Generalized Fractional Stockwell and Wavelet Transforms

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***Abstract*—Since the introduction of the fractional Fourier transform by V. Namias in 1980, the subject of fractional integral transformations has flourished and expanded in different directions. The Stockwell transform, which is widely used in frequency analysis, turns out to be related to the continuous wavelet transform. Several fractional Stockwell and wavelet transforms have been introduced in the scientific literature in the last decade. In this talk we discuss the relationship between the Stockwell and the continuous wavelet transforms and then present generalizations of these transforms using a novel convolution operation. Properties of these transforms are presented.**

***Index Terms*—Fractional transforms, fractional Stockwell transform, fractional wavelet transforms.**

I. Introduction

Integral transforms are an important mathematical tool used in many fields, in particular, in physics and engineering. Chief among these integral transforms are the Fourier and the continuous wavelet transforms, which because of their utility in modeling time-frequency and time-scale representations, have ubiquitous applications in signal and image processing, data compression, fluid dynamics, texture analysis, and tomography.

The significance of these two transforms piqued the interest of many researchers and prompted them to introduce generalizations of those transforms. Among the many generalizations of the Fourier transform is the fraction Fourier transform introduced by V. Namais in 1980 [**?**] which is regarded as the first authentic fractional generalization of the Fourier transform. This transform depends on an angle $0 \leq \theta \leq 2\pi$ and reduces to the identity transformation when $\theta = 0$ and to the standard Fourier transform when $\theta = \pi/2$. The fraction Fourier transform has numerous applications in optics and signal processing [**?**], [**?**].

In the last two decades other generalizations of the fraction Fourier transform were introduced, such as the linear canonical transform formally introduced by S. Collins in 1970 [**?**] in his study of coherent light propagation through lens system. The linear canonical transform depends on four parameters a, b, c, d with $ad - bc = 1,\ b \neq 0$, and reduces to Namias' transform for a, b, c, d being $\cos\theta, \sin\theta, -\sin\theta, \cos\theta$, respectively.

More generally, the set of real canonical transforms may be viewed as a group of unitary transformations acting on $L^2(I\!\!R)$ and it is a representation of the special linear group $SL(2, I\!\!R)$ of unimodular matrices

$$SL(2, I\!\!R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} :\ a, b, c, d \in I\!\!R \text{ such that } ad - bc = 1 \right\}.$$

The transform has been the focus of many research articles in recent years because of its applications in optics, radar system analysis, and signal processing.

A more general transform than the linear canonical transform is the Special Affine Fourier Transformation (SAFT) introduced by Abe and Sheridan [**?**], [**?**] in their study to develop an operator formalism to show how the fractional Fourier transformation of a wave function can be derived from the rotation of the corresponding Wigner distribution function in phase-space.

The SAFT depends on 6 parameters $(a, b, c, d; p, q)$ with $ad - bc = 1$. In phase space if we denote the position and wavenumber by x and k respectively, the transformation action maybe described by the equation

$$\begin{pmatrix} x' \\ k' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ k \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix}.$$

A closely related transform to the special affine Fourier transform is the *offset linear canonical transform* introduced by Pie et al [**?**]. Sampling results for the SAFT were obtained by Zayed and Bhandari [**?**].

The Windowed-Fourier Transform, or Short-time Fourier transform, of a signal $f \in L^2(I\!\!R)$ which is given by

$$F(t, w) = \frac{1}{\sqrt{2\pi}} \int_{I\!\!R} f(x) g(x - t) e^{iwx} dx,\ w, t \in I\!\!R,$$

where $g \in L^2(I\!\!R)$ is the window function, extends the Fourier transform in a different direction to the time-frequency domain. The Gabor transform is a special case of the windowed Fourier transform where the window function is a Gaussian function.

R.G. Stockwell et.al. [**?**] introduced the Stockwell transform, which is commonly referred to as the S transform, as a generalization of the windowed Fourier transform [**?**]. They employed a Gaussian window function that dilates and translates to achieve localization of a signal $f \in L^2(I\!\!R)$.

Wei and et al [**?**], [**?**] introduced a generalized Stockwell transform using a general window function $\phi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$

instead of a Gaussian function and obtained an inversion formula for this transform. The Stockwell transform with a general window function is given as:

$$S_\phi(b,\xi) = \frac{|\xi|}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f(x)\overline{\phi(\xi(x-b))}\, dx\ , \quad (1)$$

$\forall(b,\xi) \in \mathbb{R}\times\mathbb{R}^*$. The Stockwell transform is widely applied in geophysics [?] to provide more detailed information about spectral (frequency) components. For other generalizations of the Stockwell transform, see [?], [?]

As for generalizations of the wavelet transform, let us first recall the definition of the continuous wavelet transform $F(a,b)$ of $f \in L^p(I\!R), 1 \le p \le 2$, with respect to a mother wavelet ψ,

$$F(a,b) = \int_{I\!R} f(x)\psi_{a,b}(x)dx,$$

where $\psi_{a,b}(x) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{x-b}{a}\right),\ a > 0,\ b \in I\!R$, and ψ satisfies the admissibility condition

$$\int_{I\!R} \frac{|\hat{\psi}(w)|^2}{|w|} dw < \infty, \quad (2)$$

where $\hat{\psi}$ is the Fourier transform of ψ.

There is a number of fractional wavelet transforms, such as the one in which the mother wavelet ψ is replaced by

$$\psi^\theta_{a,b}(x) = \psi_{a,b}(x)\exp\left[-i(x^2-b^2)/2\cot\theta\right];$$

see also [?], [?].

The aim of this article is to introduce a more general fractional Stockwell and wavelet transforms that combine feature of these transforms and the special affine Fourier transform. The passage to this generalization is through the connection between the wavelet and the Stockwell transform which was first reported in [?] and a new convolution theorem for the Stockwell transform [?].

II. The Manin result

First, let us observe that in the wavelet transform if we replace the dilation parameter a by $1/\xi$ and set $\sqrt{\xi}\psi_{\xi,b}(x)$ as the wavelets, it becomes evident that the Stockwell transform of a function f is just the wavelet transform of the modulated version $e^{ix\xi}f(x)$ of f.

Let $\mathbf{A} = \left(\begin{array}{cc|c} a & b & p \\ c & d & q \end{array}\right)$. Furthermore, for simplification, we make use of the following notations, $\boldsymbol{\lambda} = bq - dp$,.

$$\mu_{p,b}(x) = \exp\left(i\frac{p}{b}x\right) \quad (3)$$

$$\Omega_{\mathbf{A}}(x) = \exp\left(\frac{i}{2b}[\,dx^2 + 2\boldsymbol{\lambda}x]\right), \quad (4)$$

$$\Phi_{a,b}(x) = \exp\left(i\frac{a}{2b}x^2\right) \quad (5)$$

$$\Psi_{\mathbf{A}}(x) = \exp\left(\frac{i}{2b}[\,dx^2 - 2px + 2\boldsymbol{\lambda}x]\right), \quad (6)$$

It is customary in the wavelet transform to denote the translation and dilation parameters by b and a, respectively. However, because these symbols appear as parameters in the matrix $\mathbf{A}$ we will replace them by y and ξ respectively.

For a given parameter matrix $\mathbf{A}$, $(y,\xi) \in \mathbb{R}\times\mathbb{R}^*$, and $0 \ne g \in L^1(\mathbb{R},\mathbb{C}) \cap L^2(\mathbb{R},\mathbb{C})$, we let

$$g^{\mathbf{A}}_{y,\xi}(x) = |\xi|\ g(\xi(x-y))\ \Phi_{a,b}(y)\ \overline{\Phi_{a,b}(x)} \quad (7)$$

$$\times\ \mu_{p-\boldsymbol{\lambda},b}(y-x)\ \mu_{\xi,b}(x)\forall x \in \mathbb{R} \quad (8)$$

and let $\boldsymbol{\lambda} = bq - dp$.

Definition 1: Let $0 \ne g \in L^1(\mathbb{R},\mathbb{C}) \cap L^2(\mathbb{R},\mathbb{C})$ and $f \in L^2(\mathbb{R},\mathbb{C})$. For a parameter matrix $\mathbf{A}$, and mother wavelet ψ, we define the generalized affine wavelet transform (GAWT) of f as follows.

$$\begin{aligned} &(\mathcal{W}^s_{\mathbf{A},\psi}f)(y,\xi) \\ =\ &\frac{|\xi|}{\sqrt{2\pi|b|}}\int_{\mathbb{R}} f(x)\overline{\psi(\xi(x-y))} \\ \times\ &\exp\left(\frac{-i}{2b}\left[a(y^2-x^2) + 2((p-\boldsymbol{\lambda})(y-x))\right]\right)dx \\ &\forall(y,\xi) \in \mathbb{R}\times\mathbb{R}^*. \end{aligned}$$

Similarly, we define the generalized fractional Stockwell transform (GFST) of f as follows

$$\begin{aligned} &(\mathcal{S}^s_{\mathbf{A},\psi}f)(y,\xi) = \frac{1}{\sqrt{2\pi|b|}}\int_{\mathbb{R}} f(x)\overline{\psi^{\mathbf{A}}_{y,\xi}(x)}\,dx, \\ =\ &\frac{|\xi|}{\sqrt{2\pi|b|}}\int_{\mathbb{R}} f(x)\overline{\psi(\xi(x-y))} \\ \times\ &\exp\left(\frac{-i}{2b}\left[a(y^2-x^2) + 2((p-\boldsymbol{\lambda})(y-x)+\xi x)\right]\right)dx \\ &\forall(y,\xi) \in \mathbb{R}\times\mathbb{R}^*. \end{aligned}$$

As we pointed out above that the Stockwell transform of a function f is essentially the wavelet transform of a modulated version of f, in the above definition similar feature is retained.

Definition 2: [?] Suppose $f \in L^p(\mathbb{R},\mathbb{C})$, $p = 1$ or 2, $g \in L^1(\mathbb{R},\mathbb{C})$ and $\mathbf{A}$ is a parameter matrix. We define a novel special affine convolution or a generalized fractional convolution as follows.

$$(f \otimes_{\mathbf{A}} g)(x) = \frac{\overline{\Phi_{a,b}(x)}}{\sqrt{|b|}}\left(\Phi_{a,b}f * \Psi_{\mathbf{A}}g\right)(x),\ \forall x \in \mathbb{R},$$

where $*$ denotes the standard convolution associated with the Fourier transform, i.e.,

$$(h_1 * h_2)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h_1(y)h_2(x-y)\,dy.$$

.

It follows with some easy calculations that

$$(\mathcal{S}^s_{\mathbf{A},g}f)(y,\xi) = [\mu_{-\xi,b}f \otimes_{\mathbf{A}} \overline{\Phi_{d,b}}D_\xi\check{\bar{g}}](y),\ \forall(y,\xi) \in \mathbb{R}\times\mathbb{R}^*.$$

In the rest of the paper, we will list some properties of the generalized fractional Stockwell transform and from which with minor adjustment, one can drive the analog properties for the GAWT.

Here we list two important properties of the GFST without proof since the proofs are long and will be published somewhere else.

Theorem 1: (The Parseval's identity) Let $0 \neq g \in L^1(\mathbb{R},\mathbb{C}) \cap L^2(\mathbb{R},\mathbb{C})$ such that

$$0 \neq C_{b,g} = \frac{1}{|b|\sqrt{2\pi}} \int\limits_{\mathbb{R}} \left| \hat{g}\left(x - \frac{1}{b}\right)\right|^2 \frac{dx}{|x|} < \infty. \qquad (9)$$

Then, for $f, h \in L^2(\mathbb{R},\mathbb{C})$, we have

$$\left\langle \mathcal{S}^s_{\mathbf{A},g} f, \mathcal{S}^s_{\mathbf{A},g} h \right\rangle = C_{b,g} \left\langle f, h \right\rangle .$$

We sketch the proof but for the complete proof, see [?].

Proof:

Let $f, h \in L^2(\mathbb{R},\mathbb{C})$. Then

$$\begin{aligned}
&\left\langle \mathcal{S}^s_{\mathbf{A},g} f, \mathcal{S}^s_{\mathbf{A},g} h \right\rangle \\
= \;& \frac{1}{2\pi} \int\limits_{\mathbb{R}^*} \int\limits_{\mathbb{R}} (\mathcal{S}^s_{\mathbf{A},g} f)(y,\xi) \overline{(\mathcal{S}^s_{\mathbf{A},g} f)(y,\xi)}\, dy\; \frac{d\xi}{|\xi|} \\
= \;& \frac{1}{2\pi} \int\limits_{\mathbb{R}^*} \int\limits_{\mathbb{R}} [\mu_{-\xi,b} f \otimes_{\mathbf{A}} \overline{\Phi_{d,b}} D_\xi \breve{\bar{g}}](y) \\
\times \;& \overline{[\mu_{-\xi,b} h \otimes_{\mathbf{A}} \overline{\Phi_{d,b}} D_\xi \breve{\bar{g}}](y)}\, dy\; \frac{d\xi}{|\xi|} \\
= \;& \frac{1}{2\pi} \int\limits_{\mathbb{R}^*} \int\limits_{\mathbb{R}} [\mathcal{S}_{\mathbf{A}}(\mu_{-\xi,b} f \otimes_{\mathbf{A}} \overline{\Phi_{d,b}} D_\xi \breve{\bar{g}})](t) \\
\times \;& \overline{[\mathcal{S}_{\mathbf{A}}(\mu_{-\xi,b} h \otimes_{\mathbf{A}} \overline{\Phi_{d,b}} D_\xi \breve{\bar{g}})](t)}\, dt\; \frac{d\xi}{|\xi|} \\
= \;& \frac{1}{2\pi} \int\limits_{\mathbb{R}^*} \int\limits_{\mathbb{R}} \overline{\Omega_{\mathbf{A}}(t)} [\mathcal{S}_{\mathbf{A}}(\mu_{\xi,b} f)](t) [\mathcal{S}_{\mathbf{A}}(\overline{\Phi_{d,b}} D_\xi \breve{\bar{g}})_{\mathbf{A}}](t) \\
& \qquad \overline{\overline{\Omega_{\mathbf{A}}(t)} [\mathcal{S}_{\mathbf{A}}(\mu_{\xi,b} h)](t) [\mathcal{S}_{\mathbf{A}}(\overline{\Phi_{d,b}} D_\xi \breve{\bar{g}})_{\mathbf{A}}](t)}\, dt\; \frac{d\xi}{|\xi|} \\
= \;& \frac{1}{2\pi} \int\limits_{\mathbb{R}^*} \int\limits_{\mathbb{R}} [\mathcal{S}_{\mathbf{A}} f](t+\xi) \\
\times \;& \overline{[\mathcal{S}_{\mathbf{A}} h](t+\xi)} |[\mathcal{S}_{\mathbf{A}}(\overline{\Phi_{d,b}} D_\xi \breve{\bar{g}})_{\mathbf{A}}](t)|^2\, dt\; \frac{d\xi}{|\xi|} \\
= \;& \frac{1}{2\pi} \int\limits_{\mathbb{R}^*} \int\limits_{\mathbb{R}} [\mathcal{S}_{\mathbf{A}} f](t) \overline{[\mathcal{S}_{\mathbf{A}} h](t)} \\
\times \;& |[\mathcal{S}_{\mathbf{A}}(\overline{\Phi_{d,b}} D_\xi \breve{\bar{g}})_{\mathbf{A}}](t-\xi)|^2\, dt\; \frac{d\xi}{|\xi|} \\
= \;& \frac{1}{2\pi} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}^*} [\mathcal{S}_{\mathbf{A}} f](t) \overline{[\mathcal{S}_{\mathbf{A}} h](t)} \frac{1}{|b|} \left| \hat{g}\left(\frac{t-\xi-\lambda}{b\xi}\right)\right|^2 \frac{d\xi}{|\xi|}\, dt \\
= \;& \frac{1}{2\pi} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}^*} [\mathcal{S}_{\mathbf{A}} f](t) \overline{[\mathcal{S}_{\mathbf{A}} h](t)} \frac{1}{|b|} \left| \hat{g}\left(x - \frac{1}{b}\right)\right|^2 \frac{dx}{|x|}\, dt \\
= \;& \frac{C_{b,g}}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} [\mathcal{S}_{\mathbf{A}} f](t) \overline{[\mathcal{S}_{\mathbf{A}} h](t)}\, dt \\
= \;& \frac{C_{b,g}}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} f(y) \overline{h(y)}\, dy \quad \text{(by Parseval's identity for SAFT)} \\
= \;& C_{b,g} \left\langle f, h \right\rangle .
\end{aligned}$$

Here we have used the relation

$$[\mathcal{S}_{\mathbf{A}} f](y-t) = \exp\left(\frac{i}{2b}[d(t^2 - 2yt) - 2\boldsymbol{\lambda} t]\right) [\mathcal{S}_{\mathbf{A}}(\mu_{t,b} f)](y),$$

which can be verified with some computations.

The inversion formula for the generalized fraction Stockwell transform is given in the next theorem without proof. For the complete proof, see [?].

Theorem 2: (The Inversion Formula) If $0 \neq g \in L^1(\mathbb{R},\mathbb{C}) \cap L^2(\mathbb{R},\mathbb{C})$ satisfies the admissibility condition (??), then for each $f \in L^2(\mathbb{R},\mathbb{C})$, we have

$$f(x) = \frac{1}{C_{b,g}\sqrt{2\pi}} \int\limits_{\mathbb{R}^*} \mu_{\xi,b}(x) [\mathcal{S}^s_{\mathbf{A},g} f \otimes_{\mathbf{A}} \overline{\Phi_{d,b}} D_\xi g](x)\, \frac{d\xi}{|\xi|},$$

weakly in $L^2(\mathbb{R},\mathbb{C})$.

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