OPT2025: 17th Annual Workshop on Optimization for Machine Learning

First Provable Guarantees for Practical Private FL: Beyond Restrictive Assumptions

Egor Shulgin EGOR.SHULGIN@KAUST.EDU.SA
Grigory Malinovsky
Sarit Khirirat
Peter Richtárik
KAUST, Saudi Arabia

Abstract

Federated Learning (FL) enables collaborative training on decentralized data. Differential Privacy (DP) is crucial for FL, but current private methods often rely on unrealistic assumptions (e.g., bounded gradients or heterogeneity), hindering practical application. Existing works that relax these assumptions typically neglect practical FL mainstays like partial client participation or multiple local updates. We introduce Fed-α-NormEC, the first differentially private FL framework providing provable convergence and DP guarantees under standard assumptions while fully supporting these practical elements. Fed-α-NormEC integrates local updates (full and incremental gradient steps), separate server and client stepsizes, and, crucially, partial client participation—essential for real-world deployment and vital for privacy amplification. Our theoretical guarantees are corroborated by experiments on private deep learning tasks.

1. Introduction

Federated Learning (FL) [33, 44] enables collaborative training across multiple devices or organizations without centralized data collection. Despite its advantages, FL faces key challenges: communication bottlenecks due to slow or unreliable networks [6], partial client participation caused by scale and intermittent availability [8], and heterogeneous local datasets [26, 50]. These issues have motivated specialized distributed optimization algorithms to improve communication, handle partial participation, and mitigate heterogeneity [25, 63].

Although FL avoids raw data exchange, it does not ensure full privacy. Model updates may still leak sensitive information to servers or adversaries [4, 47]. To address this, Differential Privacy (DP) [13] has become the standard framework, typically enforced via gradient clipping with added noise, as in DP-SGD (Abadi et al., 2016). However, clipping introduces bias that can hinder convergence [9, 32], e.g., FedAvg with clipping fails on convex quadratics [67]. Existing convergence guarantees rely on restrictive assumptions such as bounded gradients [37, 39, 66] or bounded heterogeneity [35, 51], leaving the general case unresolved.

Error Compensation (EC) [17], also known as Error Feedback (EF) [54, 61], mitigates clipping bias by tracking and reusing errors, ensuring convergence without privacy noise [29]. Recent works combine EF with DP: Shulgin et al. [60] use EF with local momentum, while Islamov et al. [24] replace clipping with smoothed normalization [5, 65], which is more robust to parameter choices. These approaches achieve strong convergence and privacy guarantees under standard assumptions, without bounded gradients or limited heterogeneity. Yet, they remain restricted to distributed opti-

mization, lacking core FL features such as partial client participation and local training steps. Thus, rigorous analysis of private FL methods under realistic settings is still largely open.

2. Preliminaries

Federated optimization problem. Consider an FL setting with the server being connected with M clients over the network. Each client $i \in [1, M]$ has a private dataset. The objective is to determine the model parameters $x \in \mathbb{R}^d$ that solves the following optimization problem:

$$\min_{x \in \mathbb{R}^d} f(x) = \frac{1}{M} \sum_{i=1}^{M} f_i(x), \; f_i(x) := \frac{1}{N} \sum_{j=1}^{N} f_{i,j}(x). \tag{1}$$

Here, $f_{i,j}(x)$ is the loss of the model x on training data $j \in [1, N]$ of client $i \in [1, M]$. Also, we assume that the objective functions f, f_i, and $f_{i,j}$ satisfy the following conditions.

Assumption 1 *Consider Problem (1). Assume that each individual function $f_{i,j}(x)$ is L-smooth and bounded below by $f_{i,j}^{\inf} > -\infty$; that each local function $f_i(x)$ is bounded below by $f_i^{\inf} > -\infty$; and that the global objective $f(x)$ is bounded below by $f^{\inf} > -\infty$.*

DP-FedAvg. A basic FL method for Problem (1) is DP-FedAvg [46], which alternates local client updates with server aggregation:

$$x^{k+1} = x^k - \frac{\eta}{B} \left[\sum_{i \in S^k} \Psi(x^k - \mathcal{T}_i(x^k)) + z_i^k \right],$$

where S^k is a client subset of size B, $\Psi(\cdot)$ is a bounding operator (e.g., clipping or normalization), $\mathcal{T}_i(x^k)$ is the local update of client i, and $z_i^k \in \mathbb{R}^d$ is DP noise. Since $\Psi(\cdot)$ limits update magnitude, the variance of z_i^k can be calibrated to guarantee privacy. Subsampling further reduces required noise variance via privacy amplification.

Bias from Clipping or Normalization. Clipping/normalization introduce bias, so DP-FedAvg may not converge even without noise; e.g., Zhang et al. [67] show FedAvg with clipping fails on convex quadratics. Most analyses avoid this issue by assuming bounded gradients [37, 39, 66, 67]. Das et al. [12] study convergence without this assumption, but only for convex smooth problems with step sizes depending on inaccessible constants.

3. Fed-α-NormEC

Now, we describe Fed-α-NormEC for solving federated optimization under privacy and communication constraints. The method proceeds in communication rounds $k = 0, 1, \ldots, K$. At each round, the server broadcasts the global model x^k to a subset of clients. Each client computes a local update $\mathcal{T}_i(x^k)$ (e.g., via gradient descent) and maintains a memory vector v_i^k for error feedback. This vector is updated as

$$v_i^{k+1} = v_i^k + \beta \mathrm{Norm}_\alpha \left(\frac{x^t - \mathcal{T}_i(x^k)}{\gamma} - v_i^k \right),$$

where $\beta > 0$ controls error compensation and $\gamma > 0$ is a local stepsize. The smoothed normalization operator is defined as $\mathrm{Norm}_\alpha(g) := \frac{1}{\alpha+\|g\|}g$, ensuring bounded sensitivity since $\|\mathrm{Norm}_\alpha(g)\| \leq 1$ for all $g \in \mathbb{R}$.

Algorithm 1 (DP-)Fed-α-NormEC

Input: Tuning parameters $\gamma > 0$, $\beta > 0$, and $\eta \in (0,1)$; normalization parameter $\alpha > 0$; initialized vectors $x^0, v_i^0 \in \mathbb{R}^d$ for $i \in [1, M]$ and $\hat{v}^0 = \frac{1}{M}\sum_{i=1}^M v_i^0$; local fixed-point operators $\mathcal{T}_i(\cdot)$; probability of transmitting the client's local vector to the server $p \in [0,1]$; Gaussian noise with zero mean and σ_{DP}^2-variance $z_i^k \in \mathbb{R}^d$.
for each iteration $k = 0, 1, \ldots, K$ **do**
  for each client $i = 1, 2, \ldots, M$ in parallel **do**
    Compute local updating $\mathcal{T}_i(x^k)$
    Compute $\Delta_i^k = \mathrm{Norm}_\alpha\left(\frac{x^k - \mathcal{T}_i(x^k)}{\gamma} - v_i^k\right)$
    Update $v_i^{k+1} = v_i^k + \beta\Delta_i^k$
    Choose $q_i^k = 1/p$ with prob. p and 0 otherwise
    Non-private setting: Transmit $\hat{\Delta}_i^k = q_i^k\Delta_i^k$
    Private setting: Transmit $\hat{\Delta}_i^k = q_i^k(\Delta_i^k + z_i^k)$
  end for
  Server computes $\hat{v}^{k+1} = \hat{v}^k + \frac{\beta}{M}\sum_{i=1}^M \hat{\Delta}_i^k$
  Server updates $x^{k+1} = x^k - \frac{\eta}{\|\hat{v}^{k+1}\|}\left(\hat{v}^{k+1}\right)$
end for
Output: x^{K+1}

Each client sends an update $\hat{\Delta}_i^k$ with probability p, modeling partial participation. In the non-private case, $\hat{\Delta}_i^k := q_i^k\mathrm{Norm}_\alpha\left(\frac{x^t - \mathcal{T}_i(x^k)}{\gamma} - v_i^k\right)$, while in the private setting it is $\hat{\Delta}_i^k := q_i^k\left(\mathrm{Norm}_\alpha\left(\frac{x^t - \mathcal{T}_i(x^k)}{\gamma} - v_i^k\right) + z_i^k\right)$, with $q_i^k = 1/p$ w.p. p and 0 otherwise. The noise z_i^k is Gaussian with mean 0 and variance σ_{DP}^2.

The server aggregates client updates, forming the global memory vector $\hat{v}^k$, and updates the model:

$$\hat{v}^{k+1} = \hat{v}^k + \frac{\beta}{M}\sum_{i=1}^M \hat{\Delta}_i^k, \quad x^{k+1} = x^k - \frac{\eta}{\|\hat{v}^{k+1}\|}\hat{v}^{k+1},$$

with server stepsize $\eta > 0$. The complete procedure is summarized in Algorithm 1.

Now, we provide the convergence result for Fed-α-NormEC that incorporates multiple local gradient descent (GD) steps and partial participation in a differentially private setting.

Theorem 2 *Consider Fed-α-NormEC for solving Problem (1) where Theorem 1 holds. Let $\mathcal{T}_i(x^k) = x^k - \frac{\gamma}{T}\sum_{j=0}^{T-1}\nabla f_i(x_i^{k,j})$, where the sequence $\{x_i^{k,j}\}$ is generated by $x_i^{k,j+1} = x_i^{k,j} - \frac{\gamma}{T}\nabla f_i(x_i^{k,j})$, for $j = 0, 1, \ldots, T-1$, given that $x_i^{k,0} = x^k$. Furthermore, let $\beta, \alpha > 0$ be chosen such that $\frac{\beta}{\alpha+R} < 1$ with*

$R = \max_{i\in[1,M]} \left\| v_i^0 - \frac{x^0 - \mathcal{T}_i(x^0)}{\gamma} \right\|$. *If* $\eta\gamma \leq \frac{1}{K+1}\frac{\Delta^{\inf}}{4L\sqrt{2L}}$, $0 < \eta \leq \frac{\gamma}{2}\frac{\beta R}{\alpha+R}$, *and* $0 < \gamma \leq \frac{1}{2L}$, *then*

$$\min_{k\in[0,K]} \mathrm{E}\left[\left\|\nabla f(x^k)\right\|\right] \leq \frac{3}{K+1}\frac{f(x^0)-f^{\inf}}{\eta} + 2R + 2\sqrt{\frac{\beta^2 B}{M}(K+1)} + \gamma \cdot \mathbb{I}_{T\neq 1}\left[8L\sqrt{2L}\sqrt{\Delta^{\inf}}\right] + \eta \cdot \frac{L}{2},$$

for $B = 2\frac{(p-1)^2}{p} + 2\frac{\sigma_{\mathrm{DP}}^2}{p}$, $\Delta^{\inf} = f^{\inf} - \frac{1}{M}\sum_{i=1}^M f_i^{\inf} > 0$.

From Theorem 2, Fed-α-NormEC with multiple local GD steps achieves sub-linear convergence, with additive constants due to smoothed normalization R, partial participation and DP noise $B = 2\frac{(p-1)^2}{p} + \frac{2\sigma_{\mathrm{DP}}^2}{p}$, and data heterogeneity $\Delta^{\inf}$. Unlike Shulgin et al. [60], our result holds under partial participation. It also supports local steps without bounded heterogeneity assumptions, unlike Li et al. [35], Noble et al. [51].

Fed-α-NormEC with One Local Step. We further study the case $\mathcal{T}_i(x) = x - \gamma\nabla f_i(x)$ for $i \in [1, M]$ to isolate the effects of smoothed normalization, participation, and DP noise on convergence.

Full participation and non-private setting. When $T = 1$, $p = 1$, and $\sigma_{\mathrm{DP}} = 0$, the additive constants $\mathbb{I}_{T\neq 1}\left[8L\sqrt{2L}\sqrt{\Delta^{\inf}}\right]$ and B vanish. The bound then reduces to three terms: $\frac{3}{K+1}\frac{f(x^0)-f^{\inf}}{\eta} + 2R + \frac{\eta L}{2}$. In this case, Fed-$\alpha$-NormEC recovers the convergence of α-NormEC. Similar to Corollary 1 of Shulgin et al. [60], with tuned η, β, R, the rate nearly matches standard gradient descent at $\mathcal{O}(1/\sqrt{K+1})$ in gradient norm.

For constant σ_{DP}, careful tuning of η, β is required to guarantee convergence of Fed-α-NormEC, as shown below:

Corollary 3 *Consider* Fed-α-NormEC *for solving Problem* (1) *under the same setting as Theorem 2. Let* $T = 1$ *and* $N = 0$ *(one local GD step). If* $v_i^0 \in \mathbb{R}^d$ *is chosen such that* $\gamma = \frac{1}{2L}$, $\max_{i\in[1,M]} \left\| \frac{x^0 - \mathcal{T}_i(x^0)}{\gamma} - v_i^0 \right\| = \frac{D_1}{(K+1)^{1/6}}$ *with* $D_1 > 0$, *and* $\beta = \frac{D_2}{(K+1)^{2/3}}$ *with* $D_2 > 0$, *and* $\eta \leq \frac{LD_1D_2}{2(\alpha+D_1)(K+1)^{5/6}}$, *then*

$$\min_{k\in[0,K]} \mathrm{E}\left[\left\|\nabla f(x^k)\right\|\right] \leq \frac{A_1}{(K+1)^{1/6}} + \frac{A_2}{(K+1)^{5/6}},$$

where $A_1 = \frac{f(x^0)-f^{\inf}}{\eta_0} + 2D_1 + 2D_2\sqrt{\frac{2p(1-1/p)^2}{M} + \frac{2\sigma_{DP}^2}{p}}$, $A_2 = \frac{L\eta_0}{2}$, *and* $\eta_0 = \frac{LD_1D_2}{2(\alpha+D_1)}$.

Theorem 3 shows convergence of Fed-α-NormEC under partial participation with constant variance σ_{DP}. Unlike α-NormEC, its bound includes an additional term from client sampling, $B = 2\frac{(p-1)^2}{p} + 2\sigma_{\mathrm{DP}}^2/p$. Smaller p lowers communication but increases B. In practice, p often varies with client availability. For full participation ($p = 1$), Fed-α-NormEC matches the $\mathcal{O}\left(\frac{1}{(K+1)^{1/6}}\right)$ rate of α-NormEC in the private setting with constant σ_{DP}.

DP utility bound with privacy amplification. Fed-α-NormEC satisfies (ϵ, δ)-DP with noise calibrated via Abadi et al. [1], setting $\sigma_{\mathrm{DP}} = c \cdot p\sqrt{(K+1)\log(1/\delta)}/\epsilon$ for constant $c > 0$ and $0 < p \leq 1$. Due to subsampling amplification, σ_{DP} depends more weakly on p, leading to the following utility guarantee:

Corollary 4 *Consider* Fed-α-NormEC *for solving Problem* (1) *under the same setting as Theorem 2. Let* $T = 1$ *(one local GD step), let* $\sigma_{\text{DP}} = cp\sqrt{(K+1)\log(1/\delta)}/\epsilon$ *with* $c > 0$, *and let* $p = \frac{\hat{B}}{M}$ *for* $\hat{B} \in [1, M]$. *If* $\beta = \frac{\hat{\beta}}{K+1}$ *with* $\hat{\beta} = \sqrt{\frac{3(f(x^0)-f^{\inf})}{\gamma}}\sqrt[4]{\frac{M}{B_2}}$, $\gamma < \frac{\Delta^{\inf}(\alpha+R)}{\sqrt{2L}\hat{\beta}R}$ $\alpha = R = \mathcal{O}\left(\sqrt[4]{d}\frac{\sqrt{f(x^0)-f^{\inf}}}{\sqrt{\gamma}}\sqrt[4]{\frac{B_2}{M}}\right)$ *with* $B_2 = 2c^2\frac{\hat{B}}{M}\frac{\log(1/\delta)}{\epsilon^2}$, *and* $\eta = \frac{1}{K+1}\frac{\gamma}{2}\frac{\hat{\beta}R}{\alpha+R}$, *then*

$$\min_{k\in[0,K]} \mathrm{E}\left[\left\|\nabla f(x^k)\right\|\right] \leq \mathcal{O}\left(\Delta\sqrt[4]{\frac{d\hat{B}}{M^2}\frac{\log(1/\delta)}{\epsilon^2}}\right),$$

where $\Delta = \max(\alpha, 2)\sqrt{L}\sqrt{f(x^0) - f^{\inf}}$.

Theorem 4 gives the utility bound of Fed-α-NormEC under partial participation and privacy. With $p = \hat{B}/M$, where $\hat{B}$ clients are sampled per round, Fed-α-NormEC attains $\mathcal{O}\left(\Delta\sqrt[4]{d\frac{B}{M^2}\frac{\log(1/\delta)}{\epsilon^2}}\right)$, improving over α-NormEC 's $\mathcal{O}\left(\Delta\sqrt[4]{d\frac{1}{M}\frac{\log(1/\delta)}{\epsilon^2}}\right)$ via privacy amplification by subsampling. For $p = 1$, Fed-α-NormEC recovers the same bound as α-NormEC.

Our results extend to Fed-α-NormEC with multiple local GD or IG steps; see Appendix D.

4. Experiments

We evaluate the performance of Fed-α-NormEC on solving a non-convex optimization task involving deep neural network training. Following the experimental setup from prior work [60] common for DP training, we use the CIFAR-10 dataset [34] and the ResNet20 architecture [22]. Detailed settings and additional results are provided in the Appendix. We analyze the performance of Fed-α-NormEC in the differentially private setting by setting the variance of added noise at $p\beta\sqrt{K\log(1/\delta)}\epsilon^{-1}$ for $\epsilon = 8, \delta = 10^{-5}$ and vary β to simulate different privacy levels. The step size γ is tuned for every combination of parameters p and β. The behavior of test accuracy is shown in Figure 1, with the corresponding training loss depicted in fig. 4.

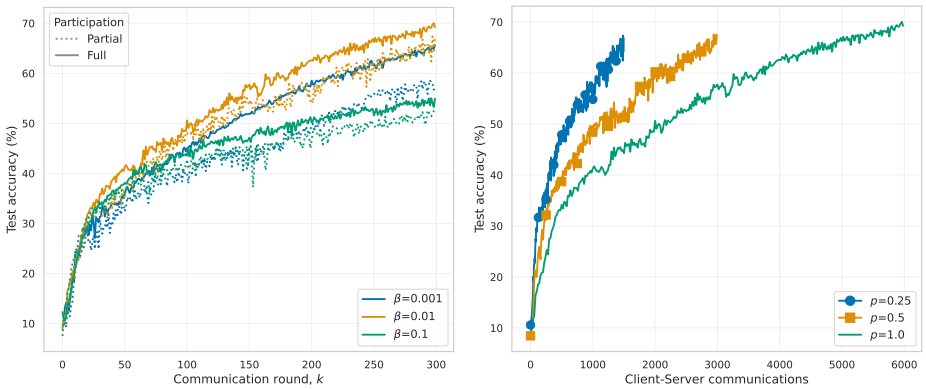


Figure 1: Left. Convergence of Fed-α-NormEC under Full [solid] and Partial participation [dotted] for $p = 0.25$. Right. Fed-α-NormEC under varying participation rates; x-axis shows total client-to-server transmissions.

Acknowledgments

The research reported in this publication was supported by funding from King Abdullah University of Science and Technology (KAUST): i) KAUST Baseline Research Scheme, ii) Center of Excellence for Generative AI, under award number 5940, iii) SDAIA-KAUST Center of Excellence in Artificial Intelligence and Data Science.

References

[1] Martin Abadi, Andy Chu, Ian Goodfellow, H Brendan McMahan, Ilya Mironov, Kunal Talwar, and Li Zhang. Deep learning with differential privacy. In *Proceedings of the 2016 ACM SIGSAC conference on computer and communications security*, pages 308–318, 2016.

[2] Hongyan Bao, Pengwen Chen, Ying Sun, and Zhize Li. EFSkip: A new error feedback with linear speedup for compressed federated learning with arbitrary data heterogeneity. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 39, pages 15489–15497, 2025.

[3] Dimitri P Bertsekas et al. Incremental gradient, subgradient, and proximal methods for convex optimization: A survey. *Optimization for Machine Learning*, 2010(1-38):3, 2011.

[4] Franziska Boenisch, Adam Dziedzic, Roei Schuster, Ali Shahin Shamsabadi, Ilia Shumailov, and Nicolas Papernot. When the curious abandon honesty: Federated learning is not private. In *2023 IEEE 8th European Symposium on Security and Privacy (EuroS&P)*, pages 175–199. IEEE, 2023.

[5] Zhiqi Bu, Yu-Xiang Wang, Sheng Zha, and George Karypis. Automatic clipping: Differentially private deep learning made easier and stronger. *Advances in Neural Information Processing Systems*, 36:41727–41764, 2023.

[6] Sebastian Caldas, Jakub Konečny, H Brendan McMahan, and Ameet Talwalkar. Expanding the reach of federated learning by reducing client resource requirements. *arXiv preprint arXiv:1812.07210*, 2018.

[7] Zachary Charles and Jakub Konečnỳ. On the outsized importance of learning rates in local update methods. *arXiv preprint arXiv:2007.00878*, 2020.

[8] Wenlin Chen, Samuel Horvath, and Peter Richtarik. Optimal client sampling for federated learning. *arXiv preprint arXiv:2010.13723*, 2020.

[9] Xiangyi Chen, Steven Z Wu, and Mingyi Hong. Understanding gradient clipping in private SGD: A geometric perspective. *Advances in Neural Information Processing Systems*, 33: 13773–13782, 2020.

[10] Sélim Chraibi, Ahmed Khaled, Dmitry Kovalev, Peter Richtárik, Adil Salim, and Martin Takáč. Distributed fixed point methods with compressed iterates. *arXiv preprint arXiv:1912.09925*, 2019.

[11] Michael Crawshaw, Yajie Bao, and Mingrui Liu. Episode: Episodic gradient clipping with periodic resampled corrections for federated learning with heterogeneous data. *arXiv preprint arXiv:2302.07155*, 2023.

[12] Rudrajit Das, Abolfazl Hashemi, Sujay Sanghavi, and Inderjit S Dhillon. On the convergence of differentially private federated learning on non-lipschitz objectives, and with normalized client updates. *arXiv preprint arXiv:2106.07094*, 2021.

[13] Cynthia Dwork, Aaron Roth, et al. The algorithmic foundations of differential privacy. *Foundations and Trends® in Theoretical Computer Science*, 9(3–4):211–407, 2014.

[14] Ilyas Fatkhullin, Alexander Tyurin, and Peter Richtárik. Momentum provably improves error feedback! *Advances in Neural Information Processing Systems*, 36:76444–76495, 2023.

[15] Yuan Gao, Rustem Islamov, and Sebastian Stich. EControl: Fast distributed optimization with compression and error control. *arXiv preprint arXiv:2311.05645*, 2023.

[16] Robin C Geyer, Tassilo Klein, and Moin Nabi. Differentially private federated learning: A client level perspective. *arXiv preprint arXiv:1712.07557*, 2017.

[17] Eduard Gorbunov, Dmitry Kovalev, Dmitry Makarenko, and Peter Richtárik. Linearly converging error compensated sgd. *Advances in Neural Information Processing Systems*, 33: 20889–20900, 2020.

[18] Eduard Gorbunov, Filip Hanzely, and Peter Richtárik. Local sgd: Unified theory and new efficient methods. In *International Conference on Artificial Intelligence and Statistics*, pages 3556–3564. PMLR, 2021.

[19] Kaja Gruntkowska, Alexander Tyurin, and Peter Richtárik. EF21-P and friends: Improved theoretical communication complexity for distributed optimization with bidirectional compression. In *International Conference on Machine Learning*, pages 11761–11807. PMLR, 2023.

[20] Farzin Haddadpour, Mohammad Mahdi Kamani, Aryan Mokhtari, and Mehrdad Mahdavi. Federated learning with compression: Unified analysis and sharp guarantees. In *International Conference on Artificial Intelligence and Statistics*, pages 2350–2358. PMLR, 2021.

[21] Jeff Haochen and Suvrit Sra. Random shuffling beats SGD after finite epochs. In *International Conference on Machine Learning*, pages 2624–2633. PMLR, 2019.

[22] Kaiming He, Xiangyu Zhang, Shaoqing Ren, and Jian Sun. Deep residual learning for image recognition. In *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition (CVPR)*, pages 770–778, 2016.

[23] Yerlan Idelbayev. Proper ResNet implementation for CIFAR10/CIFAR100 in PyTorch. https://github.com/akamaster/pytorch_resnet_cifar10. Accessed: 2024-12-31.

[24] Rustem Islamov, Samuel Horvath, Aurelien Lucchi, Peter Richtarik, and Eduard Gorbunov. Double momentum and error feedback for clipping with fast rates and differential privacy. *arXiv preprint arXiv:2502.11682*, 2025.

[25] Peter Kairouz, H Brendan McMahan, Brendan Avent, Aurélien Bellet, Mehdi Bennis, Arjun Nitin Bhagoji, Kallista Bonawitz, Zachary Charles, Graham Cormode, Rachel Cummings, et al. Advances and open problems in federated learning. *Foundations and trends® in machine learning*, 14(1–2):1–210, 2021.

[26] Sai Praneeth Karimireddy, Satyen Kale, Mehryar Mohri, Sashank Reddi, Sebastian Stich, and Ananda Theertha Suresh. Scaffold: Stochastic controlled averaging for federated learning. In *International conference on machine learning*, pages 5132–5143. PMLR, 2020.

[27] Ahmed Khaled and Peter Richtárik. Gradient descent with compressed iterates. *arXiv preprint arXiv:1909.04716*, 2019.

[28] Ahmed Khaled, Konstantin Mishchenko, and Peter Richtárik. Tighter theory for local sgd on identical and heterogeneous data. In *International conference on artificial intelligence and statistics*, pages 4519–4529. PMLR, 2020.

[29] Sarit Khirirat, Eduard Gorbunov, Samuel Horváth, Rustem Islamov, Fakhri Karray, and Peter Richtárik. Clip21: Error feedback for gradient clipping. *arXiv preprint arXiv:2305.18929*, 2023.

[30] Anastasia Koloskova, Nicolas Loizou, Sadra Boreiri, Martin Jaggi, and Sebastian Stich. A unified theory of decentralized sgd with changing topology and local updates. In *International conference on machine learning*, pages 5381–5393. PMLR, 2020.

[31] Anastasia Koloskova, Nikita Doikov, Sebastian U Stich, and Martin Jaggi. On convergence of incremental gradient for non-convex smooth functions. *arXiv preprint arXiv:2305.19259*, 2023.

[32] Anastasia Koloskova, Hadrien Hendrikx, and Sebastian U Stich. Revisiting gradient clipping: Stochastic bias and tight convergence guarantees. In *International Conference on Machine Learning*, pages 17343–17363. PMLR, 2023.

[33] Jakub Konečnỳ, H Brendan McMahan, Felix X Yu, Peter Richtárik, Ananda Theertha Suresh, and Dave Bacon. Federated learning: Strategies for improving communication efficiency. *arXiv preprint arXiv:1610.05492*, 2016.

[34] Alex Krizhevsky, Geoffrey Hinton, et al. Learning multiple layers of features from tiny images. Technical report, University of Toronto, Toronto, 2009.

[35] Bo Li, Xiaowen Jiang, Mikkel N Schmidt, Tommy Sonne Alstrøm, and Sebastian U Stich. An improved analysis of per-sample and per-update clipping in federated learning. In *The Twelfth International Conference on Learning Representations*, 2024.

[36] Hanmin Li, Kirill Acharya, and Peter Richtárik. The power of extrapolation in federated learning. *arXiv preprint arXiv:2405.13766*, 2024.

[37] Zhize Li, Haoyu Zhao, Boyue Li, and Yuejie Chi. SoteriaFL: A unified framework for private federated learning with communication compression. *Advances in Neural Information Processing Systems*, 35:4285–4300, 2022.

[38] Mingrui Liu, Zhenxun Zhuang, Yunwen Lei, and Chunyang Liao. A communication-efficient distributed gradient clipping algorithm for training deep neural networks. *Advances in Neural Information Processing Systems*, 35:26204–26217, 2022.

[39] Andrew Lowy, Ali Ghafelebashi, and Meisam Razaviyayn. Private non-convex federated learning without a trusted server. In *International Conference on Artificial Intelligence and Statistics*, pages 5749–5786. PMLR, 2023.

[40] Grigory Malinovskiy, Dmitry Kovalev, Elnur Gasanov, Laurent Condat, and Peter Richtarik. From local sgd to local fixed-point methods for federated learning. In *International Conference on Machine Learning*, pages 6692–6701. PMLR, 2020.

[41] Grigory Malinovsky and Peter Richtárik. Federated random reshuffling with compression and variance reduction. *arXiv preprint arXiv:2205.03914*, 2022.

[42] Grigory Malinovsky, Samuel Horváth, Konstantin Burlachenko, and Peter Richtárik. Federated learning with regularized client participation. *arXiv preprint arXiv:2302.03662*, 2023.

[43] Grigory Malinovsky, Konstantin Mishchenko, and Peter Richtárik. Server-side stepsizes and sampling without replacement provably help in federated optimization. In *Proceedings of the 4th International Workshop on Distributed Machine Learning*, pages 85–104, 2023.

[44] Brendan McMahan, Eider Moore, Daniel Ramage, Seth Hampson, and Blaise Aguera y Arcas. Communication-efficient learning of deep networks from decentralized data. In *Artificial intelligence and statistics*, pages 1273–1282. PMLR, 2017.

[45] H Brendan McMahan, Daniel Ramage, Kunal Talwar, and Li Zhang. Learning differentially private recurrent language models. *arXiv preprint arXiv:1710.06963*, 2017.

[46] H Brendan McMahan, Daniel Ramage, Kunal Talwar, and Li Zhang. Learning differentially private recurrent language models. In *International Conference on Learning Representations*, 2018.

[47] Luca Melis, Congzheng Song, Emiliano De Cristofaro, and Vitaly Shmatikov. Exploiting unintended feature leakage in collaborative learning. In *2019 IEEE Symposium on Security and Privacy (SP)*, pages 691–706, 2019.

[48] Konstantin Mishchenko, Ahmed Khaled, and Peter Richtárik. Random reshuffling: Simple analysis with vast improvements. *Advances in Neural Information Processing Systems*, 33: 17309–17320, 2020.

[49] Konstantin Mishchenko, Ahmed Khaled, and Peter Richtárik. Proximal and federated random reshuffling. In *International Conference on Machine Learning*, pages 15718–15749. PMLR, 2022.

[50] Konstantin Mishchenko, Grigory Malinovsky, Sebastian Stich, and Peter Richtárik. Proxskip: Yes! local gradient steps provably lead to communication acceleration! finally! In *International Conference on Machine Learning*, pages 15750–15769. PMLR, 2022.

[51] Maxence Noble, Aurélien Bellet, and Aymeric Dieuleveut. Differentially private federated learning on heterogeneous data. In *International conference on artificial intelligence and statistics*, pages 10110–10145. PMLR, 2022.

[52] Kumar Kshitij Patel, Margalit Glasgow, Ali Zindari, Lingxiao Wang, Sebastian U Stich, Ziheng Cheng, Nirmit Joshi, and Nathan Srebro. The limits and potentials of local sgd for distributed heterogeneous learning with intermittent communication. In *The Thirty Seventh Annual Conference on Learning Theory*, pages 4115–4157. PMLR, 2024.

[53] Sashank Reddi, Zachary Charles, Manzil Zaheer, Zachary Garrett, Keith Rush, Jakub Konečnỳ, Sanjiv Kumar, and H Brendan McMahan. Adaptive federated optimization. *arXiv preprint arXiv:2003.00295*, 2020.

[54] Peter Richtárik, Igor Sokolov, and Ilyas Fatkhullin. EF21: a new, simpler, theoretically better, and practically faster error feedback. *Advances in Neural Information Processing Systems*, 34: 4384–4396, 2021.

[55] Abdurakhmon Sadiev, Grigory Malinovsky, Eduard Gorbunov, Igor Sokolov, Ahmed Khaled, Konstantin Burlachenko, and Peter Richtárik. Federated optimization algorithms with random reshuffling and gradient compression. *arXiv preprint arXiv:2206.07021*, 2022.

[56] Itay Safran and Ohad Shamir. How good is SGD with random shuffling? In *Conference on Learning Theory*, pages 3250–3284. PMLR, 2020.

[57] Itay Safran and Ohad Shamir. Random shuffling beats SGD only after many epochs on ill-conditioned problems. *Advances in Neural Information Processing Systems*, 34:15151–15161, 2021.

[58] Frank Seide, Hao Fu, Jasha Droppo, Gang Li, and Dong Yu. 1-bit stochastic gradient descent and its application to data-parallel distributed training of speech dnns. In *Interspeech*, volume 2014, pages 1058–1062. Singapore, 2014.

[59] Egor Shulgin and Peter Richtárik. Shifted compression framework: Generalizations and improvements. In *Uncertainty in Artificial Intelligence*, pages 1813–1823. PMLR, 2022.

[60] Egor Shulgin, Sarit Khirirat, and Peter Richtárik. Smoothed normalization for efficient distributed private optimization. *arXiv preprint arXiv:2502.13482*, 2025.

[61] Sebastian U Stich and Sai Praneeth Karimireddy. The error-feedback framework: Better rates for sgd with delayed gradients and compressed communication. *arXiv preprint arXiv:1909.05350*, 2019.

[62] Aleksei Triastcyn and Boi Faltings. Federated learning with bayesian differential privacy. In *2019 IEEE International Conference on Big Data (Big Data)*, pages 2587–2596. IEEE, 2019.

[63] Jianyu Wang, Zachary Charles, Zheng Xu, Gauri Joshi, H Brendan McMahan, Maruan Al-Shedivat, Galen Andrew, Salman Avestimehr, Katharine Daly, Deepesh Data, et al. A field guide to federated optimization. *arXiv preprint arXiv:2107.06917*, 2021.

[64] Lingxiao Wang, Bargav Jayaraman, David Evans, and Quanquan Gu. Efficient privacy-preserving stochastic nonconvex optimization. In *Uncertainty in Artificial Intelligence*, pages 2203–2213. PMLR, 2023.

[65] Chulhee Yun, Suvrit Sra, and Ali Jadbabaie. Can single-shuffle SGD be better than reshuffling SGD and GD? *arXiv preprint arXiv:2103.07079*, 2021.

[66] Xin Zhang, Minghong Fang, Jia Liu, and Zhengyuan Zhu. Private and communication-efficient edge learning: A sparse differential gaussian-masking distributed sgd approach. In *Proceedings of the Twenty-First International Symposium on Theory, Algorithmic Foundations, and Protocol Design for Mobile Networks and Mobile Computing*, pages 261–270, 2020.

[67] Xinwei Zhang, Xiangyi Chen, Mingyi Hong, Zhiwei Steven Wu, and Jinfeng Yi. Understanding clipping for federated learning: Convergence and client-level differential privacy. In *International Conference on Machine Learning, ICML 2022*, 2022.

Appendix A. Related work and contributions

Clipping. Two popular clipping operators for FL algorithms are per-sample clipping and per-update clipping. Per-sample clipping [38] bounds the norm of the local gradient being used to update the local model parameters on each client, and ensures example-level privacy [1]. Per-update clipping [16] limits the bound of the local model update, and preserves user-level privacy [16, 67], which provides stronger privacy guarantee than example-level privacy. The convergence of FL algorithms, such as FedAvg [44] and SCAFFOLD [26], with per-sample and/or per-update clipping was analyzed by [35, 38, 51, 64, 67]. In this paper, we leverage per-update smoothed normalization, introduced by Bu et al. [5] as an alternative to clipping, to design FL algorithms that accommodate local training and differential privacy.

Federated learning with clipping and privacy. A simple yet popular FL algorithm, FedAvg [44], has been adapted to provide differential privacy (DP) by clipping model updates and injecting random noise [16, 45, 62]. These DP-FedAvg algorithms were outperformed by DP-SCAFFOLD [51], a DP version of SCAFFOLD [26]. However, these existing results require restrictive assumptions that do not hold in practice, especially in deep neural network training, such as uniformly bounded stochastic noise [11, 38], bounded gradients [37, 39, 66, 67] (which effectively ignores the impact of clipping bias), and/or bounded heterogeneity [35, 51]. To the best of our knowledge, there has been a recent work by Das et al. [12] that provides convergence guarantees for DP-FedAvg without these restrictive assumptions, but their results are limited to convex, smooth problems and require a stepsize to depend on an inaccessible constant $\Delta_i := f_i(x^\star) - \min_{x\in\mathbb{R}^d} f_i(x)$, where $x^\star = \arg\min_{x\in\mathbb{R}^d} f(x)$. In this paper, we provide convergence guarantees for private FL algorithms with smoothed normalization and error feedback. In particular, our guarantees do not rely on the restrictive assumptions commonly found in prior work, and our theoretical stepsizes can be implemented in practice.

Communcation efficiency. The most common and natural way to reduce communication is by skipping rounds through the use of local updates, which has become a standard approach in federated learning. This strategy has been extensively studied [18, 28, 30, 40, 52]

Another common biased estimator, besides clipping and normalization, is compression, which improves communication efficiency by reducing message size. The convergence of FL algorithms with compression—such as FedAvg [20], local gradient descent [27, 59], and fixed-point methods [10]—has been studied, but typically under the assumption of unbiased compression. While biased compression of local updates has been explored [19], it often requires integration with other techniques for effective gradient tracking. To our knowledge, no FL method to date uses biased compression to both address data heterogeneity and enhance communication efficiency.

Server and local stepsizes. The use of separate server and local stepsizes has been shown to be crucial in federated learning [7, 42, 53]. This separation provides greater flexibility in optimization. The local stepsize helps mitigate the impact of data heterogeneity and controls the variance from local updates [43], while the global (server-side) stepsize manages the aggregation process and stabilizes extrapolation during model updates [36].

Random reshuffling. Random reshuffling, a without-replacement sampling strategy, is widely used in SGD and often outperforms sampling with replacement. Its convergence properties have been extensively studied [21, 48, 57, 65], including in FL settings [41, 49, 55]. Other without-replacement strategies include Shuffle-Once [56] and Incremental Gradient methods [3, 31]. In this

work, Fed-α-NormEC can be extended to support IG updates, partial participation, and differential privacy with provable convergence guarantees.

Error feedback. Error feedback, also known as error compensation, has proven effective in enhancing the convergence of distributed gradient algorithms with compressed communication, leading to faster convergence and improved solution accuracy. Popular error feedback mechanisms include EF14 [58], EF21 [54], EF21-SGDM [14], EControl [15], and EFSkip [2]. Beyond compression, error feedback has been adapted by substituting compression with other operators. EF21 has inspired the development of Clip21 [29] (using clipping instead of compression) and α-NormEC [60] (employing smoothed normalization). In this paper, we contribute by adapting α-NormEC to the FL setting, resulting in Fed-α-NormEC.

Contributions. We describe our contributions below.

• **A practical method private Federated Learning.** We introduce Fed-α-NormEC —a federated learning algorithm that integrates smoothed normalization and the error feedback mechanism EF21 into clients' local updates. Unlike previous approaches, Fed-α-NormEC enables partial client participation and local training through multiple gradient-type steps. It also leverages separate server and local step sizes, offering flexibility in managing the effects of local updates and global aggregation. To reduce the need for full gradient computations, the algorithm incorporates a cyclic incremental gradient method.

• **Convergence guarantees for non-convex, smooth problems under standard assumptions** We establish the convergence of Fed-α-NormEC for minimizing non-convex, smooth objectives without relying on commonly imposed but restrictive assumptions such as bounded gradients or bounded heterogeneity. Our analysis encompasses both local gradient descent and incremental gradient updates. Notably, in the special case of full client participation with a single local gradient step, we recover the convergence guarantees of α-NormEC. For the more practical scenario involving multiple local steps, we provide— to the best of our knowledge—the first convergence analysis of differentially private federated learning methods incorporating local training. Furthermore, by introducing a server-side step size, we are able to disentangle the effects of data heterogeneity and server aggregation, leading to a clearer characterization of their individual contributions to the optimization error.

• **Differential privacy guarantees with amplification via partial participation.** We provide a privacy analysis of the proposed method for both single and multiple local update steps. Specifically, we consider an independent client sampling scheme, where each client participates in each round with probability p, independently of others. Our analysis shows that this partial participation setup enables significant reduction in differential privacy (DP) noise variance via privacy amplification through subsampling.

• **Empirical validations of Fed-α-NormEC on image classification.** We demonstrate the effectiveness of Fed-α-NormEC by applying it to the image classification task on the CIFAR-10 dataset using the ResNet20 architecture. Experiments highlight the impact of key algorithm parameters and client participation levels, corroborating our theoretical insights on convergence and privacy trade-offs. Notably, we show that partial participation, by leveraging privacy amplification, can achieve target accuracy with significantly improved communication efficiency compared to full participation, showcasing Fed-α-NormEC's utility for real-world private deep learning.

Appendix B. Conclusion

This paper presented Fed-α-NormEC, the first differentially private federated learning algorithm to offer provable convergence for nonconvex, smooth problems without resorting to unrealistic assumptions such as bounded gradients or heterogeneity. Fed-α-NormEC uniquely combines smoothed normalization and error compensation with essential practical FL components: local updates, distinct server/client learning rates, partial client participation (vital for privacy amplification), and DP noise. Our contributions pave the way for more reliable and deployable private FL systems. Finally, we verify the effectiveness of Fed-α-NormEC by experiments on private deep neural network training.

Appendix C. Notations

We use $[a, b]$ for the set $\{a, a+1, \ldots, b\}$ for integers a, b such that $a \leq b$, $\mathrm{E}[u]$ for the expectation of a random variable u, and $f(x) = \mathcal{O}(g(x))$ if $f(x) \leq Ag(x)$ for some $A > 0$ for functions $f, g : \mathbb{R}^d \to \mathbb{R}$. Finally, for vectors $x, y \in \mathbb{R}^d$, $\langle x, y\rangle$ denotes their inner product, and $\|x\|$ denotes the Euclidean norm of x.

Appendix D. Fed-α-NormEC with Multiple Local Steps

In this section, we present the convergence of Fed-α-NormEC with multiple local steps in a partial participation and private setting.

Local GD steps. We obtain the convergence of Fed-α-NormEC with local GD steps in Theorem 2. The convergence bound comprises an error term due to data heterogeneity $\mathbb{I}_{T\neq 1}\left[8L\sqrt{2L}\cdot\sqrt{\Delta^{\inf}}\right]$ where $\Delta^{\inf} = f^{\inf} - \frac{1}{M}\sum_{i=1}^{M} f_i^{\inf}$. Our theorem does not assume bounded heterogeneity that is imposed by Li et al. [35], Noble et al. [51]. Notably, if all clients share the same infimum, i.e., $f_1^{\inf} = f_2^{\inf} = \ldots = f_M^{\inf}$, this data heterogeneity error term vanishes. Furthermore, this error term is proportional to the local step size γ, due to the presence of a separate server update and distinct server- and client-side step sizes. These theoretical results highlight that the less heterogeneous the client data is, the more effective Fed-α-NormEC becomes.

Local IG steps. To avoid full gradient computations in the clients, we also introduce a variant of Fed-α-NormEC that uses cyclic incremental gradient (IG) steps. In particular, for each client, local updates are performed using gradient steps of the individual loss functions $f_{i,j}$ for each client, applied in a cyclic manner over the local dataset. The local fixed-point operators $\mathcal{T}_i(\cdot)$ are defined as $\mathcal{T}_i(x^k) = x^k - \gamma\cdot\frac{1}{N}\sum_{j=0}^{N-1}\nabla f_{i,j}(x_i^{k,j})$. Here, we focus on the deterministic version of the algorithm, avoiding high-probability analyses that are typically required for methods involving clipping or normalization. Generalization to random reshuffling and arbitrary numbers of epochs is left for future work. Further note that using cyclic incremental gradient updates introduces an additional error term of $\gamma\cdot 4L\sqrt{2L}\cdot\sqrt{\frac{1}{M}\sum_{i=1}^{M}\Delta_i^{\inf}}$, where $\Delta_i^{\inf} = f^{\inf} - \frac{1}{N}\sum_{j=1}^{N} f_{i,j}^{\inf}$. This error vanishes if all functions $f_{i,j}$ share the same infimum $f_i^{\inf}$, in which case we recover the previous result for the local GD setting.

A more detailed discussion of convergence and privacy for the method with local steps, along with formal statements of the theorems, is presented in the supplementary materials.

Appendix E. Additional experiments and details

The convergence behavior of Fed-α-NormEC as a function of communication rounds is depicted in Figure 1. The plots illustrate performance for Full ($p = 1.0$, solid lines) and Partial client participation ($p = 0.25$, dotted lines) across three settings for the hyperparameter β. The choice of β markedly influences performance. Empirically, $\beta = 0.01$ (orange lines) consistently delivers the best results, achieving the lowest training loss and highest test accuracy for both full and partial participation. For instance, with full participation, $\beta = 0.01$ leads to approximately 70% test accuracy, while $\beta = 0.1$ (green lines) results in the poorest performance (around 55-60% accuracy). Our theory (Theorem 2) supports this sensitivity, as β influences both error feedback and the DP noise term (since $\sigma_{\text{DP}} \propto p\beta$). The convergence bound includes a term $\sqrt{\beta^2 B(K+1)/M}$, implying an optimal β balances error compensation and noise.

Per communication round, Full participation ($p = 1.0$) outperforms Partial participation ($p = 0.25$) for a fixed β. This is consistent with Theorem 2: the client sampling variance component of B ($(p-1)^2/p$) is zero for $p = 1$ but positive for $p = 0.25$. Although the DP noise contribution to B ($\sigma_{\text{DP}}^2/p \propto p\beta^2$) is smaller for $p = 0.25$, the client sampling variance appears more dominant in round-wise performance. These results underscore the trade-offs in selecting β and the impact of client participation on round-wise performance.

?? further analyzes Fed-α-NormEC's performance against the total number of client-server communications (i.e., $k \times p \times M$). This visualization offers direct insights into communication efficiency. Notably, configurations with smaller client participation probabilities ($p = 0.25$ and $p = 0.5$) achieve target performance levels with significantly fewer total client-server transmissions compared to full participation ($p = 1.0$). For instance, to reach approximately 65% test accuracy, $p = 0.25$ (blue circles) requires about 1200 total communications, whereas $p = 1.0$ (green line) needs nearly 4500.

Additional details. All methods are run using a constant learning rate, without auxiliary techniques such as learning rate schedules, warm-up phases, or weight decay. The CIFAR-10 dataset is partitioned into 90% for training and 10% for testing. Training samples are randomly shuffled and evenly distributed across $n = 20$ workers, each using a local batch size of 32. We use a fixed random seed (42) to ensure reproducibility. Our implementation builds upon the publicly available GitHub repository of Idelbayev [23], and all experiments are conducted on a single NVIDIA GeForce RTX 3090 GPU.

We use a fixed smoothed normalization parameter $\alpha = 0.01$, as it was shown to have an insignificant effect on convergence [60]. Server normalization (Line 12 in Algorithm 1) is not used, as omitting it empirically improves final performance [60]. All methods are evaluated across the following hyperparameter combinations: step size $\gamma \in \{0.001, 0.01, 0.1\}$ and sensitivity threshold $\beta \in \{0.001, 0.01, 0.1\}$. We analyze the performance of Fed-α-NormEC in the differentially private setting by training the model for 300 communication rounds.

E.1. Fed-α-NormEC vs FedAvg

We compare the performance of our Algorithm 1 (Fed-α-NormEC) with the standard FedAvg approach, as defined in Section 2:

$$x^{k+1} = x^k - \frac{\eta}{B}\left[\sum_{i \in S^k} \Psi(x^k - \mathcal{T}_i(x^k)) + z_i^k\right],$$

where Ψ is the smoothed normalization operator, $\mathcal{T}_i(x) = x - \gamma \nabla f_i(x)$ is the local gradient mapping, $\eta = \gamma$, and $p = 1$ in the Differentially Private (DP) setting. We follow the same experimental setup as described in Section 4.

Figure 2 presents the convergence of training loss and test accuracy for both methods across different values of the sensitivity parameter β. The results demonstrate that the Error Compensation (EC) mechanism in Fed-α-NormEC consistently accelerates convergence and improves test accuracy compared to FedAvg, across all privacy levels (i.e., all tested values of β). Notably, Fed-α-NormEC achieves its best performance for $\beta = 0.01$, which aligns with the findings in Section 4.

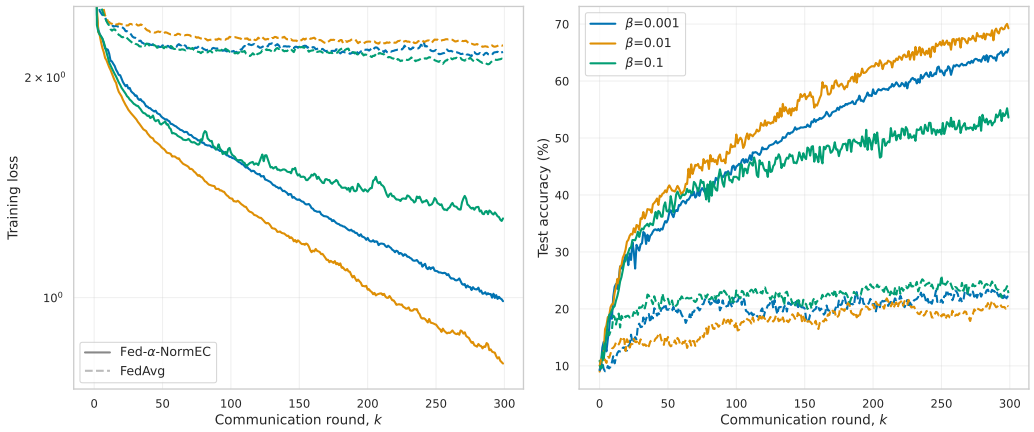


Figure 2: Error Compensation (EC) provides significant benefits across various β values.

To further analyze the effect of hyperparameters, Figure 3 shows the highest test accuracy achieved by FedAvg for each (β, γ) pair. The optimal performance for FedAvg is observed at $\beta = 0.1$, while the best results are generally found along the diagonal, where the product $\beta \cdot \gamma = 0.001$.

Importantly, prior work [29, 60] has shown that FedAvg with clipping or normalization may fail to converge in certain settings, whereas Fed-α-NormEC remains robust and convergent. Our results further support this observation, highlighting the effectiveness of the Error Compensation mechanism in improving both convergence speed and final accuracy, especially in privacy-constrained federated learning scenarios.

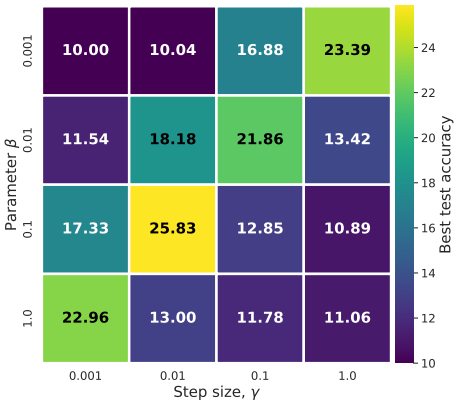
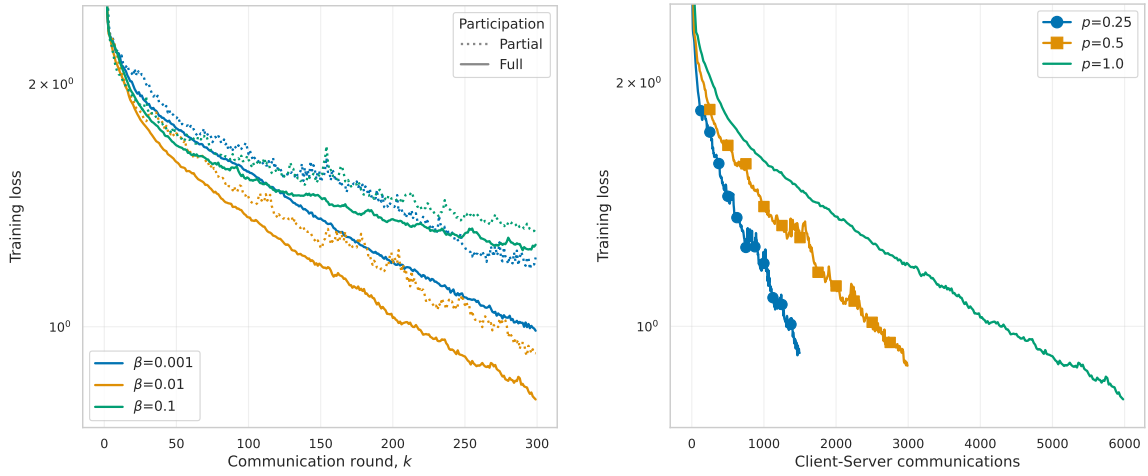


Figure 3: The highest test accuracy achieved by FedAvg for different β and γ parameters.



(a) Convergence of Fed-α-NormEC under Full [solid] and Partial participation [dotted] for $p = 0.25$.

(b) Fed-α-NormEC under varying participation rates; x-axis shows total client-to-server transmissions.

Figure 4: Training loss convergence of Fed-α-NormEC corresponding to the test accuracy plots shown in the main text.

Appendix F. Useful Lemmas

We introduce useful lemmas for our convergence analysis.

First, Theorem 5 establishes the bounds for $\left\|\frac{x^k-\mathcal{T}_i(x^k)}{\gamma} - v_i^k\right\|$ and $\left\|\frac{x^k-\mathcal{T}_i(x^k)}{\gamma} - v_i^{k+1}\right\|$, two quantities that will be applied in the induction proof to establish the first convergence step of Fed-α-NormEC.

Lemma 5 *Let $v_i^k \in \mathbb{R}^d$ be governed by*

$$v_i^{k+1} = v_i^k + \beta \mathrm{Norm}_\alpha\left(\frac{x^k - \mathcal{T}_i(x^k)}{\gamma} - v_i^k\right), \text{ for } i \in [1, M] \text{ and } k \geq 0,$$

and let the fixed-point operator $\mathcal{T}_i(\cdot)$ satisfy

$$\|\mathcal{T}_i(x) - \mathcal{T}_i(y)\| \leq \rho \|x - y\|, \text{ for } \rho > 0 \text{ and } x, y \in \mathbb{R}^d.$$

If $\left\|\frac{x^k - \mathcal{T}_i(x^k)}{\gamma} - v_i^k\right\| \leq C$ *for some* $C > 0$, $\left\|x^{k+1} - x^k\right\| \leq \eta$, $\frac{\beta}{\alpha + C} < 1$, *and* $\eta \leq \frac{\gamma\beta C}{(1+\rho)(\alpha+C)}$, *then* $\left\|\frac{x^{k+1} - \mathcal{T}_i(x^{k+1})}{\gamma} - v_i^{k+1}\right\| \leq C$ *and* $\left\|\frac{x^k - \mathcal{T}_i(x^k)}{\gamma} - v_i^{k+1}\right\| \leq C$.

Proof From the definition of the Euclidean norm,

$$\begin{aligned}\left\|P_i(x^{k+1}) - v_i^{k+1}\right\| &\overset{v_i^{k+1}}{=} \left\|P_i(x^{k+1}) - v_i^k - \beta N_\alpha(P_i(x^k) - v_i^k)\right\| \\ &\leq \left\|P_i(x^{k+1}) - P_i(x^k)\right\| + \left\|P_i(x^k) - v_i^k - \mathrm{Norm}_\alpha\left(P_i(x^k) - v_i^k\right)\right\|,\end{aligned}$$

where $P_i(x) = (x - \mathcal{T}_i(x))/\gamma$.

Next, by the triangle inequality and by the fact that $\|\mathcal{T}_i(x) - \mathcal{T}_i(y)\| \leq \rho \|x - y\|$ for $\rho > 0$ and $x, y \in \mathbb{R}^d$, we bound the first term:

$$\begin{aligned}\left\|P_i(x^{k+1}) - P_i(x^k)\right\| &= \left\|\frac{x^{k+1} - \mathcal{T}_i(x^{k+1})}{\gamma} - \frac{x^k - \mathcal{T}_i(x^k)}{\gamma}\right\| \\ &\leq \frac{1}{\gamma}\left(\left\|x^{k+1} - x^k\right\| + \left\|\mathcal{T}_i(x^k) - \mathcal{T}_i(x^{k+1})\right\|\right) \\ &\leq \frac{1}{\gamma}(1+\rho)\left\|x^{k+1} - x^k\right\|.\end{aligned}$$

Therefore,

$$\left\|P_i(x^{k+1}) - v_i^{k+1}\right\| \leq \frac{1}{\gamma}(1+\rho)\left\|x^{k+1} - x^k\right\| + \left\|P_i(x^k) - v_i^k - \mathrm{Norm}_\alpha\left(P_i(x^k) - v_i^k\right)\right\|.$$

Next, from Lemma 1 of [60], we can bound the second term:

$$\left\|P_i(x^k) - v_i^k - \mathrm{Norm}_\alpha\left(P_i(x^k) - v_i^k\right)\right\| \leq \left|1 - \frac{\beta}{\alpha + \left\|\frac{x^k - \mathcal{T}_i(x^k)}{\gamma} - v_i^k\right\|}\right| \left\|\frac{x^k - \mathcal{T}_i(x^k)}{\gamma} - v_i^k\right\|.$$

If $\left\|\frac{x^k - \mathcal{T}_i(x^k)}{\gamma} - v_i^k\right\| \leq C$ for some $C > 0$, and $\frac{\beta}{\alpha + C} < 1$, then

$$\begin{aligned}\left\|P_i(x^k) - v_i^k - \mathrm{Norm}_\alpha\left(P_i(x^k) - v_i^k\right)\right\| &\leq \left|1 - \frac{\beta}{\alpha + C}\right| C \\ &\leq \left(1 - \frac{\beta}{\alpha + C}\right) C.\end{aligned}$$

Hence, we obtain

$$\left\|P_i(x^{k+1}) - v_i^{k+1}\right\| \leq \frac{1}{\gamma}(1+\rho)\left\|x^{k+1} - x^k\right\| + \left(1 - \frac{\beta}{\alpha + C}\right) C.$$

If $\left\|x^{k+1}-x^k\right\| \leq \eta$, then

$$\left\|P_i(x^{k+1})-v_i^{k+1}\right\| \leq \frac{1}{\gamma}(1+\rho)\eta + \left(1-\frac{\beta}{\alpha+C}\right)C.$$

If $\eta \leq \frac{\gamma}{1+\rho}\frac{\beta C}{(\alpha+C)}$, then $\left\|P_i(x^{k+1})-v_i^{k+1}\right\| \leq C$. Furthermore, we can show that

$$\begin{aligned}
\left\|P_i(x^k)-v_i^{k+1}\right\| &\overset{v_i^{k+1}}{=} \left\|P_i(x^k)-v_i^k-\beta\mathrm{Norm}_\alpha\left(P_i(x^k)-v_i^k\right)\right\| \\
&\overset{\text{Lemma 1 of [60]}}{\leq} \left|1-\frac{\beta}{\alpha+\left\|\frac{x^k-\mathcal{T}_i(x^k)}{\gamma}-v_i^k\right\|}\right|\left\|\frac{x^k-\mathcal{T}_i(x^k)}{\gamma}-v_i^k\right\| \\
&\overset{\beta/(\alpha+C)<1}{\leq} \left(1-\frac{\beta}{\alpha+C}\right)C \\
&\leq C.
\end{aligned}$$

■

Next, Theorem 6 bounds $\left\|e^k\right\|$ under the recursion of $e^{k+1}=e^k+\beta\frac{1}{M}\sum_{i=1}^M z_i^k$, where z_i^k is the random vector, and by utilizing Theorem 6, we obtain Theorem 7, which bounds $\left\|\frac{1}{M}\sum_{i=1}^M v_i^k-\hat{v}^k\right\|$, the quantity that will be applied to conclude the convergence of Fed-α-NormEC.

Lemma 6 *Let $e^k \in \mathbb{R}^d$ be governed by*

$$e^{k+1}=e^k+\beta z^k, \quad \textit{for } 0 \leq k \leq K,$$

where $z^k=\frac{1}{M}\sum_{i=1}^M z_i^k$ and each $z_i^k \in \mathbb{R}^d$ is an independent random vector satisfying

$$\mathrm{E}\left[z_i^k\right]=0, \quad \textit{and} \quad \mathrm{E}\left[\left\|z_i^k\right\|^2\right] \leq \sigma^2.$$

Then,

$$\mathrm{E}\left[\left\|e^{k+1}\right\|\right] \leq \mathrm{E}\left[\left\|e^0\right\|\right]+\sqrt{\frac{\beta^2(K+1)\sigma^2}{M}}.$$

Proof By applying the recursion of e^{k+1} recursively,

$$e^{k+1}=e^0+\beta\sum_{l=0}^k z^l.$$

From the definition of the Euclidean norm, and next by the triangle inequality and by taking the expectation,

$$\mathrm{E}\left[\left\|e^{k+1}\right\|\right] \leq \mathrm{E}\left[\left\|e^0\right\|\right]+\mathrm{E}\left[\left\|\beta\sum_{l=0}^k z^l\right\|\right].$$

By Jensen's inequality,

$$\begin{aligned}\mathrm{E}\left[\left\|e^{k+1}\right\|\right] &\leq \mathrm{E}\left[\left\|e^0\right\|\right] + \sqrt{\mathrm{E}\left[\left\|\beta\sum_{l=0}^{k} z^l\right\|^2\right]} \\ &= \mathrm{E}\left[\left\|e^0\right\|\right] + \sqrt{\beta^2\sum_{l=0}^{k}\mathrm{E}\left[\left\|z^l\right\|^2\right] + \beta^2\sum_{i\neq j}\mathrm{E}\left[\langle z^i, z^j\rangle\right]}.\end{aligned}$$

Since z_i^k is independent of one another, we obtain $\mathrm{E}\left[\left\langle z^i, z^j\right\rangle\right] = 0$ for $i \neq j$, and $\mathrm{E}\left[\left\|z^k\right\|^2\right] = \frac{1}{M}\sum_{i=1}^{M}\mathrm{E}\left[\left\|z_i^k\right\|^2\right] \leq \sigma^2/n$. Therefore,

$$\mathrm{E}\left[\left\|e^{k+1}\right\|\right] \leq \mathrm{E}\left[\left\|e^0\right\|\right] + \sqrt{\beta^2\frac{(K+1)\sigma^2}{M}}.$$

■

Lemma 7 *Consider* Fed-α-NormEC *with any local updating operator* $\mathcal{T}_i(\cdot)$ *for solving Problem* (1)*, where Theorem 1 holds. Then,*

$$\mathrm{E}\left[\left\|\frac{1}{M}\sum_{i=1}^{M} v_i^{k+1} - \hat{v}^{k+1}\right\|\right] \leq \sqrt{\frac{\beta^2 B}{M}(K+1)},$$

where $B = 2p(1-1/p)^2 + 2(1-p) + 2\sigma_{\mathrm{DP}}^2/p$.

Proof Define $e^k := \frac{1}{M}\sum_{i=1}^{M} v_i^k - \hat{v}^k$. Then,

$$\begin{aligned}e^{k+1} &= \frac{1}{M}\sum_{i=1}^{M} v_i^{k+1} - \hat{v}^{k+1} \\ &\overset{v_i^{k+1},\hat{v}^{k+1}}{=} \frac{1}{M}\sum_{i=1}^{M} v_i^k - \hat{v}^k + \beta n^k \\ &= e^k + \beta n^k,\end{aligned}$$

where $n^k = \frac{1}{M}\sum_{i=1}^{M} n_i^k$ and $n_i^k = (1-q_i^k)\mathrm{Norm}_\alpha\left(\frac{x^k - \mathcal{T}_i(x^k)}{\gamma} - v_i^k\right) - q_i^k z_i^k$.

Next, as q_i^k and z_i^k are independent random vectors, n_i^k is also independent of one another, and satisfies $\mathrm{E}\left[n_i^k\right] = 0$ and

$$\begin{aligned}
\mathrm{E}\left[\left\|n_i^k\right\|^2\right] &= \mathrm{E}\left[\left\|(1-q_i^k)\mathrm{Norm}_\alpha\left(\frac{x^k - \mathcal{T}_i(x^k)}{\gamma} - v_i^k\right) - q_i^k z_i^k\right\|^2\right] \\
&\leq 2\mathrm{E}\left[(1-q_i^k)^2\left\|\mathrm{Norm}_\alpha\left(\frac{x^k - \mathcal{T}_i(x^k)}{\gamma} - v_i^k\right)\right\|^2\right] + 2\mathrm{E}\left[(q_i^k)^2\left\|z_i^k\right\|^2\right] \\
&\overset{\|\mathrm{Norm}_\alpha(\cdot)\|\leq 1}{\leq} 2\mathrm{E}\left[(1-q_i^k)^2\right] + 2\mathrm{E}\left[(q_i^k)^2\left\|z_i^k\right\|^2\right] \\
&\overset{q_i^k \text{ and } z_i^k \text{ are independent}}{=} 2\mathrm{E}\left[(1-q_i^k)^2\right] + 2\mathrm{E}\left[(q_i^k)^2\right]\mathrm{E}\left[\left\|z_i^k\right\|^2\right] \\
&\leq 2p(1-1/p)^2 + 2(1-p) + 2p/p^2 \cdot \sigma_{\mathrm{DP}}^2.
\end{aligned}$$

Therefore, from Theorem 6 with $z^k = n^k$ and $z_i^k = n_i^k$, we obtain

$$\mathrm{E}\left[\left\|e^{k+1}\right\|\right] \leq \mathrm{E}\left[\left\|e^0\right\|\right] + \sqrt{\frac{\beta^2(K+1)\cdot B}{M}},$$

where $B = 2p(1-1/p)^2 + 2(1-p) + 2\sigma_{\mathrm{DP}}^2/p$. Finally, since $\hat{v}^0 = \frac{1}{n}\sum_{i=1}^n v_i^0$, we obtain $\left\|e^0\right\| = 0$, and complete the proof. ■

Finally, Theorem 8 provides the descent inequality for $f(x^k) - f^{\inf}$ in normalized gradient descent. From these established descent inequalities, and Theorem 9 derives the sublinear convergence up to constants.

Lemma 8 *Let $f : \mathbb{R}^d \to \mathbb{R}$ be lower-bounded by $f^{\inf} > -\infty$ and L-smoooth, and let $x^k \in \mathbb{R}^d$ be governed by*

$$x^{k+1} = x^k - \gamma\frac{G^k}{\|G^k\|},$$

where $\gamma > 0$. Then,

$$f(x^{k+1}) - f^{\inf} \leq f(x^k) - f^{\inf} - \gamma\left\|\nabla f(x^k)\right\| + 2\gamma\left\|\nabla f(x^k) - G^k\right\| + \frac{L\gamma^2}{2}.$$

Proof By the lower-bound and smoothess of $f(\cdot)$, and by the definition of x^{k+1},

$$\begin{aligned}
f(x^{k+1}) - f^{\inf} &\leq f(x^k) - f^{\inf} - \frac{\gamma}{\|G^k\|}\left\langle \nabla f(x^k), G^k\right\rangle + \frac{L\gamma^2}{2} \\
&\leq f(x^k) - f^{\inf} - \frac{\gamma}{\|G^k\|}\left\langle G^k, G^k\right\rangle + \frac{\gamma}{\|G^k\|}\left\langle G^k - \nabla f(x^k), G^k\right\rangle + \frac{L\gamma^2}{2} \\
&= f(x^k) - f^{\inf} - \gamma\left\|G^k\right\| + \frac{\gamma}{\|G^k\|}\left\langle G^k - \nabla f(x^k), G^k\right\rangle + \frac{L\gamma^2}{2}.
\end{aligned}$$

By Cauchy-Scwartz inequality, i.e. $\langle x, y\rangle \leq \|x\| \, \|y\|$ for $x, y \in \mathbb{R}^d$,

$$f(x^{k+1}) - f^{\inf} \quad \leq \quad f(x^k) - f^{\inf} - \gamma \left\|G^k\right\| + \gamma \left\|\nabla f(x^k) - G^k\right\| + \frac{L\gamma^2}{2}.$$

Finally, by the triangle inequality,

$$f(x^{k+1}) - f^{\inf} \quad \leq \quad f(x^k) - f^{\inf} - \gamma \left\|\nabla f(x^k)\right\| + 2\gamma \left\|\nabla f(x^k) - G^k\right\| + \frac{L\gamma^2}{2}.$$

■

Lemma 9 *Let $\{V^k\}$, $\{W^k\}$ be non-negative sequences satisfying*

$$V^{k+1} \leq (1 + b_1\gamma^2)V^k - b_2\gamma W^k + b_3\gamma.$$

Then,

$$\min_{k\in[0,K]} W^k \leq \frac{\exp(b_1\gamma^2(K+1))}{K+1}\frac{V^0}{b_2\gamma} + \frac{b_3}{b_2}.$$

Proof Define $w^k := \frac{w^{k+1}}{1+b_1\gamma^2}$ for all $k \geq 0$. Then,

$$\begin{aligned} w^k W^k \quad &\leq \quad \frac{w^k(1+b_1\gamma^2)V^k}{b_2\gamma} - \frac{w^k V^{k+1}}{b_2\gamma} + \frac{b_3}{b_2} \\ &= \quad \frac{w^{k-1}V^k - w^k V^{k+1}}{b_2\gamma} + \frac{b_3}{b_2}. \end{aligned}$$

By summing the inequality over $k = 0, 1, \ldots, K$,

$$\begin{aligned} \sum_{k=0}^K w^k W^k \qquad &\leq \qquad \frac{w^{-1}V^0 - w^K V^{K+1}}{b_2\gamma(K+1)} + \frac{b_3}{b_2}\sum_{k=0}^K w^k \\ &\overset{w^k, V^k \geq 0}{\leq} \quad \frac{w^{-1}V^0}{b_2\gamma(K+1)} + \frac{b_3}{b_2}\sum_{k=0}^K w^k. \end{aligned}$$

Therefore,

$$\begin{aligned} \min_{k\in[0,K]} W^k \quad &\leq \quad \frac{1}{\sum_{k=0}^K w^k}\sum_{k=0}^K w^k W^k \\ &\leq \quad \frac{w^{-1}V^0}{b_2\gamma(K+1)\sum_{k=0}^K w^k} + \frac{b_3}{b_2}. \end{aligned}$$

Next, since

$$\begin{aligned} \sum_{k=0}^K w^k \quad &\geq \quad (K+1)\min_{k\in[0,K]} w^k \\ &= \quad (K+1)w^{K+1} \\ &= \quad \frac{(K+1)w^{-1}}{(1+b_1\gamma^2)^{K+1}}, \end{aligned}$$

we get

$$\min_{k\in[0,K]} W^k \leq \frac{(1+b_1\gamma^2)^{K+1}V^0}{b_2\gamma(K+1)} + \frac{b_3}{b_2}.$$

Finally, since $1+x \leq \exp(x)$, we have $(1+b_1\gamma^2)^{K+1} \leq \exp(b_1\gamma^2(K+1))$. Hence, we obtain the final result.

■

Appendix G. Multiple Local GD Steps

We derive the convergence theorem of Fed-α-NormEC using multiple local gradient descent (GD) steps (Theorem 2).

G.1. Key Lemmas

We begin by introducing key lemmas for analyzing Fed-α-NormEC using multiple local GD steps. Theorem 10 bounds $\frac{1}{M}\sum_{i=1}^{M}\|\nabla f_i(x)\|$, while Theorem 11 proves the properties of local GD steps.

Lemma 10 *Let f be bounded from below by $f^{\inf} > -\infty$, and each f_i be bounded from below by $f_i^{\inf} > -\infty$ and L-smooth. Then,*

$$\frac{1}{M}\sum_{i=1}^{M}\|\nabla f_i(x)\| \leq \sqrt{\frac{2L}{\Delta^{\inf}}}[f(x) - f^{\inf}] + \sqrt{2L\Delta^{\inf}},$$

where $\Delta^{\inf} = f^{\inf} - \frac{1}{M}\sum_{i=1}^{M} f_i^{\inf} > 0$.

Proof Let f be bounded from below by $f^{\inf} > -\infty$, and each f_i be bounded from below by $f_i^{\inf} > -\infty$ and L-smooth. Then,

$$\|\nabla f_i(x)\|^2 \leq 2L[f_i(x) - f_i^{\inf}].$$

Therefore,

$$\frac{1}{M}\sum_{i=1}^{M}\|\nabla f_i(x)\|^2 \leq A[f(x) - f^{\inf}] + B,$$

where $A = 2L$, $B = 2L\Delta^{\inf}$, and $\Delta^{\inf} = f^{\inf} - \frac{1}{M}\sum_{i=1}^{M} f_i^{\inf} > 0$. Thus, we obtain

$$\begin{aligned}
\frac{1}{M}\sum_{i=1}^{M}\|\nabla f_i(x)\| &\overset{\text{Jensen's inequality}}{\leq} \sqrt{\frac{1}{M}\sum_{i=1}^{M}\|\nabla f_i(x)\|^2} \\
&\leq \sqrt{A[f(x) - f^{\inf}] + B} \\
&= \frac{A[f(x) - f^{\inf}] + B}{\sqrt{A[f(x) - f^{\inf}] + B}} \\
&\overset{f(x)\geq f^{\inf}}{\leq} \frac{A}{\sqrt{B}}[f(x) - f^{\inf}] + \sqrt{B}.
\end{aligned}$$

■

Lemma 11 *Let each f_i be L-smooth, and let $\mathcal{T}_i(x^k) = x^k - \frac{\gamma}{T}\sum_{j=0}^{T-1}\nabla f_i(x_i^{k,j})$, where the sequence $\{x_i^{k,l}\}$ is generated by*

$$x_i^{k,l+1} = x_i^{k,l} - \frac{\gamma}{T}\nabla f_i(x_i^{k,l}), \quad \text{for} \quad l = 0, 1, \ldots, T-1,$$

given that $x_i^{k,0} = x^k$. If $\gamma \leq \frac{1}{2L}$, and $\left\|x^{k+1} - x^k\right\| \leq \eta$ with $\eta > 0$, then

1. $x_i^{k,l} = x^k - \frac{\gamma}{T}\sum_{j=0}^{l-1} \nabla f_i(x_i^{k,l})$.
2. $\frac{1}{T}\sum_{j=0}^{T-1} \left\|x_i^{k+1,j} - x_i^{k,j}\right\| \leq 2\eta$.
3. $\frac{1}{T}\sum_{j=0}^{T-1} \left\|x^k - x_i^{k,j}\right\| \leq 2\gamma \left\|\nabla f_i(x^k)\right\|$.
4. $\left\|\mathcal{T}_i(x^{k+1}) - \mathcal{T}_i(x^k)\right\| \leq 2\eta$.
5. $\left\|(x^k - \gamma\nabla f_i(x^k)) - \mathcal{T}_i(x^k)\right\| \leq 2L\gamma^2 \left\|\nabla f_i(x^k)\right\|$.

Proof We prove the first statement by recursively applying the equation for $x_i^{k,j+1}$ for $j = 0, 1, \ldots, l-1$.

Next, we prove the second statement. From the definition of the Euclidean norm, by the triangle inequality, and by the L-smoothness of $f_i(\cdot)$,

$$\begin{aligned}
\left\|x_i^{k+1,l} - x_i^{k,l}\right\| &\overset{x_i^{k,j}}{=} \left\|x^{k+1} - x^k - \frac{\gamma}{T}\sum_{j=0}^{l-1}(\nabla f_i(x_i^{k+1,j}) - \nabla f_i(x_i^{k,j}))\right\| \\
&\leq \left\|x^{k+1} - x^k\right\| + \frac{\gamma}{T}\sum_{j=0}^{l-1}\left\|\nabla f_i(x_i^{k+1,j}) - \nabla f_i(x_i^{k,j})\right\| \\
&\leq \left\|x^{k+1} - x^k\right\| + \frac{L\gamma}{T}\sum_{j=0}^{l-1}\left\|x_i^{k+1,j} - x_i^{k,j}\right\|.
\end{aligned}$$

If $\left\|x^{k+1} - x^k\right\| \leq \eta$ with $\eta > 0$, then

$$\begin{aligned}
\left\|x_i^{k+1,l} - x_i^{k,l}\right\| &\leq \eta + \frac{L\gamma}{T}\sum_{j=0}^{l-1}\left\|x_i^{k+1,j} - x_i^{k,j}\right\| \\
&\overset{l\leq T}{\leq} \eta + \frac{L\gamma}{T}\sum_{j=0}^{T-1}\left\|x_i^{k+1,j} - x_i^{k,j}\right\|
\end{aligned}$$

Therefore,

$$\sum_{j=0}^{T-1}\left\|x_i^{k+1,j} - x_i^{k,j}\right\| \leq \eta T + L\gamma\sum_{j=0}^{T-1}\left\|x_i^{k+1,j} - x_i^{k,j}\right\|.$$

If $\gamma \leq \frac{1}{2L}$, then $L\gamma \leq 1/2$, and

$$\frac{1}{T}\sum_{j=0}^{T-1}\left\|x_i^{k+1,j} - x_i^{k,j}\right\| \leq 2\eta.$$

Next, we prove the third statement. From the definition of the Euclidean norm, and of $x_i^{k,l}$ from the first statement,

$$\begin{aligned}\left\|x^k - x_i^{k,j}\right\| &= \left\|\frac{\gamma}{T}\sum_{j=0}^{l-1}\nabla f_i(x_i^{k,j})\right\| \\ &= \left\|\frac{\gamma}{T}\sum_{j=0}^{l-1}[\nabla f_i(x_i^{k,j}) - \nabla f_i(x^k) + \nabla f_i(x^k)]\right\|.\end{aligned}$$

By the triangle inequality, and by the L-smoothness of $f_i(\cdot)$,

$$\begin{aligned}\left\|x^k - x_i^{k,j}\right\| &\leq \frac{\gamma}{T}\sum_{j=0}^{l-1}\left\|\nabla f_i(x_i^{k,j}) - \nabla f_i(x^k)\right\| + \frac{\gamma}{T}\sum_{j=0}^{l-1}\left\|\nabla f_i(x^k)\right\| \\ &\leq \frac{L\gamma}{T}\sum_{j=0}^{l-1}\left\|x_i^{k,j} - x^k\right\| + \frac{\gamma}{T}\sum_{j=0}^{l-1}\left\|\nabla f_i(x^k)\right\|.\end{aligned}$$

By the fact that $l \leq T$ and that $\|x\| \geq 0$ for $x \in \mathbb{R}^d$,

$$\left\|x^k - x_i^{k,j}\right\| \leq \frac{L\gamma}{T}\sum_{j=0}^{T-1}\left\|x_i^{k,j} - x^k\right\| + \gamma\left\|\nabla f_i(x^k)\right\|.$$

Therefore,

$$\sum_{j=0}^{T-1}\left\|x^k - x_i^{k,j}\right\| \leq L\gamma\sum_{j=0}^{T-1}\left\|x_i^{k,j} - x^k\right\| + \gamma T\left\|\nabla f_i(x^k)\right\|.$$

If $\gamma \leq \frac{1}{2L}$, then $L\gamma \leq 1/2$, and

$$\sum_{j=0}^{T-1}\left\|x^k - x_i^{k,j}\right\| \leq 2\gamma T\left\|\nabla f_i(x^k)\right\|.$$

Next, we prove the fourth statement. From the definition of $\mathcal{T}_i(x^k)$,

$$\left\|\mathcal{T}_i(x^{k+1}) - \mathcal{T}_i(x^k)\right\| = \left\|x^{k+1} - x^k - \frac{\gamma}{T}\sum_{j=0}^{l-1}[\nabla f_i(x_i^{k,l+1}) - \nabla f_i(x_i^{k,l})]\right\|.$$

By the triangle inequality, and by the L-smoothness of $f_i(\cdot)$,

$$\begin{aligned}\left\|\mathcal{T}_i(x^{k+1}) - \mathcal{T}_i(x^k)\right\| &\leq \left\|x^{k+1} - x^k\right\| + \frac{\gamma}{T}\sum_{j=0}^{l-1}\left\|\nabla f_i(x_i^{k,l+1}) - \nabla f_i(x_i^{k,l})\right\| \\ &\leq \left\|x^{k+1} - x^k\right\| + \frac{L\gamma}{T}\sum_{j=0}^{l-1}\left\|x_i^{k,l+1} - x_i^{k,l}\right\|.\end{aligned}$$

By the fact that $\left\|x^{k+1}-x^k\right\| \leq \eta$, that $l \leq T$, and that $\sum_{j=0}^{T-1}\left\|x_i^{k+1,j}-x_i^{k,j}\right\| \leq 2\eta T$,

$$\left\|\mathcal{T}_i(x^{k+1})-\mathcal{T}_i(x^k)\right\| \leq \eta + L\gamma\cdot 2\eta \overset{L\gamma\leq 1/2}{\leq} 2\eta.$$

Finally, we prove the fifth statement. From the definition of $\mathcal{T}_i(x^k)$,

$$\left\|(x^k-\gamma\nabla f_i(x^k))-\mathcal{T}_i(x^k)\right\| = \left\|\left(x^k-\frac{\gamma}{T}\sum_{l=0}^{T-1}\nabla f_i(x^k)\right)-\left(x^k-\frac{\gamma}{T}\sum_{l=0}^{T-1}\nabla f_i(x_i^{k,l})\right)\right\|.$$

By the triangle inequality, the L-smoothness of $f_i(\cdot)$, and the fact that

$$\sum_{j=0}^{T}\left\|x^k-x_i^{k,j}\right\| \leq 2\gamma T\left\|\nabla f_i(x^k)\right\|,$$

we obtain

$$\begin{aligned}\left\|(x^k-\gamma\nabla f_i(x^k))-\mathcal{T}_i(x^k)\right\| &\leq \frac{\gamma}{T}\sum_{l=0}^{T-1}\left\|\nabla f_i(x^k)-\nabla f_i(x_i^{k,l})\right\| \\ &\leq \frac{L\gamma}{T}\sum_{l=0}^{T-1}\left\|x^k-x_i^{k,l}\right\| \\ &\leq 2L\gamma^2\left\|\nabla f_i(x^k)\right\|.\end{aligned}$$

■

G.2. Proof of Theorem 2

Now we are ready to prove the convergence rate of Fed-α-NormEC using multiple local GD steps.

Theorem 12 (Fed-α-NormEC with local GD steps) *Consider Fed-α-NormEC for solving Problem (1) where Theorem 1 holds. Let $\mathcal{T}_i(x^k) = x^k - \gamma\frac{1}{T}\sum_{j=0}^{T-1}\nabla f_i(x_i^{k,j})$, where the sequence $\{x_i^{k,j}\}$ is generated by $x_i^{k,j+1} = x_i^{k,j} - \frac{\gamma}{T}\nabla f_i(x_i^{k,j})$, for $j = 0,1,\ldots,T-1$, given that $x_i^{k,0} = x^k$. Furthermore, let $\beta,\alpha>0$ be chosen such that $\frac{\beta}{\alpha+R}<1$ with $R = \max_{i\in[1,M]}\left\|v_i^0 - \frac{x^0-\mathcal{T}_i(x^0)}{\gamma}\right\|$. If $\eta\gamma \leq \frac{1}{K+1}\frac{\Delta^{\inf}}{4L\sqrt{2L}}$, $0<\eta\leq\frac{\gamma}{3}\frac{\beta R}{\alpha+R}$, and $0<\gamma\leq\frac{1}{2L}$, then*

$$\begin{aligned}\min_{k\in[0,K]}\mathrm{E}\left[\left\|\nabla f(x^k)\right\|\right] \leq& \frac{3}{K+1}\frac{f(x^0)-f^{\inf}}{\eta} + 2R + 2\sqrt{\frac{\beta^2 B}{M}(K+1)} \\ &+\gamma\cdot\mathbb{I}_{T\neq 1}\left[8L\sqrt{2L}\sqrt{\Delta^{\inf}}\right] + \eta\cdot\frac{L}{2},\end{aligned}$$

where $B = 2p(1-1/p)^2 + 2(1-p) + 2\sigma_{\mathrm{DP}}^2/p$, and $\Delta^{\inf} = f^{\inf} - \frac{1}{M}\sum_{i=1}^{M} f_i^{\inf} > 0$.

Proof

We prove the result in the following steps.

Step 1) Bound $\left\|v_i^k - \frac{x^k - \mathcal{T}_i(x^k)}{\gamma}\right\|$ by induction, and bound $\left\|v_i^{k+1} - \frac{x^k - \mathcal{T}_i(x^k)}{\gamma}\right\|$. We prove that $\left\|v_i^k - \frac{x^k - \mathcal{T}_i(x^k)}{\gamma}\right\| \leq \max_{i\in[1,M]} \left\|v_i^0 - \frac{x^0 - \mathcal{T}_i(x^0)}{\gamma}\right\|$ by induction. It is trivial to show the condition when $k = 0$. Next, suppose that $\left\|v_i^k - \frac{x^k - \mathcal{T}_i(x^k)}{\gamma}\right\| \leq \max_{i\in[1,M]} \left\|v_i^0 - \frac{x^0 - \mathcal{T}_i(x^0)}{\gamma}\right\|$ holds. From Theorem 11, $\mathcal{T}_i(x^k) = x^k - \frac{\gamma}{T}\sum_{j=0}^{T-1} \nabla f_i(x_i^{k,j})$ satisfies

$$\left\|\mathcal{T}_i(x^{k+1}) - \mathcal{T}_i(x^k)\right\| \leq 2\eta.$$

Therefore, from Lemma 5 with $\rho = 2$, $C = R = \max_{i\in[1,M]} \left\|v_i^0 - \frac{x^0 - \mathcal{T}_i(x^0)}{\gamma}\right\|$, we can prove that by choosing $\frac{\beta}{\alpha+R} < 1$ and $\eta \leq \frac{\gamma\beta R}{(1+\rho)(\alpha+R)}$, $\left\|v_i^{k+1} - \frac{x^{k+1} - \mathcal{T}_i(x^{k+1})}{\gamma}\right\| \leq R$. We complete the induction proof.

Next, from Lemma 5, $\left\|v_i^{k+1} - \frac{x^k - \mathcal{T}_i(x^k)}{\gamma}\right\| \leq \max_{i\in[1,M]} \left\|v_i^0 - \frac{x^0 - \mathcal{T}_i(x^0)}{\gamma}\right\|$.

Step 2) Bound $f(x^k) - f^{\inf}$. From Theorem 8 with $G^k = \hat{v}^{k+1}$,

$$\begin{aligned} f(x^{k+1}) - f^{\inf} &\leq f(x^k) - f^{\inf} - \eta\left\|\nabla f(x^k)\right\| + 2\eta\left\|\nabla f(x^k) - \hat{v}^{k+1}\right\| + \frac{L\eta^2}{2} \\ &\overset{\text{triangle inequality}}{\leq} f(x^k) - f^{\inf} - \eta\left\|\nabla f(x^k)\right\| + 2\eta\left\|\nabla f(x^k) - v^{k+1}\right\| \\ &\quad + 2\eta\left\|\hat{v}^{k+1} - v^{k+1}\right\| + \frac{L\eta^2}{2}, \end{aligned}$$

where $v^{k+1} = \frac{1}{M}\sum_{i=1}^M v_i^{k+1}$. Next, since

$$\begin{aligned} \left\|\nabla f(x^k) - v^{k+1}\right\| &= \left\|\nabla f(x^k) - \frac{1}{M}\sum_{i=1}^M v_i^{k+1}\right\| \\ &\overset{\text{triangle inequality}}{\leq} \frac{1}{M}\sum_{i=1}^M \left\|v_i^{k+1} - \nabla f_i(x^k)\right\| \\ &\overset{\text{triangle inequality}}{\leq} \frac{1}{M}\sum_{i=1}^M \left\|v_i^{k+1} - \frac{x^k - \mathcal{T}_i(x^k)}{\gamma}\right\| \\ &\quad + \frac{1}{M}\sum_{i=1}^M \left\|\frac{x^k - \mathcal{T}_i(x^k)}{\gamma} - \nabla f_i(x^k)\right\|, \end{aligned}$$

where $\mathcal{T}_i(x^k) = x^k - \frac{\gamma}{T}\sum_{j=0}^{T-1} \nabla f_i(x_i^{k,j})$, we get

$$\left\|\nabla f(x^k) - v^{k+1}\right\| \leq \frac{1}{M}\sum_{i=1}^M \left\|v_i^{k+1} - \frac{x^k - \mathcal{T}_i(x^k)}{\gamma}\right\| + \frac{1}{\gamma}\frac{1}{M}\sum_{i=1}^M \left\|x^k - \mathcal{T}_i(x^k) - \gamma\nabla f_i(x^k)\right\|.$$

Plugging the upperbound for $\left\|\nabla f(x^k) - v^{k+1}\right\|$ into the main inequality in $f(x^k) - f^{\inf}$, we obtain

$$\begin{aligned} f(x^{k+1}) - f^{\inf} &\leq f(x^k) - f^{\inf} - \eta \left\|\nabla f(x^k)\right\| + 2\eta \frac{1}{M}\sum_{i=1}^{M} \left\| v_i^{k+1} - \frac{x^k - \mathcal{T}_i(x^k)}{\gamma} \right\| \\ &\quad + \frac{2\eta}{\gamma}\frac{1}{M}\sum_{i=1}^{M} \left\|(x^k - \gamma\nabla f_i(x^k)) - \mathcal{T}_i(x^k)\right\| + 2\eta \left\|\hat{v}^{k+1} - v^{k+1}\right\| + \frac{L\eta^2}{2}. \end{aligned}$$

By the fact that $\left\| v_i^{k+1} - \frac{x^k - \mathcal{T}_i(x^k)}{\gamma} \right\| \leq R$ from Step 1),

$$\begin{aligned} f(x^{k+1}) - f^{\inf} &\leq f(x^k) - f^{\inf} - \eta \left\|\nabla f(x^k)\right\| + 2\eta R \\ &\quad + \frac{2\eta}{\gamma}\frac{1}{M}\sum_{i=1}^{M} \left\|(x^k - \gamma\nabla f_i(x^k)) - \mathcal{T}_i(x^k)\right\| + 2\eta \left\|\hat{v}^{k+1} - v^{k+1}\right\| + \frac{L\eta^2}{2}. \end{aligned}$$

To complete the proof, we consider two possible cases for $\mathcal{T}_i(x^k)$: 1) when $T = 1$ and 2) when $T \neq 1$.

Case 1) $\mathcal{T}_i(x^k)$ with $T = 1$. When $\mathcal{T}_i(x^k)$ with $T = 1$, $\left\|(x^k - \gamma\nabla f_i(x^k)) - \mathcal{T}_i(x^k)\right\| = 0$, and

$$f(x^{k+1}) - f^{\inf} \leq f(x^k) - f^{\inf} - \eta \left\|\nabla f(x^k)\right\| + 2\eta R + 2\eta \left\|\hat{v}^{k+1} - v^{k+1}\right\| + \frac{L\eta^2}{2}.$$

Case 2) $\mathcal{T}_i(x^k)$ with $T > 1$. When $\mathcal{T}_i(x^k)$ with $T > 1$, from Theorem 11,

$$\begin{aligned} f(x^{k+1}) - f^{\inf} &\leq f(x^k) - f^{\inf} - \eta \left\|\nabla f(x^k)\right\| + 2\eta R \\ &\quad + 4L\gamma\eta \frac{1}{M}\sum_{i=1}^{M} \left\|\nabla f_i(x^k)\right\| + 2\eta \left\|\hat{v}^{k+1} - v^{k+1}\right\| + \frac{L\eta^2}{2}. \end{aligned}$$

Therefore, from two cases, we obtain the descent inequality,

$$\begin{aligned} f(x^{k+1}) - f^{\inf} &\leq f(x^k) - f^{\inf} - \eta \left\|\nabla f(x^k)\right\| + 2\eta R \\ &\quad + 4L\gamma\eta \frac{1}{M}\sum_{i=1}^{M} \left\|\nabla f_i(x^k)\right\| + 2\eta \left\|\hat{v}^{k+1} - v^{k+1}\right\| + \frac{L\eta^2}{2}. \end{aligned}$$

Next, from Theorem 10,

$$\begin{aligned} f(x^{k+1}) - f^{\inf} &\leq \left(1 + \frac{4L\sqrt{2L}}{\sqrt{\Delta^{\inf}}}\gamma\eta\right)(f(x^k) - f^{\inf}) - \eta \left\|\nabla f(x^k)\right\| + 2\eta R \\ &\quad + 4L\sqrt{2L}\gamma\eta\sqrt{\Delta^{\inf}} + 2\eta \left\|\hat{v}^{k+1} - v^{k+1}\right\| + \frac{L\eta^2}{2}. \end{aligned}$$

Since

$$\begin{aligned} \mathrm{E}\left[\left\|\hat{v}^{k+1} - v^{k+1}\right\|\right] &\leq \frac{1}{\gamma}\mathrm{E}\left[\left\|\frac{1}{M}\sum_{i=1}^{M} v_i^{k+1} - \hat{v}^{k+1}\right\|\right] \\ &\overset{\text{Theorem 7}}{\leq} \frac{1}{\gamma}\sqrt{\frac{\beta^2 B}{M}(K+1)}, \end{aligned}$$

by taking the expectation,

$$\mathrm{E}\left[f(x^{k+1})-f^{\inf}\right] \leq \left(1+\frac{4L\sqrt{2L}}{\sqrt{\Delta^{\inf}}}\gamma\eta\right)\mathrm{E}\left[f(x^k)-f^{\inf}\right]-\eta\mathrm{E}\left[\left\|\nabla f(x^k)\right\|\right]+2\eta R$$
$$+8L\sqrt{2L}\gamma\eta\sqrt{\Delta^{\inf}}+2\eta\sqrt{\frac{\beta^2 B}{M}(K+1)}+\frac{L\eta^2}{2}.$$

By applying Theorem 9 with $\eta\gamma \leq \frac{1}{K+1}\frac{\Delta^{\inf}}{4L\sqrt{2L}}$ and using the fact $(1+\eta\gamma\frac{4L\sqrt{2L}}{\Delta^{\inf}})^{K+1} \leq \exp(\eta\gamma\frac{4L\sqrt{2L}}{\Delta^{\inf}}(K+1)) \leq \exp(1) \leq 3$ we finalize the proof. ■

G.3. Corollaries for Fed-α-NormEC with multiple local GD steps from Theorem 2

Corollary 13 (Convergence bound for Fed-α-NormEC with multiple local GD steps) *Consider Fed-α-NormEC for solving Problem (1) under the same setting as Theorem 2. Let $T>1$ (multiple local GD steps). If $\gamma=\frac{1}{2L(K+1)^{1/8}}$, $v_i^0\in\mathbb{R}^d$ is chosen such that $\max_{i\in[1,M]}\left\|\frac{x^0-\mathcal{T}_i(x^0)}{\gamma}-v_i^0\right\|=\frac{D_1}{(K+1)^{1/8}}$ with $D_1>0$, and $\beta=\frac{D_2}{(K+1)^{5/8}}$ with $D_2>0$, and $\eta=\frac{\hat{\eta}}{(K+1)^{7/8}}$ with $\hat{\eta}=\min\left(\frac{\Delta^{\inf}}{2\sqrt{2L}},\frac{D_1D_2}{4L(\alpha+D_1)}\right)$, then*

$$\min_{k\in[0,K]}\mathrm{E}\left[\left\|\nabla f(x^k)\right\|\right]\leq\frac{A_1}{(K+1)^{1/8}}+\frac{A_2}{(K+1)^{7/8}},$$

where $A_1=3\frac{f(x^0)-f^{\inf}}{\hat{\eta}}+2D_1+\frac{2\sqrt{B}D_2}{\sqrt{M}}+4\sqrt{2L}\sqrt{\Delta^{\inf}}$ and $A_2=\hat{\eta}L/2$.

Proof Let $T>1$. Then, from Theorem 2,

$$\min_{k\in[0,K]}\mathrm{E}\left[\left\|\nabla f(x^k)\right\|\right]\leq\frac{3}{K+1}\frac{f(x^0)-f^{\inf}}{\eta}+2R+2\sqrt{\frac{\beta^2B}{M}(K+1)}+\eta\cdot\frac{L}{2}$$
$$+\gamma\cdot\left[8L\sqrt{2L}\sqrt{\Delta^{\inf}}\right],$$

where $B=2p(1-1/p)^2+2(1-p)+2\sigma_{\mathrm{DP}}^2/p$.

Next, suppose that

- $\gamma=\frac{1}{2L(K+1)^{1/8}}$ to guarantee that $\gamma\leq 1/(2L)$
- $v_i^0\in\mathbb{R}^d$ such that $\max_{i\in[1,M]}\left\|\frac{x^0-\mathcal{T}_i(x^0)}{\gamma}-v_i^0\right\|=R=\frac{D_1}{(K+1)^{1/8}}$ with $D_1>0$
- $\beta=\frac{D_2}{(K+1)^{5/8}}$ with $D_2>0$.

Then, we choose $\eta=\frac{\hat{\eta}}{(K+1)^{7/8}}$ with $\hat{\eta}=\min\left(\frac{\Delta^{\inf}}{2\sqrt{2L}},\frac{D_1D_2}{4L(\alpha+D_1)}\right)$ to ensure that $\eta\gamma\leq\frac{1}{K+1}\frac{\Delta^{\inf}}{4L\sqrt{2L}}$ and $\eta\leq\frac{\gamma}{2}\frac{\beta R}{\alpha+R}$. Therefore,

$$\min_{k\in[0,K]}\mathrm{E}\left[\left\|\nabla f(x^k)\right\|\right]\leq\frac{A_1}{(K+1)^{1/8}}+\frac{A_2}{(K+1)^{7/8}},$$

where $A_1 = 3\frac{f(x^0)-f^{\inf}}{\hat{\eta}} + 2D_1 + \frac{2\sqrt{B}D_2}{\sqrt{M}} + 4\sqrt{2L}\sqrt{\Delta^{\inf}}$ and $A_2 = \hat{\eta}L/2$. ■

Corollary 14 (Utility bound for Fed-α-NormEC with multiple local GD steps) *Consider Fed-α-NormEC for solving Problem (1) under the same setting as Theorem 2. Let $T > 1$ (multiple local GD steps), let $\sigma_{\text{DP}} = c\frac{p\sqrt{(K+1)\log(1/\delta)}}{\epsilon}$ with $c > 0$ (privacy with subsampling amplification), and let $p = \frac{\hat{B}}{M}$ for $\hat{B} \in [1, M]$ (client subsampling). If $\beta = \frac{\hat{\beta}}{K+1}$ with $\hat{\beta} = \sqrt{\frac{3(f(x^0)-f^{\inf})}{\gamma}}\sqrt[4]{\frac{M}{B_2}}$, $\gamma < \frac{\Delta^{\inf}(\alpha+R)}{\sqrt{2L}\hat{\beta}R}$, $\alpha = R = \mathcal{O}\left(\sqrt[4]{d}\frac{\sqrt{f(x^0)-f^{\inf}}}{\sqrt{\gamma}}\sqrt[4]{\frac{B_2}{M}}\right)$ with $B_2 = 2c^2\frac{\hat{B}}{M}\frac{\log(1/\delta)}{\epsilon^2}$, and $\eta = \frac{1}{K+1}\frac{\gamma}{2}\frac{\hat{\beta}R}{\alpha+R}$, then*

$$\min_{k\in[0,K]} \mathrm{E}\left[\left\|\nabla f(x^k)\right\|\right] \leq \mathcal{O}\left(\Delta\sqrt[4]{\frac{d\hat{B}}{M^2}\frac{\log(1/\delta)}{\epsilon^2}} + \sqrt{L}\sqrt{\Delta^{\inf}}\right),$$

where $\Delta = \max(\alpha, 2)\sqrt{L}\sqrt{f(x^0) - f^{\inf}}$.

Proof Let $T > 1$. Then, from Theorem 2,

$$\min_{k\in[0,K]} \mathrm{E}\left[\left\|\nabla f(x^k)\right\|\right] \leq \frac{3}{K+1}\frac{f(x^0)-f^{\inf}}{\eta} + 2R + 2\sqrt{\frac{\beta^2 B}{M}(K+1)} + \eta\cdot\frac{L}{2} + \gamma\cdot\left[8L\sqrt{2L}\sqrt{\Delta^{\inf}}\right],$$

where $B = 2p(1-1/p)^2 + 2(1-p) + 2\sigma_{\text{DP}}^2/p$.

Also, let $\sigma_{\text{DP}} = c\frac{p\sqrt{(K+1)\log(1/\delta)}}{\epsilon}$ with $c > 0$, and let $p = \frac{\hat{B}}{M}$ for $\hat{B} \in [1, M]$ is the number of clients being sampled on each round. Then, $B = \frac{2\hat{B}}{M}\left(1-\frac{M}{\hat{B}}\right)^2 + 2\left(1-\frac{\hat{B}}{M}\right) + 2\frac{c\sqrt{K+1}\log(1/\delta)}{\epsilon}$, and

$$\min_{k\in[0,K]} \mathrm{E}\left[\left\|\nabla f(x^k)\right\|\right] \leq \frac{3}{K+1}\frac{f(x^0)-f^{\inf}}{\eta} + 2R + 2\beta\sqrt{\frac{B_1}{M}(K+1)} + 2\beta\sqrt{\frac{B_2}{M}(K+1)} + \eta\cdot\frac{L}{2} + \gamma\cdot\left[8L\sqrt{2L}\sqrt{\Delta^{\inf}}\right],$$

where $B_1 = \frac{2\hat{B}}{M}\left[\left(1-\frac{M}{\hat{B}}\right)^2 + \frac{M}{\hat{B}} - 1\right]$ and $B_2 = 2c^2\frac{\hat{B}}{M}\frac{\log(1/\delta)}{\epsilon^2}$.

If $\beta = \frac{\hat{\beta}}{K+1}$ with $\hat{\beta} > 0$, then

$$\min_{k\in[0,K]} \mathrm{E}\left[\left\|\nabla f(x^k)\right\|\right] \leq \frac{3}{K+1}\frac{f(x^0)-f^{\inf}}{\eta} + 2R + 2\hat{\beta}\sqrt{\frac{B_1}{M(K+1)}} + 2\hat{\beta}\sqrt{\frac{B_2}{M}} + \eta\cdot\frac{L}{2} + \gamma\cdot\left[8L\sqrt{2L}\sqrt{\Delta^{\inf}}\right].$$

Since $\beta = \frac{\hat{\beta}}{K+1}$, we obtain

$$\eta \leq \frac{1}{K+1} \min\left(\frac{\Delta^{\inf}}{2\sqrt{2L}}, \frac{\gamma}{2}\frac{\hat{\beta}R}{\alpha+R}\right).$$

If $\Delta^{\inf} > \frac{\gamma\sqrt{2L}\hat{\beta}R}{\alpha+R}$, then

$$\eta \leq \frac{1}{K+1}\frac{\gamma}{2}\frac{\hat{\beta}R}{\alpha+R}.$$

If $\eta = \frac{1}{K+1}\frac{\gamma}{2}\frac{\hat{\beta}R}{\alpha+R}$, then

$$\begin{aligned}\min_{k\in[0,K]} \mathrm{E}\left[\left\|\nabla f(x^k)\right\|\right] \leq& \frac{6\alpha(f(x^0)-f^{\inf})}{\gamma\hat{\beta}R} + \frac{6(f(x^0)-f^{\inf})}{\gamma\hat{\beta}} + 2R + 2\hat{\beta}\sqrt{\frac{B_2}{M}} \\ &+ 2\hat{\beta}\sqrt{\frac{B_1}{M(K+1)}} + \frac{1}{K+1}\cdot\frac{\gamma L\hat{\beta}R}{4(\alpha+R)} \\ &+ \gamma\cdot\left[8L\sqrt{2L}\sqrt{\Delta^{\inf}}\right].\end{aligned}$$

If $\hat{\beta} = \sqrt{\frac{3(f(x^0)-f^{\inf})}{\gamma}}\sqrt[4]{\frac{M}{B_2}}$, then

$$\begin{aligned}\min_{k\in[0,K]} \mathrm{E}\left[\left\|\nabla f(x^k)\right\|\right] \leq& \frac{2\sqrt{3}\alpha\sqrt{f(x^0)-f^{\inf}}}{\sqrt{\gamma}R}\sqrt[4]{\frac{B_2}{M}} + \frac{4\sqrt{3}\sqrt{f(x^0)-f^{\inf}}}{\sqrt{\gamma}}\sqrt[4]{\frac{B_2}{M}} + 2R \\ &+ 2\hat{\beta}\sqrt{\frac{B_1}{M(K+1)}} + \frac{1}{K+1}\cdot\frac{\gamma L\hat{\beta}R}{4(\alpha+R)} + \gamma\cdot\left[8L\sqrt{2L}\sqrt{\Delta^{\inf}}\right].\end{aligned}$$

If $\alpha = R = \mathcal{O}\left(\sqrt[4]{d}\frac{\sqrt{f(x^0)-f^{\inf}}}{\sqrt{\gamma}}\sqrt[4]{\frac{B_2}{M}}\right)$, then

$$\begin{aligned}\min_{k\in[0,K]} \mathrm{E}\left[\left\|\nabla f(x^k)\right\|\right] \leq& \mathcal{O}\left(\Delta\frac{\sqrt{f(x^0)-f^{\inf}}}{\sqrt{\gamma}}\sqrt[4]{d\frac{B_2}{M}}\right) + \mathcal{O}\left(\frac{1}{\sqrt{K+1}}\right) + \mathcal{O}\left(\frac{1}{K+1}\right) \\ &+ \gamma\cdot\left[8L\sqrt{2L}\sqrt{\Delta^{\inf}}\right] \\ \leq& \mathcal{O}\left(\Delta\frac{\sqrt{f(x^0)-f^{\inf}}}{\sqrt{\gamma}}\sqrt[4]{d\frac{B_2}{M}} + \gamma\cdot\left[8L\sqrt{2L}\sqrt{\Delta^{\inf}}\right]\right) \\ &+ \mathcal{O}\left(\frac{1}{\sqrt{K+1}}\right) + \mathcal{O}\left(\frac{1}{K+1}\right),\end{aligned}$$

where $\Delta = 2\sqrt{3}\max(\alpha, 2)$. Finally, if $\gamma = 1/(2L)$, then we complete the proof. ■

G.4. Proof of Theorem 3

Corollary 15 (Convergence bound for Fed-α-NormEC with one local GD step) *Consider Fed-α-NormEC for solving Problem (1) under the same setting as Theorem 2. Let $T = 1$ and $N = 0$ (one local GD step). If $\gamma = \frac{1}{2L}$, $v_i^0 \in \mathbb{R}^d$ is chosen such that $\max_{i\in[1,M]} \left\| \frac{x^0 - \mathcal{T}_i(x^0)}{\gamma} - v_i^0 \right\| = \frac{D_1}{(K+1)^{1/6}}$ with $D_1 > 0$, and $\beta = \frac{D_2}{(K+1)^{2/3}}$ with $D_2 > 0$, and $\eta = \frac{\hat{\eta}}{(K+1)^{5/6}}$ with $\hat{\eta} = \frac{D_1 D_2}{4L(\alpha+D_1)}$, then*

$$\min_{k\in[0,K]} \mathrm{E}\left[\left\|\nabla f(x^k)\right\|\right] \leq \frac{A_1}{(K+1)^{1/6}} + \frac{A_2}{(K+1)^{5/6}},$$

where $A_1 = 3\frac{f(x^0)-f^{\inf}}{\hat{\eta}} + 2D_1 + \frac{2\sqrt{B}D_2}{\sqrt{M}}$ and $A_2 = \hat{\eta}L/2$.

Proof Let $T = 1$. Then, from Theorem 2,

$$\min_{k\in[0,K]} \mathrm{E}\left[\left\|\nabla f(x^k)\right\|\right] \leq \frac{3}{K+1}\frac{f(x^0)-f^{\inf}}{\eta} + 2R + 2\sqrt{\frac{\beta^2 B}{M}(K+1)} + \eta\cdot\frac{L}{2},$$

where $B = 2p(1-1/p)^2 + 2(1-p) + 2\sigma_{\mathrm{DP}}^2/p$.

Next, suppose that

- $\gamma = \frac{1}{2L}$
- $v_i^0 \in \mathbb{R}^d$ such that $\max_{i\in[1,M]} \left\| \frac{x^0 - \mathcal{T}_i(x^0)}{\gamma} - v_i^0 \right\| = R = \frac{D_1}{(K+1)^{1/6}}$ with $D_1 > 0$
- $\beta = \frac{D_2}{(K+1)^{2/3}}$ with $D_2 > 0$.

Then, we choose $\eta = \frac{\hat{\eta}}{(K+1)^{5/6}}$ with $\hat{\eta} = \frac{D_1 D_2}{4L(\alpha+D_1)}$ to ensure that $\eta \leq \frac{\gamma}{2}\frac{\beta R}{\alpha+R}$. Therefore,

$$\min_{k\in[0,K]} \mathrm{E}\left[\left\|\nabla f(x^k)\right\|\right] \leq \frac{A_1}{(K+1)^{1/6}} + \frac{A_2}{(K+1)^{5/6}},$$

where $A_1 = 3\frac{f(x^0)-f^{\inf}}{\hat{\eta}} + 2D_1 + \frac{2\sqrt{B}D_2}{\sqrt{M}}$ and $A_2 = \hat{\eta}L/2$.

■

G.5. Proof of Theorem 4

Corollary 16 (Utility bound for Fed-α-NormEC with one local GD step) *Consider Fed-α-NormEC for solving Problem (1) under the same setting as Theorem 2. Let $T = 1$ (one local GD step), let $\sigma_{\mathrm{DP}} = c\frac{p\sqrt{(K+1)\log(1/\delta)}}{\epsilon}$ with $c > 0$ (privacy with subsampling amplification), and let $p = \frac{\hat{B}}{M}$ for $\hat{B} \in [1, M]$ (client subsampling). If $\beta = \frac{\hat{\beta}}{K+1}$ with $\hat{\beta} = \sqrt{\frac{3(f(x^0)-f^{\inf})}{\gamma}}\sqrt[4]{\frac{M}{B_2}}$, $\gamma < \frac{\Delta^{\inf}(\alpha+R)}{\sqrt{2L}\hat{\beta}R}$, $\alpha = R = \mathcal{O}\left(\sqrt[4]{d}\frac{\sqrt{f(x^0)-f^{\inf}}}{\sqrt{\gamma}}\sqrt[4]{\frac{B_2}{M}}\right)$ with $B_2 = 2c^2\frac{\hat{B}}{M}\frac{\log(1/\delta)}{\epsilon^2}$, and $\eta = \frac{1}{K+1}\frac{\gamma}{2}\frac{\hat{\beta}R}{\alpha+R}$, then*

$$\min_{k\in[0,K]} \mathrm{E}\left[\left\|\nabla f(x^k)\right\|\right] \leq \mathcal{O}\left(\Delta\sqrt[4]{\frac{d\hat{B}}{M^2}\frac{\log(1/\delta)}{\epsilon^2}}\right),$$

where $\Delta = \max(\alpha, 2)\sqrt{L}\sqrt{f(x^0) - f^{\inf}}$.

Proof Let $T = 1$. Then, from Theorem 2,

$$\min_{k\in[0,K]} \mathrm{E}\left[\left\|\nabla f(x^k)\right\|\right] \leq \frac{3}{K+1}\frac{f(x^0)-f^{\inf}}{\eta} + 2R + 2\sqrt{\frac{\beta^2 B}{M}(K+1)} + \eta\cdot\frac{L}{2},$$

where $B = 2p(1-1/p)^2 + 2(1-p) + 2\sigma_{\mathrm{DP}}^2/p$.

Also, let $\sigma_{\mathrm{DP}} = c\frac{p\sqrt{(K+1)\log(1/\delta)}}{\epsilon}$ with $c > 0$, and let $p = \frac{\hat{B}}{M}$ for $\hat{B} \in [1, M]$ is the number of clients being sampled on each round. Then, $B = \frac{2\hat{B}}{M}\left(1-\frac{M}{\hat{B}}\right)^2 + 2\left(1-\frac{\hat{B}}{M}\right) + 2\frac{c\sqrt{K+1}\log(1/\delta)}{\epsilon}$, and

$$\min_{k\in[0,K]} \mathrm{E}\left[\left\|\nabla f(x^k)\right\|\right] \leq \frac{3}{K+1}\frac{f(x^0)-f^{\inf}}{\eta} + 2R + 2\beta\sqrt{\frac{B_1}{M}(K+1)} + 2\beta\sqrt{\frac{B_2}{M}}(K+1) + \eta\cdot\frac{L}{2},$$

where $B_1 = \frac{2\hat{B}}{M}\left[\left(1-\frac{M}{\hat{B}}\right)^2 + \frac{M}{\hat{B}} - 1\right]$ and $B_2 = 2c^2\frac{\hat{B}}{M}\frac{\log(1/\delta)}{\epsilon^2}$.

If $\beta = \frac{\hat{\beta}}{K+1}$ with $\hat{\beta} > 0$, then

$$\min_{k\in[0,K]} \mathrm{E}\left[\left\|\nabla f(x^k)\right\|\right] \leq \frac{3}{K+1}\frac{f(x^0)-f^{\inf}}{\eta} + 2R + 2\hat{\beta}\sqrt{\frac{B_1}{M(K+1)}} + 2\hat{\beta}\sqrt{\frac{B_2}{M}} + \eta\cdot\frac{L}{2}.$$

Since $\beta = \frac{\hat{\beta}}{K+1}$, we obtain

$$\eta \leq \frac{1}{K+1}\min\left(\frac{\Delta^{\inf}}{2\sqrt{2L}}, \frac{\gamma}{2}\frac{\hat{\beta}R}{\alpha+R}\right).$$

If $\Delta^{\inf} > \frac{\gamma\sqrt{2L}\hat{\beta}R}{\alpha+R}$, then

$$\eta \leq \frac{1}{K+1}\frac{\gamma}{2}\frac{\hat{\beta}R}{\alpha+R}.$$

If $\eta = \frac{1}{K+1}\frac{\gamma}{2}\frac{\hat{\beta}R}{\alpha+R}$, then

$$\begin{aligned}\min_{k\in[0,K]} \mathrm{E}\left[\left\|\nabla f(x^k)\right\|\right] \leq& \frac{6\alpha(f(x^0)-f^{\inf})}{\gamma\hat{\beta}R} + \frac{6(f(x^0)-f^{\inf})}{\gamma\hat{\beta}} + 2R + 2\hat{\beta}\sqrt{\frac{B_2}{M}} \\ &+ 2\hat{\beta}\sqrt{\frac{B_1}{M(K+1)}} + \frac{1}{K+1}\cdot\frac{\gamma L\hat{\beta}R}{4(\alpha+R)}.\end{aligned}$$

If $\hat{\beta} = \sqrt{\frac{3(f(x^0)-f^{\inf})}{\gamma}}\sqrt[4]{\frac{M}{B_2}}$, then

$$\begin{aligned}\min_{k\in[0,K]} \mathrm{E}\left[\left\|\nabla f(x^k)\right\|\right] \leq& \frac{2\sqrt{3}\alpha\sqrt{f(x^0)-f^{\inf}}}{\sqrt{\gamma}R}\sqrt[4]{\frac{B_2}{M}} + \frac{4\sqrt{3}\sqrt{f(x^0)-f^{\inf}}}{\sqrt{\gamma}}\sqrt[4]{\frac{B_2}{M}} + 2R \\ &+ 2\hat{\beta}\sqrt{\frac{B_1}{M(K+1)}} + \frac{1}{K+1}\cdot\frac{\gamma L\hat{\beta}R}{4(\alpha+R)}.\end{aligned}$$

If $\alpha = R = \mathcal{O}\left(\sqrt[4]{d}\frac{\sqrt{f(x^0)-f^{\inf}}}{\sqrt{\gamma}}\sqrt[4]{\frac{B_2}{M}}\right)$, then

$$\min_{k\in[0,K]} \mathrm{E}\left[\left\|\nabla f(x^k)\right\|\right] \leq \mathcal{O}\left(\Delta\frac{\sqrt{f(x^0)-f^{\inf}}}{\sqrt{\gamma}}\sqrt[4]{d\frac{B_2}{M}}\right) + \mathcal{O}\left(\frac{1}{\sqrt{K+1}}\right) + \mathcal{O}\left(\frac{1}{K+1}\right),$$

where $\Delta = 2\sqrt{3}\max(\alpha, 2)$. Finally, if $\gamma = 1/(2L)$, then we complete the proof. ■

Appendix H. Multiple Local IG steps

In this section, we derive the convergence theorem of Fed-α-NormEC with multiple local steps using the Incremental Gradient (IG) method. The IG method has the following update rule.

$$\mathcal{T}_i^{IG}(x^k) = x^k - \gamma \frac{1}{N} \sum_{j=0}^{N-1} \nabla f_{i,j}(x_i^{k,j}), \tag{2}$$

where $x_i^{k,j}$ is updated according to:

$$x_i^{k,j+1} = x_i^{k,j} - \frac{\gamma}{N} \nabla f_{i,j}(x_i^{k,j}) \quad \text{for} \quad j = 0, 1, \ldots, T-1.$$

In the update rule of the IG method, the number of local steps is equal to the size of the local data set. This implies that each client performs local updates $\mathcal{T}_i^{IG}(\cdot)$ using their entire local dataset. Furthermore, the IG method employs a fixed, deterministic permutation for its cyclic updates, unlike the well-known Random Reshuffling method.

H.1. Key Lemmas

First, we introduce key lemmas for analyzing Fed-α-NormEC using multiple local IG steps. Theorem 17 bounds $\frac{1}{M}\sum_{i=1}^{M}\frac{1}{N}\sum_{j=0}^{N-1}\left\|x_i^{k,j} - x^k\right\|$ while Theorem 18 proves the properties of local IG steps.

Lemma 17 *Consider the local IG method updates in (2). Let f be bounded from below by $f^{\inf} > -\infty$, let each f_i be bounded from below by $f_i^{\inf} > -\infty$, and let each $f_{i,j}$ be bounded from below by $f_{i,j}^{\inf}$ and L-smooth. Then,*

$$\frac{1}{M}\sum_{i=1}^{M}\frac{1}{N}\sum_{j=0}^{N-1}\left\|x_i^{k,j} - x^k\right\| \leq \frac{2\sqrt{2}L\gamma(f(x^k) - f^{\inf})}{\sqrt{L\Delta^{\inf}}} + \sqrt{2}\gamma\sqrt{L\Delta^{\inf}} + 2\gamma\sqrt{L\frac{1}{M}\sum_{i=1}^{M}\Delta_i^{\inf}},$$

where $\Delta^{\inf} = f^{\inf} - \frac{1}{M}\sum_{i=1}^{M} f_i^{\inf}$ and $\Delta_i^{\inf} = f^{\inf} - \frac{1}{N}\sum_{j=1}^{N} f_{i,j}^{\inf}$

Proof Applying Lemma 6 from [43] for the local IG method updates in (2), we have

$$\frac{1}{M}\sum_{i=1}^{M}\frac{1}{N}\sum_{j=0}^{N-1}\left\|x_i^{k,j} - x^k\right\|^2 \leq 4L\gamma^2\left(f(x^k) - f^{\inf}\right) + 2\gamma^2 L\Delta^{\inf} + 2\gamma^2 L\frac{1}{M}\sum_{i=1}^{M}\Delta_i^{\inf}.$$

Next, by Jensen's inequality,

$$\begin{aligned}
\frac{1}{M}\sum_{i=1}^{M}\frac{1}{N}\sum_{j=0}^{N-1}\left\|x_i^{k,j} - x^k\right\| &\leq \sqrt{\frac{1}{M}\sum_{i=1}^{M}\frac{1}{N}\sum_{j=0}^{N-1}\left\|x_i^{k,j} - x^k\right\|^2} \\
&\leq \sqrt{4L\gamma^2\left(f(x^k) - f^{\inf}\right) + 2\gamma^2 L\Delta^{\inf} + 2\gamma^2 L\frac{1}{M}\sum_{i=1}^{M}\Delta_i^{\inf}} \\
&\leq \sqrt{4L\gamma^2\left(f(x^k) - f^{\inf}\right) + 2\gamma^2 L\Delta^{\inf}} + \sqrt{2\gamma^2 L\frac{1}{M}\sum_{i=1}^{M}\Delta_i^{\inf}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{M}\sum_{i=1}^{M}\frac{1}{N}\sum_{j=0}^{N-1}\left\|x_i^{k,j}-x^k\right\| &\leq \frac{4L\gamma^2\left(f(x^k)-f^{\inf}\right)+2\gamma^2L\Delta^{\inf}}{\sqrt{4L\gamma^2\left(f(x^k)-f^{\inf}\right)+2\gamma^2L\Delta^{\inf}}}+2\gamma\sqrt{L\frac{1}{M}\sum_{i=1}^{M}\Delta_i^{\inf}}\\ &\leq \frac{4L\gamma^2\left(f(x^k)-f^{\inf}\right)+2\gamma^2L\Delta^{\inf}}{\sqrt{2\gamma^2L\Delta^{\inf}}}+2\gamma\sqrt{L\frac{1}{M}\sum_{i=1}^{M}\Delta_i^{\inf}}\\ &\leq \frac{2\sqrt{2}L\gamma(f(x^k)-f^{\inf})}{\sqrt{L\Delta^{\inf}}}+\sqrt{2}\gamma\sqrt{L\Delta^{\inf}}+2\gamma\sqrt{L\frac{1}{M}\sum_{i=1}^{M}\Delta_i^{\inf}}.\end{aligned}$$

■

Lemma 18 *Let each f_i be L-smooth, and let $\mathcal{T}_i(x^k)=x^k-\frac{\gamma}{N}\sum_{j=0}^{N-1}\nabla f_{i,j}(x_i^{k,j})$, where the sequence $\{x_i^{k,l}\}$ is generated by*

$$x_i^{k,l+1}=x_i^{k,l}-\frac{\gamma}{N}\nabla f_{i,j}(x_i^{k,l}),\quad \textit{for}\quad l=0,1,\ldots,N-1,$$

given that $x_i^{k,0}=x^k$. If $\gamma\leq\frac{1}{2L}$, and $\left\|x^{k+1}-x^k\right\|\leq\eta$ with $\eta>0$, then

1. $x_i^{k,l}=x^k-\frac{\gamma}{N}\sum_{j=0}^{l-1}\nabla f_{i,j}(x_i^{k,l})$.
2. $\frac{1}{N}\sum_{j=0}^{N-1}\left\|x_i^{k+1,j}-x_i^{k,j}\right\|\leq 2\eta$.
3. $\left\|\mathcal{T}_i(x^{k+1})-\mathcal{T}_i(x^k)\right\|\leq 2\eta$.
4. $\frac{1}{M}\sum_{i=1}^{M}\left\|\mathcal{T}_i(x^k)-\left(x^k-\gamma\nabla f_i(x^k)\right)\right\|\leq\gamma L\frac{1}{M}\sum_{i=1}^{M}\frac{1}{N}\sum_{j=0}^{N-1}\left\|x_i^{k,j}-x^k\right\|$

Proof The first statement derives from unrolling the recursion for $x_i^{k,j+1}$.

Next, we prove the second statement. Let us consider

$$\begin{aligned}\left\|x_i^{k+1,j}-x_i^{k,j}\right\| &= \left\|x^{k+1}-\gamma\frac{1}{N}\sum_{l=0}^{j-1}\nabla f_{i,l}(x_i^{k+1,l})-\left(x^k-\gamma\frac{1}{N}\sum_{l=0}^{j-1}\nabla f_{i,l}(x_i^{k,l})\right)\right\|\\ &\leq \|x^{k+1}-x^k\|+\gamma L\frac{1}{N}\sum_{l=0}^{j-1}\|x_i^{k+1,l}-x_i^{k,l}\|\\ &\leq \|x^{k+1}-x^k\|+\gamma L\frac{1}{N}\sum_{j=0}^{N-1}\|x_i^{k+1,j}-x_i^{k,j}\|.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{N}\sum_{j=0}^{N-1}\left\|x_i^{k+1,j}-x_i^{k,j}\right\| &\leq \frac{1}{N}\sum_{j=0}^{N-1}\left(\|x^{k+1}-x^k\|+\gamma\frac{1}{N}\sum_{j=0}^{N-1}\|x_i^{k+1,j}-x_i^{k,j}\|\right)\\ &\leq \|x^{k+1}-x^k\|+\gamma L\frac{1}{N}\sum_{j=0}^{N-1}\|x_i^{k+1,j}-x_i^{k,j}\|.\end{aligned}$$

If $\gamma \leq \frac{1}{2L}$, then

$$\begin{aligned}\frac{1}{N}\sum_{j=0}^{N-1}\left\|x_{i,j}^{k+1}-x_{i,j}^{k}\right\| &\leq \frac{1}{1-\gamma L}\|x^{k+1}-x^{k}\| \\ &\leq 2\|x^{k+1}-x^{k}\| \\ &= 2\eta.\end{aligned}$$

Next, we prove the third statement. Let us consider

$$\begin{aligned}\left\|\mathcal{T}_i(x^{k+1})-\mathcal{T}_i(x^{k})\right\| &= \left\|x^{k+1}-\gamma\frac{1}{N}\sum_{j=0}^{N-1}\nabla f_{i,j}(x_i^{k+1,j})-\left(x^{k}-\gamma\frac{1}{N}\sum_{j=0}^{N-1}\nabla f_{i,j}(x_i^{k,j})\right)\right\| \\ &\leq \|x^{k+1}-x^{k}\|+\gamma\frac{1}{N}\sum_{j=0}^{N-1}\left\|\nabla f_{i,j}(x_i^{k+1,j})-\nabla f_{i,j}(x_i^{k,j})\right\| \\ &\leq \|x^{k+1}-x^{k}\|+\gamma L\frac{1}{N}\sum_{j=0}^{N-1}\|x_i^{k+1,j}-x_i^{k,j}\|.\end{aligned}$$

By the fact that $\left\|x^{k+1}-x^{k}\right\| \leq \eta$ and that $\gamma \leq \frac{1}{2L}$, and by the second statement,

$$\left\|\mathcal{T}_i(x^{k+1})-\mathcal{T}_i(x^{k})\right\| \leq 2\eta.$$

Finally, we prove the fourth statement. Let us consider

$$\begin{aligned}\left\|\mathcal{T}_i(x^{k})-\left(x^{k}-\gamma\nabla f_i(x^{k})\right)\right\| &= \left\|x^{t}-\gamma\frac{1}{N}\sum_{j=0}^{N-1}\nabla f_{i,j}(x_i^{k,j})-(x^{t}-\gamma\nabla f_i(x^{k})\right\| \\ &= \left\|\gamma\left(\frac{1}{N}\sum_{i=0}^{N-1}\nabla f_{i,j}(x_i^{k,j})-\nabla f_i(x^{k})\right)\right\| \\ &= \left\|\gamma\left(\frac{1}{N}\sum_{i=0}^{N-1}\nabla f_{i,j}(x_i^{k,j})-\frac{1}{N}\sum_{i=0}^{N-1}\nabla f_{i,j}(x^{k})\right)\right\| \\ &= \gamma\frac{1}{N}\sum_{i=0}^{N-1}\left\|\nabla f_{i,j}(x_i^{k,j})-\nabla f_{i,j}(x^{k})\right\| \\ &\leq \gamma L\frac{1}{N}\sum_{i=0}^{N-1}\left\|x_i^{k,j}-x^{k}\right\|.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{M}\sum_{i=1}^{M}\left\|\mathcal{T}_i(x^{k})-\left(x^{k}-\gamma\nabla f_i(x^{k})\right)\right\| &\leq \frac{1}{M}\sum_{i=1}^{M}\gamma L\frac{1}{N}\sum_{i=0}^{N-1}\left\|x_i^{k,j}-x^{k}\right\| \\ &\leq \gamma L\frac{1}{M}\sum_{i=1}^{M}\frac{1}{N}\sum_{j=0}^{N-1}\left\|x_i^{k,j}-x^{k}\right\|.\end{aligned}$$

■

H.2. Convergence Theorem for Fed-α-NormEC with local IG steps

Theorem 19 (Fed-α-NormEC with local IG steps) *Consider Fed-α-NormEC for solving Problem (1) where Theorem 1 holds. Let $\mathcal{T}_i(x^k) = x^k - \gamma \frac{1}{N}\sum_{j=0}^{N-1} \nabla f_{i,j}(x_i^{k,j})$, where the sequence $\{x_i^{k,j}\}$ is generated by $x_i^{k,j+1} = x_i^{k,j} - \frac{\gamma}{N}\nabla f_{i,j}(x_i^{k,j})$ for $j = 0, 1, \ldots, T-1$, given that $x_i^{k,0} = x^k$. Furthermore, let $\beta, \alpha > 0$ be chosen such that $\frac{\beta}{\alpha+R} < 1$ with $R = \max_{i\in[1,M]} \left\|v_i^0 - \frac{x^0 - \mathcal{T}_i(x^0)}{\gamma}\right\|$. If $\eta\gamma \leq \frac{1}{K+1}\frac{\Delta^{\inf}}{4L\sqrt{2L}}$, $0 < \eta \leq \frac{\gamma}{3}\frac{\beta R}{\alpha+R}$, and $0 < \gamma \leq \frac{1}{2L}$, then*

$$\min_{k\in[0,K]} \mathrm{E}\left[\left\|\nabla f(x^k)\right\|\right] \leq \frac{3}{K+1}\frac{f(x^0) - f^{\inf}}{\eta} + 2R + 2\sqrt{\frac{\beta^2 B}{M}(K+1)} + \gamma \cdot 8L\sqrt{2L}\sqrt{\Delta^{\inf}} + \gamma \cdot 4L\sqrt{2L}\sqrt{\frac{1}{M}\sum_{i=1}^{M}\Delta_i^{\inf}} + \eta \cdot \frac{L}{2},$$

where $B = 2p(1-1/p)^2 + 2(1-p) + 2\sigma_{\mathrm{DP}}^2/p$, and $\Delta^{\inf} = f^{\inf} - \frac{1}{M}\sum_{i=1}^{M} f_i^{\inf} > 0$, and $\Delta_i^{\inf} = f^{\inf} - \frac{1}{N}\sum_{j=1}^{N} f_{i,j}^{\inf} > 0$

Proof We prove the result in the following steps.

Step 1) Bound $\left\|v_i^k - \frac{x^k - \mathcal{T}_i(x^k)}{\gamma}\right\|$ by induction, and bound $\left\|v_i^{k+1} - \frac{x^k - \mathcal{T}_i(x^k)}{\gamma}\right\|$. We prove $\left\|v_i^k - \frac{x^k - \mathcal{T}_i(x^k)}{\gamma}\right\| \leq \max_{i\in[1,M]} \left\|v_i^0 - \frac{x^0 - \mathcal{T}_i(x^0)}{\gamma}\right\|$ by induction. We can easily show the condition when $k = 0$. Next, let $\left\|v_i^k - \frac{x^k - \mathcal{T}_i(x^k)}{\gamma}\right\| \leq \max_{i\in[1,M]} \left\|v_i^0 - \frac{x^0 - \mathcal{T}_i(x^0)}{\gamma}\right\|$. Then, from Theorem 18, $\mathcal{T}_i(x^k)$ satisfies

$$\left\|\mathcal{T}_i(x^{k+1}) - \mathcal{T}_i(x^k)\right\| \leq 2\eta.$$

Therefore, from Lemma 5 with $\rho = 2$, $C = R = \max_{i\in[1,M]} \left\|v_i^0 - \frac{x^0 - \mathcal{T}_i(x^0)}{\gamma}\right\|$, we can prove that by choosing $\frac{\beta}{\alpha+R} < 1$ and $\eta \leq \frac{\gamma\beta R}{(1+\rho)(\alpha+R)}$, $\left\|v_i^{k+1} - \frac{x^{k+1} - \mathcal{T}_i(x^{k+1})}{\gamma}\right\| \leq R$. We complete the proof.

Next, from Lemma 5, $\left\|v_i^{k+1} - \frac{x^k - \mathcal{T}_i(x^k)}{\gamma}\right\| \leq \max_{i\in[1,M]} \left\|v_i^0 - \frac{x^0 - \mathcal{T}_i(x^0)}{\gamma}\right\|$.

Step 2) Bound $f(x^k) - f^{\inf}$. From Theorem 8 with $G^k = \hat{v}^{k+1}$,

$$\begin{aligned} f(x^{k+1}) - f^{\inf} &\leq f(x^k) - f^{\inf} - \eta\left\|\nabla f(x^k)\right\| + 2\eta\left\|\nabla f(x^k) - \hat{v}^{k+1}\right\| + \frac{L\eta^2}{2} \\ &\overset{\text{triangle inequality}}{\leq} f(x^k) - f^{\inf} - \eta\left\|\nabla f(x^k)\right\| + 2\eta\left\|\nabla f(x^k) - v^{k+1}\right\| \\ &\quad + 2\eta\left\|\hat{v}^{k+1} - v^{k+1}\right\| + \frac{L\eta^2}{2}, \end{aligned}$$

where $v^{k+1} = \frac{1}{M}\sum_{i=1}^{M} v_i^{k+1}$. Next, since

$$\begin{aligned}
\left\|\nabla f(x^k) - v^{k+1}\right\| &= \left\|\nabla f(x^k) - \frac{1}{M}\sum_{i=1}^{M} v_i^{k+1}\right\| \\
&\overset{\text{triangle inequality}}{\leq} \frac{1}{M}\sum_{i=1}^{M}\left\|v_i^{k+1} - \nabla f_i(x^k)\right\| \\
&\overset{\text{triangle inequality}}{\leq} \frac{1}{M}\sum_{i=1}^{M}\left\|v_i^{k+1} - \frac{x^k - \mathcal{T}_i(x^k)}{\gamma}\right\| \\
&\quad + \frac{1}{M}\sum_{i=1}^{M}\left\|\frac{x^k - \mathcal{T}_i(x^k)}{\gamma} - \nabla f_i(x^k)\right\|,
\end{aligned}$$

where $\mathcal{T}_i(x^k) = x^k - \gamma\frac{1}{N}\sum_{j=0}^{N-1}\nabla f_{i,j}(x_i^{k,j})$, we get

$$\left\|\nabla f(x^k) - v^{k+1}\right\| \leq \frac{1}{M}\sum_{i=1}^{M}\left\|v_i^{k+1} - \frac{x^k - \mathcal{T}_i(x^k)}{\gamma}\right\| + \frac{1}{\gamma}\frac{1}{M}\sum_{i=1}^{M}\left\|x^k - \mathcal{T}_i(x^k) - \gamma\nabla f_i(x^k)\right\|.$$

Plugging the upperbound for $\left\|\nabla f(x^k) - v^{k+1}\right\|$ into the main inequality in $f(x^k) - f^{\inf}$, we obtain

$$\begin{aligned}
f(x^{k+1}) - f^{\inf} &\leq f(x^k) - f^{\inf} - \eta\left\|\nabla f(x^k)\right\| + 2\eta\frac{1}{M}\sum_{i=1}^{M}\left\|v_i^{k+1} - \frac{x^k - \mathcal{T}_i(x^k)}{\gamma}\right\| \\
&\quad + \frac{2\eta}{\gamma}\frac{1}{M}\sum_{i=1}^{M}\left\|(x^k - \gamma\nabla f_i(x^k)) - \mathcal{T}_i(x^k)\right\| + 2\eta\left\|\hat{v}^{k+1} - v^{k+1}\right\| + \frac{L\eta^2}{2}.
\end{aligned}$$

By the fact that $\left\|v_i^{k+1} - \frac{x^k - \mathcal{T}_i(x^k)}{\gamma}\right\| \leq R$ from Step 1),

$$\begin{aligned}
f(x^{k+1}) - f^{\inf} &\leq f(x^k) - f^{\inf} - \eta\left\|\nabla f(x^k)\right\| + 2\eta R \\
&\quad + \frac{2\eta}{\gamma}\frac{1}{M}\sum_{i=1}^{M}\left\|(x^k - \gamma\nabla f_i(x^k)) - \mathcal{T}_i(x^k)\right\| + 2\eta\left\|\hat{v}^{k+1} - v^{k+1}\right\| + \frac{L\eta^2}{2}.
\end{aligned}$$

From Theorem 18,

$$\begin{aligned}
f(x^{k+1}) - f^{\inf} &\leq f(x^k) - f^{\inf} - \eta\left\|\nabla f(x^k)\right\| + 2\eta R \\
&\quad + \frac{2\eta}{\gamma}\gamma L\frac{1}{M}\sum_{i=1}^{M}\frac{1}{N}\sum_{j=0}^{N-1}\left\|x_i^{k,j} - x^k\right\| + 2\eta\left\|\hat{v}^{k+1} - v^{k+1}\right\| + \frac{L\eta^2}{2}.
\end{aligned}$$

Next, from Theorem 17,

$$f(x^{k+1}) - f^{\inf} \leq \left(1 + \frac{4L\sqrt{2L}}{\sqrt{\Delta^{\inf}}}\gamma\eta\right)(f(x^k) - f^{\inf}) - \eta\left\|\nabla f(x^k)\right\| + 2\eta R$$
$$+4L\sqrt{2L}\gamma\eta\sqrt{\Delta^{\inf}} + 4L\sqrt{2L}\gamma\eta\sqrt{\frac{1}{M}\sum_{i=1}^{M}\Delta_i^{\inf}}$$
$$+2\eta\left\|\hat{v}^{k+1} - v^{k+1}\right\| + \frac{L\eta^2}{2}.$$

Since

$$\mathrm{E}\left[\left\|\hat{v}^{k+1} - v^{k+1}\right\|\right] \leq \frac{1}{\gamma}\mathrm{E}\left[\left\|\frac{1}{M}\sum_{i=1}^{M}v_i^{k+1} - \hat{v}^{k+1}\right\|\right]$$
$$\overset{\text{Theorem 7}}{\leq} \frac{1}{\gamma}\sqrt{\frac{\beta^2 B}{M}(K+1)},$$

by taking the expectation,

$$\mathrm{E}\left[f(x^{k+1}) - f^{\inf}\right] \leq \left(1 + \frac{4L\sqrt{2L}}{\sqrt{\Delta^{\inf}}}\gamma\eta\right)\mathrm{E}\left[f(x^k) - f^{\inf}\right] - \eta\mathrm{E}\left[\left\|\nabla f(x^k)\right\|\right] + 2\eta R$$
$$+8L\sqrt{2L}\gamma\eta\sqrt{\Delta^{\inf}} + 4L\sqrt{2L}\gamma\eta\sqrt{\frac{1}{M}\sum_{i=1}^{M}\Delta_i^{\inf}}$$
$$+2\eta\sqrt{\frac{\beta^2 B}{M}(K+1)} + \frac{L\eta^2}{2}.$$

By applying Theorem 9 with $\eta\gamma \leq \frac{1}{K+1}\frac{\Delta^{\inf}}{4L\sqrt{2L}}$ and using the fact $(1 + \eta\gamma\frac{4L\sqrt{2L}}{\Delta^{\inf}})^{K+1} \leq \exp(\eta\gamma\frac{4L\sqrt{2L}}{\Delta^{\inf}}(K+1)) \leq \exp(1) \leq 3$ we finalize the proof. ■

H.3. Corollaries for Fed-α-NormEC with multiple local IG steps from Theorem 19

Corollary 20 (Convergence bound for Fed-α-NormEC with multiple local IG steps) *Consider Fed-α-NormEC for solving Problem (1) under the same setting as Theorem 19. Let $T > 1$ (multiple local GD steps). If $\gamma = \frac{1}{2L(K+1)^{1/8}}$, $v_i^0 \in \mathbb{R}^d$ is chosen such that $\max_{i\in[1,M]}\left\|\frac{x^0 - \mathcal{T}_i(x^0)}{\gamma} - v_i^0\right\| = \frac{D_1}{(K+1)^{1/8}}$ with $D_1 > 0$, and $\beta = \frac{D_2}{(K+1)^{5/8}}$ with $D_2 > 0$, and $\eta = \frac{\hat{\eta}}{(K+1)^{7/8}}$ with $\hat{\eta} = \min\left(\frac{\Delta^{\inf}}{2\sqrt{2L}}, \frac{D_1 D_2}{4L(\alpha+D_1)}\right)$, then*

$$\min_{k\in[0,K]}\mathrm{E}\left[\left\|\nabla f(x^k)\right\|\right] \leq \frac{A_1}{(K+1)^{1/8}} + \frac{A_2}{(K+1)^{7/8}},$$

where $A_1 = 3\frac{f(x^0) - f^{\inf}}{\hat{\eta}} + 2D_1 + \frac{2\sqrt{B}D_2}{\sqrt{M}} + 8\sqrt{2L}\sqrt{\Delta^{\inf}} + 4\sqrt{2L}\sqrt{\frac{1}{M}\sum\limits_{m=1}^{M}\Delta_i^{\inf}}$ and $A_2 = \hat{\eta}L/2$.

Proof The proof is analogous to the proof of Corollary 13. ■

Corollary 21 (Utility bound for Fed-α-NormEC with multiple local IG steps) *Consider Fed-α-NormEC for solving Problem (1) under the same setting as Theorem 19. Let $T > 1$ (multiple local GD steps), let $\sigma_{\mathrm{DP}} = c\frac{p\sqrt{(K+1)\log(1/\delta)}}{\epsilon}$ with $c > 0$ (privacy with subsampling amplification), and let $p = \frac{\hat{B}}{M}$ for $\hat{B} \in [1, M]$ (client subsampling). If $\beta = \frac{\hat{\beta}}{K+1}$ with $\hat{\beta} = \sqrt{\frac{3(f(x^0)-f^{\inf})}{\gamma}}\sqrt[4]{\frac{M}{B_2}}$, $\gamma < \frac{\Delta^{\inf}(\alpha+R)}{\sqrt{2L}\hat{\beta}R}$, $\alpha = R = \mathcal{O}\left(\sqrt[4]{d}\frac{\sqrt{f(x^0)-f^{\inf}}}{\sqrt{\gamma}}\sqrt[4]{\frac{B_2}{M}}\right)$ with $B_2 = 2c^2\frac{\hat{B}}{M}\frac{\log(1/\delta)}{\epsilon^2}$, and $\eta = \frac{1}{K+1}\frac{\gamma}{2}\frac{\hat{\beta}R}{\alpha+R}$, then*

$$\min_{k\in[0,K]} \mathrm{E}\left[\left\|\nabla f(x^k)\right\|\right] \leq \mathcal{O}\left(\Delta\sqrt[4]{\frac{d\hat{B}}{M^2}\frac{\log(1/\delta)}{\epsilon^2}} + \sqrt{L}\sqrt{\Delta^{\inf}} + \sqrt{L}\sqrt{\frac{1}{M}\sum_{i=1}^{M}\Delta_i^{\inf}}\right),$$

where $\Delta = \max(\alpha, 2)\sqrt{L}\sqrt{f(x^0) - f^{\inf}}$.

Proof The proof is analogous to the proof of Corollary 14. ■