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Design Strategies for Stabilization and Tracking of Positive Singular Systems

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Abstract— **This paper considers the problem of positive stabilization and tracking of linear continuous-time singular systems. First, preliminary results on singular systems are provided. Then, the class of positive singular systems is defined through its equivalent input derivative systems. An elimination procedure for input derivatives is given, which transfers the derivative terms from the state equation to the output equation allowing stabilization by state feedback to be performed using its standard representation. It is also shown that the positive stabilization can be achieved by proportional derivative (PD) state feedback by two steps of normalization and subsequent stabilization using LMI. Finally, the proportional integral (PI) feedback is considered for tracking design of positive systems. Numerical examples are included to support the theoretical results.**

I. INTRODUCTION

Singular systems, also known as descriptor systems or generalized state space systems is an important class of dynamic systems and appear in various applications. Analysis and control design of singular systems have been investigated by researchers for many years. A few of them set the foundation of the major results, which can be found in survey papers and books [1]–[7]. The renewed interest of singular systems tied to the further mathematical development and its alternative representations. This facilitated analysis and design of control system for singular systems [8]–[15]. One of the most interesting connections of singular systems is through its input derivative systems, which has recently been identified [16]. The input derivative representation of singular systems also allows to define the class of positive singular systems.

Positive singular systems have their own identity due to the fact that positive systems appear in many domains of science and engineering [17]–[21]. The response of positive systems to a positive initial condition and positive input remains in positive orthant of state space. The available results on stabilization and control of positive systems can be applied to the class of positive singular systems by appropriate modifications. The main goal of this paper is to show the possibility of using new approaches to achieve this goal.

After a preliminary background on stability, regularity, and admissibility of singular systems, the state feedback problem for singular systems is reviewed. The existing development for the solution of stabilization is outlined showing their limitations due to non-strictness of LMI and computational complexity. Also, it is not trivial to impose the positivity in the available stabilization results of singular systems. We demonstrate that the well-known positive stabilization technique based on LMI for conventional linear systems can be applied to singular systems represented by its equivalent input derivative form. An elimination procedure for input derivatives transfers the system to an equivalent standard representation. This allows positive stabilization to be performed. Motivated by the results of [5], which were used for regular singular systems, we show that the positive stabilization can be achieved for positive singular systems by proportional derivative (PD) feedback using normalization and subsequent positive stabilization. Finally, the proportional integral (PI) feedback is considered for tracking design of positive systems. Numerical examples are also included to support our results.

II. PRELIMINARY RESULTS ON SINGULAR SYSTEMS

Consider the linear singular system described by

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are state, input, and output vectors, respectively. Obviously, the system matrices A, B, and C are of appropriate dimensions, and the matrix $E \in \mathbb{R}^{n\times n}$ is singular with $\text{rank}(E) = r < n$.

We also define the generalized spectral abscissa for the pair $\{E, A\}$ as $\alpha(E, A) = \max\{\Re(\lambda)\}$, where $\lambda \in \{s : \det(sE - A) = 0\}$. If $E = I$, then $\alpha(I, A) = \alpha(A)$ reduces to the conventional spectral abscissa of standard systems.

Definition 1. The singular system (1), (2) is called *Regular* if $\det(sE - A) \neq 0$ for some $s \in \mathbb{C}$, *Impulse-free* if $\deg(\det(sE - A)) = \text{rank}(E) = r$, and *Stable* if all roots of $\det(sE - A) = 0$ lie in $\mathbb{C}^-$. In addition, we say that a singular system is *Admissible* if it is regular, impulse-free, and stable.

Using definition 1, we have the following lemma [2], [5].

Lemma 1. *Let the singular system (1), (2) be regular. Then, there exists nonsingular matrices L and R such that*

$$\begin{aligned} &\bar{E} = LER = diag\{I_{n_1}, N\}, \quad \bar{A} = LAR = diag\{A_1, I_{n_2}\} \\ &\bar{B} = LB = \begin{bmatrix} B_1^T & B_2^T \end{bmatrix}^T, \quad \bar{C} = CR = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \end{aligned} \tag{3}$$

where $n_1 = \deg(\det(sE - A)) \leq r$, $n_2 = n - n_1$, $A_1 \in \mathbb{R}^{n_1\times n_1}$, and $N \in \mathbb{R}^{n_2\times n_2}$ is a nilpotent matrix with index

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μ (i.e. $N^{\mu} = 0$, $N^{\mu-1} \neq 0$). Furthermore, assume that the pair $\{E, A\}$ is regular, then the singular system

1) *is impulse free iff $N = 0$.*
2) *is stable iff $\alpha(A_1) < 0$.*
3) *is admissible iff $N = 0$, $\alpha(A_1) < 0$.*

Theorem 1. *The pair $\{E, A\}$ associated with (1) is admissible if and only if there exists a matrix P such that*

$$E^T P = P^T E \succeq 0 \tag{4}$$

$$P^T A + A^T P \prec 0 \tag{5}$$

The equality constraint (4) imposes the Lyapunov inequality (5) to be non-strict LMI, which is not desirable. Therefore, it is preferable to have a strict LMI by integrating (4) in (5). Theorem 1 and the following theorem were introduced in [14], [22] to facilitate the process of stability and stabilization of singular systems.

Theorem 2. *The pair $\{E, A\}$ associated with (1) is admissible if and only if there exist $P \succ 0$ and Q such that*

$$(PE + SQ)^T A + A^T (PE + SQ) \prec 0 \tag{6}$$

where $S \in \mathbb{R}^{r\times(n-r)}$ is any full rank matrix satisfying $E^T S = 0$. Furthermore, (6) can equivalently be written in terms of $\{E^T, A^T\}$ by replacing E with E^T and A with A^T in (6), and the side constraint $ES = 0$.

Using the above theorem, the stabilization of singular systems can be performed by integrating state feedback controller $u(t) = Kx(t)$. Let the stabilizability condition rank$[sE - A \quad B] = n$ be satisfied, then the closed-loop system $E\dot{x}(t) = (A + BK)x(t)$ is admissible if and only if there exist matrices $P \succ 0$, Q, and V such that

$$W^T(P,Q)A^T + AW(P,Q) + BV + V^T B^T \prec 0 \tag{7}$$

provided that $W(P,Q) = PE^T + SQ$ is a nonsingular matrix with any S satisfying $ES = 0$. Furthermore, the feedback gain matrix K is obtained by $K = VW^{-1}$. Additional effort was made to simplify LMI (7). However, the proposed solutions are inconvenient, especially for the class of positive singular systems. Thus, it is desirable to provide alternative solutions for stabilization and control of positive singular systems.

III. POSITIVE SINGULAR SYSTEMS AND EQUIVALENT REPRESENTATION

Let $E = I$ in (1). Then, the singular system reduces to a standard system and it is called internally positive if and only if for any positive initial condition $x(0) \in \mathbb{R}^n_+$ and every positive input $u \in \mathbb{R}^m_+$, the state and output responses remain positive i.e. $x(t) \in \mathbb{R}^n_+$ and $y(t) \in \mathbb{R}^p_+$. The following results are well-known and we state them without proofs.

Lemma 2. *The system (1), (2) with $E = I$ (Metzlerian System) is internally positive if and only if matrix A is Metzler i.e. $A \in M_n$, $B \in \mathbb{R}^{n\times m}_+$, and $C \in \mathbb{R}^{p\times n}_+$ are nonnegative matrices.*

Lemma 3. *A Metzlerian system is asymptotically stable if and only if the following equivalent conditions is satisfied:*

1) *The matrix A is strictly diagonal dominant i.e. $|a_{ii}| > \sum_{j=1,j\neq i}^{n} |a_{ij}|$.*
2) *There exists a positive diagonal matrix P such that $A^T P + PA \prec 0$.*
3) *There exists a vector $v \in \mathbb{R}^n_+$ such that $Av < 0$.*

Now, let us consider the singular system (1), (2) and analyze its positivity and stability.

Definition 2. The singular system (1), (2) is called weakly positive if and only if $E \in \mathbb{R}^{n\times n}_+$, $B \in \mathbb{R}^{n\times m}_+$, and $A \in \mathbb{M}_n$ is a Metzler matrix.

This definition is a natural generalization of positivity as defined for standard positive systems. Unfortunately, it does not guarantee the strong positivity of singular systems, which has been proven in the literature.

There are three ways to check the strong positivity of singular systems. Using lemma 1, one can represent the singular systems by slow and fast subsystems. After combining the state equations of two subsystems, one can obtain an input derivative representation of singular systems and detect positivity [9].

The singular system can also be decomposed to dynamic and static parts by applying singular value decomposition (SVD) on the matrix E. Then, one can use the so-called Shuffle algorithm [23] to derive an equivalent standard system with input derivatives. Performing SVD on the matrix E, the singular system can be rewritten as

$$\hat{E}\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t) \tag{8}$$

$$y(t) = \hat{C}\hat{x}(t) \tag{9}$$

where

$$\hat{A} = U^T A V = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \hat{B} = U^T B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

$$\hat{E} = U^T E V = \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{C} = CV = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

with $\Sigma_r = \text{diag}\{\sigma_i, i = 1, \ldots, r\}$ defining the nonzero singular values and orthogonal pair of matrices $\{U, V\}$.

The transformation to SVD form (8),(9) prepares the system for initial step of Shuffle algorithm by defining

$$\hat{E} = \begin{bmatrix} E_1 \\ 0 \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{10}$$

where $E_1 = [\Sigma_r \quad 0], A_1 = [A_{11} \quad A_{12}]$, and $A_2 = [A_{21} \quad A_{22}]$. Therefore, we have

$$E_1\dot{x} = A_1 x + B_1 u \tag{11}$$

$$0 = A_2 x + B_2 u \tag{12}$$

Taking the derivative of (12) and combine it with (11) yields

$$\begin{bmatrix} E_1 \\ A_2 \end{bmatrix} \dot{x} = \begin{bmatrix} A_1 \\ 0 \end{bmatrix} x + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -B_2 \end{bmatrix} \dot{u} \tag{13}$$

If $\begin{bmatrix} E_1^T & A_2^T \end{bmatrix}^T$ is nonsingular, one can multiply (13) from left by its inverse and define the input derivative system. If $\begin{bmatrix} E_1^T & A_2^T \end{bmatrix}^T$ is singular, then one should apply the steps of Shuffle algorithm $\mu - 1$ steps until a regular pencil is obtained where $\begin{bmatrix} E_{\mu-1}^T & A_{\mu-1}^T \end{bmatrix}^T$ becomes nonsingular. Thus, the singular system (1),(2) represented by SVD form (8), (9) is transformed to the input derivative system as follows

$$\dot{x}(t) = \tilde{A}x(t) + \sum_{i=0}^{\mu-1} \tilde{B}_i u^{(i)}(t) \tag{14}$$

$$y(t) = \tilde{C}x(t) \tag{15}$$

The third way to obtain an equivalent representation of singular systems is based on the Drazin inverse [10]. Here, it is sufficient to use the input derivative representation (14), (15) since it is numerically preferable.

Definition 3. The singular system (1), (2) is internally positive if and only if its equivalent input derivative system (14), (15) is internally positive i.e. for every consistent initial condition $x_0 \in \mathbb{R}_+^n$ and every nonnegative input $u(t) \in \mathbb{R}_+^m$ such that $u^{(i)}(t) \in \mathbb{R}_+^m$ for $i = 0, 1, ..., \mu - 1$; $x(t) \in \mathbb{R}_+^n$, $y(t) \in \mathbb{R}_+^p$ for $t \geq 0$ where $u^{(i)}(t) = d^i u(t)/dt$.

Theorem 3. *The singular system (1), (2) is internally positive if and only if its equivalent input derivative system (14), (15) satisfies* $\tilde{A} \in \mathbb{M}_n$ *i.e. a Metzler matrix,* $\tilde{B}_i \in \mathbb{R}_+^{n\times m}$, $\tilde{C} \in \mathbb{R}_+^{p\times n}$ *for all* $i = 0, 1, ..., \mu - 1$. *Furthermore, it is asymptotically stable if and only if* $\tilde{A}$ *is a stable Metzler matrix satisfying one of the equivalent conditions of lemma 3.*

IV. POSITIVE STABILIZATION OF SINGULAR SYSTEMS BY PROPORTIONAL STATE FEEDBACK

A. Elimination of Input Derivatives

Let us consider the input derivative representation of singular system (14), (15) and define the number of input derivatives by $\ell = \mu - 1$. Starting with $\ell = 1$, we have

$$\dot{x}(t) = \tilde{A}x(t) + \tilde{B}_0 u(t) + \tilde{B}_1 \dot{u}(t) \tag{16}$$

$$y(t) = \tilde{C}x(t) \tag{17}$$

The elimination of derivative inputs can be done by algebraic transformation. To apply the process for one input derivative (16), let $z(t) = x(t) - \tilde{B}_1 u(t)$ and obtain

$$\dot{z}(t) = \tilde{A}z(t) + (\tilde{B}_0 + \tilde{A}\tilde{B}_1)u(t) \tag{18}$$

$$y(t) = \tilde{C}z(t) + \tilde{C}\tilde{B}_1 u(t) \tag{19}$$

For $\ell = 2$, we perform twice the algebraic transformation and obtain

$$\dot{z}(t) = \tilde{A}z(t) + (\tilde{B}_0 + \tilde{A}\tilde{B}_1 + \tilde{A}^2\tilde{B}_2)u(t) \tag{20}$$

$$y(t) = \tilde{C}z(t) + \tilde{C}(\tilde{B}_1 + \tilde{A}\tilde{B}_2)u(t) + \tilde{C}\tilde{B}_2\dot{u}(t) \tag{21}$$

Note that (18) remains positive if $\tilde{A}\tilde{B}_1 \geq 0$, and (20) remains positive if $\tilde{A}\tilde{B}_1 \geq 0$, $\tilde{A}^2\tilde{B}_2 \geq 0$ or their sum is positive. Continuing the elimination process for $\ell > 2$, we obtain the following closed-form expression for ℓ derivative inputs,

$$\dot{z}(t) = \tilde{A}z(t) + \tilde{\tilde{B}}_0 u(t) \tag{22}$$

$$y(t) = \tilde{C}z(t) + \tilde{C}\sum_{j=1}^{\ell} \tilde{\tilde{B}}_j u^{(j-1)}(t) \tag{23}$$

where

$$\tilde{\tilde{B}}_j = \sum_{i=j}^{\ell} \tilde{A}^{i-j}\tilde{B}_i, \quad \tilde{\tilde{B}}_0 = \sum_{i=0}^{\ell} \tilde{A}^i\tilde{B}_i, \quad i, j = 0, 1, \ldots, \ell \tag{24}$$

It should be pointed out that if the singular system (1), (2) is internally positive using its equivalent input derivative system (14), (15) as stated in theorem 3, then (22), (23) remains positive if $\tilde{\tilde{B}}_0 > 0$ and $\tilde{C}\tilde{\tilde{B}}_j > 0$. This is evident since the pair $\{\tilde{A}, \tilde{C}\}$ is not changing after elimination process.

Remark 1. The condition $\tilde{\tilde{B}}_j \in \mathbb{R}_+^{n\times m}$ is not required for the special subset of Metzler matrices $\tilde{A} \in \mathbb{R}_+^{n\times n} \subset \mathbb{M}_n$. In this case, the condition $\tilde{\tilde{B}}_j \in \mathbb{R}_+^{n\times m}$ in the above theorem can be replaced by $\tilde{B}_i \in \mathbb{R}_+^{n\times m}$.

B. Positive Stabilization by State Feedback

Before starting the main result of this section, we need the stabilizability condition using the equivalent representation (22), (23).

Lemma 4. *The singular system (1), (2) is controllable if and only if one of the following equivalent conditions is satisfied*

(a) *rank*$[sE - A \quad B] = n$ *for all* $s \in \mathbb{C}$ *and rank*$[E \quad B] = n$.

(b) *rank*$[\tilde{\tilde{B}}_0 \quad \tilde{A}\tilde{\tilde{B}}_0 \quad \ldots \quad \tilde{A}^{n-1}\tilde{\tilde{B}}_0; \tilde{\tilde{B}}_1, \tilde{\tilde{B}}_2, \ldots \tilde{\tilde{B}}_\ell] = n$ *with respect to (22), (23).*

Theorem 4. *Let the singular system (1),(2) or equivalently (22), (23) be controllable. Then, the state feedback control law* $u = K_z z$ *stabilizes (22) and the closed-loop system becomes* $\dot{z}(t) = A_z z(t)$ *where* $A_z = \tilde{A} + \tilde{\tilde{B}}_0 K_z$ *if and only if the following LMI has a feasible solution*

$$Z\tilde{A}^T + Y^T\tilde{\tilde{B}}_0{}^T + \tilde{A}Z + \tilde{\tilde{B}}_0 Y \prec 0 \tag{25}$$

$$(\tilde{A}Z + \tilde{\tilde{B}}_0 Y)_{ij} \geq 0, \quad i \neq j \tag{26}$$

whereby K_z *is obtained by* $K_z = YZ^{-1}$ *and* $Z \succ 0$. *Moreover, the feedback gain matrix* K_x *associated with the control law* $u = v + Kx$ *that stabilizes the positive singular system (1) is given by*

$$K_x = K_z[I + \sum_{i=j}^{\ell} \tilde{\tilde{B}}_j K_z A_z{}^{j-1}]^{-1} \tag{27}$$

where x *and* z *are related by*

$$x = [I + \sum_{i=j}^{\ell} \tilde{\tilde{B}}_j K_z A_z{}^{j-1}]z, \quad j = 1, \ldots, \ell \tag{28}$$

Proof: The proof of this theorem is constructive by applying positive stabilization of regular system (22) using LMI (25), (26). Here, the Lyapunov stability condition can be employed with respect to $A_z = \widetilde{A} + \widetilde{\bar{B}}_0 K_z$. Then, applying congruent transformation and change of variables, the LMI (25), (26) follows.

V. POSITIVE STABILIZATION OF SINGULAR SYSTEMS BY PROPORTIONAL DERIVATIVE (PD) FEEDBACK

The procedure of previous section requires two steps of transforming the singular system to equivalent input derivative system followed by an algebraic transformation. This section is devoted to apply direct procedures using a combined proportional plus derivative feedback.

A. State Derivative Feedback

Consider the singular system (1) and suppose it is desired to obtain the state derivative feedback of the form

$$u(t) = -K_D \dot{x}(t) \tag{29}$$

such that the closed-loop system becomes positive and stable. Although it is possible to positively stabilize the singular systems using (29), we use it for the purpose of normalization. Note that we are using (29) in connection to proportional derivative (PD) feedback design in the next section.

Definition 4. The singular system (1) is called normalizable if there exists a feedback controller (29) such that the closed-loop system

$$(E + BK_D)\dot{x}(t) = Ax(t) \tag{30}$$

becomes standard, i.e.

$$\det(E + BK_D) \neq o \tag{31}$$

Based on the above definition, as long as the condition (31) is satisfied, the closed-loop system (30) reduces to

$$\dot{x}(t) = (E + BK_D)^{-1} Ax(t) \tag{32}$$

The following lemma establishes the condition for normalizability of singular systems.

Lemma 5. *The singular system (1) is normalizable if and only if rank*$[E \quad B] = n$.

Proof: Assume that the system (1) is normalizable. Then, by the above definition, there exists a matrix K_D such that $\det(E + BK_D) \neq 0$ i.e.

$$\text{rank}[E + BK_D] = \text{rank}\{[E \quad B] \begin{bmatrix} I \\ K_D \end{bmatrix}\} = n$$

Thus, $\text{rank}[E \quad B] = n$ guarantees the normalizability. The converse can be established through standard decomposition of singular system.

Note that if we assume for simplicity that the pair $\{E, B\}$ is controllable, then one can always find K_D such that the eigenvalues of $E + BK_D$ are located arbitrary in complex plane guaranteeing the nonsingularity of $E + BK_D$.

B. Proportional Derivative Feedback

consider again the singular system (1) and let the controller of the form

$$u(t) = K_P x(t) - K_D \dot{x}(t) \tag{33}$$

be used so that the closed-loop system becomes

$$(E + BK_D)\dot{x}(t) = (A + BK_P)x(t) \tag{34}$$

The state derivative feedback procedure of previous section allows normalization of (34) i.e. one can obtain K_D such that $\det(E + BK_D) \neq 0$. Then, we can write (34) as

$$\dot{x}(t) = (E + BK_D)^{-1}(A + BK_P)x(t) \tag{35}$$

and state the main result of positive stabilization by PD feedback control law (33).

Theorem 5. *Let the singular system (1) be stabilizable and normalizable with* K_D. *Also, let the pairs* $\{E, B\}$ *and {A, B} be controllable. Then, there exists a PD state feedback control law (33) such that the closed-loop system (35) is positive and asymptotically stable, if and only if the following LMI has a feasible solution with respect to variables* Z *and* Y*,*

$$Z\hat{A}^T + Y^T\hat{B}^T + \hat{A}Z + \hat{B}Y \prec 0 \tag{36}$$

$$(\hat{A}Z + \hat{B}Y)_{ij} \geq 0 \quad fori \neq j \tag{37}$$

where $Z \succ 0$ *is positive definite diagonal matrix,* $\hat{A} = (E + BK_D)^{-1}A$ *and* $\hat{B} = (E + BK_D)^{-1}B$*. Furthermore,* K_P *is obtained from* $K_P = YZ^{-1}$*.*

Proof: Since the pair $\{E, B\}$ is controllable, the singular system is normalizable and we can find K_D such that $\det(E + BK_D) \neq 0$, as outlined in the proof of lemma 5. The nonsingularity of $E + BK_D$ and controllability of the pair $\{A, B\}$ allow the pair $\{\hat{A}, \hat{B}\}$ to be controllable where $\hat{A} = (E + BK_D)^{-1}A$ and $\hat{B} = (E + BK_D)^{-1}B$. This guarantees the stabilizability condition for LMI (36), (37). Thus, one can employ the conventional steps of applying Lyapunov inequality to $\hat{A} + \hat{B}K_p$ with the pair of Z and Y to obtain (36) along with the structural constraint of Metzler matrix (37).

VI. POSITIVE TRACKING OF SINGULAR SYSTEMS BY PROPORTIONAL INTEGRAL (PI) FEEDBACK

The conventional tracking problem for standard controllable systems

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{38}$$

$$y(t) = Cx(t) \tag{39}$$

requires the output vector $y(t)$ to track the command reference input $r(t)$ in the case of steady-state response,

$$\lim_{t \to \infty} y(t) = r(t) \tag{40}$$

To achieve this goal, the design method consists of the addition of a vector comparator and integrator which satisfies

$$\dot{z}(t) = r(t) - y(t) = r(t) - Cx(t) \tag{41}$$

The composite open-loop system is therefore governed by the augmented state and output equations formed from (38) and (41)

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}}_{\bar{A}} \begin{bmatrix} x \\ z \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\bar{B}} u + \begin{bmatrix} 0 \\ I \end{bmatrix} r \tag{42}$$

$$y = \underbrace{\begin{bmatrix} C & 0 \end{bmatrix}}_{\bar{C}} \begin{bmatrix} x \\ z \end{bmatrix} \tag{43}$$

The state feedback control law to be used is

$$u = K_P x + K_I z = \begin{bmatrix} K_P & K_I \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \tag{44}$$

Thus, the closed-loop system becomes

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A + BK_P & BK_I \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} r \tag{45}$$

The condition for stabilization requires $\{\bar{A}, \bar{B}\}$ to be controllable. It can be shown that this pair is controllable if $\{A, B\}$ is controllable, which has been assumed for (38), (39), and

$$\text{rank} \begin{bmatrix} B & A \\ 0 & -C \end{bmatrix} = n + p \tag{46}$$

Although the tracking can be accomplished under the above assumption, it is not possible to positively stabilize the system and achieve tracking due to zero block in (45).

The remedy is to assume the properness of the system so that (39) is replaced by

$$y(t) = Cx(t) + Du(t) \tag{47}$$

and (41) becomes

$$\dot{z}(t) = r(t) - Cx(t) - Du(t) \tag{48}$$

Consequently, the closed-loop system matrix is modified to

$$\bar{A} + \bar{B}\bar{K} = \begin{bmatrix} A + BK_P & BK_I \\ -(C + DK_P) & -DK_I \end{bmatrix} \tag{49}$$

which allows positive stabilization and tracking.

The inclusion of the direct term D can be done by an artificial inclusion in tracking system or with the assumption of properness on the system. In fact, with respect to the singular system (1), (2), it is interesting to point out that its equivalent input derivative system with $\ell = 1$ after elimination of $\dot{u}$ leads to (18), (19). This causes the direct term to be generated by $\widetilde{D} = \widetilde{C}\widetilde{B}_1$ and one can apply positive stabilization and tracking. It is well-known from classical servomechanism of SISO feedback system that the necessary conditions (i) $D - CA^{-1}B \neq 0$ and (ii) the steady-state control signal $u_{ss} = r/D - CA^{-1}B$ to be satisfied. Obviously, for positive stable system $D - CA^{-1}B > 0$. One can also generalize the result to MIMO systems, which will be elaborated in a future paper.

VII. ILLUSTRATIVE EXAMPLES

Example 1: Consider a simple second order singular system

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

which is unstable with eigenvalues at 1.

Applying the Shuffle Algorithm one step with the aid of (10)-(13), we get the equivalent input derivative positive system

$$\dot{x} = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{\widetilde{A}} x + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\widetilde{B}_0} u + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\widetilde{B}_1} \dot{u}.$$

After applying algebraic transformation and subsequent elimination of the input derivative, we obtain (18) where $\widetilde{\widetilde{B}}_0 = \widetilde{B}_0 + \widetilde{A}\widetilde{B}_1 = \begin{bmatrix} 2 & 0 \end{bmatrix}^T$. The state feedback control law $u = K_z z$ results in stable closed-loop system $\dot{z} = A_z z$, where $A_z = \widetilde{A} + \widetilde{\widetilde{B}}_0 K_z$ with $K_z = \begin{bmatrix} -2 & 0 \end{bmatrix}$. Finally, K_x for the original singular system can be obtained from (27) as $K_x = K_z(I + \widetilde{B}_1 K_z)^{-1} = \begin{bmatrix} -2 & 0 \end{bmatrix}$. To check the positivity and stability of the closed-loop singular system, we let $u = v + K_x x$ and after application of shuffle algorithm one step, we get the positive system

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \dot{v}$$

The stability of the closed-loop singular system can be verified by $\det(sE - A_c) = 0$, where $A_c = A + BK_x$ with eigenvalues of -3. Note that other feasible solution is possible, for example if $K_z = \begin{bmatrix} -2 & -1/4 \end{bmatrix}$, then $K_x = \begin{bmatrix} -8/3 & -1/3 \end{bmatrix}$.

Example 2: Consider the regular singular system

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{E} \dot{x} = \underbrace{\begin{bmatrix} -2 & 0 & 2 & -0.1 \\ 0 & -2 & 0 & 3 \\ 1 & 0 & 1 & -0.4 \\ 0 & 1 & 0 & -1 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 4 & 0 \\ 0 & 2 \end{bmatrix}}_{B} u$$

with $n = 4$ and $\text{rank}(E) = r = 2$. Since the roots of $\det(sE - A) = 0$ are given by $\{-4, 1\}$, the system (1) is unstable. After the steps of transformation to the equivalent input derivative system and application of algebraic transformation, one can solve the modified LMI (25),(26) to obtain K_x.

The equivalent input derivative system is obtained as

$$\widetilde{A} = \begin{bmatrix} -2 & 0 & 2 & -0.1 \\ 0 & -2 & 0 & 3 \\ 2 & -0.8 & -2 & 1.3 \\ 0 & -2 & 0 & 3 \end{bmatrix},$$

$$\widetilde{B}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0.4 \\ 0 & 1 \end{bmatrix}, \quad \widetilde{B}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -0.4 & 0.8 \\ 0 & 2 \end{bmatrix}$$

After algebraic transformation, we obtain

$$\dot{z} = \widetilde{A} + \underbrace{\left(\widetilde{B}_0 + \widetilde{A}\widetilde{B}_1\right)}_{\widetilde{\bar{B}}_0} u$$

and its stabilizer can be found by applying theorem 4 with

$$u = K_z z, \quad \dot{z} = \underbrace{\left(\widetilde{A} + \widetilde{\bar{B}}_0 K_z\right)}_{A_z} z$$

$$K_z = YZ^{-1} = \begin{bmatrix} -0.0833 & 0.0222 & 0.3333 & -0.1111 \\ 0 & 0.1667 & 0 & -0.3333 \end{bmatrix}$$

where

$$Y = \begin{bmatrix} -0.11 & 0.0569 & 1.15 & -0.6267 \\ 0 & 0.4267 & 0 & -1.88 \end{bmatrix}$$

and $Z = \text{diag}\{1.32, 2.56, 3.45, 5.64\}$. Finally, we obtain,

$$K_x = K_z \left(I + \widetilde{B}_1 K_z\right)^{-1} = \begin{bmatrix} 0.25 & 0 & -1 & 0.2 \\ 0 & 0.5 & 0 & -1 \end{bmatrix}$$

This feedback gain matrix stabilizes the singular system and the closed-loop system matrix $A_x = A + BK_x$ becomes stable and Metzler. The roots of $\det(sE - A_x) = 0$ will be $\{-1.0833, -0.1667\}$.

Example 3: Consider again Example 2. Applying positive stabilization by PD state feedback control procedure of section VII, we get the following parameters after normalization

$$\hat{A} = \begin{bmatrix} -2.25 & 0 & 1.75 & 0 \\ 0 & -2.5 & 0 & 3.5 \\ 1 & 0 & 1 & -0.4 \\ 0 & 1 & 0 & -1 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 4 & 0 \\ 0 & 2 \end{bmatrix}$$

where

$$K_D = \begin{bmatrix} 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}, \quad K_P = \begin{bmatrix} 0 & 0 & -1 & 0.25 \\ 0 & 0 & 0.5 & -2 \end{bmatrix}$$

are obtained from LMI of theorem 5. Finally, the closed-loop system matrix A_k is obtained as

$$A_k = \begin{bmatrix} -2.5 & 0 & 1.75 & 0 \\ 0 & -2.5 & 0 & 3.5 \\ 1 & 0 & -3 & 0.6 \\ 0 & 1 & 1 & -5 \end{bmatrix}$$

which is a stable Metzler matrix and the roots of $\det(sE - A_k) = 0$ are given by $\{-6.173, -3.897, -1.084, -1.597\}$.

VIII. CONCLUSION

This paper considered the problem of positive stabilization and tracking of continuous-time singular systems. After a summary of available results on singular systems, the positivity of singular systems was defined in terms of its equivalent input derivative representation. Then, an algebraic transformation was used to eliminate the derivative inputs, reducing the positive singular systems to standard form. Consequently, it was possible to apply positive stabilization by state feedback. It was also shown that stabilization can be achieved by Proportional Derivative (PD) state feedback by normalization of singular system and subsequent positive stabilization using LMI. Finally, the proportional Integral (PI) feedback design was provided for tracking design of positive systems.

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