algorithm 1; this ensures $\Delta = O(k)$, which then yields the desired guarantees. (See Algorithm 2.)

**Algorithm 2** Cluster-then-Greedy Augmentation

```
Input: Graph G = (V, E), parameters k, δ
S ← random sample of O(log n / δ) vertices
for each vertex s ∈ S do
    P = Approx solution of KMP(G, s, 2k)
    Let F_s be the edges {{s, p} : p ∈ P} and let G_s = G + F_s
    for t = 1, 2, ..., 2ck log k do
        Let {s, v} be the edge that minimizes the objective R_{G_s+{s,v}}(s) = Tr(L_{G_s+{s,v}}[s]^{-1})
        F_s ← F_s + {s, v}
        G_s ← G_s + {s, v}
    end for
    Compute Kirchhoff index K_{G_s}
end for
Let s* = arg min_{s∈V} K_{G_s}
Output: Edge (multi-)set F_{s*}
```

As in Section 3, we first show that one can restrict to adding edges from a small set of centers.

**Lemma 13.** *Let $s$ be any vertex in $G$ and let $F_{opt}$ and $G'$ be defined as above. There exists a (multi-)set of $2k$ edges $F'_{opt}$ all incident to $s$ such that $\mathcal{K}_{G+F'_{opt}} \leq 2\mathcal{K}_{G'}$. Further, we also have $R_{G+F'_{opt}}(s) \leq 2R_{G'}(s)$.*

We appeal to the simple fact that for any $u, v$

$$(e_u - e_v)(e_u - e_v)^\top \preceq 2\big[(e_u - e_s)(e_u - e_s)^\top + (e_v - e_s)(e_v - e_s)^\top\big]$$

which implies that $\mathcal{L}_{G'} \preceq 2\mathcal{L}_{G+F}$, where $F$ is the (multi-)set of edges obtained by connecting $s$ to each of the end-points of the edges of $F_{\text{opt}}$. This then turns out to imply the lemma (see Appendix A.6).

Next, we use Lemma 6 to conclude that with probability $\geq 1 - \frac{1}{n}$, there exists an $s \in \mathcal{S}$ (in the algorithm) such that $R_{G'}(s) \leq \frac{2+\delta}{n}\mathcal{K}_{G'}$. In what follows, we fix such an $s$ and argue that $\mathcal{K}_{G_s}$ found by the algorithm satisfies the desired guarantee.

For a multi-set of edges $F$ *all incident to* $s$, we define

$$\phi(F) := R_{G+F}(s) = \mathrm{Tr}\left(\mathcal{L}_{G+F}[s]^{-1}\right)$$

where $\mathcal{L}_G[s]$ refers to the principal minor of the Laplacian obtained by removing the $s$-th row and column [Izmailian et al., 2013]. For a connected graph $G$, $\mathcal{L}_G[s]$ turns out to be invertible, and moreover:

**Theorem 14.** *[Shan et al., 2018] The function $\phi(F)$ defined above is monotone decreasing and supermodular. I.e., for $D \subset F$ and any edge $\{s, x\}$,*

1. $\phi(D) \geq \phi(F)$, *and*
2. $\phi(D) - \phi(D + \{s, x\}) \geq \phi(F) - \phi(F + \{s, x\})$.

(As a technical aside, [Shan et al., 2018] only show the Theorem for sets of edges and not multi-sets, but the argument extends easily. For the sake of completeness, we include a proof in Appendix A.7.)

The rest of the proof can be summarized as follows. First, we show that by adding the set of edges $P$ in Step 4 of the algorithm (by solving $2k$-median with penalties), we end up with $F_s$ that satisfies $\phi(F_s) \leq O(k) \cdot R_{G'}(s)$, by our argument from Section 3. Now, Lemma 13 implies that there exists some $F$ (specifically, $F'_{\text{opt}}$) with $R_{G+F}(s) \leq 2R_{G'}(s)$ and $|F| \leq 2k$. The supermodularity property now implies that if we add $2k$ edges using a greedy procedure, the gap between $\phi(F_s)$ and $\phi(F'_{\text{opt}})$ drops by a constant factor (specifically a $(1 - \frac{1}{e})$ factor, e.g., [Nemhauser et al., 1978]). Since the gap is $O(k)$ to start with (i.e., after adding $P$ from Step 4 of the algorithm), we can conclude that after $2ck \log k$ greedy edge additions (for an appropriately chosen constant $c$), we end up with $F_s$ that satisfies $\phi(F_s) \leq (2 + \epsilon)R_{G'}(s)$.

Finally, using the fact that $R_{G'}(s) \leq \frac{2+\delta}{n}\mathcal{K}_{G'}$ and the triangle inequality of effective resistances (see Eq. (5)), we obtain that

$$\mathcal{K}_{G+F_s} \leq (2 + \epsilon)(2 + \delta)\mathcal{K}_{G'}.$$

This completes the proof of Theorem 2. A detailed proof is deferred to Appendix A.8.

## 5 LIMITS OF THE CONVEX RELAXATION

In this section, we show that there is an integrality gap of $\Omega(\frac{1}{c} \cdot \sqrt{\frac{n}{k}})$ between the solution of discrete problem of adding $ck$ (multi-)edges to improve the effective resistance of graph, and its natural convex relation (1), that adds weighted edges such that the sum of weights is $k$. The integrality gap highlights the challenge in deriving approximation guarantees from convex relaxations, even if we allow bi-criteria edge additions.

For this section, given a graph $G$ and parameter $k$, define $\mathcal{K}_{\text{discrete}}(G, k)$ to be the minimum Kirchoff index of a graph obtained by adding $k$ (multi-)edges to $G$. Next, define $\mathcal{K}_{\text{conv}}(G, k)$ to be the optimum objective value of the relaxation (1) with parameter $k$ on graph $G$. Our result is the following:

**Theorem 15.** *For any even $n$, parameters $k < n$ and $c \geq 1$, there is a connected undirected $G$ such that*

$$\mathcal{K}_{discrete}(G, ck) \geq \Omega\left(\frac{1}{c} \cdot \sqrt{\frac{n}{k}}\right)\mathcal{K}_{conv}(G, k)$$

*Proof.* The graph we consider is simply $C_n$, the undirected cycle on $n$ vertices. Since $n$ is even, the eigenvalues of $C_n$ are well-known to be $z_p = 4\sin^2(\frac{\pi p}{n})$ for $p \in [0, \frac{n}{2}]$. Moreover, for $p \geq 1$, each of the eigenvalues has multiplicity 2. Thus, for $i = 2, 4, \ldots, n$ (even integers), the $i$th smallest eigenvalue $\lambda_i$ is $4\sin^2(\frac{\pi i}{2n})$.

Using this fact, we can lower bound $\mathcal{K}_{\text{discrete}}(G, ck)$ as follows: any discrete solution adds at most $ck$ edges. From the perspective of the Laplacian, each edge corresponds to a rank 1 matrix, and thus by using the interlacing property of eigenvalues, if $\mu_i$ is the $i$th smallest Laplacian eigenvalue of the graph obtained by adding $ck$ edges to $C_n$, we have, for $i \leq n - ck$,

$$\mu_i \leq \lambda_{ck+i} \leq 4\sin^2\left(\frac{\pi(ck + i)}{2n}\right).$$

Now using the fact that $\sin x \leq x$, we have that

$$\sum_{i=2}^{n} \frac{1}{\mu_i} \geq \sum_{i=2}^{n-ck} \frac{1}{\mu_i} \geq \frac{n^2}{\pi^2} \sum_{i=2}^{n-ck} \frac{1}{(ck + i)^2}.$$

Note that for any $m$, we have $\sum_{i \geq 1} \frac{1}{(m+i)^2} \geq \frac{1}{(m+1)}$.[3] Setting $m = ck + 1$ and using $\mathcal{K}_{\text{discrete}} = n \cdot \sum_{i=2}^{n} \frac{1}{\mu_i}$, we obtain

$$\mathcal{K}_{\text{discrete}} \geq \frac{n^3}{\pi^2(ck + 1)}.$$

**Upper bound for $\mathcal{K}_{\text{conv}}$.** A fractional solution to the convex program (1) could add the edges of a Ramanujan expander graph of degree $d$ (that consists of $\frac{nd}{2}$ edges in total) [Cohen, 2016, Marcus et al., 2018], with weight $\frac{2k}{nd}$ spread on each edge. This ensures that the total weight of the edges added is exactly $k$. Since the eigenvalues of the Laplacian of a Ramanujan graph lie in range $(d - 2\sqrt{d-1}, d + 2\sqrt{d-1})$, by choosing $d = 5$, the eigenvalues of the Laplacian of newly added weighted edges will lie in the interval $(\frac{2k}{5n}, \frac{18k}{5n})$.

Let $\mu_i$ be the $i$-th smallest eigenvalue of Laplacian obtained by adding the fractional weighted edges above to the cycle $C_n$. By Weyl's inequality, for any $i \geq 2$,

$$\mu_i \geq \frac{2k}{5n} + \lambda_i \implies \sum_{i=2}^{n} \frac{1}{\mu_i} \leq \sum_{i} \frac{1}{\lambda_i + \frac{2k}{5n}}. \quad (6)$$

Now let us split the terms in summation based on an index $t$ to be chosen later.

$$\sum_{i=2}^{n} \frac{1}{\mu_i} \leq \sum_{i=2}^{t} \frac{1}{\lambda_i + \frac{2k}{5n}} + \sum_{i=t+1}^{n} \frac{1}{\lambda_i + \frac{2k}{5n}}.$$

[3] This follows by observing that $\frac{1}{(m+i)^2} \geq \frac{1}{m+i} - \frac{1}{m+i+1}$, followed by telescoping.

Now for $i \in [1,t]$, we use the inequality $\lambda_i + \frac{2k}{5n} \geq \frac{2k}{5n}$ and for $i \in [t+1, n-1]$, $\lambda_i + \frac{2k}{n} \geq \lambda_i$, Hence,

$$\sum_{i=2}^{n} \frac{1}{\mu_i} \leq \frac{5nt}{2k} + \sum_{i=t+1}^{n} \frac{1}{\lambda_i}.$$

Now using the fact that $\sin x \geq \frac{2x}{\pi}$ for $x \in [0, \pi/2]$,

$$\lambda_i = 4\sin^2(\frac{\pi i}{2n}) \geq 4\left(\frac{2}{\pi}\right)^2 \left(\frac{\pi i}{2n}\right)^2 \geq \frac{4i^2}{n^2}.$$

Plugging this into the bound above,

$$\begin{aligned}\sum_{i=2}^{n} \frac{1}{\mu_i} &\leq \frac{5nt}{2k} + \sum_{i=t+1}^{n} \frac{n^2}{4i^2} \\ &\leq \frac{5nt}{2k} + \frac{n^2}{4} \sum_{i=t+1}^{n} \frac{1}{i-1} - \frac{1}{i} \\ &\leq \frac{5nt}{2k} + \frac{n^2}{4t}\end{aligned}$$

Setting $t = \sqrt{kn}$, the above term can be bounded by $O\left(\frac{n^{1.5}}{k^{0.5}}\right)$. Using $\mathcal{K}_{\text{conv}} = n \cdot \sum_{i=2}^{n} \frac{1}{\mu_i}$ in turn implies that

$$\mathcal{K}_{\text{conv}} \leq O\left(\frac{n^{2.5}}{k^{0.5}}\right),$$

thus completing the proof of the integrality gap. □

## 6 EXPERIMENTS

In this section, we compare the performance of our clustering methods, specifically Algorithm 1 and an additional heuristic based on the $k$-median algorithm, to the natural greedy algorithm on small graphs to assess their empirical effectiveness. Note that prior works have shown that greedy addition performs quite well in practice [Predari et al., 2023, Zhou et al., 2025], although we do not know of a good analysis. We will use sampled versions of datasets from the SNAP repository (with roughly 1.5k nodes and 20k edges) [Leskovec and Sosič, 2016], as well as on a synthetic dataset constructed as follows: we first generate small, well-connected clusters of vertices and then connect them via long paths. This construction yields large effective resistance values across clusters in the original graph.

Here are the five algorithms on which our experiments are performed:

- $\underline{C_k}$: This corresponds to outputting a solution that adds $k$ edges using clustering based method in Algorithm 1.
- $\underline{C_{2k}}$: This corresponds to outputting a solution that adds $2k$ edges using Algorithm 1. This is the regime most naturally aligned with our theoretical guarantees that outputs a solution corresponding to $2k$ edges added.
- $\underline{C_{2k} + G_k}$: This corresponds to an algorithm that is a hybrid of a clustering-based method and a greedy method. Here we output a solution by first running a simple $2k$-median clustering algorithm based on local search to obtain $2k$ candidate centers and then greedily adding $k$ edges between these centers by restricting attention to the $\binom{2k}{2}$ possible edges among them.
- $\underline{G_k}$: This corresponds to outputting a solution that adds $k$ edges using a greedy algorithm that adds the edge with the greatest improvement in each step.
- $\underline{G_{2k}}$.This corresponds to outputting a solution that adds $2k$ edges using a greedy algorithm that adds the edge with the greatest improvement in each step.

Now, we compare the performance of the clustering-based algorithms to that of greedy algorithms. We report the *average* pairwise effective resistance of the graph after $k$ edge additions (which is proportional to $\mathcal{K}_G$, since $R_{avg} \cdot \binom{v}{2} = \mathcal{K}_G$). The results indicate that Algorithm 1 with $2k$ edges ($C_{2k}$) performed slightly better than the greedy algorithm that adds $k$ edges ($G_k$) in most of the cases, indicating the effectiveness of the bi-criteria approximation that has been proved for Algorithm 1. Additionally, the heuristic ($C_{2k} + G_k$) performs nearly as well as the greedy algorithm ($G_k$) while offering significant computational advantages. Specifically, the greedy algorithm requires recomputing the Laplacian pseudoinverse for each potential edge addition across $k$ rounds, resulting in $O(kn^2)$ recomputations. In contrast, our heuristic algorithm ($C_{2k} + G_k$) computes the Laplacian pseudoinverse *once at the start*, and the greedy process takes only $O(k^3)$ recomputations instead of $O(kn^2)$. Computationally, Algorithm 1 (in $C_k, C_{2k}$) does even better by calculating the Laplacian pseudoinverse only once, but it needs to add $2k$ edges to compete with the performance of $G_k$.

Unfortunately, even with small graphs, the convex relaxation turns out to be infeasible to run, so we do not have a baseline for performance. The comparison with greedy shows that (i) the greedy algorithm consistently performs as well as clustering-based algorithms on real-world graphs, making it an interesting open direction to study its performance; (ii) if Greedy is close to optimal, it implies that our approximation factors are much better than the $O(k)$ factor suggested by the theory. Exploring larger graphs and making our techniques scalable is an interesting future direction.

| $k$ | $C_k$ | $C_{2k}$ | $C_{2k} + G_k$ | $G_k$ | $G_{2k}$ |
|---|---|---|---|---|---|
| 0 | 1.495 | 1.495 | 1.495 | 1.495 | 1.495 |
| 5 | 1.467 | 1.437 | 1.447 | 1.442 | 1.407 |
| 10 | 1.437 | 1.393 | 1.414 | 1.407 | 1.353 |
| 15 | 1.419 | 1.359 | 1.387 | 1.378 | 1.308 |
| 20 | 1.393 | 1.335 | 1.364 | 1.353 | 1.269 |
| 25 | 1.378 | 1.305 | 1.342 | 1.330 | 1.235 |

Table 1: ca-GrQc: Arxiv collaboration network

| $k$ | $C_k$ | $C_{2k}$ | $C_{2k} + G_k$ | $G_k$ | $G_{2k}$ |
|---|---|---|---|---|---|
| 0 | 0.581 | 0.581 | 0.581 | 0.581 | 0.581 |
| 5 | 0.577 | 0.573 | 0.576 | 0.576 | 0.571 |
| 10 | 0.573 | 0.565 | 0.571 | 0.571 | 0.560 |
| 15 | 0.569 | 0.558 | 0.565 | 0.566 | 0.551 |
| 20 | 0.565 | 0.550 | 0.561 | 0.560 | 0.458 |
| 25 | 0.561 | 0.543 | 0.556 | 0.556 | 0.531 |

Table 2: wiki-Vote: Wikipedia adminship votes

| $k$ | $C_k$ | $C_{2k}$ | $C_{2k} + G_k$ | $G_k$ | $G_{2k}$ |
|---|---|---|---|---|---|
| 0 | 0.312 | 0.312 | 0.312 | 0.312 | 0.312 |
| 5 | 0.306 | 0.301 | 0.304 | 0.303 | 0.296 |
| 10 | 0.301 | 0.290 | 0.297 | 0.296 | 0.284 |
| 15 | 0.297 | 0.280 | 0.291 | 0.290 | 0.272 |
| 20 | 0.290 | 0.274 | 0.285 | 0.284 | 0.260 |
| 25 | 0.288 | 0.265 | 0.279 | 0.279 | 0.251 |

Table 3: email-Enron: Email network from Enron

| $k$ | $C_k$ | $C_{2k}$ | $C_{2k} + G_k$ | $G_k$ | $G_{2k}$ |
|---|---|---|---|---|---|
| 0 | 1.298 | 1.298 | 1.298 | 1.298 | 1.298 |
| 5 | 1.279 | 1.270 | 1.274 | 1.273 | 1.257 |
| 10 | 1.270 | 1.258 | 1.262 | 1.258 | 1.230 |
| 15 | 1.263 | 1.242 | 1.248 | 1.243 | 1.205 |
| 20 | 1.258 | 1.230 | 1.237 | 1.230 | 1.182 |
| 25 | 1.250 | 1.215 | 1.226 | 1.217 | 1.160 |

Table 4: P2P-Gnutella: Gnutella p2p network

| $k$ | $C_k$ | $C_{2k}$ | $C_{2k} + G_k$ | $G_k$ | $G_{2k}$ |
|---|---|---|---|---|---|
| 0 | 27.76 | 27.76 | 27.76 | 27.76 | 27.76 |
| 5 | 4.338 | 1.589 | 3.791 | 3.790 | 1.416 |
| 10 | 1.589 | 1.279 | 1.414 | 1.416 | 0.737 |
| 15 | 1.394 | 1.113 | 0.913 | 0.911 | 0.560 |
| 20 | 1.279 | 0.984 | 0.746 | 0.737 | 0.458 |
| 25 | 1.206 | 0.901 | 0.647 | 0.636 | 0.390 |

Table 5: Synthetic I: Clusters connected by paths

# 7 CONCLUSION

We studied the problem of how to augment a graph (by adding a few edges) in order to minimize the Kirchhoff Index, a popular metric for connectivity in graphs. Prior works both in optimization and in the network design communities studied heuristics. We gave the first algorithms with approximation guarantees, leveraging a connection to a variant of $k$-median clustering. Our experiments lead to the open problem of developing guarantees for the natural greedy heuristic.

**Acknowledgements**

The authors acknowledge the support from the National Science Foundation, via the award NSF CCF-2047288.

**References**

[Achterberg and Kooij, 2025] Achterberg, M. A. and Kooij, R. E. (2025). On the non-submodularity of the problem of adding links to minimize the effective graph resistance. *arXiv preprint arXiv:2501.03363*.

[Allen-Zhu et al., 2017] Allen-Zhu, Z., Li, Y., Singh, A., and Wang, Y. (2017). Near-optimal design of experiments via regret minimization. In *International Conference on Machine Learning*, pages 126–135. PMLR.

[Bhaskara et al., 2025] Bhaskara, A., Crane, A., Jain, S., Mazumder, M. M. H. U., Sullivan, B. D., and Yalamanchili, P. (2025). Optimizing probabilistic propagation in graphs by adding edges.

[Bilò et al., 2012] Bilò, D., Gualà, L., and Proietti, G. (2012). Improved approximability and non-approximability results for graph diameter decreasing problems. *Theoretical Computer Science*, 417:12–22.

[Boyd and Vandenberghe, 2004] Boyd, S. P. and Vandenberghe, L. (2004). *Convex optimization*. Cambridge university press.

[Cohen, 2016] Cohen, M. B. (2016). Ramanujan graphs in polynomial time. In *2016 IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 276–281. IEEE.

[Crescenzi et al., 2015] Crescenzi, P., d'Angelo, G., Severini, L., and Velaj, Y. (2015). Greedily improving our own centrality in a network. In *International symposium on experimental algorithms*, pages 43–55. Springer.

[De la Calle-Arroyo et al., 2023] De la Calle-Arroyo, C., Amo-Salas, M., López-Fidalgo, J., Rodríguez-Aragón, L., and Wong, W. (2023). A methodology to d-augment experimental designs. *Chemometrics and Intelligent Laboratory Systems*, 237:104822.

[Demaine and Zadimoghaddam, 2010] Demaine, E. D. and Zadimoghaddam, M. (2010). Minimizing the diameter of a network using shortcut edges. In *Scandinavian Workshop on Algorithm Theory*, pages 420–431. Springer.

[Doyle and Snell, 1984] Doyle, P. G. and Snell, J. L. (1984). *Random Walks and Electric Networks*. Number Book 22 in Carus Mathematical Monographs. Mathematical Association of America, Washington, DC.

[Fedorov, 2013] Fedorov, V. V. (2013). *Theory of optimal experiments*. Elsevier.

[Ghosh et al., 2008] Ghosh, A., Boyd, S., and Saberi, A. (2008). Minimizing effective resistance of a graph. *SIAM review*, 50(1):37–66.

[Izmailian et al., 2013] Izmailian, N. S., Kenna, R., and Wu, F. (2013). The two-point resistance of a resistor network: a new formulation and application to the cobweb network. *Journal of Physics A: Mathematical and Theoretical*, 47(3):035003.

[Jambulapati et al., 2025] Jambulapati, A., Sachdeva, S., Sidford, A., Tian, K., and Zhao, Y. (2025). Eulerian graph sparsification by effective resistance decomposition. In *Proceedings of the 2025 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 1607–1650. SIAM.

[Jambulapati and Sidford, 2025] Jambulapati, A. and Sidford, A. (2025). Ultrasparse ultrasparsifiers and faster laplacian system solvers. *ACM Transactions on Algorithms*, 21(3):1–49.

[Klein and Randic, 1993] Klein, D. J. and Randic, M. (1993). Resistance distance. *Journal of Mathematical Chemistry*, 12:81–95.

[Kolla et al., 2010] Kolla, A., Makarychev, Y., Saberi, A., and Teng, S.-H. (2010). Subgraph sparsification and nearly optimal ultrasparsifiers. In *Proceedings of the forty-second ACM symposium on Theory of computing*, pages 57–66.

[Kooij and Achterberg, 2023] Kooij, R. E. and Achterberg, M. A. (2023). Minimizing the effective graph resistance by adding links is np-hard. *Operations Research Letters*, 51(6):601–604.

[Lau and Zhou, 2022] Lau, L. C. and Zhou, H. (2022). A local search framework for experimental design. *SIAM Journal on Computing*, 51(4):900–951.

[Leskovec and Sosič, 2016] Leskovec, J. and Sosič, R. (2016). Snap: A general-purpose network analysis and graph-mining library. *ACM Transactions on Intelligent Systems and Technology (TIST)*, 8(1):1–20.

[Madan et al., 2019] Madan, V., Singh, M., Tantipongpipat, U., and Xie, W. (2019). Combinatorial algorithms for optimal design. In *Conference on Learning Theory*, pages 2210–2258. PMLR.

[Madry et al., 2014] Madry, A., Straszak, D., and Tarnawski, J. (2014). Fast generation of random spanning trees and the effective resistance metric. In *Proceedings of the twenty-sixth annual ACM-SIAM symposium on Discrete algorithms*, pages 2019–2036. SIAM.

[Marcus et al., 2018] Marcus, A. W., Spielman, D. A., and Srivastava, N. (2018). Interlacing families iv: Bipartite ramanujan graphs of all sizes. *SIAM Journal on computing*, 47(6):2488–2509.

[Meyer, 1973] Meyer, Jr, C. D. (1973). Generalized inversion of modified matrices. *Siam journal on applied mathematics*, 24(3):315–323.

[Meyerson and Tagiku, 2009] Meyerson, A. and Tagiku, B. (2009). Minimizing average shortest path distances via shortcut edge addition. In *International Workshop on Approximation Algorithms for Combinatorial Optimization*, pages 272–285. Springer.

[Nemhauser et al., 1978] Nemhauser, G. L., Wolsey, L. A., and Fisher, M. L. (1978). An analysis of approximations for maximizing submodular set functions—i. *Mathematical programming*, 14(1):265–294.

[Nikolov et al., 2022] Nikolov, A., Singh, M., and Tantipongpipat, U. (2022). Proportional volume sampling and approximation algorithms for a-optimal design. *Mathematics of Operations Research*, 47(2):847–877.

[Plemmons, 1977] Plemmons, R. J. (1977). M-matrix characterizations. i—nonsingular m-matrices. *Linear Algebra and its applications*, 18(2):175–188.

[Predari et al., 2023] Predari, M., Berner, L., Kooij, R., and Meyerhenke, H. (2023). Greedy optimization of resistance-based graph robustness with global and local edge insertions. *Social Network Analysis and Mining*, 13(1):130.

[Pukelsheim, 2006] Pukelsheim, F. (2006). *Optimal design of experiments*. SIAM.

[Shan et al., 2018] Shan, L., Yi, Y., and Zhang, Z. (2018). Improving the information centrality of a node in complex networks by adding edges. In *Proceedings of the Twenty-Seventh International Joint Conference on Artificial Intelligence, IJCAI-18*, pages 3535–3541. International Joint Conferences on Artificial Intelligence Organization.

[Spielman and Srivastava, 2008] Spielman, D. A. and Srivastava, N. (2008). Graph sparsification by effective resistances. In *Proceedings of the fortieth annual ACM symposium on Theory of computing*, pages 563–568.

[Summers et al., 2015] Summers, T., Shames, I., Lygeros, J., and Dörfler, F. (2015). Topology design for optimal network coherence. In *2015 European Control Conference (ECC)*, pages 575–580.

[Sviridenko et al., 2017] Sviridenko, M., Vondrák, J., and Ward, J. (2017). Optimal approximation for submodular and supermodular optimization with bounded curvature. *Mathematics of Operations Research*, 42(4):1197–1218.

[Yang et al., 2019] Yang, C., Mao, J., Qian, X., and Wei, P. (2019). Designing robust air transportation networks via minimizing total effective resistance. *IEEE Transactions on Intelligent Transportation Systems*, 20(6):2353–2366.

[Zhou et al., 2025] Zhou, X., Zehmakan, A. N., and Zhang, Z. (2025). Efficient algorithms for minimizing the kirchhoff index via adding edges. *IEEE Transactions on Knowledge and Data Engineering*.

## Checklist

1. For all models and algorithms presented, check if you include:
   (a) A clear description of the mathematical setting, assumptions, algorithm, and/or model. [Yes]
   (b) An analysis of the properties and complexity (time, space, sample size) of any algorithm. [Yes]
   (c) (Optional) Anonymized source code, with specification of all dependencies, including external libraries. [Yes]

2. For any theoretical claim, check if you include:
   (a) Statements of the full set of assumptions of all theoretical results. [Yes]
   (b) Complete proofs of all theoretical results. [Yes]
   (c) Clear explanations of any assumptions. [Yes]

3. For all figures and tables that present empirical results, check if you include:
   (a) The code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL). [Yes]
   (b) All the training details (e.g., data splits, hyperparameters, how they were chosen). [Not Applicable]
   (c) A clear definition of the specific measure or statistics and error bars (e.g., with respect to the random seed after running experiments multiple times). [Not Applicable]
   (d) A description of the computing infrastructure used. (e.g., type of GPUs, internal cluster, or cloud provider). [Not Applicable]

4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets, check if you include:
   (a) Citations of the creator If your work uses existing assets. [Not Applicable]
   (b) The license information of the assets, if applicable. [Not Applicable]
   (c) New assets either in the supplemental material or as a URL, if applicable. [Not Applicable]
   (d) Information about consent from data providers/curators. [Not Applicable]
   (e) Discussion of sensible content if applicable, e.g., personally identifiable information or offensive content. [Not Applicable]

5. If you used crowdsourcing or conducted research with human subjects, check if you include:
   (a) The full text of instructions given to participants and screenshots. [Not Applicable]
   (b) Descriptions of potential participant risks, with links to Institutional Review Board (IRB) approvals if applicable. [Not Applicable]
   (c) The estimated hourly wage paid to participants and the total amount spent on participant compensation. [Not Applicable]

# Appendix

## A Missing Proofs

### A.1 Proof of Lemma 6

*Proof.* By definition of the Kirchhoff index and the symmetry of effective resistance, we have:

$$\mathcal{K}_{G'} = \sum_{u<v} R_{G'}(u,v) = \frac{1}{2}\sum_{u\in V} R_{G'}(u).$$

Thus,

$$\sum_{u\in V} R_{G'}(u) = 2\mathcal{K}_{G'}.$$

This in turn implies:

$$\mathbb{E}_{s\sim V}[R_{G'}(s)] = \frac{2\mathcal{K}_{G'}}{n}$$

By averaging (formally, Markov's inequality) with $\delta \leq 2$, we have:

$$\Pr_{s\sim V}[R_{G'}(s) > \frac{(2+\delta)\mathcal{K}_{G'}}{n}] \leq \frac{2}{2+\delta} \leq 1 - \frac{\delta}{4}.$$

By the independence of samples being picked in the set $\mathcal{S}$, we have

$$\Pr[\min_{s\in\mathcal{S}} R_{G'}(s) > \frac{(2+\delta)\mathcal{K}_{G'}}{n}] \leq \left(1-\frac{\delta}{4}\right)^{|\mathcal{S}|}$$

Using inequality $1 - x \leq e^{-x}$ with $|\mathcal{S}| = \frac{4c\cdot\log n}{\delta}$

$$\Pr[\min_{s\in\mathcal{S}} R_{G'}(s) > \frac{(2+\delta)\mathcal{K}_{G'}}{n}] \leq e^{-\frac{\delta|\mathcal{S}|}{4}} \leq \frac{1}{n^c}$$

Thus the probability that $\min_{s\in\mathcal{S}} R_{G'}(s) \leq \frac{(2+\delta)}{n}\mathcal{K}_{G'}$ is at least $1 - \frac{1}{n^c}$ (with $\delta \leq 2$), completes the proof.

□

### A.2 Proof of Lemma 7

*Proof.* We will show that the Lemma holds for $C$ being the set of endpoints of the $k$ edges in $F_{\text{opt}}$. (Thus we clearly have $|C| \leq 2k$.)

Fix any vertex $v$, and consider a unit flow $f^*$ from $s$ to $v$ achieving $R_{G'}(s,v) = \sum_e (f_e^*)^2$, as guaranteed by Lemma 3. Since any flow can be decomposed in paths, there exist a collection of simple $s - v$ paths $\{P_1, P_2, \ldots, P_m\}$, each carrying flow $f_{P_i} > 0$, such that

$$\sum_{i=1}^{m} f_{P_i} = 1 \quad \text{and} \quad f^* = \sum_{i=1}^{m} f_{P_i} P_i.$$

Each of these paths uses a (possibly empty) subset of the edges from $F_{\text{opt}}$. We can thus partition the paths into $(k+1)$ equivalence classes $\{Q_j\}_{j=1}^{k+1}$, based on the *last* edge from $F_{\text{opt}}$ used in the path. These yield $k$ classes; the $(k+1)$th class consists of paths that do not use any edges in $F_{\text{opt}}$. Additionally, note that for any edge $\{u, v\}$, all the paths $P_i$ that use the edge use the same "orientation" (either they all send flow from $u$ to $v$ or vice versa). Now, let $F_j = \sum_{P_i \in Q_j} f_{P_i}$ denote the total flow in class $Q_j$. For each $j$ with $F_j > 0$, define the normalized flow

$$f^{(j)} = \frac{1}{F_j} \sum_{P_i \in Q_j} f_{P_i} P_i,$$

which is a unit flow from $s$ to $v$ using paths in $Q_j$.

For each edge $e$, let $f_e^*$ denote the flow passing through edge $e$ in unit flow $f^*$, and $f_e^*(j)$ be the flow passing through edge $e$ by using paths in class $Q_j$ (so $f_e^* = \sum_{j=1}^{k+1} f_e^*(j)$). Now consider the energy (sum of squared flows) of each unit flow $f^{(j)}$:

$$\mathcal{E}(f^{(j)}) = \sum_e \frac{f_e^*(j)^2}{F_j^2}.$$

By using the inequality $\forall a_i, b_i \geq 0$, $\min_i(\frac{a_i}{b_i}) \leq \frac{\sum_i a_i}{\sum_i b_i}$. We are interested in bounding $\min_j \mathcal{E}(f_j)$

$$\begin{aligned}
\min_j \mathcal{E}(f^{(j)}) &= \min_j \left( \frac{\sum_e f_e^*(j)^2}{F_j^2} \right) \\
&\leq \frac{\sum_{j=1}^{k+1} \sum_e f_e^*(j)^2}{\sum_{j=1}^{k+1} F_j^2} \\
&\leq \frac{\sum_e \sum_{j=1}^{k+1} f_e^*(j)^2}{\sum_{j=1}^{k+1} F_j^2}
\end{aligned}$$

Since $f_e^*(j) \geq 0$ implies that $\sum_{j=1}^{k+1} f_e^*(j)^2 \leq \left(\sum_{j=1}^{k+1} f_e^*(j)\right)^2 \leq f_e^2$ and also $\sum_{j=1}^{k+1} F_j = 1$ implies that from Cauchy–Schwarz inequality $\sum_{j=1}^{k+1} F_j^2 \geq \frac{1}{k+1}$.

$$\min_j \mathcal{E}(f^{(j)}) \leq \frac{\sum_e (f_e^*)^2}{\frac{1}{k+1}} \leq (k+1) \cdot R_{G'}(s, v). \tag{7}$$

Let $j$ be the index at which the minimum is attained above, and let $Q_j$ be the corresponding equivalence class. If $j = (k+1)$, the paths in $Q_j$ have no edges from $F_{\text{opt}}$, so by using the definition of effective resistance from Lemma 3 that $R_G(s, v)$ is the minimum energy possible in a $(s, v)$ unit flow in graph $G$. Since $f^{(j)}$ is a candidate $s - v$ unit flow in $G$, we have

$$R_G(s, v) \leq \mathcal{E}(f^{(j)}) \leq (k+1) \cdot R_{G'}(s, v).$$

If $j \leq k$ , the paths in $Q_j$ corresponds to paths whose last new edge is some $e = (v_p, v_q) \in F_{opt}$ with flow flowing in direction $v_p \to v_p$ in $f^*$. Then we will prove that

$$R_G(v_q, v) \leq (k+1) \cdot R_{G'}(s, v) - 1$$

Let $p_1, p_2, p_3 ... p_r$ be the paths that are part of equivalence group $Q_j$ and let $f_{p_1}, f_{p_2}, ... f_{p_r}$ be the corresponding amounts of flow that pass through each of these paths in flow-$f^{(j)}$ following $\sum_i f_{p_1} = 1$. Now using the sub-paths $p_1[v_q, v], p_2[v_q, v], ... p_r[v_q, v]$ that start at $v_q$ and end at $v$ (since every $p_i$ go through last new edge $v_p \to v_q$ and then to $v$), we create a new unit flow $f^{(j')}$ from $v_q$ to $v$ with same flow amounts $f_{p_1}, f_{p_2}, ... f_{p_r}$.

Since the flow $f^{(j')}$ is a sub flow of $f^{(j)}$, which implies: $\mathcal{E}(f^{(j')}) \leq \mathcal{E}(f^{(j)})$ and in addition to this, we also know that last new edge $e = (v_p, v_q)$ is present in every path $p_i$ of $f^{(j)}$ and not part of any path $p_i[v_q, v]$ in flow $f^{(j')}$, hence this implies a much stronger inequality that

$$\mathcal{E}(f^{(j')}) \leq \mathcal{E}(f^{(j)}) - (f_e^{(j)})^2.$$

Now using the fact that $f_e^{(j)} = 1$ (since edge $e$ is present in every path $p_i$ of flow $f^{(j)}$) and $\mathcal{E}(f^{(j)}) \leq (k+1) \cdot R_{G'}(s, v)$ from inequality (7), we have

$$\mathcal{E}(f^{(j')}) \leq (k+1) \cdot R_{G'}(s, v) - 1$$

Now observe that $f^{(j')}$ is a candidate unit flow from $v_d$ to $v$ in graph $G$ (since $p_i[v_q, v]$ does not contain any new edges from $F_{opt}$). By using this with Lemma 3, we have

$$R_G(v_q, v) \leq \mathcal{E}(f^{(j')}) \leq (k+1) \cdot R_{G'}(s, v) - 1$$

Hence we proved that for every vertex $v$ in graph $G$ either $R_G(s, v) \leq (k+1) \cdot R_{G'}(s, v)$ is true or there exist vertex $v_q \in C$ that follows $R_G(v_q, v) \leq (k+1) \cdot R_{G'}(s, v) - 1$. Since $|C| \leq 2k$ this proves the lemma.

□

### A.3 Proof of Observation 8

*Proof.* Non-negativity, identity, and symmetry follow trivially from the definition of metric space. For the triangle inequality, observe that

$$d(x, y) + d(y, z) = 2 + R_G(x, y) + R_G(y, z).$$

Using the fact that $R$ is a metric with the above equation, we have

$$d(x, y) + d(y, z) \geq 2 + R_G(x, z),$$

which in turn implies

$$\begin{aligned} d(x, y) + d(y, z) &\geq 1 + \big(1 + R_G(x, z)\big) \\ &\geq 1 + d(x, z) \\ &\geq d(x, z). \end{aligned}$$

Hence, $d$ satisfying the triangle inequality completes the proof that $d$ is a metric. □

### A.4 Proof of Lemma 10

*Proof.* Firstly, we observe from our choice of $d$ and $p_j = R_G(s, j)$ and $\ell = 2k$ the objective KMP$(G, s, 2k)$ in Definition 9 is to return a subset $T \subseteq V$ of size $\leq 2k$ minimizing the objective.

$$\sum_{j \in V \setminus s} \min(p_j, d(j, T)).$$

Now using the fact that $\forall j \in T$, $d(j, T) = 0$ and $\forall j \in V \setminus (T \cup \{s\})$, $d(j, T) = 1 + R_G(j, T)$ and with $V' = V \setminus (T \cup \{s\})$ the objective can be simplified as

$$\sum_{j \in V'} \min(R_G(s, j), 1 + R_G(j, T)) \tag{8}$$

Where $R_G(j, T) = \min_{i \in T} R_G(i, j)$ (smallest pairwise effective resistance between $j$ and any vertex in $T$). Now consider a set $C$ of size at most $2k$ that contains the endpoints of edges in $F_{\text{opt}}$. From proof of Lemma 7, we observe that for every client $j \in V$,

$$\min\big(R_G(s, j), R_G(j, C) + 1\big) \leq (k+1) \cdot R_{G'}(s, j).$$

Now summing over $j \in V'$ (where $V' = V \setminus (C \cup \{s\})$), we have

$$\sum_{j \in V'} \min(R_G(s, j), 1 + R_G(j, C)) \leq (k+1) \cdot \sum_{j \in V'} R_{G'}(s, j) \tag{9}$$

Since $C$ is a candidate set of size $\leq 2k$, the optimal objective value $\Lambda_{kmp}$ of (8) is always less than or equal to the objective value with $T = C$.

$$\Lambda_{kmp} \leq \sum_{j \in V'} \min(R_G(s, j), 1 + R_G(j, C))$$

Combining this with inequality (9), we have

$$\Lambda_{kmp} \leq (k+1) \cdot \sum_{j \in V'} R_{G'}(s,j)$$

Since $V' \subseteq V \setminus s$, we have

$$\Lambda_{kmp} \leq (k+1) \cdot \sum_{j \in V'} R_{G'}(s,j) \leq (k+1) \cdot \sum_{j \in V \setminus s} R_{G'}(s,j) \leq (k+1) \cdot R_{G'}(s)$$

which completes the proof. □

### A.5 Proof of Lemma 11

*Proof.* By $\gamma$-approximation algorithm of objective function (8), we can say that with $V' = V \setminus (C \cup \{s\})$

$$\sum_{j \in V'} \min(R_G(s,j), 1 + R_G(j,C)) \leq \gamma \cdot R_{kmp} \tag{10}$$

Now, we upper bound the value of $R_{G \cup F}(s,j)$ for each $j \in V'$ by case-based analysis.

*Case 1.* For $j$ that follows $R_G(s,j) \leq 1 + R_G(j,C)$.

Since adding edges will only reduce effective resistance in graph by the monotonicity property of effective resistance,

$$R_{G \cup F}(s,j) \leq R_G(s,j)$$

This implies that $R_{G \cup F}(s,j) \leq \min(R_G(s,j), 1 + R_G(j,C))$ for all $j$ that satisfies *Case-1*.

*Case 2.* For $j$ that follows $R_G(s,j) \geq 1 + R_G(j,C))$.

Let $i \in C$ be the vertex that has the smallest effective resistance distance from $C$ to $j$ in graph $G$, then $R_G(j,C) = R_G(j,i)$. By using the triangle inequality from effective resistance being a metric,

$$R_{G \cup F}(s,j) \leq R_{G \cup F}(s,i) + R_{G \cup F}(i,j)$$

Now using the fact that '$s$' is connected to '$i$' via direct edge in $G \cup F$ (since $s$ and $C$ form a star in $G \cup F$), we get $R_{G \cup F}(s,i) \leq 1$, which in turn implies

$$R_{G \cup F}(s,j) \leq 1 + R_{G \cup F}(i,j)$$

By using the monotonicity property of effective resistance ($R_{G \cup F}(i,j) \leq R_G(i,j)$) and also using the fact that $R_G(j,i) = R_G(j,C)$ as above mentioned, we have

$$R_{G \cup F}(s,j) \leq 1 + R_G(i,j) \leq 1 + R_G(j,C)$$

This implies that $R_{G \cup F}(s,j) \leq \min(R_G(s,j), 1 + R_G(j,C))$ for all $j$ that satisfies *Case-2*.

From proving both cases, we have for any $j \in V'$,

$$R_{G \cup F}(s,j) \leq \min(R_G(s,j), 1 + R_G(j,C))$$

By summation over all $j \in V'$

$$\sum_{j \in V'} R_{G \cup F}(s,j) \leq \sum_{j \in V'} \min(R_G(s,j), 1 + R_G(j,C))$$

By combining with inequality (10), we have

$$\sum_{j \in V'} R_{G \cup F}(s,j) \leq \gamma \cdot R_{kmp}$$

Now using the fact that $R_{G \cup F}(s,j) \leq 1$ for all vertices $j \in C$ (since $s$ and $C$ form a star in $G \cup F$) and $|C| \leq 2k$, we have

$$\sum_{j \in V \setminus s} R_{G \cup F}(s,j) \leq \sum_{j \in V'} R_{G \cup F}(s,j) + \sum_{j \in C} R_{G \cup F}(s,j)$$

$$R_{G^*}(s) \leq \gamma \cdot R_{kmp} + 2k$$

Which completes the proof. □

### A.6 Proof of Lemma 13

*Proof.* Let $F_{opt} = \{u_i, v_i\}_{i=1}^k$ be $k$-optimal edges and each edge $\{u_i, v_i\}$ is represented as a vector $f_i = e_{u_i} - e_{v_i} \in \mathbb{R}^n$, where $e_j$ denotes the $j$-th standard basis vector.

From Equation (2), the effective resistance between vertices $x$ and $y$ in the augmented graph $G' = G + F_{opt}$ can be expressed as:

$$R_{G'}(x, y) = (e_x - e_y)^T \left(\mathcal{L}_{G'}\right)^\dagger (e_x - e_y)$$

Using the fact that $\mathcal{L}_{G'} = \mathcal{L}_G + \sum_{i=1}^k f_i f_i^T$, we can write

$$R_{G'}(x, y) = (e_x - e_y)^T \left(\mathcal{L}_G + \sum_{i=1}^k f_i f_i^T\right)^\dagger (e_x - e_y)$$

Now, observe that each $f_i$ can be decomposed as:

$$f_i = (e_{u_i} - e_s) + (e_s - e_{v_i}) = g_{u_i} + g_{v_i}$$

Where $g_{u_i}$ and $g_{v_i}$ represent the edges $(s, u_i)$ and $(s, v_i)$ respectively and let $F'_{opt}$ be the multiset of all such edges from $s$, containing at most $2k$ edges (with possible repetitions).

By using the inequality that $(g_{u_i} - g_{v_i})(g_{u_i} - g_{v_i})^T \succeq 0$ (since any matrix of form $AA^T$ is positive semi definite), we have

$$f_i f_i^T = (g_{u_i} + g_{v_i})(g_{u_i} + g_{v_i})^T \preceq 2(g_{u_i} g_{u_i}^T + g_{v_i} g_{v_i}^T)$$

Which in turn implies

$$\mathcal{L}_G + \sum_{i=1}^k f_i f_i^T \preceq \mathcal{L}_G + 2\sum_{i=1}^k (g_{u_i} g_{u_i}^T + g_{v_i} g_{v_i}^T)$$

Since $\mathcal{L}_G \succeq 0$, we extend that to

$$\mathcal{L}_G + \sum_{i=1}^k f_i f_i^T \preceq 2\left(\mathcal{L}_G + \sum_{i=1}^k (g_{u_i} g_{u_i}^T + g_{v_i} g_{v_i}^T)\right)$$

Thus, we have

$$\mathcal{L}_{G'} \preceq 2\mathcal{L}_{G+F'_{opt}}$$

Now using the fact that Moore-Penrose pseudoinverse reverses the Loewner order (when restricted to their common invertible subspace), we obtain:

$$\mathcal{L}^\dagger_{G+F'_{opt}} \preceq 2\mathcal{L}^\dagger_{G'}$$

Using this with equation (2), we have

$$R_{G+F'_{opt}}(x, y) \leq 2R_{G'}(x, y) \quad \forall (x, y) \in \binom{V}{2}$$

Now summing over all pairs $x \in V \setminus s$ and $y = s$ yields

$$R_{G+F'_{opt}}(s) \leq 2R_{G'}(s)$$

Where as summing over $(x, y) \in \binom{V}{2}$ yields

$$\mathcal{K}_{G+F'_{opt}} \leq 2\mathcal{K}_{G'}$$

Which completes the proof. □

### A.7 Proof of Theorem 14

*Proof.* This proof follows directly from supermodularity argument in [Shan et al., 2018] for weighted graphs that in turn works for the multi graphs as in our case.

*Monotonicity:* Firstly we know that the effective resistance $R$ is a monotonically decreasing function from Rayleigh's Monotonicity law [Doyle and Snell, 1984], Hence $R_G(s)$, which is the sum of effective resistances of all the vertices from $s$ is also a monotonically decreasing function. This implies for any two multi-set's $D \subset F$

$$R_{G+D}(s) \geq R_{G+F}(s)$$

This in turn implies the monotonicity of $\phi$

$$\phi(D) \geq \phi(F)$$

*Supermodularity:* In order to prove that $\phi$ is supermodular, we need to show that for any two multi-sets of edges $D \subset F$ all incident to $s$ and any edge $\{s, x\}$,

$$\phi(D) - \phi(D + \{s, x\}) \geq \phi(F) - \phi(F + \{s, x\}) \tag{11}$$

To prove this, firstly we observe the fact that a multi-graph on an electrical network can be viewed as a weighted graph with the weight between two vertices being the number of parallel edges between them (Since unit weight in an undirected graph corresponds to the unit conductance resistor in an electrical network with conductance = 1/resistance and the relative conductance between two nodes is the sum of the conductance's of the resistors when the resistors are connected in parallel). Hence, we can view the Laplacian of the multi-graph as the Laplacian of a weighted graph, with the weight indicating the conductance due to parallel edges.

Now, we carefully create a function $g(t) = \phi(D + t \cdot (F - D)) - \phi(D + t \cdot (F - D) + \{s, x\})$, where $F - D$ is a multi-set of edges that are present in $F$ and not present in $D$. Here, adding $t \cdot (F - D)$ indicates that we are adding the edges with weight $t$ on each of them(Corresponds to adding a new resistor with conductance value '$t$' with respect to each edge in set $F - D$). From the definition of the function $g$, we observe that proving $g(0) \geq g(1)$ is the same as proving equation 11. Hence, proving $g(0) \geq g(1)$ is sufficient to prove the supermodularity of $\phi$.

Now, we use the fact that for any set $T$, $\phi(T) = R_{G+T}(s) = \mathrm{Tr}[\mathcal{L}_{G+T}[s]^{-1}]$ from [Izmailian et al., 2013], where $\mathcal{L}_G[s]$ refers to the principal minor of the Laplacian obtained by removing the $s$-th row and $s$-th column (for a connected graph $G$, $\mathcal{L}_G[s]$ turns out to be invertible). Using this, we have

$$g(t) = \mathrm{Tr}[\mathcal{L}_{G+t\cdot(F-D)}[s]^{-1}] - \mathrm{Tr}[\mathcal{L}_{G+t\cdot(F-D)+\{s,x\}}[s]^{-1}]$$

By using the fact that $\mathrm{Tr}[\mathcal{L}_{G+\{s,u\}}[s]^{-1}] = \mathrm{Tr}[(\mathcal{L}_G[s] + e_u e_u^T)^{-1}]$ (where $e_u$ is the $u$-th basis vector in $\mathbb{R}^{n-1}$) and a multi-set $Z = \{v, \forall (s, v) \in F - D\}$, we have

$$g(t) = \mathrm{Tr}[(\mathcal{L}_G[s] + t \sum_{j \in Z} e_j e_j^T)^{-1}] - \mathrm{Tr}[(\mathcal{L}_G[s] + e_x e_x^T + t \sum_{j \in Z} e_j e_j^T)^{-1}]$$

Now solve the derivative of function $g(t)$ using $\frac{d}{dt}\mathrm{Tr}(A(t)^{-1}) = -\mathrm{Tr}[A(t)^{-1} \frac{d}{dt}(A(t))A(t)^{-1}]$.

By defining $P = \mathcal{L}_{G+t\cdot(F-D)}[s] = \mathcal{L}_G[s] + t \cdot \sum_{j \in Z} e_j e_j^T$ and $Q = \mathcal{L}_{G+t\cdot(F-D)+\{s,x\}}[s] = \mathcal{L}_G[s] + e_x e_x^T + t \cdot \sum_{j \in Z} e_j e_j^T$, We get

$$\begin{aligned}
g'(t) &= -\mathrm{Tr}[P^{-1}(\sum_j e_j e_j^T)P^{-1}] + \mathrm{Tr}[Q^{-1}(\sum_j e_j e_j^T)Q^{-1}] \\
&= -\sum_j \mathrm{Tr}[P^{-1} e_j e_j^T P^{-1}] + \sum_j \mathrm{Tr}[Q^{-1} e_j e_j^T Q^{-1}] \\
&= -\sum_j \sum_i ([P^{-1}]_{i,j})^2 + \sum_j \sum_i ([Q^{-1}]_{i,j})^2
\end{aligned}$$

Where $[A]_{i,j}$ indicates the element in $i$-th row and $j$-th column of matrix $A$, which can further be simplified as

$$g'(t) = \sum_j \sum_i ([Q^{-1}]_{i,j})^2 - ([P^{-1}]_{i,j})^2 = \sum_j \sum_i ([Q^{-1}]_{i,j} + [P^{-1}]_{i,j})([Q^{-1}]_{i,j} - [P^{-1}]_{i,j}) \tag{12}$$

Now using the fact that $P, Q$ are principal minors of the Laplacian matrices of graphs $G + t \cdot (F - D)$ and $G + t \cdot (F - D) + \{s, x\}$, [Plemmons, 1977] implies that $P, Q$ are M-matrices. By using the property that all elements in the inverse of an M-matrix are non-negative from [Plemmons, 1977], we have

$$\forall (i,j) \quad [P^{-1}]_{i,j} \geq 0, [Q^{-1}]_{i,j} \geq 0$$

Now, we look at the matrix $Q^{-1} - P^{-1} = (P + e_x e_x^T)^{-1} - P^{-1}$ using Sherman-Morrison formula [Meyer, 1973] as below

$$Q^{-1} - P^{-1} = (P + e_x e_x^T)^{-1} - P^{-1} = -\left(\frac{P^{-1} e_x e_x^T P^{-1}}{1 + e_x^T P^{-1} e_x}\right)$$

Since we know from above that $[P^{-1}]_{i,j} \geq 0$ and $e_x$ is non-negative basis vector, this implies that all entries in the matrix $\left(\frac{P^{-1} e_x e_x^T P^{-1}}{1 + e_x^T P^{-1} e_x}\right)$ are non-negative, this in turn implies that

$$\forall (i,j) \quad [Q^{-1}]_{i,j} - [P^{-1}]_{i,j} \leq 0$$

Using this in Equation 12 implies that

$$g'(t) \leq 0$$

Hence, $g(t)$ being a decreasing function implies $g(0) \geq g(1)$, which in turn completes the proof that $\phi$ is supermodular.

□

### A.8 Proof of Theorem 2

*Proof.* Algorithm 2 constructs a set $F'$ of $O(k \log(k/\delta))$ edges that follows $\mathcal{K}_{G+F'} \leq (4 + \delta)\mathcal{K}_{G^*}$. Proof goes as follows:

Firstly, we observe that Algorithm 2 as part of **Step-1** to **Step-5** adds at most $2k$ edges by using the same steps as Algorithm 1, creating a graph $H = G + F_s$ that follows as below from inequality (4) in the analysis of Algorithm 1.

$$R_H(s) \leq (7k + 5) R_{G*}(s) \tag{13}$$

Let $F_{opt}$ be the optimal set of $k$-edges added to $G$ forming $G^* = G + F_{opt}$. From Lemma 13, there exist a set $P$ of $2k$ edges from any vertex $s$ such that $R_{G+P}(s) \leq 2R_{G^*}(s)$ and by monotonicity property of effective resistance, it also implies that

$$R_{H+P}(s) \leq R_{G+P}(s) \leq 2R_{G^*}(s)$$

Let $P_{opt}$ be the optimal multi-set of $2k$-edge additions to $H$ that gives the best single source objective $R_{H+P_{opt}}(s)$. Then, we have

$$R_{H+P_{opt}} \leq R_{H+P}(s) \leq 2R_{G^*}(s) \tag{14}$$

Now using the fact that the function $\phi(F) = R_{H+F}(s)$ is monotone and supermodular from Theorem 14 and $P_{opt}$ being the optimal set of $2k$ edges that improves the objective function. The greedy algorithm in **Step-6** to **Step-10** of Algorithm 2 that adds set $P_t$ of $t$ edges to improve $R_H(s)$ follows as below from standard greedy analysis of monotone, supermodular functions in [Nemhauser et al., 1978].

$$R_H(s) - R_{H+P_t}(s) \geq \left(1 - \left(1 - \frac{1}{2k}\right)^t\right)(R_H(s) - R_{H+P_{opt}})$$

Which in turn implies

$$\begin{aligned} R_{H+P_t}(s) - R_{H+P_{opt}}(s) &\leq \left(1 - \frac{1}{2k}\right)^t (R_H(s) - R_{H+P_{opt}}) \\ &\leq \left(1 - \frac{1}{2k}\right)^t R_H(s) \end{aligned}$$

With $\Delta = \frac{R_H(s)}{R_{H+P_{opt}}(s)}$ ($\Delta \geq 1$) and using inequality $1 - x \leq e^{-x}$, we have

$$R_{H+P_t}(s) \leq R_{H+P_{opt}}(s) \left(1 + \Delta \cdot e^{-\frac{t}{2k}}\right)$$

Now inequalities (13), (14) together imply that $\Delta \leq 3.5k + 2.5$ and separately using inequality 14, we have

$$R_{H+P_t}(s) \leq 2R_{G^*}(s) \left(1 + (3.5k + 2.5) \cdot e^{-\frac{t}{2k}}\right)$$

Now running greedy algorithm for $t = 2ck \cdot \log(\frac{k}{\delta})$ steps for an appropriate constant $c$, we can ensure that

$$R_{H+P_t}(s) \leq 2R_{G^*}(s)(1 + \frac{\delta}{8+\delta})$$

Now using the triangle inequality that $R_{H+P_t}(u,v) \leq R_{H+P_t}(s,u) + R_{H+P_t}(s,v)$ with $R_{H+P_t}(u,u) = 0$, we have

$$\begin{aligned}
\mathcal{K}_{H_{G+P_t}} &= \sum_{u<v} R_{H+P_t}(u,v) = \frac{1}{2}\sum_{u}\sum_{v} R_{G+P_t}(u,v) \\
&\leq \frac{1}{2}\sum_{u}\sum_{v} R_{H+P_t}(s,u) + R_{H+P_t}(s,v) \\
&\leq n \cdot \sum_{u} R_{H+P_t}(s,u) \\
&\leq n \cdot R_{H+P_t}(s)
\end{aligned}$$

This in turn implies

$$\mathcal{K}_{H+P_t} \leq 2n \cdot R_{G^*}(s)(1 + \frac{\delta}{8+\delta})$$

From lemma 6 (replacing $\delta$ with $\delta/4$), we know that with probability $\geq 1 - \frac{1}{n}$, for some $s \in \mathcal{S}$ in **Step-2** of Algorithm 2

$$R_{G*}(s) \leq \frac{2 + \frac{\delta}{4}}{n} \cdot \mathcal{K}_{G*}$$

Hence this implies

$$\mathcal{K}_{H+P_t} \leq 2n \cdot (1 + \frac{\delta}{8+\delta})(\frac{2+\frac{\delta}{4}}{n}) \cdot \mathcal{K}_{G*}$$

Simplifying, we obtain

$$\mathcal{K}_{H+P_t} \leq (4+\delta)\mathcal{K}_{G^*}$$

Hence the graph $G' = G + F' = H + P_t = G + F_s + F_t$ that is formed by adding a total of $|F_s| + |F_t| = 2k + 2ck \cdot \log(\frac{k}{\delta}) = O(k \cdot \log(\frac{k}{\delta}))$ edges using Algorithm 2 follows

$$\mathcal{K}_{G'} \leq (4+\delta)\mathcal{K}_{G^*}$$

Which completes the proof.

□