Sign retrieval in spaces of variable bandwidth

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***Abstract*—The aim of this paper is to get a deeper understanding of the spaces of variable bandwidth introduced by Gröchenig and Klotz (*What is variable bandwidth?* Comm. Pure Appl. Math., 70 (2017), 2039–2083). In particular, we show that when the variation of the bandwidth is modeled by a step function with a finite number of jumps, then, the sign retrieval principle applies.**

***Index Terms*—Spaces of variable bandwidth, sign retrieval, phase retrieval**

I. Introduction

The phase retrieval problem is an ubiquitous family of problems in the applied sciences when one wants to reconstruct a signal f from its modulus $|f|$ and some a priori knowledge on f that is usually expressed through the fact that f belongs to some function space $\mathcal{F}$. In other words, we are asking whether $f, g \in \mathcal{F}$ with $|f| = |g|$ implies $f = cg$ where c is a global phase factor, that is a complex number of modulus 1. When one further restricts the signals f, g to be real valued, then we ask whether $f = g$ or $f = -g$, *i.e.*, f and g are the same up to a global sign.

Our general aim is to investigate which properties of the function space lead to the sign retrieval property. To explain the general aim, let us focus first on the Paley-Wiener spaces. Fix $c > 0$ and recall that the classical Paley-Wiener space $PW_c(\mathbb{R})$ is the set of all L^2 functions whose Fourier transforms are supported on the interval $[-c, c]$, that is,

$$PW_c(\mathbb{R}) = \{f \in L^2(\mathbb{R}) \,:\, \operatorname{supp} \widehat{f} \subset [-c, c]\}.$$

Here, we use the normalized definition of the Fourier transform given by

$$\widehat{f}(\xi) = \mathcal{F}f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ix\xi} \, \mathrm{d}x.$$

Functions in $PW_c(\mathbb{R})$ are called band-limited functions and c is the bandwidth. The sign-retrieval problem is easy to solve in this space :

Lemma I.1 (Paley-Wiener sign retrieval). *Let $I \subset \mathbb{R}$ be an interval, and let $f, g \in PW_c(\mathbb{R})$ are real valued (on $\mathbb{R}$) and such that $|f(x)| = |g(x)|$ for $x \in I$. Then either $f = g$ or $f = -g$.*

Indeed, it is known that each function in $PW_c(\mathbb{R})$ extends to an entire function and this is the only property needed : If f, g are real valued on $\mathbb{R}$ and satisfy $|f(x)| = |g(x)|$ for $x \in I$ then there is a set $E \subset I$ such that $f = g$ on E and $f = -g$ on $I \setminus E$. Now at least one of E or $I \setminus E$ has an accumulation point, say E. As f and g are entire functions, $f = g$ on E implies $f = g$ on $\mathbb{C}$.

In particular, the sign retrieval is valid in any space of entire functions (or directly related to such a space) like *e.g.* de Branges spaces, or time-warping variable bandwidth spaces, *i.e.*, spaces of functions of the form $f \circ \gamma$ where $f \in PW_c(\mathbb{R})$ and γ is a strictly increasing homeomorphism of $\mathbb{R}$ as introduced in [7] (*see* also [3] and further references therein).

Further, sampling theorems are a key feature of the Paley-Wiener spaces and the next natural question is of course to know for which (discrete) sets $\Lambda \subset \mathbb{R}$, $|f(x)| = |g(x)|$ for $x \in \Lambda$ implies that $f = g$ or $f = -g$. This has been first investigated by Thakur in [9], further investigated in shift invariant spaces [4], [8]. We also refer to [5] for deep results in this direction for the Fock space.

Our general aim is to be able to extend such results to spaces of variable bandwidth. This is a natural concept in the physical sciences for which there is at this stage no clear mathematical formulation. The paper [3] by Gröchenig and Klotz offers an overview of various attempts as well as a first definition of a new family of spaces of variable bandwidth. For those spaces, the bandwidth is described by two parameters, a global bandwidth Ω and a local variation $p(x)$ so that, at some point x, the bandwidth is $c(x) = \Omega/\sqrt{p(x)}$. We will denote those spaces as $PW_\Omega(A_p)$.[1] This note stems from our will to better understand those spaces and to see whether they are adapted to phase retrieval or at least to sign retrieval, if possible in its sampled form. We conjecture that this is possible, at least when the local variation $p(x)$ is a step function with finitely many jumps. The reason we think this is possible are the following :

– $PW_\Omega(A_p)$ is a reproducing kernel Hilbert space consisting of continuous functions. Moreover, on an interval on which p is constant, the functions in $PW_\Omega(A_p)$ are restrictions of functions in $PW_c(\mathbb{R})$ to this interval.

– There are good sampling formulas in $PW_\Omega(A_p)$.

– Sign retrieval is possible in $PW_\Omega(A_p)$. This result is new and is the main result of this note. We offer two proofs, one in the so-called toy model when p is a step function with a single jump and that is based on the reproducing kernel. The second one is for general step functions.

The remaining of the paper is organized as follows : The next section is devoted to recalling the main facts from [3], reformulating the definition of $PW_\Omega(A_p)$ when p is a step

[1]We have decided to adopt a slightly different normalization, this space would be $PW_{\Omega^2}(A_p)$ in [3].

function with finitely many jumps and then proving the sign retrieval property. The last section is devoted to giving a second proof of this property in the case of a single jump, based this time on the reproducing kernel.

II. Functions of Variable Bandwidth

The starting point of Gröchenig and Klotz [3] is as follows : Band-limited functions are contained in a spectral subspace of the differential operator $-D^2 = -\frac{d^2}{dx^2}$ since this operator is diagonalized by the Fourier transform $\mathcal{F}$ with $-\mathcal{F}D^2\mathcal{F}^{-1}f(\xi) = \xi^2\widehat{f}(\xi)$. The idea in [3] is to replace $-D^2$ by the Sturm-Liouville operator τ_p given by

$$\tau_p f(x) = -(pf')'(x), \qquad x \in \mathbb{R}$$

where $p > 0$ is a bandwidth-parametrizing function. For almost every $x \in \mathbb{R}$ and $z \in \mathbb{C}$, the Sturm-Liouville equation corresponding to τ_p is given by $(\tau_p - z)f = 0$ on $\mathbb{R}$.

We will here focus on a particular case that was investigated in more depth by Celiz, Gröchenig and Klotz [2] :

Notation II.1. We will say that $p \in PC_N$ if p is a step function with N jumps. We will further write $x_0 = -\infty < x_1 < \cdots < x_N < x_{N+1} = +\infty$, ($x_1, \ldots, x_N$ are the jump points), $I_j := (x_j, x_{j+1})$, $j = 0, \ldots, N$, $p_0, \ldots, p_N > 0$ and on each I_j, $p(x) = p_j$. We will also write $q_j = p_j^{-1/2}$.

To a given τ_p corresponds the maximal operator $(A_p, \mathcal{D}(A_p))$ given by

$$\begin{array}{rcl} \mathcal{D}(A_p) & = & \{f \in L^2(\mathbb{R}) \ \ f, pf' \in AC_{loc}(\mathbb{R}),\ \tau_p \in L^2(\mathbb{R})\} \\ A_p f & = & \tau_p f,\ f \in \mathcal{D}(A_p). \end{array}$$

where $AC_{loc}(\mathbb{R})$ is the space of functions that locally are absolutely continuous (integrals of their distributional derivative).

Now, for a Sturm-Liouville operator τ_p associated to $p \in PC_N$, a solution ϕ_z of $(\tau_p - z)f = 0$ lies left in $L^2(\mathbb{R})$ if $\phi_z \in L^2(I_0)$, and lies right if $\phi_z \in L^2(I_N)$. Then, for every $z \notin \mathbb{R}$ there are two unique solutions of $(\tau_p - z)f = 0$ up to a multiplicative constant, one of which lies left and one of which lies right in $L^2(\mathbb{R})$ and the corresponding maximal operator A_p is self-adjoint. We will need the following :

Theorem II.2 ([3], Theorem 2.3)**.** *Let $p \in PC_N$ and A_p be the self-adjoint realization of τ_p. If $\Phi(\lambda, x) = (\phi_+(\lambda, x), \phi_-(\lambda, x))$, for $\lambda, x \in \mathbb{R}$, is a fundamental system of solutions of $(\tau_p - \lambda)\phi = 0$ that continuously depends on λ, then there exists a 2×2 matrix measure μ such that the operator $\mathcal{F}_{A_p} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}, \mathrm{d}\mu)$*

$$\mathcal{F}_{A_p} f(\lambda) = \int_{\mathbb{R}} f(x)\overline{\Phi(\lambda, x)}\,\mathrm{d}x$$

is unitary and diagonalizes A_p, i.e.,

$$\mathcal{F}_{A_p} A_p \mathcal{F}_{A_p}^{-1} G(\lambda) = \lambda G(\lambda), \qquad G \in L^2(\mathbb{R}, \mathrm{d}\mu).$$

Then $\mathcal{F}_{A_p}$ is called the spectral (Fourier) transform of A_p and its inverse is given as

$$\mathcal{F}_{A_p}^{-1} G(x) = \int_{\mathbb{R}} G(\lambda) \cdot \Phi(\lambda, x)\,\mathrm{d}\mu(\lambda), \qquad G \in L^2(\mathbb{R}, \mathrm{d}\mu).$$

Now, when $p \in PC_N$, this can be made (almost) explicit. A fundamental system of solutions is given by $\Phi_\pm(z, x) = \tilde{\Phi}_\pm(\sqrt{z}, x)$ where by convention, when $z = re^{i\theta}$, $r > 0$ and $-\pi < \theta < \pi$, $\sqrt{z} = \sqrt{r}e^{i\theta/2}$, and, for $x \in I_j$,

$$\tilde{\Phi}_\pm(\zeta, x) = a_j^\pm(\zeta)e^{iq_j\zeta x} + b_j^\pm(\zeta)e^{-iq_j\zeta x}, \tag{1}$$

and the coefficients $A_j^\pm := \begin{pmatrix} a_j^\pm \\ b_j^\pm \end{pmatrix}$ are given by the induction formula $A_0^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $A_N^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $A_j^\pm = \dfrac{1}{2}L_j(\zeta)A_{j-1}^\pm$ where $L_j(\zeta)$ is the invertible matrix

$$\begin{pmatrix} \left(1 + \frac{q_j}{q_{j-1}}\right)e^{ix_j(q_j - q_{j-1})\zeta} & \left(1 - \frac{q_j}{q_{j-1}}\right)e^{-ix_j(q_j + q_{j-1})\zeta} \\ \left(1 - \frac{q_j}{q_{j-1}}\right)e^{ix_j(q_j + q_{j-1})\zeta} & \left(1 + \frac{q_j}{q_{j-1}}\right)e^{-ix_j(q_j - q_{j-1})\zeta} \end{pmatrix}.$$

It is then easy to see that $a_j^\pm, b_j^\pm$ are almost periodic trigonometric polynomials, in particular, they are bounded. An other key fact for us is that

$$\det \begin{pmatrix} a_j^+(\zeta) & a_j^-(\zeta) \\ b_j^+(\zeta) & b_j^-(\zeta) \end{pmatrix} \neq 0. \tag{2}$$

The spectral measure is also explicitly given by

$$\mathrm{d}\mu(\lambda) = \frac{1}{4\pi p_n} \begin{pmatrix} \dfrac{1}{q_0} & 0 \\ 0 & \dfrac{1}{q_n} \end{pmatrix} \frac{\mathrm{d}\lambda}{|b_n^-(\sqrt{\lambda})|^2\sqrt{\lambda}}.$$

We can now define the following spectral projections : For Λ a Borel subset of $\mathbb{R}^+$ and $f \in L^2$,

$$\begin{array}{rcl} \chi_\Lambda(A_p)f(x) & = & \displaystyle\int_\Lambda \mathcal{F}(A_p)f(\lambda) \cdot \Phi(\lambda, x)\,\mathrm{d}\mu(\lambda) \\ & = & \mathcal{F}_{A_p}^{-1}(\chi_\Lambda \mathcal{F}_{A_p} f)(x), \qquad x \in \mathbb{R}. \end{array}$$

We now define the Paley-Wiener space of variable bandwidth functions :

Definition 1 ([3])**.** Let $p \in PC_N$ and A_p be the self-adjoint realization of τ_p. For $\Omega > 0$, the Paley-Wiener space of variable bandwidth functions, denoted by $PW_\Omega(A_p)$, is the range of the spectral projection $\chi_{[0,\sqrt{\Omega}]}(A_p)$, i.e.,

$$\begin{array}{rcl} PW_\Omega(A_p) & = & \chi_{[0,\sqrt{\Omega}]}(A_p)(L^2(\mathbb{R})) \\ & = & \{f \in L^2(\mathbb{R}) :\ f = \chi_{[0,\sqrt{\Omega}]}(A_p)f\}. \end{array}$$

Note that when $p \equiv 1$, $PW_\Omega(A_p) = PW_\Omega(\mathbb{R})$. We also recall a characterization for the spaces $PW_\Omega(A_p)$ akin to the one from classical Paley-Wiener spaces :

Theorem II.3 ([3], Proposition 3.2)**.** *Let $p \in PC_N$, A_p be the self-adjoint realization of τ_p and $\Omega > 0$. Let μ be the spectral measure of A_p with corresponding spectral transform $\mathcal{F}_{A_p}$. Then the following are equivalent :*

1) *$f \in PW_\Omega(A_p)$,*
2) *$\operatorname{supp} \mathcal{F}_{A_p} f \subseteq [0, \sqrt{\Omega}]$,*
3) *there exists a function $F \in L^2([0, \sqrt{\Omega}], \mathrm{d}\mu)$ such that for almost every $x \in \mathbb{R}$,*

$$f(x) = \int_0^{\sqrt{\Omega}} F(\lambda) \cdot \Phi(\lambda, x)\,d\mu(\lambda). \tag{3}$$

More explicitly, there exist F_+, F_- such that

$$f(x) = \int_0^{\sqrt{\Omega}} \left(\frac{F_+(\lambda)\tilde{\phi}_+(\sqrt{\lambda}, x)}{q_0} + \frac{F_-(\lambda)\tilde{\phi}_-(\sqrt{\lambda}, x)}{q_n} \right) \frac{\mathrm{d}\lambda}{4\pi p_n |b_n^-(\sqrt{\lambda})|^2 \sqrt{\lambda}}.$$

Using the explicit expressions given above, this can be written as

$$f(x) = \int_0^{\Omega} \big(G_-(\zeta)\tilde{\phi}_-(\zeta, x) + G_+(\zeta)\tilde{\phi}_+(\zeta, x) \big) \,\mathrm{d}\zeta$$

after a change of variable $\zeta = \sqrt{\lambda}$ and change of unknown function, $G_-(\zeta) = \dfrac{F_-(\zeta^2)}{2\pi p_n q_n |b_n^-(\zeta)|^2}$, and a similar expression for G_+ so that $G_\pm \in L^2(0, \Omega)$. Using (1), this can further be written as

$$f(x) = \int_{-\Omega}^{\Omega} G_j(\zeta) e^{-iq_j\zeta x} \,\mathrm{d}\zeta, \quad x \in I_j$$

where

$$G_j(\zeta) = \begin{cases} b_j^-(\zeta)G_-(\zeta) + b_j^+(\zeta)G_+(\zeta), & \text{for } \zeta > 0 \\ a_j^-(-\zeta)G_-(-\zeta) + a_j^+(-\zeta)G_+(-\zeta), & \text{for } \zeta < 0 \end{cases}.$$

As $G_\pm \in L^2(0, \Omega)$ and $a_j^\pm, b_j^\pm$ are bounded, $G_j \in L^2(-\Omega, \Omega)$. We thus obtain the following, which was already partially in [3, Proposition 3.3(i), Proposition 3.5],

Corollary II.4. *Let $p \in PC_N$, A_p be the self-adjoint realization of τ_p and $\Omega > 0$. Then $PW_\Omega(A_p)$ is a closed subspace of $L^2(\mathbb{R})$ consisting of continuous functions. Moreover, on each I_j, f is the restriction to I_j of a function in $PW_{\Omega q_j}(\mathbb{R})$.*

The main difference is that in [3], the authors prove that f is locally in the larger Bernstein space. On the other hand, their result is valid for more general p. This corollary also leads to the following question :

Question II.5. *Let $p \in PC_N$ and, for each j, let f_j be the restriction to I_j of a function in $PW_{\Omega q_j}(\mathbb{R})$. Under which condition is the function $f = \sum_{j=0}^{N} f_j \mathbb{1}_{I_j} \in PW_\Omega(A_p)$?*

Of course, f has to be continuous, but this is not enough. So far, we have only been able to answer this question in the single jump case. This will be done in the next section and also leads to another proof of the sign retrieval question in this case. It is likely that this would also be a crucial step for phase retrieval in variable band width spaces.

This raises also a second question :

Question II.6. *Let $p, \tilde{p} \in PC_N$ be such that $p_1 \geq p_2$. Is $PW_\Omega(A_p) \subset PW_\Omega(A_{\tilde{p}})$?*

From the formula, this is clear when $\tilde{p} = \alpha p$, but not in the general case.

Our computation also shows that $PW_\Omega(A_p)$ is obtained as follows : take a pair of function $G_-, G_+ \in L^2(0, \Omega)$. For each ζ, and each j, construct G_j with the above formula. Then on I_j, take the (properly scaled) Fourier transform of G_j. Now this can be reverted. If $f \in PW_\Omega(A_p)$ is given, then its values on any I_j uniquely determine G_+, G_- as follows : As f is the restriction of a function $f_j \in PW_{c_j}$, we may (in theory) obtain f_j on $\mathbb{R}$ from its restriction to I_j. Consider its Fourier transform $\widehat{f}_j$. To obtain G_-, G_+, it remains to solve the system

$$\begin{cases} b_j^-(\zeta)G_-(\zeta) + b_j^+(\zeta)G_+(\zeta) = \widehat{f}_j(\zeta) \\ a_j^-(\zeta)G_-(\zeta) + a_j^+(\zeta)G_+(\zeta) = \widehat{f}_j(-\zeta) \end{cases}$$

for $\zeta > 0$. This system has non-zero determinant, so its solution is unique.

We are now ready to prove the main result :

Theorem II.7. *Let $p \in PC_N$, A_p be the self-adjoint realization of τ_p and $\Omega > 0$. Let $f, g \in PW_\Omega(A_p)$ be real valued and such that $|f| = |g|$. Then $f = g$ or $f = -g$.*

Proof. We first consider $|f| = |g|$ on each of the intervals I_j. As f (resp. g) is the restriction of a function f_j (resp. g_j) in $PW_{c_j}(\mathbb{R})$, according to Lemma I.1, there exists $\varepsilon_j \in \{\pm 1\}$ such that $g_j = \varepsilon_j f_j$ on I_j. Next, as just explained, f (resp. g) stems from a pair of functions $G_\pm^f$ (resp. $G_\pm^g$) and they are determined by the system

$$\begin{cases} b_j^-(\zeta)G_-^f(\zeta) + b_j^+(\zeta)G_+^f(\zeta) = \widehat{f}_j(\zeta) \\ a_j^-(\zeta)G_-^f(\zeta) + a_j^+(\zeta)G_+^f(\zeta) = \widehat{f}_j(-\zeta) \end{cases}$$

and

$$\begin{cases} b_j^-(\zeta)G_-^g(\zeta) + b_j^+(\zeta)G_+^g(\zeta) = \varepsilon_j \widehat{f}_j(\zeta) \\ a_j^-(\zeta)G_-^g(\zeta) + a_j^+(\zeta)G_+^g(\zeta) = \varepsilon_j \widehat{f}_j(-\zeta) \end{cases}.$$

But these systems are invertible, so that $(G_-^g, G_+^g) = \varepsilon_j (G_-^f, G_+^f)$. There are three cases

– either $\varepsilon_j = 1$ for all j and then $(G_-^g, G_+^g) = (G_-^f, G_+^f)$ and finally $g = f$;

– or $\varepsilon_j = -1$ for all j and then $g = -f$;

– or there are k, ℓ such that $\varepsilon_k = 1$ and $\varepsilon_\ell = -1$. But then we simultaneously have $(G_-^g, G_+^g) = (G_-^f, G_+^f)$ and $(G_-^g, G_+^g) = -(G_-^f, G_+^f)$ so that $G_\pm^g = G_\pm^f = 0$ and finally $g = f = 0$. □

III. A SECOND PROOF FOR THE TOY EXAMPLE

The toy example is simple case of piecewise constant functions with a single jump, which one can put at 0 :

$$p(x) = \begin{cases} p_-, & x \leq 0 \\ p_+ & x > 0 \end{cases}$$

with $p_-, p_+ > 0$.

In this case, we will give a second proof of the main result based on the fact that the space $PW_\Omega(A_p)$ is a reproducing kernel Hilbert space [3, Proposition 3.3(ii)] with kernel

$$k_\Omega(x, y) = \int_0^\Omega \overline{\Phi(\lambda, x)} \Phi(\lambda, y) \,\mathrm{d}\mu(\lambda).$$

This kernel is also the kernel of the projection from $L^2(\mathbb{R})$ onto $PW_\Omega(A_p)$. The main issue is that this kernel is far from being explicit excepted in the toy model. Even in the case of two jumps, the reproducing kernel is essentially untractable, as can be seen from the one full-page formula in [2].

To come back to the toy model, the reproducing kernel for the space $PW_{[0,\Omega]}(A_p)$ is given by

i) when $x, y \leq 0$
$$k(x,y) = \frac{c_-}{\pi}\operatorname{sinc}(c_-(x-y)) - (\rho_+-\rho_-)\frac{c_-}{\pi}\operatorname{sinc}(c_-(x+y));$$

ii) when $x \leq 0,\ y \geq 0$
$$k(x,y) = \frac{2c_+\rho_+}{\pi}\operatorname{sinc}(c_-x - c_+y);$$

iii) when $x, y > 0$,
$$k(x,y) = \frac{c_+}{\pi}\operatorname{sinc}(c_+(x-y)) + (\rho_+-\rho_-)\frac{c_+}{\pi}\operatorname{sinc}(c_+(x+y));$$

iv) when $x > 0,\ y \leq 0$
$$k(x,y) = \frac{2c_-\rho_-}{\pi}\operatorname{sinc}(c_+x - c_-y)$$

where
$$c_\pm = \sqrt{\frac{\Omega}{p_\pm}} \text{ and } \rho_\pm = \frac{\sqrt{p_\pm}}{\sqrt{p_+}+\sqrt{p_-}}.$$

Note that if $p_+ = p_- := p$ and $c = \sqrt{\dfrac{\Omega}{p}}$ then $k(x,y) = \dfrac{c}{\pi}\operatorname{sinc}\big(c(x-y)\big)$ and $PW_\Omega(A_p) = PW_c(\mathbb{R})$.

From these formulas, it is possible to relate $f \in PW_{[0,\Omega]}(A_p)$ to the usual Paley-Wiener spaces. To do so, recall that the orthogonal projection of a function $f \in L^2(\mathbb{R})$ onto $PW_c(\mathbb{R})$ is given by
$$\Pi_c f(x) = \frac{c}{\pi}\int_{\mathbb{R}} \operatorname{sinc}(c(x-y))f(y)\,dy,\ x\in\mathbb{R}.$$

For $a > 0$, let δ_a be the dilation of a function $f \in L^2(\mathbb{R})$ given by $\delta_a f(x) = \sqrt{a}f(ax)$ so that $\|\delta_a f\|_2 = \|f\|_2$. It is easy to see that $\widehat{\delta_a f} = \delta_{1/a}\widehat{f}$, thus $f \in PW_c(\mathbb{R})$ if and only if $\delta_a f \in PW_{ac}(\mathbb{R})$, and that $\delta_a \Pi_c[\delta_{1/a} f] = \Pi_{ac} f$.

Now, for $f \in PW_{[0,\Omega]}(A_p)$, write $f_\pm = f\mathbb{1}_{\mathbb{R}^\pm}$. Then, for $x \in \mathbb{R}^-$,
$$\begin{aligned}
f(x) &= \frac{1}{\pi}\int_{-\infty}^0 c_-\operatorname{sinc}(c_-(x-y))\,f_-(y)\,\mathrm{d}y \\
&\quad - \frac{\rho_+-\rho_-}{\pi}\int_0^\infty c_-\operatorname{sinc}(c_-(x-y))f_-(-y)\,\mathrm{d}y \\
&\quad - \frac{2\rho_+}{\pi}\int_0^\infty c_+\operatorname{sinc}\left(c_+\left(\frac{c_-}{c_+}x - y\right)\right) f_+(y)\,\mathrm{d}y \\
&= \Pi_{c_-}\left[f_-\cdot\mathbb{1}_{\mathbb{R}_-} - (\rho_+-\rho_-)\check{f}_-\cdot\mathbb{1}_{\mathbb{R}_+}\right](x) \\
&\quad + 2\rho_+\sqrt{\frac{c_+}{c_-}}\delta_{c_-/c_+}\Pi_{c_+}[f_+\cdot\mathbb{1}_{\mathbb{R}_+}](x),
\end{aligned}$$

where $\check{\varphi}(x) = \varphi(-x)$. Using the scaling property, we obtain
$$\begin{aligned}
f_- = \mathbb{1}_{\mathbb{R}_-}\Pi_{c_-}\Big[&f_-\cdot\mathbb{1}_{\mathbb{R}_-} - (\rho_+-\rho_-)\check{f}_-\cdot\mathbb{1}_{\mathbb{R}_+} \\
&+ 2\rho_+\sqrt{\frac{c_+}{c_-}}(\delta_{c_-/c_+}f_+))\cdot\mathbb{1}_{\mathbb{R}_+}\Big].
\end{aligned} \tag{4}$$

Similarly, the positive part is given by
$$\begin{aligned}
f_+ = \mathbb{1}_{\mathbb{R}_+}\Pi_{c_+}\Big[&f_+\cdot\mathbb{1}_{\mathbb{R}_+} + (\rho_+-\rho_-)\check{f}_+\cdot\mathbb{1}_{\mathbb{R}_-} \\
&+ 2\rho_-\sqrt{\frac{c_-}{c_+}}(\delta_{c_+/c_-}f_-))\cdot\mathbb{1}_{\mathbb{R}_-}\Big].
\end{aligned} \tag{5}$$

Proposition III.1 (Sign Retrieval)**.** *Assume that $p \in PC_1$. Let $f, g \in PW_{[0,\Omega]}(A_p)$ be* real-valued *and such that $|f| = |g|$ on $\mathbb{R}$, then $f = g$ of $f = -g$.*

Proof. Now, let $f, g \in PW_{[0,\Omega]}(A_p)$ be *real valued* and such that $|f| = |g|$ on $\mathbb{R}$, Write $f_\pm = f\mathbb{1}_{\mathbb{R}_\pm}$ and $g_\pm = g\mathbb{1}_{\mathbb{R}_\pm}$. Each of $f_\pm$ or $g_\pm$ extends holomorphically to $\mathbb{C}$ so that $|f_\pm| = |g_\pm|$ implies that there are $\varepsilon_-, \varepsilon_+ \in \{-1, 1\}$ such that $f_\pm = \varepsilon_\pm g_\pm$. We want to show that $\varepsilon_- = \varepsilon_+$. Assume this is not the case. Without loss of generality, we may assume that $\varepsilon_- = 1$ and $\varepsilon_+ = -1$ that is $f_- = g_-$ and $f_+ = -g_+$. Let us plug this into (4) to obtain
$$\begin{aligned}
f_- &= \mathbb{1}_{\mathbb{R}_-}\Pi_{c_-}\Big[f_-\cdot\mathbb{1}_{\mathbb{R}_-} - (\rho_+-\rho_-)\check{f}_-\cdot\mathbb{1}_{\mathbb{R}_+} \\
&\qquad + 2\rho_+\sqrt{\frac{c_+}{c_-}}(\delta_{c_-/c_+}f_+)\cdot\mathbb{1}_{\mathbb{R}_+}\Big] \\
= g_- &= \mathbb{1}_{\mathbb{R}_-}\Pi_{c_-}\Big[f_-\cdot\mathbb{1}_{\mathbb{R}_-} - (\rho_+-\rho_-)\check{f}_-\cdot\mathbb{1}_{\mathbb{R}_+} \\
&\qquad - 2\rho_+\sqrt{\frac{c_+}{c_-}}(\delta_{c_-/c_+}f_+)\cdot\mathbb{1}_{\mathbb{R}_+}\Big].
\end{aligned}$$

Comparing both expressions of f_- we obtain that
$$\mathbb{1}_{\mathbb{R}_-}\Pi_{c_-}\left[(\delta_{c_-/c_+}f_+)\cdot\mathbb{1}_{\mathbb{R}_+}\right] = 0.$$

As $\Pi_{c_-}[\varphi]$ is holomorphic, if it vanishes on $\mathbb{R}_-$, it is zero everywhere thus
$$\delta_{c_-/c_+}\Pi_{c_+}[f_+\cdot\mathbb{1}_{\mathbb{R}_+}] = \Pi_{c_-}\left[(\delta_{c_-/c_+}f_+)\cdot\mathbb{1}_{\mathbb{R}_+}\right] = 0$$

that is $\Pi_{c_+}[f_+\cdot\mathbb{1}_{\mathbb{R}_+}] = 0$. This means that $\varphi := f_+\cdot\mathbb{1}_{\mathbb{R}_+}$ has Fourier transform supported in $\mathbb{R}\setminus[-c_+, c_+]$. A version of the uncertainty principle (*see* [6, p36]) shows that this can only happen if $\varphi = 0$. Now that we now that $f_+ = 0$ thus $g_+ = 0$, it follows that
$$f_- = g_- = \mathbb{1}_{\mathbb{R}_-}\Pi_{c_-}\left[f_-\cdot\mathbb{1}_{\mathbb{R}_-} - (\rho_+-\rho_-)\check{f}_-\cdot\mathbb{1}_{\mathbb{R}_+}\right].$$

Further, from the expression of f_+, we obtain that
$$\begin{aligned}
0 &= 2\rho_-\sqrt{\frac{c_-}{c_+}}\mathbb{1}_{\mathbb{R}_+}\Pi_{c_+}\left[(\delta_{c_+/c_-}f_-)\cdot\mathbb{1}_{\mathbb{R}_-}\right] \\
&= \rho_-\sqrt{\frac{c_-}{c_+}}\mathbb{1}_{\mathbb{R}_+}\Pi_{c_-}\left[f_-\mathbb{1}_{\mathbb{R}_-}\right]
\end{aligned}$$

that is, $\Pi_{c_-}\left[f_-\mathbb{1}_{\mathbb{R}_-}\right]$ vanishes on $\mathbb{R}^+$ which, as for f_+ implies that $f_-\mathbb{1}_{\mathbb{R}_-} = 0$ so that, finally, $f_+ = f_- = 0$ and thus $f = g = 0$. □

IV. Some future directions of research

As we mentioned in the introduction, it is natural to ask whether sign retrieval is possible from sampled measurements. This is possible in the classical Paley-Wiener space [9], but also in certain shift-invariant spaces [4], [8]. For variable band-width spaces, sampling theorems are one of the key questions addressed in [4], [2]. At this stage, we have unfortunately been unable to give an answer to the following question:

Question IV.1. *Fix $p \in PC_N$. Under which conditions on a discrete set $\Lambda \subset \mathbb{R}$ is it true that $f, g \in PW(A_p)$ real valued with $|f(\lambda)| = |g(\lambda)|$ for every $\lambda \in \Lambda$ implies that $g = \pm f$?*

Of course, one may replace the condition $|f(\lambda)| = |g(\lambda)|$ with $f(\lambda)^2 = g(\lambda)^2$. The difficulty of this question is that f^2, g^2 do not seem to belong to a variable bandwidth space (there is no convolution theorem here). It is thus difficult to establish a sampling theorem for those functions.

One conclusion of our research is that the variable band-width spaces introduced by Gröchenig and Klotz provide a deep theoretical family of spaces but with the drawback that practical questions in those spaces are hard to tackle. Recently Andreolli and Gröchenig [1] introduced a new family of variable band-width spaces based on Wilson bases. Those spaces are easier to handle as their elements have a rather explicit description. Nevertheless the question of sign-retrieval does not seem simple in them and calls for further research.

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