Multiscale network renormalization

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Extended Abstract

The representation of the internal architecture of all physical systems depends on the resolution at which they are observed and described. In spatially embedded systems, geometric coordinates provide a natural way to change the resolution level, indicating how to coarse-grain both the real system and any model of the latter, allowing consistent mappings across scales that lie at the foundation of the renormalization group (RG). By contrast, for complex networks with no explicit spatial embedding, multiple renormalization schemes exist [1], resulting in non-unique representations of the same system across different scales.

The Multiscale Network Renormalization approach [2, 3] has been recently introduced as a way to design random graph models that can represent the same network consistently under arbitrary aggregations of nodes. It seeks a probability distribution over graphs that is invariant under node aggregation (see Fig. 1), thereby representing a fixed point of a nontrivially generalized RG flow. The model comes with node variables that are additive upon node aggregation. It successfully replicates, at multiple hierarchical levels, the properties of several real-world networks [2, 4, 5] and lends itself naturally to the renormalization of directed graphs [3], which is otherwise problematic. It also allows one to infer the structural properties of a network at a hierarchical level that is different from the one at which empirical observations are available, opening new avenues for cross-scale network reconstruction [4].

Moreover, the approach can be applied to Machine Learning algorithms that take a graph as input and encode its structure onto output vectors that represent nodes in an abstract space [5, 6]. In particular, under arbitrary coarse-grainings of the input graph, the multiscale method ensures statistical consistency of the embedding vector of a block-node with the sum of the embedding vectors of its constituent nodes. This guarantee enables the interpretable application of the basic properties of vector spaces (i.e. sum of vectors and multiplication of a vector by a scalar) to the latent space where node embeddings are identified. It turns out that several key network properties, including a large number of triangles, are successfully replicated already from embeddings of very low dimensionality, allowing for the generation of faithful replicas of the original networks at arbitrary resolution levels [5, 6].

Finally, a purely abstract, annealed version of the model leads to infinite-mean node variables and spontaneously replicates, without any fitting parameters, several real-world network properties such as power-law degree distribution, finite local clustering coefficient, and disassortativity profiles [2, 7]. Moreover, since node aggregation invariance is a form of discrete scale invariance [8], several unique properties emerge in the spectrum of the adjancency matrix, such as log-periodicity and complex scaling exponents [9].

In this talk, a synthesis of the various aspects of the Multiscale Network Renormalization approach and its relationship with other coarse-graining approaches will be discussed.

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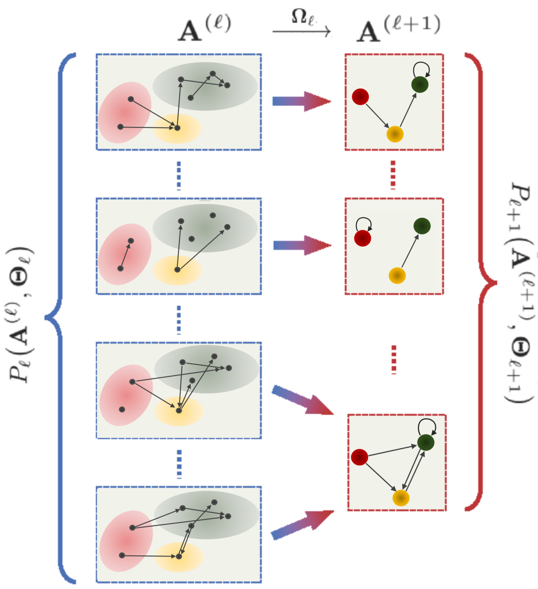

Figure 1: **Renormalization of random graph ensembles under coarse-graining.** Given a probability distribution $P_\ell\left(\mathbf{A}^{(\ell)}, \Theta_\ell\right)$ of graphs with adjacency matrix $\mathbf{A}^{(\ell)}$ (left), a given node partition Ω_ℓ is used to map sets of nodes onto 'block-nodes' of the resulting coarse-grained graphs with adjacency matrix $\mathbf{A}^{(\ell+1)}$ (right). A directed edge from $i_{\ell+1}$ to $j_{\ell+1}$ is drawn if an edge from i_ℓ to j_ℓ is present, for any $i_\ell \in i_{\ell+1}, j_\ell \in j_{\ell+1}$. Multiple realizations of the graph at level ℓ end up in the same realization of the graph at level $\ell+1$. This coarse-graining induces a new probability distribution $P_{\ell+1}\left(\mathbf{A}^{(\ell+1)}, \Theta_{\ell+1}\right)$. The Multiscale Network Renormalization approach seeks for the random graph model described by a distribution such that $P_\ell\left(\mathbf{A}^{(\ell)}, \Theta_\ell\right)$ and $P_{\ell+1}\left(\mathbf{A}^{(\ell+1)}, \Theta_{\ell+1}\right)$ have the same functional form, up to a rescaling of the parameters.