Sampling theorems with derivatives in shift-invariant spaces generated by exponential B-splines

Karlheinz Gröchenig
Faculty of Mathematics, University of Vienna
Oskar-Morgenstern Platz 1,
1090 Vienna, AT
Email: karlheinz.groechenig@univie.ac.at

Irina Shafkulovska
Faculty of Mathematics, University of Vienna
Oskar-Morgenstern Platz 1,
1090 Vienna, AT
Email: irina.shafkulovska@univie.ac.at

***Abstract*—We derive sufficient conditions for sampling with derivatives in shift-invariant spaces generated by an exponential B-spline. The sufficient conditions are expressed by a new notion of measuring the gap between consecutive points. As a consequence, we can construct sampling sets arbitrarily close to necessary conditions.**

Keywords. Shift-invariant spaces, splines, Chebyshev B-splines, nonuniform sampling, Gabor frames
AMS subject classifications. primary: 42C15; secondary: 41A15, 42C40, 94A20.

I. Introduction

We consider shift-invariant spaces: given a generator $\varphi \in L^p(\mathbb{R})$, we denote its integer translates by $T_\ell\varphi(x) = \varphi(x-\ell)$ and with $V^p(\varphi) \subseteq L^p(\mathbb{R})$ the subspace

$$V^p(\varphi) := \Big\{ \sum_{\ell\in\mathbb{Z}} c_\ell T_\ell \varphi \in L^p(\mathbb{R}),\, c \in \ell^p(\mathbb{Z}) \Big\}. \tag{I.1}$$

If $\varphi = \mathrm{sinc}$, then $V^2(\mathrm{sinc})$ is the Paley-Wiener space $PW^2(\mathbb{R})$ of band-limited functions. In this sense, shift-invariant spaces are a generalization of the Paley-Wiener space. In applications in signal processing, shift-invariant spaces are nowadays often used instead of band-limited functions since the generator can be tuned to the specific problem at hand. For instance, a fast decay of the generator φ is reflected in a fast convergence rate of the local series representation of a function $f \in V^p(\varphi)$. This is in stark contrast to $PW^2(\mathbb{R})$, since sinc only decays as x^{-1}. From the numerics standpoint, ideally, φ is compactly supported, in which case each point evaluation can be computed as a finite sum. Additional local properties of $f \in V^p(\varphi)$, such as differentiability and analyticity, can be implemented by imposing them on the generator φ. The focus here is on compactly supported right-continuous generators with only finitely many jump discontinuities. In this case, the following characterization holds, cf. [10, Thm. 3.5], [13, Thm. 29].

Proposition I.1. *Let φ be a bounded compactly supported function. There exist positive constants $0 < A_p \leq B_p$ such that for all $c \in \ell^p(\mathbb{Z})$ holds*

$$A_p \left\|c\right\|_p \leq \Big\| \sum_{\ell\in\mathbb{Z}} c_\ell T_\ell \varphi \Big\|_p \leq B_p \left\|c\right\|_p \tag{I.2}$$

if and only for all $\omega \in \mathbb{R}$ holds

$$0 < \sum_{\ell\in\mathbb{Z}} \left|\hat{\varphi}(\omega+\ell)\right|^2 . \tag{I.3}$$

If the inequalities hold, then we say that φ has stable integer translates.

We now define the sampling problem. We are given a compactly supported $\varphi \in C^{S-1}(\mathbb{R})$, with a right-continuous $\varphi^{(S)}$ with finitely many jump discontinuities. Let $X \subseteq \mathbb{R}$ be a separated set, and $\mu_X : X \to \{0,\dots,S\}$ its *multiplicity function*. We call (X,μ_X) a *sampling set with multiplicities* for $V^p(\varphi)$ if there exists constants $0 < A_p \leq B_p$ such that for all $f \in V^p(\varphi)$ holds

$$A_p \left\|f\right\|_p^p \leq \sum_{x\in X} \sum_{s=0}^{\mu_X(x)} \left| f^{(s)}(x) \right|^p \leq B_p \left\|f\right\|_p^p . \tag{I.4}$$

Sampling with derivatives is interesting in many applications because it allows us to incorporate additional features such as local trends (first derivative) or convexity conditions (second derivative)

For band-limited functions, the problem has been studied in [11] and some preliminary results for shift-invariant spaces can be found in [9], [16].

II. Exponential B-splines

A. EB-splines as generators

We begin by settling the technical prerequisites for the investigated generators. We refer to [3], [12] for further details.

Definition II.1. An exponential B-spline (EB-spline) $\mathcal{E}_{m,\alpha} : \mathbb{R} \longrightarrow \mathbb{R}$ of order m for $\alpha \in \mathbb{R}^m$ is a function of the form

$$\mathcal{E}_{m,\alpha}(x) := \prod_{s=1}^{m}{}^{*} e^{\alpha_s x} \chi_{[0,1)}(x), \tag{II.1}$$

where $\prod^{*}$ denotes the convolution product.

Lemma II.2. *Let $\varphi = \mathcal{E}_{m,\alpha}$ be an EB-spline of order m. Then φ has stable integer translates.*

Proof. An EB-spline is supported on $[0, m]$. The Fourier transform of an EB-spline's is given by

$$\widehat{\mathcal{E}}_{m,\alpha}(\omega) = \prod_{s=1}^{m} \frac{e^{\alpha_s - 2\pi i\omega} - 1}{\alpha_s - 2\pi i\omega}. \tag{II.2}$$

One can easily verify that this has no 2π-periodic zeros. The claim follows from Proposition I.1 ☐

B. Schoenberg-Whitney conditions

The name *EB-spline* comes from an equivalent construction of $\varphi = \mathcal{E}_{m,\alpha}$ as a B-spline with respect to an associated *extended complete Chebyshev system* (ECC-system) with the exponential weights $w_1(x) = e^{\alpha_1 x}$, $w_s = e^{(\alpha_s - \alpha_{s-1})x}$, $2 \leq s \leq m$, through divided differences [15, Sec. 9.4]. ECC-systems have associated differential operators. We define $D_0 f = f$ and

$$D_s f := \tfrac{d}{dx} \tfrac{f}{w_s}, \quad L_s := D_s D_{s-1} \dots D_0, \tag{II.3}$$

$0 \leq s \leq m$. By Leibniz's rule, L_s are just perturbations of the standard derivatives. An EB-spline φ belongs to $C^{m-2}(\mathbb{R})$ and is piecewise C^{m-1}. The only discontinuities of $\varphi^{(m-1)}$ are the integer points $0, 1, \dots, m$ and they are jump discontinuities. For the jump discontinuities $i \in \mathbb{Z}$, we define $L_{m-1}\varphi(i)$ to be the right-sided limit

$$L_{m-1}\varphi(i) := \lim_{x \searrow i} L_{m-1}\varphi(x). \tag{II.4}$$

Chebychev systems are essentially systems which admit unique interpolation. Extended Chebychev systems extend this property to Hermite interpolation. For the associated EB-splines, this property holds as follows.

Theorem II.3. *Let φ be an EB-spline of order m. Further let $t_0 \leq t_1 \leq \dots \leq t_D$ and set*

$$d_i := \max\{\ell : t_i = \dots = t_{i-\ell}\}, \quad 0 \leq i \leq D. \tag{II.5}$$

The collocation matrix

$$M\begin{pmatrix} t_0, \dots, t_D \\ \varphi, \dots, T_D\varphi \end{pmatrix} := \Big(L_{d_i} T_{\ell-1}\varphi(t_i)\Big)_{0 \leq i,\ell \leq D} \tag{II.6}$$

has a non-negative determinant. The collocation matrix is invertible if and only if

$$t_i \in \begin{cases} (i, i+m), & d_i < m-1 \\ [i, i+m), & d_i = m-1, \end{cases} \tag{II.7}$$

for all $0 \leq i \leq D$.

The conditions (II.7) are referred to as *Schoenberg-Whitney conditions*, cf. [14]. The collocation matrix is the matrix that describes the Hermite interpolation problem

$$f = \sum_{\ell=0}^{D} c_\ell u_\ell, \quad L_{d_i} f(t_i) = \xi_i, \quad 0 \leq i \leq D. \tag{II.8}$$

The Schoenberg-Whitney conditions characterize when the Hermite interpolation problem has a unique solution.

III. Sampling results

We state the two main results here. The proofs can be found in [6].

Theorem III.1. *Let φ be an EB-spline of order $m \in \mathbb{N}$ supported on $[0, m]$. Assume $X \subseteq \mathbb{R}$ with multiplicity function μ_X satisfies the following properties:*

(i) X is $\delta-$separated for some $\delta > 0$.
(ii) $S := \max_{x \in X} \mu_X(x) \leq m - 1$.
(iii) There exist integers $M, L > 0$ and $\varepsilon > 0$, such that for every $k \in \mathbb{Z}$, there exist points $x_1^k < x_2^k < \dots < x_{N(k)}^k$ in $X \cap I_{M,L}(k)$ and nonnegative integers $\mu_1^k, \dots, \mu_{N(k)}^k \in \mathbb{N}_0$ with the following three properties:

$$\mu_j^k \leq \mu_X(x_j^k) \quad \forall k \in \mathbb{Z}, 1 \leq j \leq N(k), \tag{III.1}$$

$$\sum_{j=1}^{N(k)} \left(1 + \mu_j^k\right) = L + m - 1 \quad \forall k \in \mathbb{Z}, \tag{III.2}$$

$$x_j^k \in \Big(\big[\mu_j^k - m + 1 + \varepsilon, 1 - \varepsilon\big] + M + kL + \sum_{n=1}^{j-1} \left(1 + \mu_n^k\right) \Big). \tag{III.3}$$

Then (X, μ_X) is a sampling set for $V^p(\varphi)$ for all $p \in [1, \infty]$.

An example of a good local point distribution is depicted in Figure 1. The theorem relies on compactness arguments and the Schoenberg-Whitney

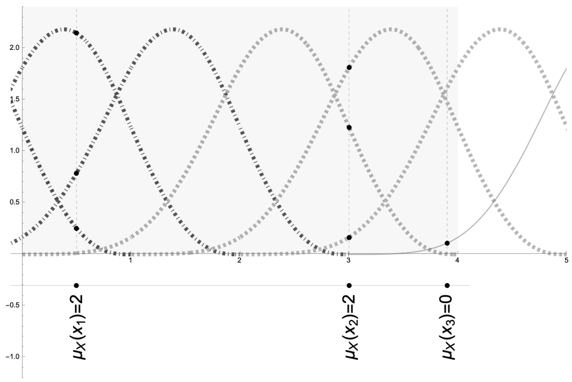

Fig. 1. Non-vanishing shifts of $\varphi(x) = \prod_{j=1}^{*\,4} e^x \chi_{[0,1)}(x)$ on $[0,4]$. The sampling points are $x_1 = 0.5$, $x_2 = 3$, $x_3 = 3.9$, with multiplicities $\mu_X(x_1) = \mu_X(x_2) = 2$, $\mu_X(x_3) = 0$. The first sampling point lies in the support of the first three shifts of φ (dot-dashed), the second point is in the support of the next three shifts of φ (dashed), and the last point - in the support of the last shift of φ (solid).

conditions. Our second result replaces compactness with a weighted density condition and relies on methods in [1], [8], [9].

Theorem III.2 (Maximum Gap Theorem)**.** *Let φ be an EB-spline of order m and let $X \subseteq \mathbb{R}$ be a separated set with $\mu_X : X \to \{0,\dots,m-1\}$. If the multiplicity function satisfies*

$$\operatorname{dist}\left(\{x \in X : \mu_X(x) = m-1\},\ \mathbb{Z}\right) > 0 \tag{III.4}$$

and the weighted maximum gap satisfies

$$\mathfrak{mg}(X,\mu_X) := \sup_{j\in\mathbb{Z}} \frac{x_{j+1}-x_j}{1+\min\{\mu_X(x_j),\mu_X(x_{j+1})\}} < 1, \tag{III.5}$$

then (X,μ_X) is a sampling set for $V^p(\varphi)$.

If the last discontinuous derivative is not sampled, then we obtain the following corollary.

Corollary III.3. *Let φ be an EB-spline of order m and let $X \subseteq \mathbb{R}$ be a separated set with multiplicity function $\mu_X : X \to \{0,\dots,m-2\}$. If the weighted maximum gap satisfies*

$$\mathfrak{mg}(X,\mu_X) < 1, \tag{III.6}$$

then (X,μ_X) is a sampling set for $V^p(\varphi)$.

Remark. If the multiplicity function is constant, i.e., $\mu_X \equiv S$, the weighted maximum gap is a multiple of the classical maximum gap:

$$\mathfrak{mg}(X) := \sup_{x,y\in X, x\neq y} |x-y| = S \cdot \mathfrak{mg}(X,\mu_X). \tag{III.7}$$

In the case of ordinary sampling, i.e., $\mu_X \equiv 0$, the assumption $\mathfrak{mg}(X) = \mathfrak{mg}(X,\mu_X) < 1$ in Theorem III.2 implies that the conditions of Theorem III.1, and the reverse implication does not hold. Interestingly enough, if we allow non-trivial multiplicities, there are sampling sets with multiplicities which satisfy the conditions of only one of the theorems III.1 and III.2. Furthermore, the results are optimal in the sense that both theorems provide examples of sampling sets with (weighted) lower Beurling density $\delta = D^-(X,\mu_X)$ for any $\delta > 1$ and a necessary condition for a pair (X,μ_X) to be sampling is $D^-(X,\mu_X) \geq 1$ [9, Prop. 3.7.].

Corollary III.4. *Let φ be an EB-spline of order m and let $X \subseteq \mathbb{R}$ be a separated set with a constant multiplicity function $\mu_X \equiv S < m-1$. If the maximum gap satisfies*

$$\mathfrak{mg}(X) < S, \tag{III.8}$$

then X is a sampling set of multiplicity S for $V^p(\varphi)$.

IV. Implications for Gabor systems

We denote with $\pi(x,\omega)$, $(x,\omega) \in \mathbb{R}^2$, the time-frequency shift (operator) acting on functions as $\pi(x,\omega)f(t) = e^{2\pi i\omega t}f(t-x)$. In terms of stable expansions, given a *window function* φ and a discrete set $\Lambda \subseteq \mathbb{R}^2$, one could ask the question when a *Gabor system* $\mathcal{G}(\varphi,\Lambda)$, defined as

$$\mathcal{G}(g,\Lambda) = \{\pi(\lambda)\varphi : \lambda \in \Lambda\}, \tag{IV.1}$$

is a *(Gabor) frame*, i.e., it satisfies the frame inequality

$$A\,\|f\|_2^2 \leq \sum_{\lambda\in\Lambda} |\langle f,\pi(\lambda)\varphi\rangle|^2 \leq B\,\|f\|_2^2, \tag{IV.2}$$

with frame bounds $0 < A \leq B < \infty$ independent of $f \in L^2(\mathbb{R})$. From the general frame theory, the inequality implies a stable reconstruction formula

$$f = \sum_{\lambda\in\Lambda} \langle f,\pi(\lambda)\varphi\rangle\,\psi_\lambda, \quad f \in L^2(\mathbb{R}) \tag{IV.3}$$

with $(\langle f,\pi(\lambda)\varphi\rangle)_{\lambda\in\Lambda}$, and $\psi_\lambda \in L^2(\mathbb{R})$ for all $\lambda \in \Lambda$. Therefore, given φ, we are interested in determining those discrete sets $\Lambda \subseteq \mathbb{R}^2$ whose Gabor system $\mathcal{G}(\varphi,\Lambda)$ is a frame. The popular methods for lattices $\Lambda = A\mathbb{Z}^2$, $A \in \mathrm{GL}(2,\mathbb{R})$, concern duality concepts. For an overview of the current stand, we refer to the survey [5]. Time-frequency analysis is well-covered in the textbooks [2], [4], [7].

While there is no manageable tool available for arbitrary point configurations, a stepping stone in that direction are semi-regular sets, e.g., of type $\Lambda = X \times \mathbb{Z}$. However, even if X is a lattice up to a few additional or a few missing points, one has to tackle the problem in a completely different way. For their work on sampling in shift-invariant spaces generated by totally positive functions of Gaussian type, a new connection between sampling in shift-invariant spaces and Gabor frames was established in [8, Thm. 3.1, Thm. 3.3].

Theorem IV.1. *Assume that* $\varphi \in C(\mathbb{R})$ *satisfies* $\sum_{\ell\in\mathbb{Z}} \left(\sup_{x\in[0,1]} |\varphi(x+\ell)| \right) < \infty$ *and has stable integer shifts. Let* $X \subseteq \mathbb{R}$ *be a separated set. Then the following are equivalent:*

(i) The family $\mathcal{G}(\varphi, (-X) \times \mathbb{Z})$ *is a frame for* $L^2(\mathbb{R})$.
(ii) X *is a sampling set of the space* $V^p(\varphi)$ *for some* $p \in [1, \infty]$.
(iii) X *is a sampling set of the space* $V^p(\varphi)$ *for all* $p \in [1, \infty]$.

Applying this to EB-splines, we obtain the following corollary.

Corollary IV.2. *Let* φ *be an EB-spline of order* $m \geq 2$. *Assume* $X \subseteq \mathbb{R}$ *is a discrete set satisfying*

(i) X *is* $\delta-$*separated for some* $\delta > 0$.
(ii) There exist integers $M, L > 0$ *and* $\varepsilon > 0$, *such that for every* $k \in \mathbb{Z}$, *there exist points* $x_1^k < x_2^k < \cdots < x_{L+m-1}^k$ *in* $X \cap I_{M,L}(k)$ *with*

$$x_j^k \in [M + kL + j - m + \varepsilon, M + kL + j - \varepsilon]. \tag{IV.4}$$

Then $\mathcal{G}(\varphi, (-X)\times\mathbb{Z})$ *is a Gabor frame. In particular,* $\mathcal{G}(\varphi, X\times\mathbb{Z})$ *is a Gabor frame if* X *is separated and* $\mathfrak{mg}(X) < 1$.

Proof. We proved that φ has stable integer shifts in the proof of Theorem III.1. We require $m \geq 2$ because the PEB-spline of order 1 is not continuous. For all higher orders, φ is continuous and compactly supported, so the previous theorem applies. The sampling property of X follows from Theorem III.1, applied to $(X, 0)$, and Theorem III.2. The claim follows from Theorem IV.1. □

A direct consequence of the last corollary is the lattice case.

Corollary IV.3. *Let* φ *be an EB-spline of order* $m \geq 2$. *Then* $\mathcal{G}(\varphi, a\mathbb{Z} \times \mathbb{Z})$ *is a Gabor frame if and only if* $0 < a < 1$.

Proof. The sufficient part is due to Corollary IV.2. The necessary part is due to the Balian-Low theorem. □

Acknowledgements

K. G. was supported in part by the Austrian Science Fund (FWF) project P31887-N32. I.S. was supported by the Austrian Science Fund (FWF) project P33217 and by the Austrian Academy of Sciences (ÖAW) through an internship and a visiting research stay at the Acoustics Research Institute (ARI).

References

[1] Aldroubi, A., and Gröchenig, K. Nonuniform Sampling and Reconstruction in Shift-Invariant Spaces. *SIAM Rev. 43*, 4 (2001), 585–620. doi.org/10.1137/s0036144501386986.

[2] Christensen, O. *An Introduction to Frames and Riesz Bases*, 2 ed. Springer International Publishing, 2016. doi.org/10.1007/978-3-319-25613-9.

[3] Christensen, O., and Massopust, P. Exponential B-splines and the partition of unity property. *Adv. Comput. Math. 37*, 3 (2011), 301–318. doi.org/10.1007/s10444-011-9208-1.

[4] Folland, G. B. *Harmonic Analysis in Phase Space. (AM-122)*. Princeton University Press, 1989. doi.org/10.1515/9781400882427.

[5] Gröchenig, K., and Koppensteiner, S. Gabor frames: characterizations and coarse structure. In *New trends in applied harmonic analysis. Vol. 2—harmonic analysis, geometric measure theory, and applications*, Appl. Numer. Harmon. Anal. Birkhäuser/Springer, Cham, [2019] ©2019, pp. 93–120. doi.org/10.1007/978-3-030-32353-0_4.

[6] Gröchenig, K., and Shafkulovska, I. Sampling with derivatives in shift-invariant spaces generated by periodic exponential B-splines. *In preparation* (2023).

[7] Gröchenig, K. *Foundations of Time-Frequency Analysis*. Birkhäuser Boston, 2001. doi.org/10.1007/978-1-4612-0003-1.

[8] Gröchenig, K., Romero, J. L., and Stöckler, J. Sampling theorems for shift-invariant spaces, Gabor frames, and totally positive functions. *Invent. Math. 211*, 3 (2017), 1119–1148. doi.org/10.1007/s00222-017-0760-2.

[9] Gröchenig, K., Romero, J. L., and Stöckler, J. Sharp Results on Sampling with Derivatives in Shift-Invariant Spaces and Multi-Window Gabor Frames. *Constr. Approx. 51*, 1 (2019), 1–25. doi.org/10.1007/s00365-019-09456-3.

[10] Jia, R.-Q., and Micchelli, C. A. Using the Refinement Equations for the Construction of Pre-Wavelets II: Powers of Two. In *Curves and Surfaces*. Elsevier, 1991, pp. 209–246. doi.org/10.1016/b978-0-12-438660-0.50036-4.

[11] Razafinjatovo, H. N. Iterative reconstructions in irregular sampling with derivatives. *J. Fourier Anal. Appl. 1*, 3 (1995), 281–295. doi.org/10.1007/s00041-001-4013-8.

[12] Ron, A. *On Exponential Box Splines and Other Types of Non-Polynomial B-Splines*. PhD thesis, Tel Aviv University, 1987.

[13] Ron, A. Introduction to shift-invariant spaces. Linear independence. In *Multivariate Approximation and Applications*. Cambridge University Press, 2001, pp. 112–151. doi.org/10.1017/cbo9780511569616.006.

[14] Schoenberg, I. J. *Cardinal Spline Interpolation*. Society for Industrial and Applied Mathematics, 1973. doi.org/0.1137/1.9781611970555.

[15] Schumaker, L. *Spline Functions: Basic Theory*, third ed. Cambridge University Press, 2007. doi.org/10.1017/cbo9780511618994.

[16] SELVAN, A. A., AND GHOSH, R. Sampling with derivatives in periodic shift-invariant spaces. *Numer. Funct. Anal. Optim. 43*, 13 (2022), 1591–1615. doi.org/10.1080/01630563.2022.2114087.