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FINDING KEYS TO THE PEANO CURVE

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Abstract. The purpose of this paper is to give a fairly complete analysis of Peano's space filling curve. We begin with a synopsis of the history of the curve, but then begin an analysis using the geometric methods that Hilbert developed. We show that this inherent geometry can be viewed as governed by the action of the Klein Four Group and continue to give a rather full arithmetization of the Peano Curve.

1. Introduction

Although not an earthquake, Giuseppi Peano's discovery of a space-filling curve in 1891 could certainly be classified as a strong aftershock. The earthquake that brought mathematics to an abrupt halt was due to Georg Cantor's discovery in 1878 that there was a 1-to-1 correspondence between the set of numbers between 0 and 1 and the set of ordered pairs of those numbers; see [2]. Geometrically, this meant that any line segment, or indeed an entire line had the same cardinality as any filled square or any filled cube or even any manifold for that matter. The distinction between dimensions had seemed so intuitively obvious that there really was no need for any clear fundamental foundation stones, until Cantor's quite crisp proof otherwise. Cantor was to pay a steep professional and personal price for his impurtenence. Netto's proof, [11] that no such function could be continuous calmed the waters somewhat, but then in 1890 in [13], Peano proved there was a space-filling curve, that is, a continuous map from the continuum, $[0,1]$ to the unit square, $[0,1]^2$. This again fueled the fire of confusion that confronted mathematics. The function, $f_P\colon [0,1] \to [0,1]^2$ that he defined was remarkably easy to define, but simultaneously remarkably complex and came in two parts. First he defined an auxilliary function that could

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very well have been inspired by Cantor's proof. All expansions should be be understood as base 3.

$$\text{If } t = .a_1 a_2 \ldots \quad \text{then } \phi(t) = .(2-a_1)(2-a_2)\ldots.$$

He then defined $f_P(t) = (x_p(t), y_P(t))$ where

$$(1) \qquad \begin{cases} x_P(t) = .a_1 \phi^{a_2}(a_3)\phi^{a_2+a_4}(a_5)\ldots, \\ y_P(t) = .\phi^{a_1}(a_2)\phi^{a_1+a_3}(a_4),\ldots \end{cases}$$

Despite the fact that Peano was a well established and respected mathematician, skeptics took solace in the fact that the proof he gave was both complicated and not quite complete. Too, there was no apparent geometry, and certainly Peano didn't provide any nor may have seen no reason to do so. There were, however, plenty of mathematicians who realized that, as with Cantor, there was much that was completely misunderstood and there was a good deal to be learned and thoroughly analyzed. During the next decade or so many prominent mathematicians contributed to a new and vibrant industry creating space-filling curves. Hilbert, [6] was among the first, and although he does not explicitly mention it, it seems likely that he plunged into Peano's proof with the explicit aim of understanding the geometry of Peano's remarkable curve. In essence he invented a method or tool in [6] by which the geometry of f_p could become apparent. In 1891, in [6] he published his own space-filling curve that became known as the Hilbert function. The "tool" he invented is inductive in nature and uses an algorithm for creating a nested sequence of linearly ordered partitions $[0,1]^2$. It also exudes the Hilbert elegance, so much so that upon reading Hilbert's paper, E. H. Moore of the University of Chicago exclaimed that Hilbert's presentation was "luminous to the geometric imagination." In the cases that Hilbert considered, the partitioning was rather straightforward, and the linear ordering was accomplished via a very few number of arrow diagrams that later became called Hilbert patterns. In 1910, and using the tools that Hilbert introduced in [6], Moore examined the Hilbert patterns for the Peano curve, and published a proof that the Peano curve could be obtained via a Hilbert type construction; see [9].

Hilbert's tools for constructing and analyzing space-filling curves can be used for more than verifying they are what they claim to be, they can also be used to give a rather complete arithmetic understanding of a given space-filling curve, something Sagan in [15] and [16] refers to as an arithmetization of the curve. This was done in [4] for the Hilbert curve. Generalizations of Hilbert's method can be found in [5].

It is the goal of this paper to present a relatively complete arithmetization of the Peano curve. We begin in Section 2 with a geometric description of the Peano curve using the tools that Hilbert introduced in [6]. In subsequent sections we use these same tools to say more about the point to point nature of the Peano curve and show that the partitioning we define in Section 2 can be accomplished via the action of the Klein 4-group at any given stage of the inductive construction. This is one aspect of the Peano curve that isn't shared by the Hilbert curve.

2. Hilbert's view of space-filling

Many, but certainly not all, space filling curves can be presented via a sequence of pictures or diagrams and this is certainly true of the Peano function; see Figures 1 and 2. As pointed out in Section 1, this method of presentation is due to Hilbert. Often, adding rigor to the diagrams is accomplished by using the diagrams to create a uniformly convergent sequence of single-variable functions $f_n\colon [0,1] \to [0,1]^2$ so that the Range(f_n) is (or at least contains) the nth diagram. Then, uniform convergence is used to show that the limit function, f, is continuous, while knowing that the nth diagram is contained in Range(f_n) is used to show that Range(f) $= [0,1]^2$. This program is fine and works, but is tedious and, without additional assumptions, does not entail a unique limit function. More importantly for this paper, this is not at all the way Peano viewed things. Peano's proof was more arithmetic and although the original proof could be verified, any inherent geometry was opaque.

For Hilbert, the diagrams did not describe a sequence of single-variable functions so much as they directly described a continuous correspondence between subintervals of $[0,1]$ and contiguous subsquares partitioning $[0,1]^2$. The existence of a continuous limit function and a proof that that limit function is onto $[0,1]^2$ is a direct consequence of that correspondence. Hilbert's geometric view was analyzed by Rose in the unpublished paper [14] and generalized by Flaten et. al. in [5]. Here, we will only need the more elementary version of this theorem found in [14] and stated below as Theorem 1.

Specifically, Rose formalizes Hilbert's ideas and argument by defining the two critical terms and then proving a theorem. In the next section we adopt this notation and examine the arithmetic generated by this approach as applied to the Peano function itself. In so doing, we refer to any horizontal segment in $[0,1]^2$ of the form $\ell = \{(x, m/3^n) : 0 \le x \le 1,\ m, n \in \mathbb{N}$ and $1 \le m < 3^n\}$ as a *horizontal triadic line segment of level* $n \in \mathbb{N}$; *vertical triadic line segments of level* n are defined similarly. If ℓ is a triadic line segment of level n, but not of level $n-1$, then ℓ is called a triadic line segment of *initial* level n. In general, the term *triadic line segment* will mean either horizontal or vertical and of no particular level. Although with some notational alterations, the bounding lines for $[0,1]^2$ could be considered triadic line segments of level 0, it will end up that these bounding segments are sufficiently special that we treat them separately.

A *level n triadic subsquare* of $[0,1]^2$ will mean a square of the form

$$[m/3^n,(m+1)/3^n]\times[k/3^n,(k+1)/3^n]$$

where $m,k=0,1,2,\dots$ and $0\le m,k\le 3^n-1$. The term *triadic subsquare* is used when referring to the generic case when level is not an issue.

In [14], Rose defines two useful conditions that were originally used by Hilbert in [6] concerning a correspondence between subintervals of $[0,1]$ and triadic subsquares of $[0,1]^2$ and subsequently proves a theorem that is useful for understanding the relationship between the Peano function and other similarly-defined curves.

Definition 1 (Adjacency Condition, [14]). Adjacent subintervals correspond to adjacent subsquares (share an edge, but do not overlap).

Definition 2 (Nesting Condition, [14]). If at the n^{th} partition, the subinterval I_{n_k} corresponds to a subsquare Q_{n_k} then at the $(n+1)^{\text{st}}$ partition the partitioning subintervals of I_{n_k} correspond with the partitioning subsquares of Q_{n_k}.

In either $[0,1]$ or $[0,1]^2$, *adjacent* means contiguous, but not overlapping. These definitions stress a correspondence between the domain and range, and become a key element in understanding the details of how the Hilbert curve behaves. Rose then proves the following theorem that we will make use of in the latter sections of the current article. Generalizations of Theorem 1 and several examples can be found in [5].

Theorem 1 [14, p. 2]. *Any correspondence between the subintervals and subsquares that satisfies the adjacency and nesting conditions determines a unique continuous function f that maps $[0,1]$ onto $[0,1]^2$.*

In the next section we show how the Peano function can be defined using these same tools. Except for notation, our description parallels that of E. H. Moore in [9], but the notation we introduce is needed for the analysis of later sections.

3. A geometric definition of the Peano curve

In the domain, things are simple. At each stage of an induction, we will be left to deal with the 9^n congruent, nonoverlapping intervals of length $1/9^n$ in $[0,1]$ and at the next stage we simply divide each of these intervals into 9 pieces (congruent and adjacent subintervals) labeling these left to right with the digits 0 to 8.

It is in the range, where the real action happens. At each stage of the induction we will be left with 9^n congruent, nonoverlapping and adjacent squares. Each of these squares will be paired with a corresponding interval of

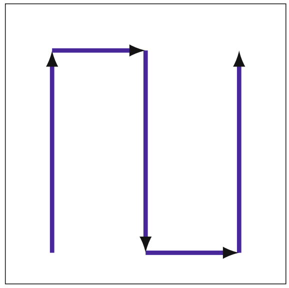

Figure 1: The square $S_\emptyset$ with its initial Peano pattern

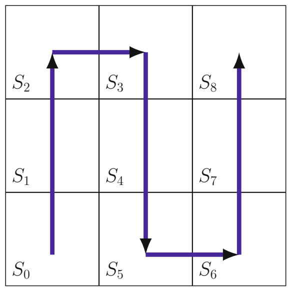

Figure 2: The enumerated level 1 squares

the nth stage in such a way that adjacent intervals are paired with adjacent squares. The matching between the domain intervals of the n^{th} stage and the range squares of the n^{th} stage generates a linear ordering of the squares. Hilbert uses arrow diagrams (or patterns) to illustrate that order.

As our interest is with the arithmetization of the Peano curve, we'll consistently use base 9 notation to describe particular points of the domain and base 9 sequences to label both domain intervals and codomain squares. Specifically, let

$$\Sigma_n^9 = \left\{ a_1 a_2 \dots a_n : a_i = 0, 1, \dots, 8 \text{ for each } i = 1, 2, \dots, n \right\}$$

denote the set of all base 9 sequences of length n ordered lexicographically and let $\Sigma_n^{9*} = \Sigma_n^9 \backslash \{8\overline{8}\}$. For $\sigma \in \Sigma_n^{9*}$ let $\sigma+$ denote the successor of σ and define $I_\sigma = [.\sigma, .\sigma+]$.

To begin the inductive matching of base 9 intervals to patterned triadic squares, define $I_\emptyset = [0, 1]$ and $S_\emptyset = [0, 1]^2$ and assign the pattern to $S_\emptyset$ as illustrated in Figure 1.

The next step of the inductive definition of the Peano curve, f_P uses the initial pattern to order the nine triadic squares of level 1; see Figure 2.

At this first inductive step, the domain intervals of level 1 are matched with corresponding triadic squares of level 1 using the first Peano pattern; see

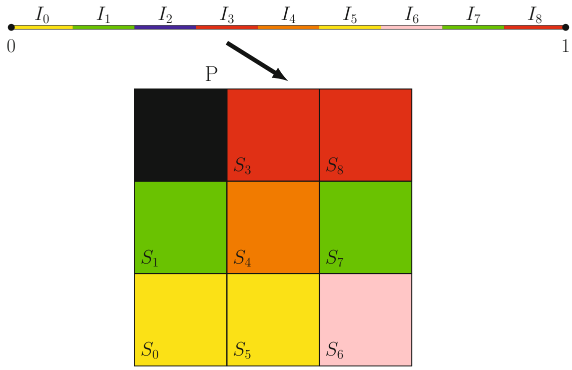

Figure 3: Correspondence between level 1 intervals and subsquares

Figure 3. When the induction is completed, this will entail that $f_P\big(I_0\big) = S_0$. In other words, any number whose initial base 9 digit is a 0 will be Peano mapped to a point in the lower left subsquare of $[0,1]^2$ and so on.

Additionally, each of the squares, S_σ, $\sigma = 0, 1, 2, \ldots, 8$ will be assigned a "Peano Pattern" that informs the next inductive step. The pattern assigned to any particular triadic square, S is then used to enumerate the nine triadic subsquares of S at the next level. Before continuing the induction, we establish some additional notation.

1. *The Peano patterns.* The four Peano patterns along with the vector notation we'll use for them is shown in Figure 4. Although the pattern assigned to a given triadic square is illustrated geometrically in Figure 4, that pattern should be thought of as simply a method by which to order the triadic squares of the next level. That ordering of the triadic subsquares of the next level is shown in Figure 5.

2. *The vector notation.* Let S denote any triadic square and suppose the pattern $\vec{u}$ has been assigned to S where $\vec{u} \in \{\langle 1,1\rangle, \langle -1,1\rangle, \langle -1,-1\rangle, \langle 1,-1\rangle\}$. Then $\vec{u}$ denotes the direction of the vector extending from the initial vertex of S as determined by the pattern to the terminal vertex of S as determined by the pattern. For example the vector, $\langle 1,1\rangle$ has an argument of $\pi/4$ and is parallel to the vector extending from the lower left corner of the left most square of Figure 4 to the upper right corner of that square. We denote the initial vertex of S (patterned with $\vec{u}$) by $v_0(S)$, the next two vertices $v_1(S)$ and $v_2(S)$ and the terminal vertex by $v_3(S)$.

Now suppose $n \ge 1$, and the indices $\sigma \in \Sigma_n^9$ have been used to label the 9^n level n triadic subsquares of $[0,1]^2$ so that S_σ has been defined for all $\sigma \in \Sigma_n^9$ and for distinct indices, the corresponding triadic subsquares are nonoverlapping. Suppose too that a Peano Pattern has been assigned to each S_σ in such a way that for all $\sigma \in \Sigma_n^9$, $v_3(S_\sigma) = v_0(S_{\sigma+})$ and S_σ and $S_{\sigma+}$ share an edge. Let $\tau \in \Sigma_{n+1}^9$ be fixed. Then $\tau = \sigma i$ for some $i = 0, 1, 2, \ldots, 8$.

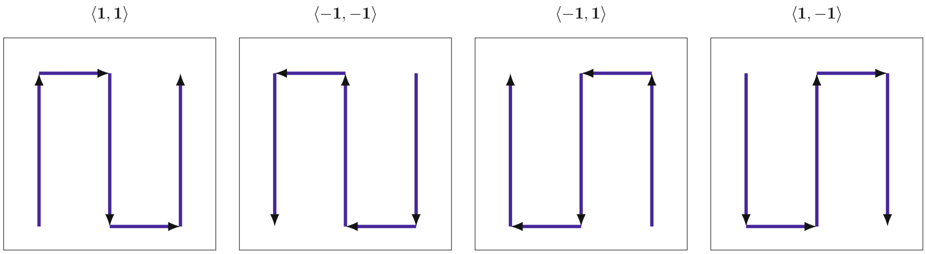

Figure 4: The four Peano patterns

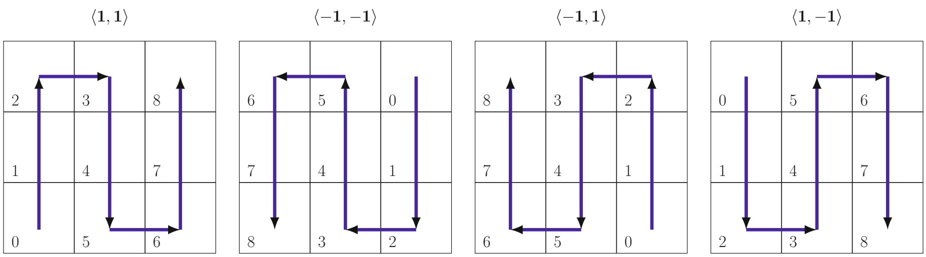

Figure 5: The patterns determine an order on the next stage triadic squares

First partition S_σ into nine congruent nonoverlapping subsquares. The pattern assigned to S_σ establishes the position of the square, S_τ using Figure 5; the pattern assigned to S_τ is then established via Table 1. This defines S_τ along with its pattern for every $\tau \in \Sigma^9_{n+1}$. An example of this pattern replacement is illustrated in Figure 6 for the case in which S_σ is patterned with $\langle -1, 1\rangle$. If S is a triadic subsquare, we denote the pattern assigned to S by, $P(S)$.

Note that in Table 1 if $P(S_\sigma) = \vec{u}$ where $\vec{u} \in \{\langle 1,1\rangle, \langle -1,1\rangle, \langle -1,-1\rangle, \langle 1,-1\rangle\}$ then all corner squares of S_σ, namely $S_{\sigma 0}$, $S_{\sigma 2}$, $S_{\sigma 6}$, and $S_{\sigma 8}$ are also patterned with $\vec{u}$. In particular, it follows that $v_0(S_\sigma) = v_0(S_{\sigma 0})$ and $v_3(S_\sigma) = v_3(S_{\sigma 8})$

We are left to verify the remaining conditions. If $i = 0, 1, 2, \dots 7$ then both S_τ and $S_{\tau+}$ lie in S_σ and the fact that $v_3(S_\tau) = v_0(S_{\tau+})$ and that S_τ shares an edge with $S_{\tau+}$ follow directly from Table 1. This is illustrated in Figure 6 for $\tau = \sigma 6$. The remaining case is when $\tau = \sigma 8$ or the upper left subsquare in Figure 6. By the induction hypothesis, $v_3(S_\sigma) = v_0(S_{\sigma+})$ so that $S_{\sigma+}$ is contiguous with S_σ and contains that common vertex. In Figure 6, $S_{\sigma+}$ would be immediately above S_σ or to the left of S_σ. In either case, $v_0(S_{\sigma+0}) = v_3(S_{\sigma+})$. Similarly, $v_3(S_{\sigma 8}) = v_3(S_\sigma)$, so that all four squares share that common vertex. Since S_σ and $S_{\sigma+}$ are contiguous triadic squares of the same level that share an edge, the same is true for $S_{\sigma 8}$ and

If $P(S_\sigma)$ is	$\langle 1,1\rangle$	$\langle -1,1\rangle$	$\langle 1,-1\rangle$	$\langle -1,-1\rangle$
$P(S_{\sigma 0})$ is	$\langle 1,1\rangle$	$\langle -1,1\rangle$	$\langle 1,-1\rangle$	$\langle -1,-1\rangle$
$P(S_{\sigma 1})$ is	$\langle -1,1\rangle$	$\langle 1,1\rangle$	$\langle -1,-1\rangle$	$\langle 1,-1\rangle$
$P(S_{\sigma 2})$ is	$\langle 1,1\rangle$	$\langle -1,1\rangle$	$\langle 1,-1\rangle$	$\langle -1,-1\rangle$
$P(S_{\sigma 3})$ is	$\langle 1,-1\rangle$	$\langle -1,-1\rangle$	$\langle 1,1\rangle$	$\langle -1,1\rangle$
$P(S_{\sigma 4})$ is	$\langle -1,-1\rangle$	$\langle 1,-1\rangle$	$\langle -1,1\rangle$	$\langle 1,1\rangle$
$P(S_{\sigma 5})$ is	$\langle 1,-1\rangle$	$\langle -1,-1\rangle$	$\langle 1,1\rangle$	$\langle -1,1\rangle$
$P(S_{\sigma 6})$ is	$\langle 1,1\rangle$	$\langle -1,1\rangle$	$\langle 1,-1\rangle$	$\langle -1,-1\rangle$
$P(S_{\sigma 7})$ is	$\langle -1,1\rangle$	$\langle 1,1\rangle$	$\langle -1,-1\rangle$	$\langle 1,-1\rangle$
$P(S_{\sigma 8})$ is	$\langle 1,1\rangle$	$\langle -1,1\rangle$	$\langle 1,-1\rangle$	$\langle -1,-1\rangle$

Table 1: The replacement patterns

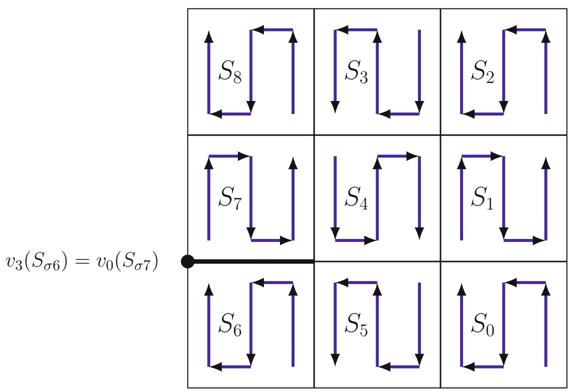

Figure 6: The replacement patterns for $\langle -1,1\rangle$

$S_{\sigma+0}$ since the latter two share a vertex. See Figure 7 in which $S_{\sigma+}$ is to the left of S_σ and S_σ is patterned with $\langle -1,1\rangle$.

This completes the induction. To define the Peano function, f_P, fix $x \in [0,1]$. Then there is a not necessarily unique, nested sequence of intervals, $\{I_{\sigma_n(x)}\}$ with $x = \cap_{n=1}^{\infty} I_{\sigma_n(x)}$. Define

$$f_P(x) = \cap_{n=1}^{\infty} S_{\sigma_n(x)}. \tag{2}$$

It is a direct consequence of Theorem 1 that f_P is uniquely defined and maps $[0,1]$ onto $[0,1]^2$. If $y_o \in [0,1]^2$ and $y_o = \cap_{n=1}^{\infty} S_{\sigma_n(x_o)}$ for some $x_o \in [0,1]$, then the sequence, $(\sigma_n(x_o))_{n\in\mathbb{N}}$ is called a *Peano sequence* of y_o. A synopsis of several features of this construction are particularly important in later sections and we group these together as Observation 1 below.

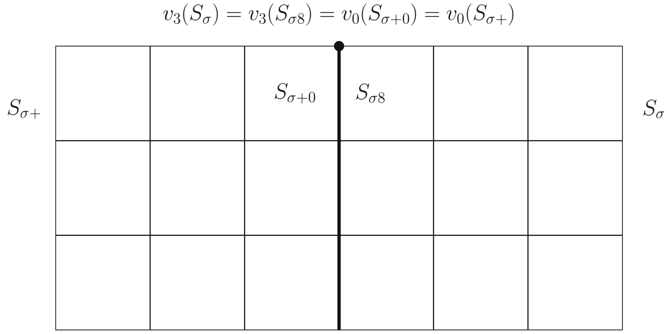

Figure 7: S_σ adjoins $S_{\sigma+}$ implies $S_{\sigma 8}$ adjoins $S_{\sigma+0}$

Observation 1. *Let $n \in \mathbb{N}$ be fixed. Then*

(1) *The set of squares of level n, $\{S_\sigma : \sigma \in \Sigma_n^9\}$ ordered lexicographically, is a linearly ordered set consistent with the Peano pattern, $P(S_\sigma)$.*

(2) *If $\sigma_1 \neq \sigma_2$, then S_{σ_1} and S_{σ_2} do not overlap, i.e. have no interior points in common.*

(3) *For each $\sigma \in \Sigma_n^{9*}$, $v_3(S_\sigma) = v_0(S_{\sigma+})$.*

(4) *For each $\sigma \in \Sigma_n^9$, $v_0(S_\sigma) = v_0(S_{\sigma 0})$ and $v_3(S_\sigma) = v_3(S_{\sigma 8})$.*

(5) *For each $\sigma \in \Sigma_n^9$, and $P(S_\sigma) = P(S_{\sigma 0})$.*

Too, the same proof that f_P maps $[0,1]$ onto $[0,1]^2$ shows that for each $n \in \mathbb{N}$, and $\sigma \in \Sigma_n^9$, f_P maps I_σ onto S_σ. The following theorem is a result of that fact.

Theorem 2. *The Peano function f_P is space filling on every subinterval $[a,b] \subset [0,1]$.*

Proof. It follows from (1) that for every triadic interval $J \subset [0,1]$, $f_p(J)$ is a triadic square. This result now follows by noting that every interval is the union of triadic intervals. $\square$

The definition of f_P given in paragraph 2 defines a 1-to-1 match between triadic subintervals of $[0,1]$ and triadic subsquares of $[0,1]^2$. Although this match is 1-to-1, it doesn't always translate to a 1-to-1 match between points $x \in [0,1]$ and points $y \in [0,1]^2$. The reason is that some points $y \in [0,1]^2$ can be determined by multiple Peano sequences not matched to equal points in $[0,1]$. The analysis of this matching of base 9 sequences of points in $[0,1]$ with Peano sequences on $[0,1]^2$ has been called the arithmetization of the Peano Curve and is the subject of the last three sections of this paper. The next section is an interlude however, showing how the Peano patterns of any given level are generated by the action of the Klein 4-group. This fact is used to simplify our point by point understanding of this arithmetization.

$B^2\vec{u} = \vec{u}$	$AB^2\vec{u} = A\vec{u}$	$A^2B^2\vec{u} = \vec{u}$
$B\vec{u}$	$AB\vec{u}$	$A^2B\vec{u} = B\vec{u}$
$\vec{u}$	$A\vec{u}$	$A^2\vec{u} = \vec{u}$

Figure 8: Using the Klein 4-group to pattern subsquares

4. The Klein 4-group plays a not so small role

Although there is a great deal of similarity between the analysis of the Peano function and that of the Hilbert function found in [4] there are also distinctions. Specifically for the Peano function, the patterning of the triadic squares of any given level is governed by the action of the Klein 4-group and the goal of this section is to describe that action. In general, the Klein 4-group is defined as the Abelian group of order four generated by two idempotents, but in our case we'll use what is occasionally called the standard presentation of that group as the following set of two by two matrices; the group operation is matrix multiplication.

$$K_4 = \big\{ I, A, B, AB \big\} \quad \text{where } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \text{ and } B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Lemma 1. *Suppose that S is a triadic square of level n and has been patterned with $\vec{u} \in \{\langle 1, 1\rangle, \langle -1, 1\rangle, \langle -1, -1\rangle, \langle 1, -1\rangle\}$. Then the patterning of Table* 1 *for the level $n + 1$ triadic subsquares of S can be found by patterning the lower left corner of S with $\vec{u}$ and applying the matrix A as you move horizontally and B whenever you move vertically. That patterning is illustrated in Figure* 8.

Proof. The proof first uses the observation that the lower left corner of S is patterned with $\vec{u}$. What remains simply amounts to checking that Table 1 and Figure 5 are compatible. □

Now suppose that $n \in \mathbb{N}$, $0 \le k, m < 3^n$ and $S = [k/3^n, (k + 1)/3^n] \times [m/3^n, (m + 1)/3^n]$ is an unlabeled triadic subsquare. The following lemma is an algorithm for determining the pattern of S without resorting to the inductive method described in Section 3.

LEMMA 2. *If* $n \in \mathbb{N}$, $0 \le k, m < 3^n$ *and* $S = [\frac{k}{3^n}, \frac{k+1}{3^n}] \times [\frac{m}{3^n}, \frac{m+1}{3^n}]$, *then* $P(S) = A^k B^m \langle 1, 1\rangle = A^{k(\mathrm{mod}\,2)} B^{m(\mathrm{mod}\,2)} \langle 1, 1\rangle$.

PROOF. We use induction. The case of $n = 1$ is the essence of Lemma 1. So assume $n \in \mathbb{N}$ and the lemma is true for all triadic subsquares of level n. Let S denote a triadic subsquare of level $n + 1$. Then there is a unique $\sigma \in \Sigma_n^9$ with $S \subset S_\sigma$. Suppose

$$S_\sigma = \left[\frac{k}{3^n}, \frac{k+1}{3^n}\right] \times \left[\frac{m}{3^n}, \frac{m+1}{3^n}\right],$$

and denote the lower left, level $n + 1$ triadic subsquare of S_σ by S^*. Then,

$$S^* = \left[\frac{3k}{3^{n+1}}, \frac{3k+1}{3^{n+1}}\right] \times \left[\frac{3m}{3^{n+1}}, \frac{3m+1}{3^{n+1}}\right],$$

Since S is a triadic subsquare of S_σ, there are $i, j \in \{0, 1, 2\}$ so that

$$S = \left[\frac{3k+i}{3^{n+1}}, \frac{3k+i+1}{3^{n+1}}\right] \times \left[\frac{3m+j}{3^{n+1}}, \frac{3m+j+1}{3^{n+1}}\right].$$

As a consequence of Lemma 1, $P(S) = A^i B^j P(S^*)$. Since S^* is a corner subsquare of S_σ, $P(S^*) = P(S_\sigma)$, and it follows from the induction hypothesis that $P(S_\sigma) = A^k B^m \langle 1, 1\rangle$. Hence,

$$P(S) = A^i B^j P(S^*) = A^i B^j P(S_\sigma) = A^i B^j A^k B^m \langle 1, 1\rangle.$$

Since both A and B are idempotent, $A^k B^m = A^{3k} B^{3m}$ and hence,

$$P(S) = A^i B^j A^{3k} B^{3m} \langle 1, 1\rangle = A^{3k+i} B^{3m+j} \langle 1, 1\rangle,$$

completing the proof. □

THEOREM 3. *Let* $n \in \mathbb{N}$ *and*

$$S = \left[\frac{k_1}{3^n}, \frac{k_1+1}{3^n}\right] \times \left[\frac{m_1}{3^n}, \frac{m_1+1}{3^n}\right] \text{ and } T = \left[\frac{k_2}{3^n}, \frac{k_2+1}{3^n}\right] \times \left[\frac{m_2}{3^n}, \frac{m_2+1}{3^n}\right]$$

be two triadic subsquares of level n. *Then*

$$P(T) = A^{|k_1 - k_2|} B^{|m_1 - m_2|} P(S). \tag{3}$$

PROOF. As a first case, suppose S and T to be as in the statement of the theorem with $k_1 \le k_2$ and $m_1 \le m_2$ so that T is above and to the right of S. Then, from Lemma 2

$$P(T) = A^{k_2} B^{m_2} \langle 1, -1\rangle = A^{k_2 - k_1} B^{m_2 - m_1} A^{k_1} B^{m_1} \langle 1, -1\rangle$$

$$= A^{k_2-k_1}B^{m_2-m_1}P(S) = A^{|k_2-k_1|}B^{|m_2-m_1|}P(S)$$

as claimed. The case in which $k_1 \geq k_2$ and $m_1 \geq m_2$ is the same, only the roles of S and T have been reversed. So what remains is to verify the conclusion when exactly one of the inequalities from the first case is not true. Suppose, for definiteness that $k_1 > k_2$ but $m_1 \leq m_2$ and set $S^* = \left[\frac{k_1}{3^n}, \frac{k_1+1}{3^n}\right] \times \left[\frac{m_2}{3^n}, \frac{m_2+1}{3^n}\right]$. Since S and S^* satisfy the hypothesis of case 1, $P(S^*) = A^0B^{m_2-m_1}P(S)$. Also, T and S^* satisfy the hypothesis of case one so that $P(S^*) = A^{k_1-k_2}B^0P(T)$. Hence, $B^{m_2-m_1}P(S) = A^{k_1-k_2}B^0P(T)$ and multiplying both sides by $A^{k_1-k_2}$ yields

$$A^{k_1-k_2}B^{m_2-m_1}P(S) = \left(A^{k_1-k_2}\right)^2P(T) = P(T),$$

completing the proof. □

Finally, corners of triadic subsquares are intersections of triadic line segments and the fact that the patterns of triadic subsquares are determined by the Klein 4-group implies that the parity of f_p at each such corner is predictable. This is discussed in the next section in which a triadic node of level n is the intersection of two triadic line segments of level n. Here it is convenient to define *triadic nodes of level 0* to be intersections of triadic line segments with the boundary of $[0,1]^2$.

5. Who maps where?

"Who maps where?" is the question we wish to consider, and it will take up the majority of the remainder of the paper to answer that question completely. In this section we analyze the point by point n-to-1 nature of f_P culminating with Theorem 4. Triadic nodes of level n are the corners of a triadic subsquares of level n and are of the form $y = (k/3^n, m/3^n)$. Such nodes will be deemed either even or odd according to whether $k+m$ is an an even or odd integer. It's easy to see that if y_o is an odd (even) triadic node of level n, then it will be an odd(even) triadic node of every level $m \geq n$, The aim of this section is to prove the following Theorem 4 in which the symbol ∂ denotes the boundary operator.

THEOREM 4. *At each fixed point $y_o \in [0,1]^2$, the Peano curve, f_P is either* 1*-to-*1*,* 2*-to-*1 *or* 4*-to-*1*. Moreover,*

(1) *f_P is* 1*-to-*1 *at*
   (a) *every $y_o \in [0,1]^2$ that is on no triadic line segment,*
   (b) *every y_o that is an even triadic node of level* 0 *on $\partial([0,1]^2)$.*

(2) *f_P is* 4*-to-*1 *at every odd triadic node $y_o \in (0,1)^2$.*

(3) *f_P is* 2*-to-*1 *at all remaining points of $[0,1]^2$.*

We prove Theorem 4 via a sequence of partial results beginning with two "Four Corner" observations.

OBSERVATION 2 (The First Four Corners). *The function* f_P *is* 1-*to*-1 *at each of the four corners of* $[0,1]^2$. *Specifically,* $f_P(0)=(0,0)$, $f_P(1/4)=(0,1)$, $f_P(3/4)=(1,0)$, *and* $f_P(1)=(1,1)$.

PROOF. The proof is similar for each one of these equalities, so we'll have a look at $f_P^{-1}((0,1))$ as an example. The location, $(0,1)$ begins in the level 1 triadic subsquare, S_2, using the original $\langle 1,1\rangle$ pattern for $[0,1]^2$, and as such is assigned the $\langle 1,1\rangle$ pattern at the second stage as well and this continues. That is,

$$(0,1)=S_2\cap S_{22}\cap S_{222}\cap\cdots$$

entailing that $f_p(1/4)=f_p(.222\ldots)=(0,1)$, and as there is no alternate base 9 representation of $1/4$, f_P is 1-to-1 at that point. □

In a similar way, we can compute a Peano sequence for the four corners of any patterned particular triadic square, S.

THEOREM 5 (A General Four Corners). *If* $S=S_\sigma$ *is a triadic square patterned with* $\vec{u}$, *then the four corners of* S *have the following Peano sequences.*

(1) $v_0(S)=\sigma00\overline{0}$
(2) $v_1(S)=\sigma22\overline{2}$
(3) $v_2(S)=\sigma66\overline{6}$
(4) $v_3(S)=\sigma88\overline{8}$

THEOREM 6. *If a point* $y_o\in[0,1]^2$ *lies on no triadic line segment, then there is a unique* $x_o\in[0,1]$ *such that* $f_P(x_o)=y_o$. *That is,* f_P *is* 1-*to*-1 *at each range value not residing on a triadic line segment.*

PROOF. As with the proof of Theorem 5, this follows directly from (1) since it is only on triadic line segments that there is any choice for the digits a_n. Where there is a choice, however, it is certainly possible for there to be several preimages of a $y_o\in[0,1]^2$. □

It immediately follows from Theorem 6 that f_P fails to be 1-to-1 on a topologically tiny subset of $[0,1]$. For the following corollaries, a set is termed *residual* if its complement is of the *first Baire Category* and a set $K\subset[0,1]$ is called a *space-filling core* for f_P if there is no proper compact subset $A\subset K$ such that $f_P(A)=[0,1]^2$; see [5].

COROLLARY 1. *The Peano function,* f_P *is* 1-*to*-1 *on a residual subset of* $[0,1]$.

COROLLARY 2. *The interval,* $[0,1]$ *is the unique space-filling core for the Peano function,* f_P.

This argument used to prove Theorem 6 can also be applied to f_P restricted to a triadic line segment.

THEOREM 7. *The Peano function f_P is 2-to-1 at points lying on exactly one triadic line segment, except those on the boundary of $[0,1]$ at which f_P is either 1-to-1 or 2-to-1.*

PROOF. Suppose a point $y_o \in [0,1]^2$ lies on a triadic line segment of level $n \in \mathbb{N}$ and no previous level. Then there are exactly two neighboring triadic squares, $S \neq T$, of level n that contain y_o on their boundaries. Suppose for instance that $S = S_{\sigma 0}$ and $T = S_{\sigma 1}$ where $\sigma \in \Sigma^9_{n-1}$. If $m > n$, then within each of these squares (S and T) there is precisely one triadic square of level m that contains y_o. Hence, y_o is addressed by two sequences

$$(\sigma 0 b_1 b_2 \dots) \quad \text{and} \quad (\sigma 1 c_1 c_2 \dots) \tag{4}$$

and only these sequences. Moreover, using Theorem 5, neither of these sequences "terminates" since y_o is not a corner of a triadic square. If $x_1 = .A0b_1b_2\dots$ and $x_2 = .A1c_1c_2\dots$, it follows that $x_1 \neq x_2$ and $y_o = f_P(x_1) = f_P(x_2)$. Other cases being entirely analogous, this completes the proof. □

Suppose that $y_o \in [0,1]^2$ is the intersection of two triadic line segments of level $n \in \mathbb{N}$. Then, y_o is the common vertex of four triadic subsquares of level n. and the union of these four squares is called the *enveloping square at y_o*, while the pattern of the lower left of these called the *determining pattern* for y_o. An example is shown in Figure 9.

THEOREM 8. *Suppose that $y_o \in [0,1]^2$ lies on the intersection of two triadic line segments. Then, f_P is either 2-to-1 or 4-to-1 at y_o depending on the determining pattern, $\vec{u}$ of y_o. Specifically,*
(1) *y_o is a 4-to-1 point if $\vec{u}$ is $\langle -1,1\rangle$ or $\langle 1,-1\rangle$,*
(2) *y_o is a 2-to-1 point if $\vec{u}$ is $\langle 1,1\rangle$ or $\langle -1,-1\rangle$.*

PROOF. We begin with the case in which the two triadic line segments intersecting at y_o are of the same initial level, $n \in \mathbb{N}$. In this case, the proofs for each of the four determining patterns rely on Theorem 5 and all proofs are basically the same. That being the case, we'll only provide the details of the case in which $\vec{u} = \langle 1,1\rangle$.

For definiteness, we'll assume $S = S_\sigma$ with $\sigma \in \Sigma^9_{n-1}$, y_o is the lower left of the four points where triadic line segments of level n intersect within S, and the determining pattern for y_o is $\langle 1,1\rangle$. The remaining Peano patterns for the enveloping square for y_o are computed using Theorem 3. See Figure 10.

Since y_o is a corner of each of the four squares comprising the enveloping square, the four Peano sequences for y_o are $(\sigma 08\overline{8})$ via $S_{\sigma 0}$, $(\sigma 10\overline{0})$ via $S_{\sigma 1}$,

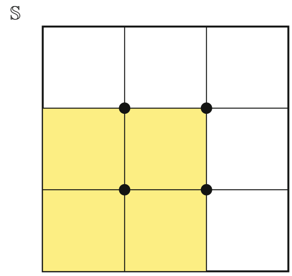

Figure 9: The 4 possible locations for $y_o \in S$

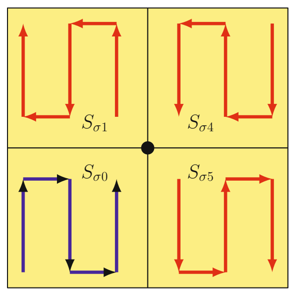

Figure 10: Case 1 where the determining pattern is $\langle 1, 1\rangle$

$(\sigma 48\overline{8})$ via $S_{\sigma 4}$, and $(\sigma 50\overline{0})$ via $S_{\sigma 5}$. As $.\sigma 08\overline{8}_9 = .\sigma 10\overline{0}_9$ and $.\sigma 48\overline{8}_9 = .\sigma 50\overline{0}_9$ there are only two preimages for y_o. The other three cases can be verified in an entirely similar fashion.

We continue with the case where y_o is the intersection of two triadic line segments of different levels. To that end, suppose that $1 \le n < m$ and 3 divides neither n nor m. Suppose for definiteness that $y_o = \left(\frac{a}{3^n}, \frac{b}{3^m}\right)$; the alternate case is analyzed the same way except for notation. Since, $3 \nmid b$ then either 3 divides $b-1$ or 3 divides $b+1$. Suppose the former which implies that $\left(\frac{a}{3^n}, \frac{b-1}{3^m}\right)$ is a triadic node of level $m-1$. Define Q^- and Q^+ respectively as the left and right halves of the level m enveloping square at y_o. Then,

$$Q^- = \left[\frac{a}{3^n} - \frac{1}{3^m}, \frac{a}{3^n}\right] \times \left[\frac{b-1}{3^m}, \frac{b+1}{3^m}\right],$$

$$Q^+ = \left[\frac{a}{3^n}, \frac{a}{3^n} + \frac{1}{3^m}\right] \times \left[\frac{b-1}{3^m}, \frac{b+1}{3^m}\right].$$

Now define

$$R^- = \left[\frac{a}{3^n} - \frac{1}{3^{m-1}}, \frac{a}{3^n}\right] \times \left[\frac{b-1}{3^m}, \frac{b+2}{3^m}\right],$$

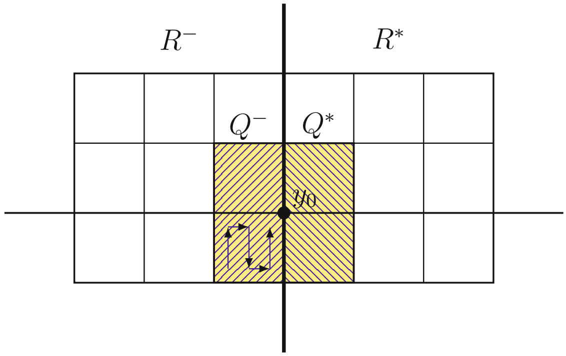

Figure 11: Case 2 where the determining pattern is $\langle 1, 1\rangle$

$$R^+ = \left[\frac{a}{3^n}, \frac{a}{3^n} + \frac{1}{3^{m-1}}\right] \times \left[\frac{b-1}{3^m}, \frac{b+2}{3^m}\right].$$

Since $\left(\frac{a}{3^n}, \frac{b-1}{3^m}\right)$ is a triadic node of level $m-1$. R^+ and R^- are triadic squares of level $m-1$, Q^- is contained in the right half of R^- adjacent to Q^+ which is contained in the left half of R^+; see Figure 11 where the enveloping square is shaded, Q^- is slashed and Q^+ is back slashed. Hence, due to Theorem 3, there are but four possible configurations of the Peano patterns for the enveloping square at y_o, each completely determined by the pattern in the lower left corner of the enveloping square at y_o. Using the same argument as was used in the proof of Theorem 8 it is a straightforward computation to check these four configurations and show that f_P is either 2-to-1 or 4-to-1 at each such point y_o. Again, refer to Figure 11 where the determining pattern at y_0 is $\langle 1, 1\rangle$ entailing that y_o is a 2-to-1 point in this instance. □

Below is a concise and enlightening way to describe the results of Theorem 8 using the notion of *even* and *odd* triadic nodes.

THEOREM 9. *For each* $n \in \mathbb{N}$, *the Peano function is* 2-*to*-1 *at every even triadic node of level* n *and* 4-*to*-1 *at every odd triadic node of level* n.

This completes the analysis of all points other than those on the boundary of $[0, 1]^2$ although we do know that f_P is 1-to-1 at each of the four corners; see Observation 2. The boundary points are different in that they are "one sided." This is reflected in our final observation.

OBSERVATION 3. *The even level* 0 *triadic nodes are* 1-*to*-1 *points of* f_P, *while the odd level* 0 *triadic nodes are* 2-*to*-1 *points of* f_P. *All remaining boundary points are* 1-*to*-1 *points of* f_P.

This completes the proof of Theorem 4 which is both our most general and specific theorem concerning the n-to-1 behavior of the Peano function. Theorem 10 below summarizes the main points a slightly different way.

THEOREM 10. *The Peano function is at most* 4*-to-*1 *at every point of the range*, $[0,1]^2$, *however*, *there are no* 3*-to-*1 *points whatsoever. The* 1*-to-*1 *points are residual in* $[0,1]^2$, *the* 2*-to-*1 *points are residual on every triadic line segment in* $[0,1]^2$, *and* 4*-to-*1 *points are countably dense in* $[0,1]^2$.

As an introduction to Section 6, note that it follows from the Peano construction presented in Section 2 that $f_P([.4,.5]) = S_4$ and that $f_P^{-1}(\text{int}(S_4))$ is an open subset of $[.4,.5]$. Consequently, $f_P^{-1}(\partial S_4)$ is a closed subset of $[.4,.5]$ and from Corollary 1 and Theorem 2 of [5] it follows that $f_P^{-1}(\partial S_4)$ is both nowhere dense and perfect (closed with no isolated points). In short, $f_P^{-1}(\partial S_4)$ is a Cantor type set. In the next section we identify and analyze the Cantor type sets arising in this way and find the results to involve be remarkably familiar sets.

6. The sets mapping to triadic line segments

To begin note that the following theorem is a direct consequence of Theorem 2 and the fact that f_P is continuous.

THEOREM 11. *If* ℓ *is a triadic line segment in* $[0,1]^2$, *then* $f_P^{-1}(\ell)$ *is a compact and nowhere dense subset of* $[0,1]$.

There is considerably more structure to these particular preimages than is described in Theorem 11, however. In addition to being compact and nowhere dense, these preimages also have the property that they contain no isolated points. That is, the preimages of the triadic line segments are all general Cantor sets, meaning they are compact, nowhere dense sets with no isolated points. The situation is complicated by the fact that every triadic line segment of level n is comprised of $2 \cdot 3^n$ shorter segments, each of which is a portion of the common boundary between two triadic squares of level n. This being the case, we'll begin our analysis with an arithmetic description of such boundaries.

To that end, let S denoted a fixed triadic subsquares of Level n. Then one of the four Peano patterns has been assigned to S and $S = S_\sigma$ for exactly one $\sigma \in \Sigma_n^9$. Each of the four edges or sides of S abuts exactly three of the level $n+1$ triadic subsquares that are contained in S and we label the edges accordingly with $S[0,1,2]$, $S[6,7,8]$, $S[2,3,8]$ and $S[0,5,6]$. See Figure 12 where S has been assigned the pattern, $\langle 1,1\rangle$, the edge $S[2,3,8]$ is bold, and $S_{\sigma 3}$ is highlighted. Suppose too that $\sigma = a_1 a_2 \dots a_n$.

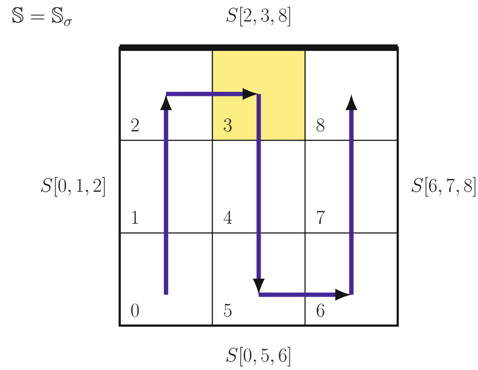

Figure 12: Labeling the edges of a $\langle 1, 1\rangle$ triadic square

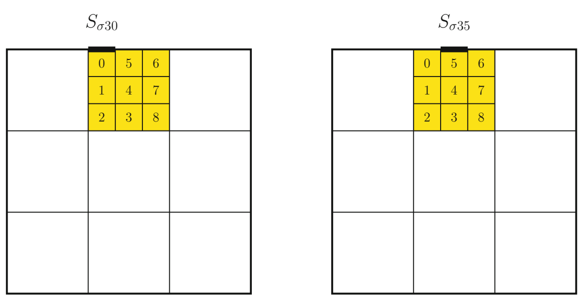

Figure 13: Left: $\sigma 3$ is followed by 0. Right: $\sigma 3$ is followed by 5

This means that if $y_o \in S_\sigma$, and y_o is on the top edge, $S[2,3,8]$ of S, then $f_P^{-1}(y_o)$ intersects each of the following three base 9 intervals on $[0,1]$,

$$f_P^{-1}(S_{\sigma 2}) = [.a_1 a_2 \dots a_n 2, .a_1 a_2 \dots a_n 3],$$
$$f_P^{-1}(S_{\sigma 3}) = [.a_1 a_2 \dots a_n 3, .a_1 a_2 \dots a_n 4],$$
$$f_P^{-1}(S_{\sigma 8}) = [.a_1 a_2 \dots a_n 8, .a_1 a_2 \dots a_n 88\overline{8}],$$

depending on which third of the top edge y_o resides. This means that any point, $x \in [0,1]$ whose base 9 expansion begins with $.\sigma$ followed by a 2, 3, or 8 will be a candidate for f_P to map to the $S[2,3,8]$ edge of S. For definiteness, suppose $y_o \in S[2,3,8]$ and that y_o is in the middle third of $S[2,3,8]$ so that the $n+1$ digit in the Peano sequence of y_o is "3." This is pictured in Figure 12 where $S_{\sigma 3}$ is shaded. The $n+2$ digit depends on the the Peano pattern assigned to the $S_{\sigma 3}$ triadic square. This is illustrated in Figure 13 where $S_{\sigma 3}$ is highlighted and y_o is on the top boundary of the labeled triadic square.

Edge Address	Edge Digits	Next Edge Choices		
$[0,1,2]$	$\{0,1,2\}$	$\{0,1,2\}(0)=[0,1,2]$	$\{0,1,2\}(1)=[6,7,8]$	$\{0,1,2\}(2)=[0,1,2]$
$[2,3,8]$	$\{2,3,8\}$	$\{2,3,8\}(2)=[2,3,8]$	$\{2,3,8\}(3)=[0,5,6]$	$\{2,3,8\}(8)=[2,3,8]$
$[0,5,6]$	$\{0,5,6\}$	$\{0,5,6\}(0)=[0,5,6]$	$\{0,5,6\}(5)=[2,3,8]$	$\{0,5,6\}(6)=[0,5,6]$
$[6,7,8]$	$\{6,7,8\}$	$\{6,7,8\}(6)=[6,7,8]$	$\{6,7,8\}(7)=[0,1,2]$	$\{6,7,8\}(8)=[6,7,8]$

Table 2: The edge sequence choices

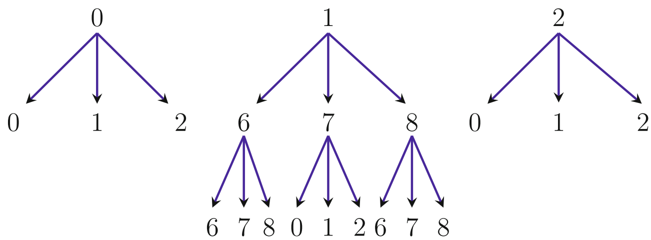

Figure 14: The edge choice tree for the $S[0,1,2]$

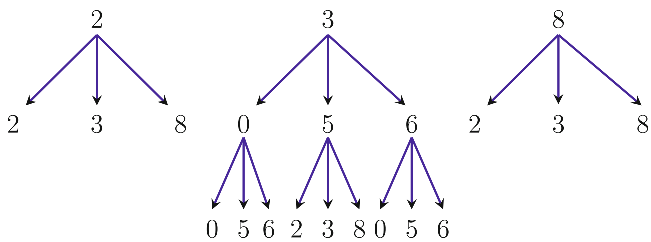

Figure 15: The edge choice tree for the $S[2,3,8]$

This is not so complicated as it might appear at first look. At any given stage, n the Peano sequences for points on the boundary of the triadic square, S_σ of stage n begin with σ followed by any base 9 digit other than 4. Once that first digit is established, subsequent digits are found using the Edge Choice Table 2 or equivalently, the four Edge Choice Trees found in Figures 14, 15, 16, and 17.

Here is the resulting theorem.

THEOREM 12. *If* $S = S_{a_1 a_2 \ldots a_n}$ *is a triadic square in* $[0,1]^2$, *then a point* y_o *is on the boundary of* S *if and only if* y_o *has the Peano sequence* $(a_1 a_2 \ldots a_n b_1 b_2 \ldots)$, *where the* b_i *are compatible with Table* 2 *or the equivalent edge choice trees.*

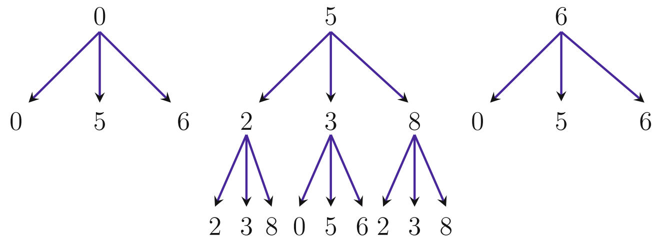

Figure 16: The edge choice tree for the $S[0,5,6]$

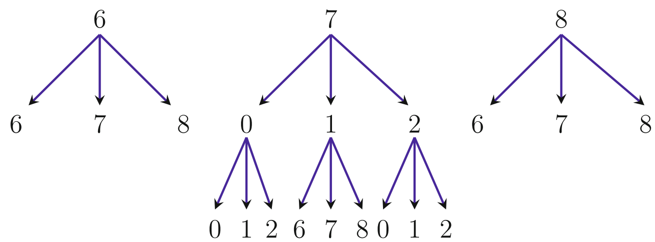

Figure 17: The edge choice tree for the $S[6,7,8]$

Theorem 12 concerns range values of f_P, but, because for every $n \in \mathbb{N}$ and every $\sigma \in \Sigma_n^9$, $f_P(I_\sigma) = S_\sigma$ there is a great deal we can conclude about the preimages of edges. For example, if $S = S_\sigma$ with $\sigma = a_1 a_2 \dots a_n$, then,

$$(5) \qquad I_\sigma \cap f_P^{-1}(S[2,3,8]) = \left\{ .a_1 a_2 \dots a_n b_1 b_2 \dots : \text{the } b_i \text{ satisfy Table 2} \right\}.$$

In this way, it follows from Theorem 12 that $I_\sigma \cap f_P^{-1}(S[2,3,8])$ is described by recursively partitioning subintervals of I_σ into nine congruent nonoveralpping subintervals and, according to the rules in Table 2, deleting six of them. The description in Theorem 12 points to the base 9 arithmetic for a geometric description of $f_P^{-1}(S[2,3,8])$ as a base 9 Cantor type set. See Figure 18 below which illustrates the first two stages of the general Cantor set preimage of $S[2,3,8]$.

The preimages of the other edges of S_σ within I_σ are found in the same way. The basic geometric nature of such general Cantor sets is well established; all are compact porous subsets of $[0,1]$ and as such they are both nowhere dense and of measure zero. Moreover, the Hausdorff dimension of each can be computed using so-called box counting and is $1/2$ which interestingly enough, is the same dimension as the preimages of edges for the Hilbert function; see [4].

Each of these observations follows directly from the fact that to compute the b_i at any stage $i > n$ one has exactly three choices out of nine and these

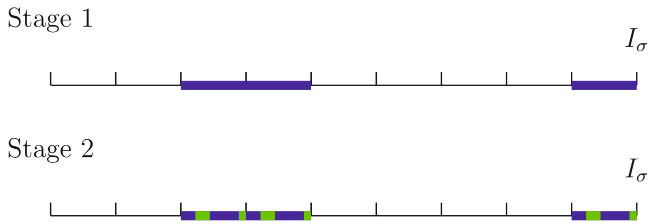

Figure 18: The interval I_σ contains the general Cantor set $f_P^{-1}(S[2,3,8])$.

choises correspond to choosing exactly three of the primary base 9 digits at that stage. All of this is summerized in Theorem 13.

THEOREM 13. *Suppose $S = S_{a_1a_2\ldots a_n}$ is a triadic square in $[0,1]^2$, I_σ is the interval of length $1/9^n$ whose initial point is $\sigma = .a_1a_2\ldots a_n$, and E is any of the four edges of S. Then $C = I_\sigma \cap f_P^{-1}(E)$ is a general Cantor set of Hausdorff dimension $1/2$ and $f_P(C) = E$.*

We conclude this discussion by returning to the original question concerning triadic line segments. Let $\ell \subset [0,1]^2$ be a fixed triadic line segment of initial level m so that $\ell = \{(a, k/3^m) : 0 \le a \le 1\}$ or $\ell = \{(k/3^m, b) : 0 \le b \le 1\}$ where $3 \nmid k$. For definiteness, we assume the former so that ℓ is horizontal. Then there are 3^m triadic squares of the mth stage below ℓ that contribute an edge to ℓ and 3^m triadic squares above ℓ that contribute an edge. Each of these $2 \cdot 3^m$ triadic squares corresponds to its own base 9 subinterval of length $1/9^m$ and although two such subintervals can intersect, they cannot overlap. Consequently, $f_P^{-1}(\ell)$ is the union of $2 \cdot 3^m$ nonoverlapping general Cantor sets each of Hausdorff dimension $1/2$; see Theorem 14.

THEOREM 14. *Let $\ell \subset [0,1]^2$ be a fixed triadic line segment of level m. Then there are nonoverlapping general Cantor sets $\{C_i : i = 1, 2, \ldots, 2\cdot 3^m\}$, each of Hausdorff dimension $1/2$ such that*

$$f_P^{-1}(\ell) = \bigcup_{k=1}^{2\cdot 3^m} (C_i).$$

7. Concluding remarks

Although the first three sections of this paper are well established in the literature, we included them, partly because the geometry was needed for the arithmetization of Section 5. But also, because many, if not most introductions to the Peano space filling curve introduce the pictures without mentioning that this view of things comes directly from Hilbert [6]. For us, Section 4 is particularly important. For one of us, a pure mathematician, this is a matter of mathematical elegance rather unique to the Peano Space filling Curve. The Klein 4-group is the symmetry group of non-square rectangles, and was introduced by Klein himself in 1884, see [7] or [8]. It has proven to be a recognizable engine of enumerably many applications in geometry, graph theory and many seemingly disparet areas of pure and applied mathematics, including music theory, see [1, p. 186] or [17] for examples. Often the algebraic structure of the Klein 4-group is the model, but in our case it is both the algebra and the action of the standard matrix model of the Klein 4-group that plays a key role in understanding the geometric patterning that Hilbert introduced inductively. The second of us is primarily interested in applications of pure mathematics to computer engineering where recursion and induction are critical theoretical tools but that can be computationally intensive. Theorem 3 and the two lemmas immediately preceding it are computational results that immediately predetermine the Hilbert geometry of the Peano curve at any specified level. Although we did not dwell on applications of space filling curves, there is a sizable body of such applications, and the second of us is particularly excited about future use of Theorem 3. See, for example, [10] where the authors use space filling curves to effectively control graph layout design.

Analogs of the Peano curve in higher dimensions will be analyzed elsewhere.

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