A/B Testing in Network Data with Covariate-Adaptive Randomization

Jialu Wang
Department of Statistics
George Washington Univ.
Washington, DC 20052, USA
jialu@gwu.edu

Ping Li
LinkedIn Ads
Linkedin Corporation
Bellevue, WA 98004, USA
pinli@linkedin.com

Feifang Hu
Department of Statistics
George Washington Univ.
Washington, DC 20052, USA
feifang@gwu.edu

Abstract

Users linked together through a network often tend to have similar behaviors. This phenomenon is usually known as network interaction. Users' characteristics, the covariates, are often correlated with their outcomes. Therefore, one should incorporate both the covariates and the network information in a carefully designed randomization to improve the estimation of the average treatment effect (ATE) in network A/B testing. In this paper, we propose a new adaptive procedure to balance both the network and the covariates. We show that the imbalance measures with respect to the covariates and the network are $O_p(1)$. We also demonstrate the relationships between the improved balances and the increased efficiency in terms of the mean square error (MSE). Numerical studies demonstrate the advanced performance of the proposed procedure regarding the greater comparability of the treatment groups and the reduction of MSE for estimating the ATE.

1. Introduction

The classical A/B testing is developed based on the *Stable Unit Treatment Value Assumption* (SUTVA)(Rubin, 1978), which claims that each user's response depends only on his or her own treatment, regardless of other users' treatments or responses. However, the SUTVA is often violated in network A/B testing because users can interact with each other, e.g., their neighbors or friends, and thus may have similar behaviors. This phenomenon, widely existing in social media platforms, online marketplaces, and economic studies, may complicate the estimation of the average treatment effect (ATE).

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This aforementioned network interaction is demonstrated by the network effect and the spill-over effect. The network effect, also known as social interference, happens when users communicate treatment information with their neighbors through a social network. It is considered as a nuisance parameter since we focus on the classical average treatment effect (ATE), which is denoted by μ (Basse and Airoldi, 2018; Zhou et al., 2020; Gui et al., 2015; Zhang and Kang, 2022). The spill-over effect is also undesirable in the ATE estimation. It describes the scenario that a user is assigned to the control group but has neighbors assigned to the treatment group. In such case, the treatment on one user may impact the neighbors and impair comparisons in the design of experiments (Jiang et al., 2016; Ugander et al., 2013).

Homophily is also used to explain network interaction. It refers to the tendency that users with similar characteristics, the covariates, are more likely to connect with each other, resulting in correlated behaviors among them. McPherson et al. (2001) explained that the network ties are structured according to the users' covariates, and the users' interacting behaviors are related to their similar covariates. Therefore, a carefully-designed randomization procedure, incorporating the information of the covariates and the observed networks, is critical for valid estimation of ATE (Manski, 1993).

1.1. Our Contributions

The imbalance of covariates and the network interaction may affect the evaluation of the ATE in network A/B testing. To address this concern, we propose an adaptive randomization algorithm to achieve the balance with respect to the covariates and the network effect simultaneously. It directly reduces the MSE through a sequential experiment without complicated post-design re-weighting. Several designs, such as complete randomization (CR), covariate-adaptive randomization procedures (CAR) (Hu and Hu, 2012), and the adaptive randomization for network A/B test (NAR) (Zhou et al., 2020), are special cases of our proposed procedure.

Moreover, we use martingale to generalize the drift conditions in Markov chains (Meyn and Tweedie, 1993) to moment conditions for sequences with negative drift (Pemantle and Rosenthal, 1999). This overcomes the challenge of the complex correlation between connected users. The results provide a theoretical guarantee by showing the two imbalance measures under our proposed procedure are $O_p(1)$ and the consistency of the linear-in-mean estimator.

In addition, via deriving the MSE of the difference-in-means estimator based on the response model (3), we develop a new network imbalance measure to further improve the efficiency for estimating the ATE. Extensive numerical studies are also presented to demonstrate the advances of our proposed procedure via the significant reductions in MSE.

1.2. Related Work

In clinical trials or economic experiments, researchers are more interested in the ATE and exclude the network interference, e.g., true drug efficiency or how well an economic strategy is. In contrast, the network effect may be part of the estimand (Jiang et al., 2016; Nandy et al., 2020; Eckles et al., 2016; Toulis and Kao, 2013). For example, in online advertising and two-sided markets, companies are more interested in the total increase in revenue rather than the proportion purely contributed by the advertisements (Nandy et al., 2020).

Furthermore, linear-in-mean models are commonly used to illustrate the relationship between users' potential outcomes and the network and/or their covariates profiles. For example, Nandy et al. (2020) modeled advertisement on social media or marketplace with connected users; Yoon (2018) discussed A/B testing in a collaboration network; Basse and Airoldi (2018) and Zhou et al. (2020) studied the application for new drug discovery; Shalizi and Thomas (2011) and Zhang and Kang (2022) provided models for general cases, etc.

1.3. Organization

The background and notation of CAR and its application in network data are summarized in Section 2. Our proposed model and randomization procedure are presented in Section 3. The theoretical results are shown in Section 4. We present the numerical studies to demonstrate the balance properties of our proposed procedure and the improvement for estimating the ATE in Section 5. Concluding remarks and future research topics are discussed in Section 6. The proofs can be found in the Appendix.

2. Covariate-Adaptive Randomization and its Application in Network Data

In clinical trials, patients usually come and get treated sequentially. Since we do not know all patients' covariates when assigning a particular patient, it's hard for stratified permuted block randomization (Zelen, 1974) to generate adequate balance over many important baseline discrete covariates. To address this issue, minimization was introduced to balance over a large number of covariates (Taves, 1974; Pocock and Simon, 1975). It sequentially assigns users to treatment or control groups based on the existing covariates information. Hu and Hu (2012) proposed a general class of covariate-adaptive randomization (CAR) to ensure the balance for multiple covariate levels, where the biased coin design (Efron, 1971) and minimization are special cases.

To introduce this class of procedures, we start with discrete covariates. Suppose the l-th covariate has m_l levels, $1 \leq l \leq L$, with a total of $m = \prod_{l=1}^{I} m_l$ strata. Let $(l; k_l)$ denote the margin formed by the users with the l-th covariates at level k_l. Let $\mathbf{k} = (k_1, \cdots, k_L)$ denote the stratum formed by users with the l-th covariates at level k_l for $1 \leq l \leq L$. Here we introduce three imbalance measures: (i) the overall imbalance D_n, which is the difference between the patients assigned to the treatment and the control groups; (ii) the marginal imbalance $D_n(l; k_l)$, which represents the difference between the two arms for a specific variable $(l; k_l)$, e.g., the difference of females between the two arms; and (iii) the stratum imbalance $D_n(\mathbf{k})$, which is the difference for a specific stratum $\mathbf{k}$, e.g., the difference of assigned smoked females between the two arms. CAR uses a weighted imbalance measure combining the 3 measures above with nonnegative weights $w_o, w_{m,l}$ and w_s with $w_o + w_s + \sum_{l=1}^{L} w_{m,l} = 1$. The choice of w_o, w_m, w_s can be based on the relative importance of covariates at the three levels in specific scenarios. Ignoring them will cause conservative type I error and reduced statistical power (Ma et al., 2015).

If the n-th user is in stratum $\mathbf{k}^* = (k_1^*, \ldots, k_L^*)$, the imbalance measure calculated for the first n users based on the n-th user's stratum $\mathbf{k}^*$ is

$$\begin{aligned} Imb_{n,cov} = w_o \left[D_n\right]^2 + \sum_{l=1}^{L} w_{m,l} \left[D_n\left(l; k_l^*\right)\right]^2 \\ + w_s \left[D_n\left(\mathbf{k}^*\right)\right]^2 . \end{aligned} \quad (1)$$

CAR assigns the users to treatment or control groups using a "biased coin" with probability $g(Imb_{n,cov})$ to sequentially minimize $Imb_{n,cov}$. In general, CAR only requires the allocation rule $g(x)$ decreasing and satisfies $g(x)+g(-x) = 1$ for symmetric.

Hu and Zhang (2020) proved that

$$\Lambda_n(\mathbf{k}) = w_o D_n + \sum_{l=1}^{L} w_{m,l} D_n(l; k_l) + w_s D_n(\mathbf{k}) = O_p(1)$$

for all $\mathbf{k}$. Consequently, D_n, $D_n(l;k_l)$, and $D_n(\mathbf{k})$ are $O_p(1)$ for $1 \le l \le L$ and $1 \le k_l \le m_l$. These results are crucial for improving the comparability and validity of CAR and also our design.

When a continuous covariate needs direct balance, one may replace the right-hand side of (1) with the Mahalanobis distance (Qin et al., 2022) or other imbalance measures for continuous covariates.

Unlike most of the clinical studies, the outcomes are often correlated through the given network in network A/B testing. It is thus desirable to improve the efficiency for estimating the ATE by using the information of the network through the design of the experiment (Basse and Airoldi, 2018; Zhou et al., 2020; Gui et al., 2015; Zhang and Kang, 2022). Zhou et al. (2020) introduced a framework to demonstrate the impact of the imbalance with respect to the network on the estimation of ATE. Based on a similar idea as the CAR procedure, they proposed an adaptive network randomization (NAR) that sequentially assigned users by minimizing the following imbalance measure

$$Imb_{n,net} = n^{-2} \left\| A_n \left(\mathbf{1}_n - 2\mathbf{T}_n \right) \right\|^2, \tag{2}$$

where A_n is the adjacent matrix of the first n users, $\mathbf{T}_n$ is the vector of their treatment assignments and $||x|| = \sqrt{x^\top x}$ denotes the L^2-norm throughout this paper. $Imb_{n,net}$ measures the difference of the numbers of neighbors in the two treatment groups for the first n users. They show that $Imb_{n,net}$ following their proposed design converges to a constant smaller than the value under complete randomization (CR), and the mean squared error (MSE) under NAR is smaller than the ones under CR. Consequently, NAR can improve the ATE estimation with improved comparability.

In summary, since users' potential outcomes are correlated with their treatment allocations, relevant covariates, and the connections to neighbors, users' covariates and the network information can be utilized to advance the design performance in network A/B testing. In the following section, we introduce a sequential randomization procedure to minimize $Imb_{n,cov}$ and $Imb_{n,net}$ simultaneously. The proposed procedure shares the advantages of both CAR and NAR, and thus can tremendously improve the efficiency for ATE estimation.

3. New Randomization Algorithm

3.1. Model Setting

We first propose a generic model to illustrate the network and covariate's effect. Suppose a network $\mathcal{G}$ with N vertices (users) is observed. Let $\mathbf{X}$ be the $N \times d$ covariates matrix and $\mathbf{X}_i$ be the d-covariates for user i. Let $\mathbf{T} \in \{0,1\}^N$ be the N-treatment assignment and T_i denote the assignment of the i-th user, i.e., the ith user is assigned to the treatment if $T_i = 1$, the ith user is assigned to the control if $T_i = 0$. The connections among nodes are represented by an undirected symmetric adjacency matrix $A_N = (A_{ij})_{1 \le i,j \le N}$, $A_{ij} \in \{0,1\}, A_{ii} = 0$. Let A_{i*} be the i-th row of adjacent matrix A which represents the connections of i-th user among the N users. Here we assume that only users with direct connections can affect each other via the network, which is typically referred to the neighborhood interference assumption in Ugander et al. (2013) and Eckles et al. (2016).

Then the outcome of the i-th subject is affected by the network effect, the spill-over effect, and the covariates as:

$$Y_i = \mu_0 + \mu_1 T_i + f(A_{i*}\mathbf{T}) + \mathbf{X}_i \boldsymbol{\beta} + \varepsilon_i, \tag{3}$$

where μ_0, μ_1, and $\boldsymbol{\beta}$ represent the effect of the baseline, the treatment effect, and the covariate effects, respectively; $f()$ is a prespecified increasing function, e.g., $f(A_{i*}\mathbf{T}) = \sqrt{A_{i*}\mathbf{T}}$, and $\varepsilon_i \sim \mathcal{N}\left(0, \sigma_\varepsilon^2\right)$ is the i.i.d. random error.

The i-th outcome Y_i depends on the treatment assignment of the i-th user T_i, the network interactions, and the corresponding covariate profile $\mathbf{X}_i$. When $f(A_{i*}\mathbf{T}) = \gamma A_{i*}\mathbf{T}$, the response model (3) assumes that Y_i is linearly affected by the number of user i's friends assigned in the treatment group by $\gamma A_{i*}\mathbf{T}$. This kind of linear outcome model is commonly studied in the literature. For instance, Basse and Airoldi (2018) considered $Y_i = \mu_0 + \mu_1 T_i + \sum_{j \in \mathcal{N}_i} X_j + \varepsilon_i$, where $\mathcal{N}_i = \{j :$ s.t. $A_{ij} = 1$ or $A_{ji} = 1\}$ is the set of the direct neighborhoods of unit i , and X_j is the covariate value of unit j. Gui et al. (2015) explained the direct network interference and the homophily by approximating the average behavior of one's neighborhood through a linear additive model. Linear outcome models are also employed in Eckles et al. (2016) and Toulis and Kao (2013).

If other types of outcome are of interest, (3) can be extended to generalized linear models by replacing Y_i with $h(\mathbb{E}Y_i)$, where $h(\cdot)$ represents the link function. For instance, $h(\cdot) = logit(\cdot)$ is considered in Section 5.2 for demonstrating the extension of our results. In numerical studies, we also relax the direct neighborhood assumption by considering the transitivity of network effect, where Y_i is not just affected by the ith row of A_N as in (3).

For covariates part, we only consider categorical variable in this paper and use all notations in section 2. Therefore, without loss of generality, we let $X_{ij} \in \{0,1\}$ and each column of $\mathbf{X}$ represents a specific stratum or a particular level of a categorical variable. Besides, continuous covariates could also be processed via direct discretization.

3.2. Algorithm for Network A/B Testing

As all users are sequentially observed, we assume that only the sub-network of first n users is revealed for treating the n-th user. Consider the following weighted imbalance measure

$$Imb_{n,w} = wImb_{n,cov} + (1-w)Imb_{n,net}, \qquad (4)$$

which combines the covariate imbalance measure $Imb_{n,cov}$ and the network imbalance measure $Imb_{n,net}$ with weight $w \in (0,1)$. The adaptive randomization in Algorithm 1 sequentially determines the values of T_i to minimize $Imb_{n,w}$. It drags large $|Imb_{n,w}|$ back to zero when it gets too large so as to achieve balance via the allocation rule (probability) $g(x)$. For instance, Hu and Hu (2012) suggested

$$g(x) = (1-q)I(x>0) + 0.5I(x=0) + qI(x<0) \quad (5)$$

with $q \in [0.75, 0.95]$ as the allocation rule.

Note that when $w=1$, $Imb_{n,w} = Imb_{n,cov}$ and the proposed procedure is equivalent to the CAR procedure proposed by Hu and Hu (2012); when $w=0$, $Imb_{n,w} = Imb_{n,net}$ and our procedure is identical to the NAR proposed by Zhou et al. (2020); when $w \in (0,1)$ is considered, the proposed procedure takes both $Imb_{n,cov}$ and $Imb_{n,net}$ into account and can control these two imbalance measures simultaneously. The choice of w is based on the relative importance of covariates and networks in specific scenarios or prior experience.

Algorithm 1 Adaptive Randomization in Network Data

Assumptions: Network is sequentially observed.
Assign $T_1 = 1$ with probability 0.5;
for *n=2 to* N **do**
- Calculate D_{n-1}, $D_{n-1}(i;k_i^*)$, for $1 \le i \le I$, and $D_{n-1}(\mathbf{k}^*)$ based on n-th user's covariate profile $\boldsymbol{X}_n$ that falls in stratum $\mathbf{k}^*$;
- Assign $\mathbf{T}_n^{(1)} \leftarrow (\mathbf{T}_{n-1}^\top, 1)$, calculate $Imb_{n,cov}^{(1)}$, $Imb_{n,net}^{(1)}$, and $Imb_{n,w}^{(1)}$;
- Assign $\mathbf{T}_n^{(2)} \leftarrow (\mathbf{T}_{n-1}^\top, 0)$, calculate $Imb_{n,cov}^{(2)}$, $Imb_{n,net}^{(2)}$, and $Imb_{n,w}^{(2)}$;
- Calculate $\Delta Imb_{n,w} = Imb_{n,w}^{(1)} - Imb_{n,w}^{(2)}$;
- Let $\mathbf{T}_n^\top = z(\mathbf{T}_{n-1}^\top, 1) + (1-z)(\mathbf{T}_{n-1}^\top, 0)$, where $z \sim Bernoulli(g(\Delta Imb_{n,w}))$

Obtain: assignment vector $\mathbf{T}$.

4. Theoretical Results

We study the asymptotic properties of our newly proposed design. We denote $D_n(\mathbf{k})$ as the true difference between the two treatment groups within stratum $\mathbf{k} = (k_1, ..., k_L)$, and denote $\Lambda_n(\mathbf{k})$ as the true weighted average of the imbalances within stratum $\mathbf{k}$ by the same way as in CAR. That is

$$\Lambda_n(\mathbf{k}) = w_o D_n + \sum_{l=1}^{L} w_{m,l} D_n(l;k_l) + w_s D_n(\mathbf{k}). \quad (6)$$

It follows from Hu and Hu (2012) that $\Lambda_n(\mathbf{k}) = O_p(1)$ under CAR procedure, which only considers covariates. However, our newly proposed design procedure also includes network influence. Our goal is to show $\Lambda_n(\mathbf{k}) = O_p(1)$ still holds for all selected $w \in (0,1)$.

4.1. The Main Theorems

Motivated by queueing networks, Pemantle and Rosenthal (1999) showed that if a sequence of random variables has negative drift when above a certain threshold and has increments bounded in L^p, then a uniform L^r bound on X_n^+ for any $r < p-1$ can be achieved. We show their Theorem 1 as an important Lemma here.

Lemma 4.1. *(Pemantle and Rosenthal, 1999) Let X_n be random variables and suppose that there exist constants $a > 0$ $J, V < \infty$, and $p > 2$, such that $X_0 \le J$, and for all n*

$$\mathbb{E}(X_n - X_{n-1} \mid \mathscr{F}_{n-1}) \le -a \text{ when } \{X_{n-1} > J\} \quad (C1)$$

and

$$\mathbb{E}(|X_n - X_{n-1}|^p \mid \mathscr{F}_{n-1}) \le V \quad (C2)$$

Then for any $r \in (0, p-1)$ there is a constant $c = c(p,a,V,J,r) > 0$ such that $\mathbb{E}(X_n^+)^r < c$ for all n.

In Appendix A.1 we show that $\Lambda_n(\mathbf{k})$ satisfies the two conditions when $w_s = 1$. Then Lemma 4.1 yields the following theorem.

Theorem 4.2. *Suppose the weights used by Algorithm 1 satisfy $w \in (0,1)$, $w_s = 1$ and $w_{m,l} = w_o = 0$ for $1 \le l \le L$ and the allocation probability function $g(x)$ is defined as (5). Then, for any $r > 0$ and any stratum $\mathbf{k} = (k_1, \ldots, k_l)$, $\mathbb{E}|\Lambda_n(\mathbf{k})|^r = \mathbb{E}|D_n(\mathbf{k})|^r = O(1)$.*

As $w_s = 1$ and $w_{m,l} = w_o = 0$, Theorem 4.2 describes the property of the procedure where only the within-stratum imbalance is considered. If we generalize Theorem 4.2 to include the overall and marginal imbalances, the condition $(C2)$ no longer holds. To prove the general case, we first propose a modified version of Lemma 4.1 as Lemma 4.3, which is the key result to prove the general case. The proof of Lemma 4.3 is shown in Appendix A.2.

Lemma 4.3. *The conclusion of lemma 4.1 still holds when $X_n - X_{n-1}$ is replaced by $(X_n - X_{n-1})' = max(X_n - X_{n-1}, C)$ in conditions $(C1)$ and $(C2)$ where C can be any negative constant such that $C \le -1$.*

To use Lemma 4.3 we further define

$$V_n = \sum_k w_s D_n^2(k) + \sum_{l=1}^{L}\sum_{k_l=1}^{m_l} w_{m,l} D_n^2\left(l; k_l\right) + w_o D_n^2$$

which leads to

$$V_n - V_{n-1} = 4\Lambda_{n-1}(\mathbf{k})\left(T_n - \frac{1}{2}\right) + 1.$$

Next, we prove that $max(V_n - V_{n-1}, -1)$ satisfies the conditions $(C1)$ and $(C2)$, and hence there exist a constant $c = (p, V, J, r) > 0$ such that $E(V_n)^r < c$.

Finally, refer to equation (6.5) in Hu and Zhang (2020) we have $\|\mathbf{\Lambda}_n\|^2 \leq mV_n$. It follows that $\sup_n \mathbb{E}\|\mathbf{\Lambda}_n\|^{2r} < \infty$, which is the conclusion of our main theorem as shown below. The whole proof of Theorem 4.4 is presented in Appendix A.3.

Theorem 4.4. *Suppose the weights used by Algorithm 1 satisfy $w_s, w_{m,l} > 0$ and $w_o + w_s + \sum_{l=1}^{L} w_{m,l} = 1$. In addition, suppose for some $p > 2$, there exists $M > 0$ such that the allocation function satisfies*

$$|x^p g(x)| \leq M \qquad \text{for all } x > 0. \tag{7}$$

Then $\mathbb{E}\left|\Lambda_n\left(\mathbf{k}\right)\right|^r = O(1)$ for any $0 < r < p - 1$ and any stratum $\mathbf{k} = \left(k_1, \ldots, k_I\right)$.

To facilitate our analysis, we add one more constraint on the allocation rule $g(x)$, which indicates that the allocation probability $g(x)$ must converge to 0 as x goes to positive infinity.

A direct and useful result from Theorem 4.4 is the following.

Corollary 4.5. *Condition Algorithm 1 satisfying the conditions in Theorem 4.4, then $D_n = O_p(1)$, $D_n\left(l; k_l\right) = O_p(1)$ for all covariates l and their level k_l, and $D_n\left(\mathbf{k}\right) = O_p(1)$ for all stratum $\mathbf{k}$.*

Note that $(2\mathbf{T} - \mathbf{1})^\top \mathbf{X} = \mathbf{T}^\top \mathbf{X} - (\mathbf{1} - \mathbf{T})^\top \mathbf{X}$ is a vector of differences between the treatment and control groups regarding any stratums or marginal of covariates, Corollary 4.5 shows that $(2\mathbf{T} - \mathbf{1})^\top \mathbf{X} = O_p(1)$.

4.2. Asymptotic Properties of the Difference-in-Mean Estimator

While the balance ensures the comparability of the treatment groups, it is also crucial to understand how such balance affects the subsequent estimation for the ATE. Consider the difference-in-means estimator,

$$W = \frac{\sum_{i=1}^N T_i Y_i}{\sum_{i=1}^N T_i} - \frac{\sum_{i=1}^N (1 - T_i) Y_i}{\sum_{i=1}^N (1 - T_i)}. \tag{8}$$

The outcome model (3) implies that the conditional mean of W can be derived as

$$\begin{aligned}\mathbb{E}(W|\mathbf{T}) =& \left(\frac{\mathbf{T}^\top}{(N + D_N)/2} - \frac{(\mathbf{1} - \mathbf{T})^\top}{(N - D_N)/2}\right) \\ &\left(\mu_0 \mathbf{1} + \mu_1 \mathbf{T} + f(A\mathbf{T}) + X\beta\right) \\ =& \mu_1 + 2\left(\frac{\mathbf{T}^\top}{N + D_N} - \frac{(\mathbf{1} - \mathbf{T})^\top}{N - D_N}\right)(f(A\mathbf{T}) + X\beta).\end{aligned}$$

As $D_n = O_p(1)$, $(N + D_n)/N$ converges to 1 in probability. Hence $\mathbb{E}(W|\mathbf{T})$ reduces to

$$\mathbb{E}(W|\mathbf{T}) = \mu_1 + \frac{2}{N}\left(2\mathbf{T} - \mathbf{1}\right)^\top\left(f(A\mathbf{T}) + \mathbf{X}\boldsymbol{\beta}\right) \tag{9}$$

as N goes to ∞.

We then introduce the following result about the MSE of W. The proof can be found in Appendix A.4.

Corollary 4.6. *Suppose Y_i follows the model (3) and f is a concave function. Using Algorithm 1 satisfying the conditions in Theorem 4.4,*

$$MSE(W) = \frac{4}{N^2}\mathbb{E}_{\mathbf{T}}\left[\{(2\mathbf{T} - \mathbf{1})^\top f(A\mathbf{T})\}^2\right] + O\left(\frac{f(N)}{N}\right).$$

Note that the choice of function f plays a key role here. If $f(A\mathbf{T})$ grows faster than $A\mathbf{T}$, W may not be a consistent estimator. In practice, however, the network effect unlikely goes to infinity as n increases as people only have finite number of friends.

To improve the efficiency for the estimation, we want to sequentially minimize $\{(2\mathbf{T} - \mathbf{1})^\top f(A\mathbf{T})\}^2$ in each assignment. It is thus desirable to control $\{(2\mathbf{T} - \mathbf{1})^\top f(A\mathbf{T})\}^2$ by using

$$Imb_{n,net}^* = \frac{1}{n^2}\left\|A_n\left(\mathbf{1}_n - 2\mathbf{T}_n\right) \cdot \mathbf{T}_n\right\|^2 \tag{10}$$

in (4), where the $\cdot$ represents the element-wise vector multiplication. This measure essentially demonstrates the imbalance of the neighbors of the users who are assigned to the treatment group. Therefore, if $Imb_{n,net}$ is further replaced with $Imb_{n,net}^*$, Algorithm 1 may further benefit the estimation when the difference-in-means estimator is used. Notice that $\left\|A_n\left(\mathbf{1}_n - 2\mathbf{T}_n\right) \cdot \mathbf{T}_n\right\|^2$ is always bounded by$\left\|A_n\left(\mathbf{1}_n - 2\mathbf{T}_n\right)\right\|^2$, so it is easy to show that $|Imb_{n,net}^{*(1)} - Imb_{n,net}^{*(2)}| \leq 4$. Consequently, Algorithm 1 using the Imb_n with $Imb_{n,net}$ replaced by $Imb_{n,net}^*$ will still satisfy the condition $C1$ and $C2$. We thus have the following corollary.

Corollary 4.7. *Suppose $Imb_{n,net}^*$ is used in (4) for Algorithm 1, the results in Theorem 4.2 and Theorem 4.4 still hold.*

5. Numerical Studies

In this section, we perform numerical studies to demonstrate the finite sample properties of our proposed adaptive randomization procedure via a hypothetical network as well as a real-world network presented in Cai et al. (2015).

In practice, for imbalance measure $Imb_{n,cov}$ and $Imb_{n,net}$ defined in (1) and (2), we usually observe

$$Imb^{(1)}_{n,cov} - Imb^{(2)}_{n,cov} = O_p(1)$$

and

$$Imb^{(1)}_{n,net} - Imb^{(2)}_{n,net} = O_p(n^{-2}).$$

However, the assignment probability $g(x)$ is defined as:

$$\begin{aligned} g(x) &= g(Imb^{(1)}_{n,w} - Imb^{(2)}_{n,w}) \\ &= g[w(Imb^{(1)}_{n,cov} - Imb^{(2)}_{n,cov}) \\ &+ (1-w)(Imb^{(1)}_{n,net} - Imb^{(2)}_{n,net})], \end{aligned}$$

so the two differences should be in the same level. Hence, for finite sample size N, we divide the original covariance imbalance measure by N^2 and get

$$\begin{aligned} Imb_{n,cov} = \frac{1}{N^2}(w_o\,[D_n]^2 + \sum_{l=1}^{L} w_{m,l}\,[D_n\,(l;k_l^*)]^2 \\ + w_s\,[D_n\,(\mathbf{k}^*)]^2). \end{aligned} \tag{11}$$

Note that $\frac{1}{N^2}$ is a fixed constant, so Theorem 4.2 and Theorem 4.4 still hold here.

For the assignment function $g(x)$, we set

$$g(x) = \begin{cases} 1-10|x|^{-2.1} & when \quad x \le -10 \\ 0.9 \quad when & -10 < x < 0 \\ 0.5 \quad when & x = 0 \\ 0.1 \quad when & 0 < x < 10 \\ 10|x|^{-2.1} & when \quad 10 \le x. \end{cases} \tag{12}$$

Hence, (7) is satisfied with $M = 10$ and $g(x) + g(-x) = 1$ holds.

5.1. Hypothetical Networks

We first evaluate the performance of different randomization schemes with hypothetical networks. The Erdös-Rényi random graph (CRG) with fixed between probability p_d and the clustering graph (CUG) are used to demonstrate the performance under different types of the network structures. The details for generating the networks are presented in the Appendix. We assume the outcome Y_i follows (3). Three different randomization schemes are evaluated and compared as follows: (1) CR, which is a baseline with T_i is i.i.d. $Bernoulli(1/2)$ and $T_i \perp \{\mathbf{X}, A\}$; (2) Algorithm 1 with the network imbalance measure $Imb_{n,net}$ (AL), which is equivalent to CAR in Hu and Hu (2012) when $w = 1$ and to NAR in Zhou et al. (2020) when $w = 0$; (3) Algorithm 1 with network imbalance measure $Imb^*_{n,net}$ (AL*), where we define NAR* when AL* has $w = 0$.

Note that since both network and covariates may affect user response, the performance of our algorithm depends on a balance between them. Specifically, $w = 1$ (CAR) suffices when the network has no impact, while $w = 0$ (NAR) is suitable when covariates have no effect. When both network and covariates affect user behavior, $w \in (0, 1)$ may yield superior results.

The performance of the randomization schemes are compared in the following two aspects. To compare the balance properties of different randomization schemes, the standard deviation of the three different levels covariate imbalances, D_n, $D_n(k; k_l)$ and $D_n(\mathbf{k})$, the mean absolute values of the two network imbalance measures $Imb_{n,net}$ and $Imb^*_{n,net}$ are evaluated. In addition, the advantages of different randomization schemes for estimating the ATE are demonstrated by presenting the bias, standard deviation (sd) and the MSE of the difference-in-means estimator W. The allocation probability $g(x)$ is defined as (12). All simulation studies are performed with 1000 runs.

We consider linear cases first,

$$Y_i = \mu_0 + \mu_1 T_i + \gamma A_{i*}\mathbf{T} + \mathbf{X}_i\boldsymbol{\beta} + \varepsilon_i$$

where $\mu_0 = 1$, $\mu_1 = 0$, $\gamma = 1$, $\beta_i = 1$ for $\forall i$, $\sigma_\varepsilon = 1$ $p_{density} = 0.05$.

Case 1: 2^2 strata. Consider an experiment with sample size $N = 200$, and 2 covariates each with 2 levels resulting in a total of 4 strata. Suppose the joint of the two covariates are i.i.d. $Multinomial$ $(p(1,1), p(1,2), p(2,1), p(2,2))$=$(0.1, 0.2, 0.3, 0.4)$. The weights used for $Imb_{n,cov}$ in CAR, and AL* are $(w_o, w_{m,1}, w_{m,2}, w_s) = (0.3, 0.1, 0.1, 0.5)$ and different w are used for AL and AL*.

Table 1 describes the advantages of the proposed procedure for balancing different types of imbalance measures. For simplicity of presentation, we only list the standard deviations of the overall imbalance D_n, the marginal imbalance $D_n(2; 2)$ and the within-stratum imbalance $D_n(2, 2)$. Also for AL and AL*, we only list the case $w = 0.7$.

First, CAR has the best performance in balancing the covariates, i.e., the smallest standard deviations for D_n, $D_n(2; 2)$ and $D_n(2, 2)$, but has larger mean absolute values of $Imb^*_{n,net}$. As NAR* considers $Imb^*_{n,net}$, it generate the best balance for $Imb^*_{n,net}$. However, it generates inadequate performance for balancing the covariates. As Algorithm 1 enjoys the advantages of the aforementioned two procedures,

Table 1. Comparison of the Bias, Standard Deviation (sd), MSE of W, Mean Absolute Value of $Imb^{(*)}_{n,net}$, and D_n, $D_n(2;2)$, and $D_n(2,2)$ under 2×2 strata.

2^2		CR	NAR	NAR*	AL	AL*	CAR
CRG	bias	0.005	-0.030	1.283	-0.021	-0.002	-0.008
	sd	0.489	0.392	0.487	0.316	0.291	0.409
	mse	0.240	0.155	1.883	0.100	0.085	0.167
	$Imb^*_{n,net}$	0.050	0.026	0.008	0.031	0.012	0.048
	D_n	11.276	4.040	44.900	0.926	0.920	0.558
	$D(2;2)$	9.401	5.926	31.336	1.110	1.036	0.818
	$D_n(2,2)$	7.098	5.748	18.270	0.884	0.849	0.640
CUG	bias	0.044	-0.069	1.294	-0.051	0.056	0.050
	sd	0.484	0.404	0.501	0.318	0.284	0.408
	mse	0.236	0.168	1.925	0.104	0.084	0.169
	$Imb^*_{n,net}$	0.053	0.027	0.009	0.028	0.010	0.050
	D_n	11.682	3.946	43.714	1.634	2.046	0.550
	$D(2;2)$	9.834	5.902	30.589	1.897	1.687	0.768
	$D_n(2,2)$	7.351	5.636	17.781	1.627	1.351	0.628

our proposed procedure have a relative good performance on both the covariate imbalance and the network imbalance, especially AL*, which directly takes $Imb_{n,net}*$ into account. As such, AL* can best facilitate the estimation of the ATE, as W following AL* has the best performance in terms of the standard deviation, and the MSE. Similar conclusion can be drawn for the experiments conducted with CUG.

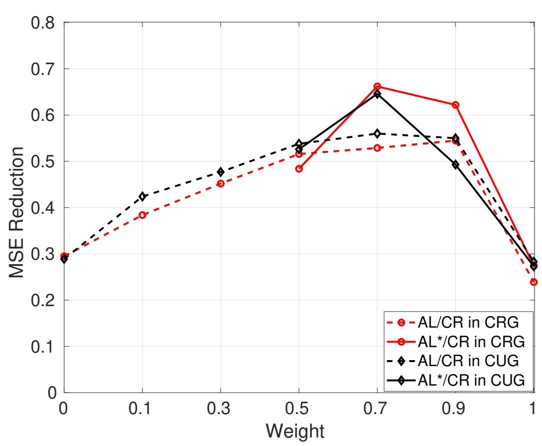

Figure 1. The reduction of MSE following AL and AL* compared with CR under hypothetical networks under the Case 1: 2^2 strata.

We also explore the MSE under different choices of w when both network and covariates effect exist. Figure 1 demonstrates a tremendous MSE reduction of W under all $w \in [0, 1]$, where the y-axis is one minus the ratio of our tested algorithms and the baseline CR in terms of MSE.

In the 2×2 case under CUG, the optimal weight is $w = 0.7$ for both AL* and AL. They clearly outperform CAR $(w = 1)$ and NAR $(w = 0)$ and AL* is better than AL here. Similar conclusions can be drawn for the experiments conducted with CRG.

Table 2. Comparison of the Bias, Standard Deviation (sd), MSE of W, Mean Absolute Value of $Imb^{(*)}_{n,net}$, and D_n, $D_n(2;2)$, and the imbalance of the stratum of 2 users under 2^{10} strata.

2^{10}		CR	NAR	NAR*	AL	AL*	CAR
CRG	bias	-0.004	-0.016	1.224	-0.037	0.005	-0.009
	sd	0.746	0.728	0.785	0.418	0.383	0.454
	mse	0.557	0.530	2.113	0.176	0.147	0.206
	$Imb^*_{n,net}$	0.223	0.162	0.089	0.170	0.104	0.217
	D_n	11.538	4.332	44.146	1.110	1.174	0.562
	$D(2;2)$	8.112	5.980	22.148	2.791	2.468	1.694
	within-strt 2pts	1.004	0.995	1.045	0.737	0.670	0.356
CUG	bias	0.017	-0.032	1.332	-0.032	-0.011	0.052
	sd	0.783	0.754	0.803	0.441	0.408	0.465
	mse	0.614	0.569	2.419	0.195	0.167	0.219
	$Imb^*_{n,net}$	0.229	0.164	0.092	0.171	0.105	0.224
	D_n	11.014	4.078	44.052	1.138	1.152	0.538
	$D(2;2)$	7.921	6.040	22.124	2.863	2.550	1.698
	within-strt 2pts	1.010	0.998	1.046	0.737	0.685	0.348

Case 2: 2^{10} strata. Consider an additional experiment with $N = 200$ and 10 covariates each with 2 levels resulting a total of 1024 strata. Consequently, the number of strata is large compared to the sample size. Suppose the 10 covariates are i.i.d. $Bernoulli(1/2)$. The weights $w_o = 0$, $w_s = 0.5$ and $w_{m,l} = 0.05$ for $l = 1, ..., 10$ with $Imb_{n,cov}$ are used for CAR, AL and AL*

From Table 2 and Figure 2, the performance of the six randomization schemes are similar to the previous scenario. Note that a larger w is needed for both AL and AL* to achieve their optimal MSEs as the larger number of covariates may increase the need for balancing the covariates. For example, the optimal w for AL and AL* are $w = 0.9$. AL* reduces 10.2% of the MSE more than AL. Thus, these results also indicate the advanced performance of our proposed procedure.

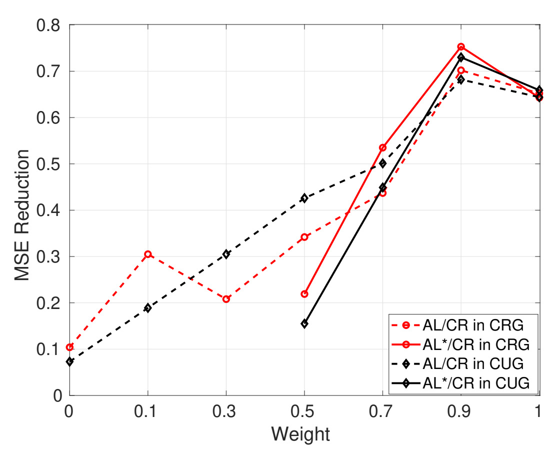

Figure 2. The reduction of the MSE following AL and AL* compared with CR under hypothetical networks under the Case 2: 2^{10} strata.

Table 3. Comparison of the Bias, Standard Deviation (sd), MSE of W, Mean Absolute Value of $Imb_{n,net}^{(*)}$, D_n, and $D(;)$ where $D(;) = E\,|D_n(l;k_l)|\,\forall l$, and the imbalance of the stratum of 2 users under the Case 1: $f(\cdot) = A_{i*}\mathbf{T}_i$.

		CR	NAR	NAR*	AL	AL*	CAR
YY	bias	0.029	-0.038	2.431	-0.035	-0.005	-0.077
	sd	0.214	0.154	1.428	0.124	0.130	0.124
	mse	0.047	0.025	7.950	0.016	0.017	0.021
	$Imb^*_{n,net}$	0.029	0.013	0.004	0.015	0.006	0.022
	D_n	11.948	5.478	65.136	0.766	0.768	0.536
	$D(;)$	7.534	5.318	27.994	1.435	1.366	1.136
	within-strt 2pts	49.1%	50.8%	44.4%	54.6%	54.6%	56.0%
DK	bias	0.011	-0.034	1.977	-0.040	-0.019	-0.053
	sd	0.196	0.160	1.011	0.126	0.127	0.141
	mse	0.039	0.027	4.929	0.018	0.017	0.023
	$Imb^*_{n,net}$	0.010	0.009	0.003	0.010	0.004	0.016
	D_n	12.864	6.268	84.268	1.022	0.960	0.586
	$D(;)$	8.175	6.169	36.172	1.606	1.440	1.063
	within-strt 2pts	50.0%	50.9%	46.9%	55.6%	53.4%	58.4%

5.2. Real Data

In this section, we redesign the experiment in Cai et al. (2015) to evaluate the performance of Algorithm 1. They studied the influence of the social network on insurance adoption by rice farmers in rural China. The original design simply randomized a subset of the farmers on multiple stages without considering the covariates and network interaction. Here we simplify it into complete randomization, treat it as a baseline, and compare it with the other five algorithms via numerical studies.

We select six discrete covariates related with the farmers: (i) dukou (DK), a village of clustering communities, and we select two addresses in it with a total population $n = 276$; (ii) yazhou and yongfeng (YY), two different villages with a total population $n = 226$, but users who live there have similarities within and between the village-friendship density around 0.05.

To generate the outcome, we reparametrize the μ and $\boldsymbol{\beta}$ based on the following generalized linear morel:

$$h(\mathbb{E}Y_i) = logit(\mathbb{E}Y_i) = \mu_0 + \mu_1 T_i + f(A_{i*}, \mathbf{T}_i).$$

We first consider the following two cases. **Case 1:** $f(A_{i*}, \mathbf{T}_i) = \gamma A_{i*}\mathbf{T}_i$, which assumes the users are linearly associated with the number of their friends assigned in the treatment group. **Case 2:** $f(A_{i*}, \mathbf{T}_i) = \gamma\sqrt{A_{i*}\mathbf{T}_i}$, an extended version of the linear-in-means model.

Table 3 shows the superiority of our proposed procedure for balancing different types of imbalance measures regardless network structures under the logit link function.

According to Figure 3 and Figure 4, the main conclusions drawn from this experiment are similar to that in the previous section. AL and AL* are both better than CAR and NAR. In Case 1, the algorithm AL* is still as good as or better than algorithm AL. However, in Case 2, AL slightly

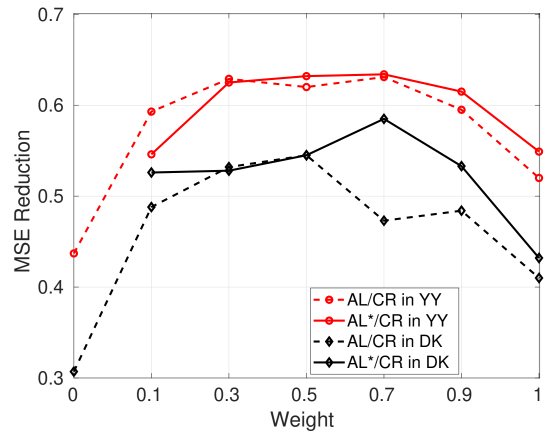

Figure 3. MSE comparison of Algorithms AL and AL* with CR in real data under the Case 1: $f(\cdot) = A_{i*}\mathbf{T}_i$.

outperforms AL*. This is because the $Imb^*_{n,net}$ in (10) is derived from the MSE given under Case 1, which indicates that AL is better than AL* in robustness.

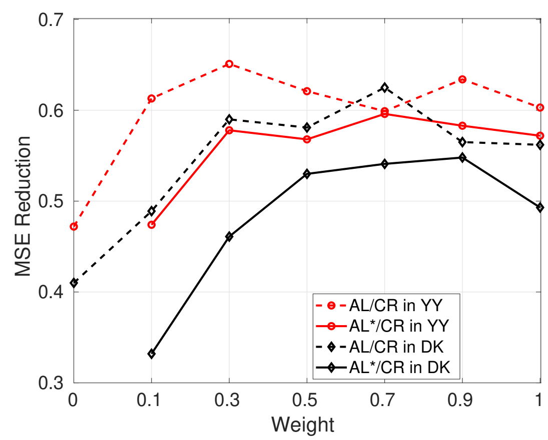

Figure 4. MSE comparison of Algorithms AL and AL* with CR in real data under the Case 2: $f(\cdot) = \sqrt{A_{i*}\mathbf{T}_i}$.

We further investigate the following two cases, where the transitivity of the network effect is taken into consideration. Here $f(A_{i*}, \mathbf{T}_i) = \gamma A_{i*}\mathbf{T}_i + \delta B_{i*}\mathbf{T}_i$, where $B_{ij} = \sum_{k\neq i,j} A_{ik}A_{jk}$. In **Case 3** we take $\delta = 0.5$, while in **Case 4** we take $\delta = 0.25$. Essentially, B_{ij} describes the number of common friends between i and j, so i may be influenced by j via their common friends, even if they may not know each other directly.

Figure 5 and Figure 6 shows that our proposed algorithms perform well even when the transitivity of network effect exists. The MSE reductions are even higher than that in Case 1.

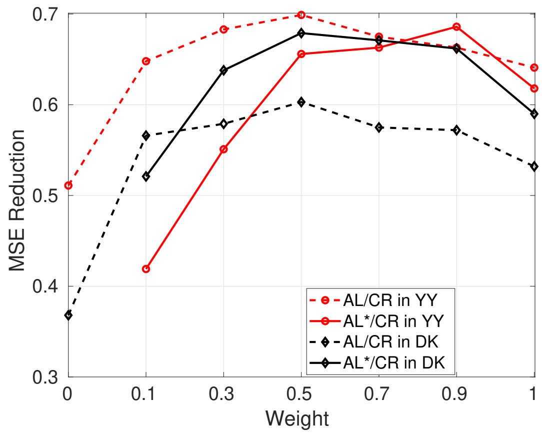

Figure 5. MSE comparison of Algorithms AL and AL* with CR in real data under the Case 3: $f(\cdot) = A_{i*}\mathbf{T}_i + 0.5B_{i*}\mathbf{T}_i$.

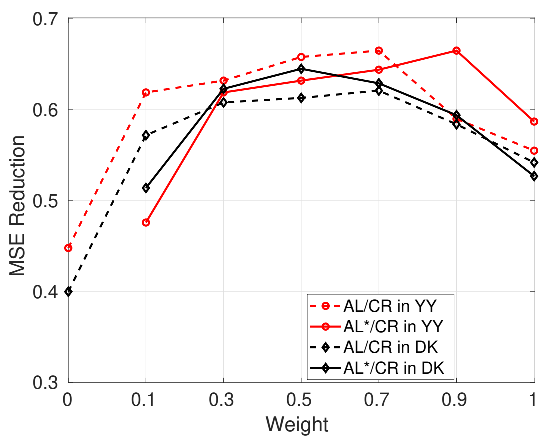

Figure 6. MSE comparison of Algorithms AL and AL* with CR in real data under the Case 4: $f(\cdot) = A_{i*}\mathbf{T}_i + 0.25B_{i*}\mathbf{T}_i$.

6. Conclusion and Future Work

In this paper, we propose a new adaptive randomization to improve the comparability of the treatment groups with respect to the covariates and the network interactions. We theoretically demonstrate the properties of our proposed procedure indicating the advanced properties of our proposed procedure, i.e., maintaining the balance for the covariates and the network simultaneously at a desirable rate. Via a simple derivation, we demonstrate how the improvement of the balance may translate into an escalation of the efficiency for estimating the ATE.

Our work can be extended in several directions. First, our work can be generalized to continuous covariates by modifying the imbalance measure. We may use Mahalanobis distance as the part of imbalance measure. Second, regression-adjusted estimators are commonly used to estimate ATE. It is important to understand how the balance may affect the performance of the regression-adjusted estimators. Third, we can treat the adjacency matrix A with data 0 and 1 as a noisy realization of a probability matrix M such that $M = E(A)$. A good estimate of M is a version of A with noise largely reduced. Fourth, the network structure may be correlated with the covariates. The connecting probability of two users may depend on their similarities of covariates. Finally, our proposed designs can be extended to multi-arms cases with the same logic as two-arms. We left these problems as our future research topics.

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A. Proofs

Here we study the asymptotic properties of our newly proposed design. Note that the covariates $Z_1, Z_2, \ldots, Z_n$ are independently and identically distributed. Because $Z_n = (k_1, \ldots, k_I)$ can take $m = \prod_{i=1}^{I} m_i$ different values, it follows a m-dimension multinomial distribution with parameter $\boldsymbol{p} = (p(k_1, \ldots, k_I))_{m\times 1}$, where each element is the probability that a unit falls within the corresponding stratum. Obviously, $p(k_1, \ldots, k_I) \geq 0$ and $\sum_{k_1,\ldots,k_I} p(k_1, \ldots, k_I) = 1$.

We denote $D_n(k_1, ..., k_I)$ as the true difference between the two treatment groups within stratum $(k_1, ..., k_I)$, and denote $\Lambda_n(k_1, \ldots, k_I)$ as the true weighted average of the imbalances within stratum $(k_1, ..., k_I)$ by the same way as in the CAR. That is

$$\Lambda_n(k_1, \ldots, k_I) = w_o D_n + \sum_{i=1}^{I} w_{m,i} D_n(i; k_i) + w_s D_n(k_1, \ldots, k_I). \tag{13}$$

Our main goal is to investigate its performance under our newly proposed design. We also let the true network imbalance measure be:

$$Imb_{n,net} = \frac{\left\|A_n(\mathbf{1}_n - 2\mathbf{T}_n)\right\|^2}{n^2}. \tag{14}$$

Comparing to the $\boldsymbol{\Lambda}_n$ under CAR procedure which only considers covariates, our newly proposed design procedure also includes network influence, balances it through $Imb_{n,net}$, and makes the $\Lambda_n(k_1, \ldots, k_I)$ under our new design procedure achieves $O_p(1)$ as well.

A.1. Proof of Theorem 4.2

Define $\mathbf{k} = (k_1, \ldots, k_I)$. When $w_s = 1$ and $w_o = w_m = 0$, the assignment rule is simplified as

$$\begin{aligned}
g(x) &= g\left(Imb_{n,w}^{(1)} - Imb_{n,w}^{(2)}\right) \\
&= g\left(w(Imb_{n,cov}^{(1)} - Imb_{n,cov}^{(2)}) + (1-w)(Imb_{n,net}^{(1)} - Imb_{n,net}^{(2)})\right) \\
&= g\left(4w \cdot \Lambda_{n-1}(k_1^*, \ldots, k_I^*) + (1-w)(Imb_{n,net}^{(1)} - Imb_{n,net}^{(2)})\right) \\
&= g\left(4w \cdot D_{n-1}(\mathbf{k}) + (1-w)(Imb_{n,net}^{(1)} - Imb_{n,net}^{(2)})\right) \\
&= \begin{cases} 1-q, & \text{if } x > 0 \\ \frac{1}{2}, & \text{if } x = 0 \\ q, & \text{if } x < 0 \end{cases}
\end{aligned}$$

Motivated by queueing networks, Pemantle and Rosenthal proves Lemma 4.1 for a sequence of random variables $\{X_n\}$. It finds a condition such that $\mathbf{E}X_n$ is bounded above by a constant independent of n and the particular sequence $\{X_n\}$, and the condition doesn't assume any special properties of the increments $X_n - X_{n-1}$. We state the Lemma here for convenience:

Lemma A.1. *(Lemma 4.1) Let X_n be random variables and suppose that there exist constants $a > 0$ $J, V < \infty$, and $p > 2$, such that $X_0 \leq J$, and for all n*

$$\mathbb{E}(X_n - X_{n-1} \mid \mathscr{F}_{n-1}) \leq -a \text{ on the event } \{X_{n-1} > J\} \quad (C1)$$

and

$$\mathbb{E}(|X_n - X_{n-1}|^p \mid \mathscr{F}_{n-1}) \leq V \quad (C2)$$

Then for any $r \in (0, p-1)$ there is a $c = c(p, a, V, J, r) > 0$ such that $\mathbb{E}(X_n^+)^r < c$ for all n.

We will use the Lemma 4.1 to prove our Theorem 4.2 via two steps. We let $X_n = |D_n(\mathbf{k})|$ be random variables based on any specific stratum $\mathbf{k}$ and if $|D_n(\mathbf{k})|$ were satisfied C1 and C2 then we can prove $\mathbb{E}(|D_n(\mathbf{k})|^r) = O(1)$.

A.1.1. SHOW C1 IS SATISFIED

In order to prove C1 is satisfied, we need the following Lemma first:

Lemma A.2. *Given A_n and $\mathcal{F}_n$, we have $G_n = Imb_{n,net}^{(1)} - Imb_{n,net}^{(2)} \leq 4$.*

Proof. Define vector $\mathbf{T}_n^{'} = 2\mathbf{T} - 1$, $T_i^{'}$ is the i-th element in $\mathbf{T}_n^{'}$. Therefore,

$$T_i^{'} = \begin{cases} 1, \; if \; T_i = 1 \\ -1, \; if \; T_i = 0 \end{cases}$$

We also define $\mathbf{T}_n^{'(1)} = (\mathbf{T}_{n-1}^{\top}, 1)$, and $\mathbf{T}_n^{'(2)} = (\mathbf{T}_{n-1}^{\top}, -1)$.
Suppose $\mathbf{C}_n$ is a $n \times 1$ vector and its i-th element is

$$C_{n(i)} = A_{i*} \cdot \mathbf{T}_n^{'} = A_{i,1:n-1} \cdot \mathbf{T}_{n-1}^{'} + A_{i,n} \cdot T_n \text{ where } i = 1, ...n$$

And $\mathbf{C}_{n-1}$ is a $(n-1) \times 1$ vector with i-th element equal

$$C_{n-1(i)} = A_{i*} \cdot \mathbf{T}_{n-1}^{'} = A_{i,1:n-1} \cdot \mathbf{T}_{n-1}^{'} \text{ where } i = 1, ...n-1$$

Define

$$C_{n(i)}^{(1)} = C_{n-1(i)} + T_n^{'(1)} = C_{n-1(i)} + 1 \; for \; i = 1, ..., n-1$$
$$C_{n(i)}^{(2)} = C_{n-1(i)} + T_n^{'(2)} = C_{n-1(i)} - 1 \; for \; i = 1, ..., n-1$$

Therefore,

$$C_{n(i)}^{2\,(1)} - C_{n(i)}^{2\,(2)} = (C_{n-1(i)}^{(1)} + 1)^2 - (C_{n-1(i)}^{(1)} - 1)^2 = 4C_{n-1(i)}$$

where $-n \leq C_{n-1(i)}^{(1)} \leq n$. The equality holds when i-th unit are connected with all (n-1) units and all the (n-1) units are assigned to treatment, where the n holds, or all the (n-1) units are assigned to control, where the -n holds.
Also define

$$C_{n(n)}^{(1)} = A_{n,1:n-1}\mathbf{T}_{n-1}^{'} + 1$$
$$C_{n(n)}^{(2)} = A_{n,1:n-1}\mathbf{T}_{n-1}^{'} - 1$$

Therefore,

$$C_{n(n)}^{2\,(1)} - C_{n(n)}^{2\,(2)} = (A_{n,1:n-1}\mathbf{T}_{n-1}^{'} + 1)^2 - (A_{n,1:n-1}\mathbf{T}_{n-1}^{'} - 1)^2 = 4A_{n,1:n-1}\mathbf{T}_{n-1}^{'}$$

where $|A_{n,1:n-1}\mathbf{T}_{n-1}^{'}| \leq n$. The equality holds when n-th unit are connected with all previous (n-1) units and all the n units are assigned to treatment or all the n units are assigned to control. Therefore,

$$|C_{n(i)}^{2\,(1)} - C_{n(i)}^{2\,(2)}| \leq 4n \; for \; i = 1, ..., n$$

Therefore we can get

$$
\begin{aligned}
|G_n| &= \left| Imb_{n,net}^{(1)} - Imb_{n,net}^{(2)} \right| \\
&= \left| \frac{\left\| A_n \left(\mathbf{1}_n - 2\mathbf{T}_n^{(1)} \right) \right\|^2 - \left\| A_n \left(\mathbf{1}_n - 2\mathbf{T}_n^{(2)} \right) \right\|^2}{n^2} \right| \\
&= \left| \frac{\left\| A_n \mathbf{T}_n^{'(1)} \right\|^2 - \left\| A_n \mathbf{T}_n^{'(2)} \right\|^2}{n^2} \right| \\
&\leq \left| \frac{1}{n^2} \left(\sum_{i=1}^{n} A_{i*} \mathbf{T}_n^{'(1)} - \sum_{i=1}^{n} A_{i*} \mathbf{T}_n^{'(2)} \right) \right| \\
&= \left| \frac{1}{n^2} \sum_{i=1}^{n} \left(C_{n(i)}^{2\,(1)} - C_{n(i)}^{2\,(2)} \right) \right| \\
&\leq \frac{1}{n^2} \sum_{i=1}^{n} \left| \left(C_{n(i)}^{2\,(1)} - C_{n(i)}^{2\,(2)} \right) \right| \\
&= \frac{1}{n^2} \sum_{i=1}^{n} 4n = 4
\end{aligned}
$$

□

Since $\left| Imb_{n,net}^{(1)} - Imb_{n,net}^{(2)} \right| \leq 4$, therefore when $D_{n-1}(\mathbf{k}) > \frac{1-w}{w}$, the nth unit will be assigned to treatment group with probability $1 - q < 0.5$ regardless of network imbalance measure.

Therefore given the nth unit has covariate profile Z_n we have:

$$
D_n(\mathbf{k}) - D_{n-1}(\mathbf{k}) = \begin{cases} 1, & \text{with probability } 1-q \\ -1, & \text{with probability } q \end{cases}
$$
$$
D_n(\mathbf{k}) - D_{n-1}(\mathbf{k}) = 0 \text{ for the other stratum where } Z_n \neq \mathbf{k}
$$

Since $\frac{1-w}{w} > 0$ and for any stratum there will be at most 1 change in each step, thus we have $D_{n-1}(\mathbf{k}) \geq 1$ and $D_n(\mathbf{k}) \geq 0$. Therefore,

$$
|D_n(\mathbf{k})| - |D_{n-1}(\mathbf{k})| = D_n(\mathbf{k}) - D_{n-1}(\mathbf{k}) = \begin{cases} 1, & \text{with probability } 1-q \\ -1, & \text{with probability } q \end{cases}
$$
$$
|D_n(\mathbf{k}) - D_{n-1}(\mathbf{k})| = D_n(\mathbf{k}) - D_{n-1}(\mathbf{k}) = 0 \text{ for the other stratum where } Z_n \neq \mathbf{k}
$$

When $D_{n-1}(\mathbf{k}) < -\frac{1-w}{w}$, the nth unit will be assigned to treatment group with probability $q > 0.5$ regardless of network imbalance measure.

Since $-\frac{1-w}{w} < 0$ and for any stratum there will be at most 1 change in each step, thus we have $D_{n-1}(\mathbf{k}) \leq -1$ and $D_n(\mathbf{k}) \leq 0$. Therefore, given the nth unit has covariate profile Z_n we have:

$$
D_n(\mathbf{k}) - D_{n-1}(\mathbf{k}) = \begin{cases} 1, & \text{with probability } 1-q \\ -1, & \text{with probability } q \end{cases}
$$
$$
D_n(\mathbf{k}) - D_{n-1}(\mathbf{k}) = 0 \text{ for the other stratum where } Z_n \neq \mathbf{k}
$$

Therefore we have

$$
|D_n(\mathbf{k})| - |D_{n-1}(\mathbf{k})| = (-D_n(\mathbf{k})) - (-D_{n-1}(\mathbf{k})) = \begin{cases} 1, & \text{with probability } 1-q \\ -1, & \text{with probability } q \end{cases}
$$
$$
|D_n(\mathbf{k}) - D_{n-1}(\mathbf{k})| = D_n(\mathbf{k}) - D_{n-1}(\mathbf{k}) = 0 \text{ for the other stratum where } Z_n \neq \mathbf{k}
$$

Therefore, When $|D_{n-1}(\mathbf{k})| > \frac{1-w}{w}$, given the nth unit has covariate profile $Z_n = (\mathbf{k})$ we have:

$$|D_n(\mathbf{k})| - |D_{n-1}(\mathbf{k})| = \begin{cases} 1, & \text{with probability } 1-q \\ -1, & \text{with probability } q \end{cases}$$
$$D_n(\mathbf{k}) - D_{n-1}(\mathbf{k}) = 0 \text{ for the other stratum where } Z_n \neq \mathbf{k}$$

Therefore,

$$\begin{aligned}\mathbb{E}\left(|D_n(\mathbf{k})| - |D_{n-1}(\mathbf{k})|\mathcal{F}_{n-1}|\right) &= \mathbb{E}_{Z_n}[\mathbb{E}\left(|D_n(\mathbf{k})| - |D_{n-1}(\mathbf{k})||\,\mathcal{F}_{n-1}, Z_n = \mathbf{k}\right)] \\ &= \mathbb{P}(Z_n = (\mathbf{k})) \cdot \mathbb{E}\left(|D_n(\mathbf{k})| - |D_{n-1}(\mathbf{k})||\,\mathcal{F}_{n-1}, Z_n = \mathbf{k}\right) \\ &+ \mathbb{P}(Z_n \neq (\mathbf{k})) \cdot \mathbb{E}\left(|D_n(\mathbf{k})| - |D_{n-1}(\mathbf{k})||\,\mathcal{F}_{n-1}, Z_n = \mathbf{k}\right) \\ &= p(\mathbf{k})(1-2q) + (1-p(\mathbf{k})) \times 0 \\ &= p(\mathbf{k})(1-2q)\end{aligned}$$

where $-1 < p(\mathbf{k})(1-2q) < 0$ because $0 < p(\mathbf{k}) < 1$ and $-1 < 1-2q < 0$.

Therefore we show that there exist constants $a = -p(\mathbf{k})(1-2q)$ which is bigger than 0, $J = \frac{w}{1-w} < \infty$, such that $|D_0(\mathbf{k})| = 0 \leq J$, and for all n

$$\mathbb{E}\left(|D_n(\mathbf{k})| - |D_{n-1}(\mathbf{k})| \mid \mathcal{F}_n\right) \leq -a \text{ on the event } \{D_{n-1}(\mathbf{k}) > J\}$$

Thus, $C1$ is satisfied.

A.1.2. Show C2 is satisfied

Since given Z_n

$$\left||D_n(\mathbf{k})| - |D_{n-1}(\mathbf{k})|\right| = \begin{cases} 1, \; if \; nth \; unit \; has \; covariate \; profile \; Z_n = \mathbf{k} \\ 0, \; otherwise \end{cases}$$

Thus,

$$\left||D_n(\mathbf{k})| - |D_{n-1}(\mathbf{k})|\right|^p \leq 1 \text{ and } E\left(\left||D_n(\mathbf{k})| - |D_{n-1}(\mathbf{k})|\right|^p|\mathcal{F}_{n-1}\right) \leq 1$$

and

$$\mathbb{E}\left(\left||D_n(\mathbf{k})| - |D_{n-1}(\mathbf{k})|\right|^p \mid \mathcal{F}_{n-1}\right) \leq V.$$

Therefore we find that there exist $V = 1 < \infty$, and $p > 2$, such that $C2$ is satisfied.

Hence, for all n for any $r \in (0, p-1)$ there is a constant $c = c(p, a, V, J, r) > 0$ such that $\mathbb{E}\left(|D_n(\mathbf{k})|^r\right) < c$ for all n, and we thus conclude that $\mathbb{E}\left(|D_n(\mathbf{k})|^r\right) = O(1)$.

A.2. Proof of Lemma 4.3

Motivated by the proof of Theorem 4.2, we propose the Lemma 4.3. It is an extension allowing the negative part of the increments to avoid the moment condition in Lemma 4.1 and will be utilized during the proof of Theorem 4.4. We state the Lemma here again for convenience:

Lemma A.3. *(Lemma 4.3) The conclusion of Lemma 4.1 will still hold when $X_n - X_{n-1}$ is replaced by $(X_n - X_{n-1})' = max(X_n - X_{n-1}, C)$ in conditions $(C1)$ and $(C2)$ where C can be any negative constant such that $C \leq -1$.*

Our proof follows the logic of the proof of Corollary 3 in Pemantle and Rosenthal (1999). They started with the following theorem and then proved a Corollary 3 from an intermediate Corollary 6.

Theorem A.4. *(Pemantle and Rosenthal, 1999) Let $\{M_n : n = 0, 1, 2, \ldots\}$ be a sequence adapted to a filtration $\{\mathcal{F}_n\}$ and let Δ_n denote $M_{n+1} - M_n$. Suppose that the sequence started at M_1 is a martingale (i.e., $\mathbb{E}\left(\Delta_n \mid F_n\right) = 0$ for $n \geq 1$), and that $M_0 \leq 0$. Suppose further that for some $p > 2$ and $b > 0$ we have*

$$\mathbb{E}\left(|\Delta_n|^p \mid \mathcal{F}_n\right) \leq b,$$

for all n including $n = 0$. Let $\tau = \inf\{n > 0 : M_n \leq n\}$. Then for any $r \in (0, p)$ there is a constant $C = C(b, p, r)$ such that

$$\mathbb{E}\left(\left(M_t^+\right)^r \mathbf{1}_{\tau > t}\right) \leq Ct^{r-p}.$$

Following their proof logic, we also start from Theorem A.4 but we modify the conditions of their Corollary 6, i.e., we further truncated random variables to a negative point from its original zero, and then propose Lemma A.5. After that, we prove Lemma 4.3, which is the bases of the proof of Theorem 4.4.

A.2.1. PROOF OF LEMMA A.5

Following the logic of the proof of the Corollary 6 in (Pemantle and Rosenthal, 1999), to prove Lemma 4.3 we need the following Lemma A.5 first:

Lemma A.5. *Let $\{Y_n\}$ be adapted to $\{\mathcal{F}_n\}$ with $Y_0 \le 0$. Suppose*

$$\mathbb{E}\left(\left|\Delta'_n \mathbf{1}_{\Delta'_n > m} + m\mathbf{1}_{\Delta'_n \le m}\right|^p |\mathcal{F}_n\right) \le B$$

for all n and any nonpositive constant m, and

$$\mathbb{E}\left(\Delta'_n \mathbf{1}_{\Delta'_n > m} + m\mathbf{1}_{\Delta'_n \le m} |\mathcal{F}_n\right) \le 0$$

for all $1 \le n < \sigma$, where $\Delta'_n = Y_{n+1} - Y_n$ and $\sigma = \inf\{n > 0 : Y_n \le n\}$. Then for $0 < r < p$, $p > 2$ there is a constant $K = K(B, p, r)$ such that

$$E\left(\left(Y_t^+\right)^r \mathbf{1}_{\sigma > t}\right) \le K t^{r-p}.$$

Proof. We denote $B_{n+1} - B_n = \Delta'_n \mathbf{1}_{\Delta'_n > m} + m\mathbf{1}_{\Delta'_n \le m}$, for any nonpositive m and $B_0 = Y_0$, then B_n is a supermartingale for all $1 \le n \le \sigma$ because $\mathbb{E}(B_{n+1} - B_n) \le 0$ when $1 \le n \le \sigma$. Also, we have $\mathbb{E}(|B_{n+1} - B_n|^p|\mathcal{F}_n) \le B$. Recall that the supermartingale may be decomposed as $B_{n\wedge\sigma} = M_n - A_n$ where $\{M_n : n \ge 1\}$ is a martingale and $\{A_n : n \ge 1\}$ is an increasing predictable process with $A_1 = 0$. Note that for $n \ge \sigma$, the increments $\Delta_n := M_{n+1} - M_n = 0$. For $n < \sigma$, we have

$$\begin{aligned}
\mathbb{E}\left(|\Delta_n|^p |\mathcal{F}_n\right) &= \mathbb{E}\left(|M_{n+1} - M_n|^p |\mathcal{F}_n\right) = \mathbb{E}\left(|M_{n+1} - E(M_{n+1}|\mathcal{F}_n)|^p |\mathcal{F}_n\right) \\
&= \mathbb{E}\left(|M_{n+1} - A_{n+1} - B_n - E(M_{n+1} - A_{n+1} - B_n|\mathcal{F}_n)|^p |\mathcal{F}_n\right) \text{ as } A_{n+1} \in \mathcal{F}_n \\
&= \mathbb{E}\left(|B_{n+1} - B_n - E(B_{n+1} - B_n|\mathcal{F}_n)|^p |\mathcal{F}_n\right)
\end{aligned}$$

Since

$$|a - b|^p \le (|a| + |b|)^p \le (2max(|a|, |b|))^p \le 2^p max(|a|, |b|)^p \le 2^p(|a|^p + |b|^p)$$

we thus have

$$\mathbb{E}\left(|\Delta_n|^p |\mathcal{F}_n)\right) \le 2^p \mathbb{E}\left(|B_{n+1} - B_n|^p + |\mathbb{E}(B_{n+1} - B_n)|^p |\mathcal{F}_n\right) \le 2^p\left(2B\right)$$

as $|\mathbb{E}X|^p \le \mathbb{E}(|X|^p)$ for $p > 1$.

Since the conditions of the Theorem A.4 are satisfied, applying Theorem A.4 to $\{M_n\}$ with $b = 2^{p+1}B$ and $M_0 := Y_0$ yields

$$\mathbb{E}\left(\left(M_t^+\right)^r \mathbf{1}_{\tau > t}\right) \le K t^{r-p}$$

Besides, since $B_{n+1} - B_n = (Y_{n+1} - Y_n)\mathbf{1}_{\Delta'_n > m} + m\mathbf{1}_{\Delta'_n \le m}$ and $B_0 = Y_0$, we have $B_{n+1} - B_n \ge Y_{n+1} - Y_n$ and thus $B_n \ge Y_n$ for all n. Therefore, we have

$$B_t^+ \mathbf{1}_{\sigma > t} \ge Y_t^+ \mathbf{1}_{\sigma > t}$$

When $\sigma > t$ it follows that $M_n \ge n + A_n$ for $1 \le n \le t$ and hence that $\tau = \inf\{n > 0 : M_n \le n\} > t$. Also, when $\sigma > t$, we know that $M_t = B_t + A_t \ge B_t \ge Y_t > n$ and therefore that

$$Y_t^+ \mathbf{1}_{\sigma > t} \le M_t^+ \mathbf{1}_{\tau > t}$$

Therefore, we have

$$\mathbb{E}\left((Y_t^+)^r \mathbf{1}_{\sigma > t}\right) = \mathbb{E}\left(Y_t^+ \mathbf{1}_{\sigma > t}\right)^r \le \mathbb{E}\left(M_t^+ \mathbf{1}_{\tau > t}\right)^r = \mathbb{E}\left((M_t^+)^r \mathbf{1}_{\tau > t}\right) \le K t^{r-p}$$

□

A.2.2. PROOF OF LEMMA 4.3

The proof of Lemma 4.3 follows the logic of the Corollary 3 in (Pemantle and Rosenthal, 1999) but changes its conditions.

Proof. First assume that $J=0$. Given $\{X_n\}$ as in the hypotheses of the Lemma 4.1, fix an $N \geq 1$; we will compute an upper bound for $E\left(X_N^{+}\right)^r$ that does not depend on N. Let $U:=\max \{k \leq N: X_k \leq 0\}$ denote the last time up to N that X takes a nonpositive value. Decompose according to the value of U :

$$\mathbb{E}\left(X_N^{+}\right)^r=\sum_{k=0}^{N-1} \mathbb{E}\left(\left(X_N^{+}\right)^r \mathbf{1}_{U=k}\right)$$

because $\mathbb{E}\left(\left(X_N^{+}\right)^r \mathbf{1}_{U=N}\right)=E(0)=0$.

To evaluate the summand, define for any $k<N$ a process $\left\{Y_n^{(k)}\right\}$ by $Y_n^{(k)}=\left(X_{k+n}+\ n\right) \mathbf{1}_{X_k \leq 0}$. In other words, if $X_k>0$ the process $\left\{Y_N^{(k)}\right\}$ is constant at zero, and otherwise it is the process $\{X_n\}$ shifted by k and compensated by adding 1 each time step. We apply lemma A.5 to the process $\left\{Y_n^{(k)}\right\}$. First, we need to show that

$$\mathbb{E}\left(\Delta_n^{(k)\prime} \mathbf{1}_{\Delta_n^{(k)\prime}>m}+m \mathbf{1}_{\Delta_n^{(k)\prime} \leq m} | \mathcal{F}_n\right) \leq 0 \tag{15}$$

for $1 \leq n \leq \sigma^{(k)}$. It suffices to show (15) holds on all $Y_n^{(k)}>n$ with some $m \leq 0$. Note that $Y_n^{(k)}>n$ implies $X_{k+n}>0$ and $X_k \leq 0$, so it suffices to show

$$\mathbb{E}\left(\left(X_{k+n+1}-X_{k+n}+1\right) \mathbf{1}_{X_{k+n+1}-X_{k+n}+1>m}+m \mathbf{1}_{X_{k+n+1}-X_{k+n}+1 \leq m} | \mathcal{F}_n\right) \leq 0 \tag{16}$$

holds whenever $X_{n+k}>0$ with some $m \leq 0$.

On the other hand, by hypothesis (C1) of Lemma 4.1, on $X_{n+k}>0$,

$$\mathbb{E}\left(\left(X_{n+k+1}-X_{n+k}\right) \mathbf{1}_{X_{n+k+1}-X_{n+k}>C}+C \mathbf{1}_{X_{n+k+1}-X_{n+k} \leq C} | \mathcal{F}_n\right) \leq-1,$$

which leads to

$$\begin{aligned}
&\mathbb{E}\left(\left(X_{n+k+1}-X_{n+k}+1\right) \mathbf{1}_{X_{n+k+1}-X_{n+k}>C}+(C+1) \mathbf{1}_{X_{n+k+1}-X_{n+k} \leq C} | \mathcal{F}_n\right) \leq 0 \\
&\mathbb{E}\left(\left(X_{n+k+1}-X_{n+k}+1\right) \mathbf{1}_{X_{n+k+1}-X_{n+k}+1>C+1}+(C+1) \mathbf{1}_{X_{n+k+1}-X_{n+k}+1 \leq C+1} | \mathcal{F}_n\right) \leq 0 .
\end{aligned}$$

Hence, (16) holds with $m=C+1 \leq 0$.

Then, we need to show

$$E\left(\left|\Delta_n^{\prime} \mathbf{1}_{\Delta_n^{\prime}>C+1}+(C+1) \mathbf{1}_{\Delta_n^{\prime} \leq C+1}\right|^p | \mathcal{F}_n\right) \leq B. \tag{17}$$

Similarly with (C2) in the Lemma 4.1 ,

$$\mathbb{E}\left(\left|\left(X_{n+k+1}-X_{n+k}\right) \mathbf{1}_{X_{n+k+1}-X_{n+k}>C}\right|^p | \mathcal{F}_n\right) \leq V<\infty$$

When $X_k>0$, $\mathbb{E}\left(\left|\Delta_n^{\prime} \mathbf{1}_{\Delta_n^{\prime}>C+1}+(C+1) \mathbf{1}_{\Delta_n^{\prime} \leq C+1}\right|^p | \mathcal{F}_n\right)=0 \leq B$. When $X_k \leq 0$ we have

$$\begin{aligned}
&\mathbb{E}\left(\left|\Delta_n^{\prime} \mathbf{1}_{\Delta_n^{\prime}>C+1}+(C+1) \mathbf{1}_{\Delta_n^{\prime} \leq C+1}\right|^p | \mathcal{F}_n\right) \\
=&\mathbb{E}\left(\left|\left(X_{n+k+1}-X_{n+k}\right) \mathbf{1}_{X_{n+k+1}-X_{n+k}>C}+1+C \mathbf{1}_{X_{n+k+1}-X_{n+k} \leq C}\right|^p | \mathcal{F}_n\right) \\
\leq&2^p \mathbb{E}\left(\left|\left(X_{n+k+1}-X_{n+k}\right) \mathbf{1}_{X_{n+k+1}-X_{n+k}>C}\right|^p+\left|1+C \mathbf{1}_{X_{n+k+1}-X_{n+k} \leq C}\right|^p | \mathcal{F}_n\right) \\
\leq&2^p\left(V+1+|C|^p\right)
\end{aligned}$$

Hence, (17) holds with $B=2^p\left(V+1+|C|^p\right)$. Therefore, based on A.4 we let $t=K-k$ and then have

$$\mathbb{E}\left(\left[\left(Y_{N-k}^{(k)}\right)^{+}\right]^r \mathbf{1}_{\sigma^{(k)}>N-k}\right) \leq K(N-k)^{r-p}$$

with $K = K(V, p, r)$. But for each k, since $U = k$ implies $\sigma^{(k)} > N - k$, and we thus get $Y_n^{(k)} > n$ for $n = 1, ..., N - k$. Since $\sigma^{(k)} > N - k$ also implies $Y_{N-k} > n$, so when $U \neq k$, $Y_{N-k}^{(k)} > 0$. In conclusion, we have

$$X_N^+ \mathbf{1}_{U=k} \leq Y_{N-k}^{(k)} \mathbf{1}_{\sigma^{(k)} > N-k}$$

and it follows that

$$\mathbb{E}\left(\left(X_N^+\right)^r \mathbf{1}_{U=k}\right) \leq K(N-k)^{r-p}$$

Now sum to get

$$\mathbb{E}\left(X_N^+\right)^r \leq \sum_{k=0}^{N-1} K(N-k)^{r-p} \leq K\zeta$$

because $N - k > 0$, $r - p < -1$ and the sum is bounded by ζ. This completes the case $J = 0$.

For the general case, let $X_n^* = X_n - J$, so we have $\mathbb{E}\left((X_n^*)^+\right)^r < \infty$

then we have

$$\mathbb{E}(X_n^+)^r \leq E\left((X_n^*)^+ + |J|\right)^r \leq 2^r(\mathbb{E}\left((X_n^*)^+\right)^r + |J|^r) < \infty$$

And therefore we have

$$\mathbb{E}(X_n)^+ \leq c(p, 1, V, J, r) := J + c(p, 1, V, 0, r)$$

Since in our theorem $V_n > 0$, so

$$\mathbb{E}(V_n) = \mathbb{E}(V_n)^+ \leq c(p, 1, V, J, r) := J + c(p, 1, V, 0, r)$$

□

A.3. Proof of Theorem 4.4 (General Case)

Here we utilize the Lemma 4.3 to prove Theorem 4.4.

Define

$$\boldsymbol{D}_n = \left[D_n\left(k_1, \ldots, k_I\right)\right]_{1 \leq k_1 \leq m_1, \ldots, 1 \leq k_I \leq m_I}$$

be an array of dimension $m_1 \times \ldots \times m_I$ which stores the current assignment differences in all strata and therefore stores the current imbalances. We also define $\boldsymbol{\Lambda}_n$ the same way. Besides, let

$$V_n = \sum_k w_s D_n^2(k) + \sum_{i=1}^{I} \sum_{k_i=1}^{m_i} w_{m,i} D_n^2\left(i; k_i\right) + w_o D_n^2$$

and we have

$$V_n - V_{n-1} = 4\Lambda_{n-1}(\mathbf{k})\left(T_n - \frac{1}{2}\right) + 1$$

where the covariate profile of the n-th unit is $Z_n = \mathbf{k}$.

The whole process will be separated into three part: (i) Show $\mathbb{E}([(V_n - V_{n-1})']^+)^p | \mathcal{F}_{n-1}]$ is bounded. (ii) Show $C1$ is satisfied. (iii) Show $\mathbb{E}([(V_n - V_{n-1})']^-)^p | \mathcal{F}_{n-1}]$ is bounded and we then combine it with (i) and have $C2$ satisfied.

A.3.1. Show $\mathbb{E}[([(V_n - V_{n-1})']^+)^p | \mathcal{F}_{n-1}]$ is bounded

Denote $G_n = Imb_{n,net}^{(1)} - Imb_{n,net}^{(2)}$, our decision rule can be expressed as

$$\mathbb{P}(T_n = 1 | Z_n = k, \mathcal{F}_{n-1}, A_n) = g(Imb_{n,cov}^{(1)} - Imb_{n,cov}^{(2)}) = g(w4\Lambda_{n-1}(\mathbf{k}) + (1-w)G_n)$$

Note that when $|\Lambda_{n-1}(k)| > \frac{w}{1-w}$,

$$(V_n - V_{n-1})' = \begin{cases} 2|\Lambda_{n-1}(k)| + 1, & \text{with probability } g(|w4\Lambda_{n-1}(\mathbf{k}) + (1-w)G_n|) \\ (-2|\Lambda_{n-1}(k)| + 1)', & \text{with probability } 1 - g(|w4\Lambda_{n-1}(\mathbf{k}) + (1-w)G_n|) \end{cases}$$

Because when $\Lambda_{n-1}(\mathbf{k}) > \frac{w}{1-w}$, we have $w4\Lambda_{n-1}(\mathbf{k}) + (1-w)G_n > 0$. So for $T_n = 1$,

$$\begin{aligned} (V_n - V_{n-1})' &= 2\Lambda_{n-1}(k) + 1 = 2|\Lambda_{n-1}(k)| + 1 \\ \text{with} \quad \mathbb{P}(T_n = 1) &= g(w4\Lambda_{n-1}(\mathbf{k}) + (1-w)G_n) = g(|w4\Lambda_{n-1}(\mathbf{k}) + (1-w)G_n|) \end{aligned}$$

and for $T_n = 0$,

$$\begin{aligned} (V_n - V_{n-1})' &= (-2\Lambda_{n-1}(k) + 1)' = (-2|\Lambda_{n-1}(k)| + 1)' \\ \text{with} \quad \mathbb{P}(T_n = 0) &= 1 - g(w4\Lambda_{n-1}(\mathbf{k}) + (1-w)G_n) = 1 - g(|w4\Lambda_{n-1}(\mathbf{k}) + (1-w)G_n|) \end{aligned}$$

When $\Lambda_{n-1}(\mathbf{k}) < -\frac{w}{1-w}$, we have $w4\Lambda_{n-1}(\mathbf{k}) + (1-w)G_n < 0$, so for $T_n = 1$,

$$\begin{aligned} (V_n - V_{n-1})' &= (2\Lambda_{n-1}(k) + 1)' = (-2|\Lambda_{n-1}(k)| + 1)' \\ \text{with} \quad \mathbb{P}(T_n = 1) &= g(w4\Lambda_{n-1}(\mathbf{k}) + (1-w)G_n) = g(-|w4\Lambda_{n-1}(\mathbf{k}) + (1-w)G_n|) \\ &= 1 - g(|w4\Lambda_{n-1}(\mathbf{k}) + (1-w)G_n|) \end{aligned}$$

and for $T_n = 0$,

$$\begin{aligned} (V_n - V_{n-1})' &= -2\Lambda_{n-1}(k) + 1 = 2|\Lambda_{n-1}(k)| + 1 \\ \text{with} \quad \mathbb{P}(T_n = 0) &= 1 - g(w4\Lambda_{n-1}(\mathbf{k}) + (1-w)G_n) = g(|w4\Lambda_{n-1}(\mathbf{k}) + (1-w)G_n|). \end{aligned}$$

Now when $|\Lambda_{n-1}(k)| > \frac{w}{1-w}$ we have

$$\begin{aligned} &\mathbb{E}[([(V_n - V_{n-1})']^+)^p | \mathcal{F}_{n-1}, Z_n = k] \\ &= (2|\Lambda_{n-1}(k)| + 1)^p g(|w4\Lambda_{n-1}(\mathbf{k}) + (1-w)G_n|) \\ &\quad + ((-2|\Lambda_{n-1}(k)| + 1)^+)^p (1 - g(|w4\Lambda_{n-1}(\mathbf{k}) + (1-w)G_n|)) \\ &\leq (2|\Lambda_{n-1}(k)| + 1)^p g(|w4\Lambda_{n-1}(\mathbf{k}) + (1-w)G_n|) + 1. \end{aligned}$$

Also, we have

$$\begin{aligned} &(2|\Lambda_{n-1}(\mathbf{k})| + 1)^p g(|w4\Lambda_{n-1}(\mathbf{k}) + (1-w)G_n|) \\ =&(2w)^{-p}(4w|\Lambda_{n-1}(\mathbf{k})| + 2w + (1-w)G_n - (1-w)G_n)^p g(|w4\Lambda_{n-1}(\mathbf{k}) + (1-w)G_n|) \\ \leq&(2w)^{-p}2^p(|4w|\Lambda_{n-1}(\mathbf{k})| + (1-w)G_n|^p + |2w - (1-w)G_n|^p) g(|w4\Lambda_{n-1}(\mathbf{k}) + (1-w)G_n|) \\ \leq&w^{-p}(M + 4^p), \quad \text{as } x^p g(x) \leq M \text{ and } |G_n| \leq 4. \end{aligned}$$

Hence,

$$\mathbb{E}[([(V_n - V_{n-1})']^+)^p | \mathcal{F}_{n-1}, Z_n = \mathbf{k}] \leq \frac{M + 4^p}{w^p} + 1 = M_0.$$

Note that when $|\Lambda_{n-1}(\mathbf{k})| \leq \frac{w}{1-w}$, $\mathbb{E}([(V_n - V_{n-1})']^+)^p | \mathcal{F}_{n-1}, Z_n = \mathbf{k}] \leq (\frac{2w}{1-w} + 1)^p$. Therefore, for any $Z_n = \mathbf{k}$,

$$\mathbb{E}[([(V_n - V_{n-1})']^+)^p | \mathcal{F}_{n-1}, Z_n = \mathbf{k}] \leq \max(M_0, (\frac{2w}{1-w} + 1)^p) = M_1.$$

Hence, $\mathbb{E}[([(V_n - V_{n-1})']^+)^p | \mathcal{F}_{n-1}] \leq M_1$.

A.3.2. Show C1 is satisfied

For any covariate profile of the n-th unit, i.e.$Z_n = \mathbf{k}$, we have:

$$\begin{aligned} \mathbb{E}\left((V_n - V_{n-1})' | \mathcal{F}_{n-1}\right) &= \mathbb{P}(Z_n = \mathbf{k})\mathbb{E}\left((V_n - V_{n-1})' | Z_n = \mathbf{k}, \mathcal{F}_{n-1}\right) + \mathbb{P}(Z_n \neq \mathbf{k})\mathbb{E}\left((V_n - V_{n-1})' | Z_n \neq \mathbf{k}, \mathcal{F}_{n-1}\right) \\ &= p_{\mathbf{k}}\mathbb{E}\left((V_n - V_{n-1})' | Z_n = \mathbf{k}, \mathcal{F}_{n-1}\right) + (1 - p_{\mathbf{k}})\mathbb{E}\left((V_n - V_{n-1})' | Z_n \neq \mathbf{k}, \mathcal{F}_{n-1}\right) \\ &\leq p_{\mathbf{k}}\mathbb{E}\left((V_n - V_{n-1})' | Z_n = \mathbf{k}, \mathcal{F}_{n-1}\right) + (1 - p_{\mathbf{k}})M_1 \end{aligned}$$

as

$$\mathbb{E}\left((V_n - V_{n-1})'|Z_n \neq \mathbf{k}, \mathcal{F}_{n-1}\right) \leq \mathbb{E}\left([(V_n - V_{n-1})']^+|Z_n \neq \mathbf{k}, \mathcal{F}_{n-1}\right) \leq M_1.$$

To show C1 is satisfied, it suffices to show that whenever $V_{n-1} > J$, there exists a $\mathbf{k}^*$ such that

$$\mathbb{E}\left((V_n - V_{n-1})'|Z_n = \mathbf{k}^*, \mathcal{F}_{n-1}\right) \leq -\left(\frac{1 + (1 - p_{\mathbf{k}^*})M_1}{p_{\mathbf{k}^*}}\right).$$

Let $p_{min} = min_{\mathbf{k}}(\mathbb{P}(S_n = \mathbf{k}))$, it's sufficient to show that whenever $V_{n-1} > J$, there exists a $\mathbf{k}^*$ such that

$$\mathbb{E}\left((V_n - V_{n-1})'|Z_n = \mathbf{k}^*, \mathcal{F}_{n-1}\right) \leq -\left(\frac{1 + (1 - p_{min})M_1}{p_{min}}\right), \quad \text{denoted as } M_2.$$

Also, for any stratum $\mathbf{k}$ with $|\Lambda_{n-1}(\mathbf{k})| > \frac{w}{1-w}$, we have

$$\begin{aligned}
&\mathbb{E}\left((V_n - V_{n-1})'|Z_n = \mathbf{k}, \mathcal{F}_{n-1}\right) \\
=&(2|\Lambda_{n-1}(\mathbf{k})| + 1)g(|w4\Lambda_{n-1}(\mathbf{k}) + (1 - w)G_n|) + (-2|\Lambda_{n-1}(\mathbf{k})| + 1)'(1 - g(|w4\Lambda_{n-1}(\mathbf{k}) + (1 - w)G_n|)) \\
\leq&(2|\Lambda_{n-1}(\mathbf{k})| + 1)^p g(|w4\Lambda_{n-1}(\mathbf{k}) + (1 - w)G_n|) + ((-2|\Lambda_{n-1}(\mathbf{k})|)' + 1)(1 - g(|w4\Lambda_{n-1}(\mathbf{k}) + (1 - w)G_n|) \\
\leq&M_0 + (-|\Lambda_{n-1}(\mathbf{k})|)'.
\end{aligned}$$

Hence, it order to make $\mathbb{E}\left((V_n - V_{n-1})'|Z_n = \mathbf{k}^*, \mathcal{F}_{n-1}\right) \leq M_2$, it's sufficient to find a $\mathbf{k}^*$ such that $|\Lambda_{n-1}(\mathbf{k}^*)| > \frac{w}{1-w}$ and $(-|\Lambda_{n-1}(\mathbf{k})|)' \leq M_2 - M_0$ whenever $V_{n-1} > J$. We simply select $C \leq M_2 - M_0$, so we need to find $\mathbf{k}^*$ such that

$$|\Lambda_{n-1}(\mathbf{k}^*)| > max(M_0 - M_2, \frac{w}{1-w}) = M_3.$$

Suppose $V_{n-1} > J$, we claim that there $\exists\ \mathbf{k}^*$ such that $|\Lambda_{n-1}(\mathbf{k}^*)| > \sqrt{\frac{J}{m'a'}}$, where m' and a' are known constants. Hence, C1 is satisfied when $J = (M_3)^2 m'a'$.

We prove the claim as following.

$$\begin{aligned}
V_{n-1} &= \sum_{\mathbf{k}} w_s D_{n-1}^2(\mathbf{k}) + \sum_{i=1}^{I}\sum_{k_i=1}^{m_i} w_{m_i} D_{n-1}^2(i;k_i) + w_o D_{n-1}^2 \\
&\leq m w_s D_{n-1}^2(\mathbf{k}_{max}) + \sum_{i=1}^{I}\sum_{k_i=1}^{m_i} w_{m_i} \prod_{j\neq i}^{m_j} D_{n-1}^2(\mathbf{k}_{max}) + w_o D_{n-1}^2(\mathbf{k}_{max}) \\
&\leq m w_s D_{n-1}^2(\mathbf{k}_{max}) + (w_{m_i})_{max} m^2 D_{n-1}^2(\mathbf{k}_{max}) + w_o m^2 D_{n-1}^2(\mathbf{k}_{max}) \\
&\leq m' D_{n-1}^2(\mathbf{k}_{max}),
\end{aligned}$$

where $D_{n-1}^2(\mathbf{k}_{max}) = max_{\mathbf{k}} D_{n-1}^2(\mathbf{k})$ and m' is a constant. Therefore $V_{n-1} > J$ implies $|D_{n-1}(\mathbf{k}_{max})| > \sqrt{\frac{J}{m'}}$. In addition, since Proposition 3.1 in Hu and Zhang (2020) shows $D_{n-1}(\mathbf{k})$ is a linear combination of $\mathbf{\Lambda}_{n-1}(\mathbf{k})$, so

$$\sqrt{\frac{J}{m'}} \leq |D_{n-1}(\mathbf{k})| = |a^\top \mathbf{\Lambda}_{n-1}| \leq \sum_{i=1}^{m}(|a_i||\Lambda_{n-1}(\mathbf{k})|),$$

where $a^\top$ is a $m \times 1$ vector with known constant $a_1, a_2, ..., a_m$.

Let $|\Lambda_{n-1}(\mathbf{k}^*)| = max_{\mathbf{k}}|\Lambda_{n-1}(\mathbf{k})|$ and $a' = (\sum_{i=1}^m |a_i|)^2$, we have

$$\sqrt{\frac{J}{m'}} \leq \sum_{i=1}^{m}(|a_i||\Lambda_{n-1}(\mathbf{k})|) \leq (\sum_{i=1}^{m}|a_i|)|\Lambda_{n-1}(\mathbf{k}^*)| \leq \sqrt{a'}|\Lambda_{n-1}(\mathbf{k}^*)|.$$

Therefore, there exists $\mathbf{k}^*$ such that $|\Lambda_{n-1}(\mathbf{k}^*)| > \sqrt{\frac{J}{m'a'}}$ whenever $V_{n-1} > J$.

A.3.3. SHOW C2 IS SATISFIED

Since $[(V_n - V_{n-1})']^- \leq |C|$, we have

$$\mathbb{E}[([(V_n - V_{n-1})']^-)^p|\mathcal{F}_{n-1}] \leq |C|^p$$

By the Minkovsky inequality,

$$\begin{aligned}
&\mathbb{E}(|(V_n - V_{n-1})'|^p|\mathcal{F}_{n-1}) \\
=&\mathbb{E}[([(V_n - V_{n-1})']^+)^p|\mathcal{F}_{n-1}] + \mathbb{E}[([(V_n - V_{n-1})']^-)^p|\mathcal{F}_{n-1}] \\
\leq&(M_1^{1/p} + |C|)^p = V.
\end{aligned}$$

Therefore, $C2$ is satisfied.

Since both $C1$ and $C2$ are satisfied, by Lemma 4.3 we have: for any $r \in (0, p-1)$ there exist a $c = (p, V, J, r) > 0$ such that $E(V_n)^r < c$ for all n.

Refer to (6.5) in Hu and Zhang (2020) we have $\|\mathbf{\Lambda}_n\|^2 \leq mV(\mathbf{\Lambda}_n)$. It follows that $\sup_n \mathbb{E}\|\mathbf{\Lambda}_n\|^{2r} < \infty$. Thus, we conclude that $\mathbb{E}\|\mathbf{\Lambda}_n\|^r = O(1)$ for all $0 < r < p-1$, which completes the proof.

A.4. Proof of Corollary 4.6

Proof. We have

$$\begin{aligned}
MSE(W) &= \mathbb{E}_{\mathbf{T}}\left[\{\mathbb{E}_\epsilon(W|\mathbf{T}) - \mu_1\}^2\right] \\
&= \frac{4}{N^2}\mathbb{E}_{\mathbf{T}}\left[\{(2\mathbf{T}-\mathbf{1})^\top f(A\mathbf{T})\}^2\right] \\
&+ \frac{4}{N^2}\mathbb{E}_{\mathbf{T}}\left[\{(2\mathbf{T}-\mathbf{1})^\top \mathbf{X}\boldsymbol{\beta}\}^2\right] \\
&+ \frac{8}{N^2}\mathbb{E}_{\mathbf{T}}\left[(2\mathbf{T}-\mathbf{1})^\top f(A\mathbf{T})(2\mathbf{T}-\mathbf{1})^\top \mathbf{X}\boldsymbol{\beta}\right].
\end{aligned} \tag{18}$$

As $(2\mathbf{T}-\mathbf{1})^\top \mathbf{X} = O_p(1)$, the second term of (18) satisfies

$$\frac{4}{N^2}\mathbb{E}_{\mathbf{T}}\left[\{(2\mathbf{T}-\mathbf{1})^\top \mathbf{X}\boldsymbol{\beta}\}^2\right] = o(1). \tag{19}$$

Similarly, the third terms of (18) satisfies

$$\begin{aligned}
&\frac{8}{N^2}\mathbb{E}_{\mathbf{T}}\left[(2\mathbf{T}-\mathbf{1})^\top f(A\mathbf{T})(2\mathbf{T}-\mathbf{1})^\top \mathbf{X}\boldsymbol{\beta}\right] \\
\leq&\frac{k}{N^2}\mathbb{E}_{\mathbf{T}}\left[(2\mathbf{T}-\mathbf{1})^\top f(A\mathbf{T})\right] \leq \frac{k}{N^2}\mathbb{E}(\mathbf{1}^\top f(A\mathbf{T})) \\
\leq&\frac{k}{N^2}(\mathbf{1}^\top f(\mathbb{E}A\mathbf{T})) \leq k\frac{f(N)}{N}.
\end{aligned}$$

B. Hypothetical Data Generation

We generate the clustering graph (CUG) according to the following model:

$$\mathbb{P}(\mathbf{A}_{ij} = 1) = \theta_i\theta_j p_b\left(1 - \left(\frac{|v_i - v_j|}{2b_{scale}}\right)^2\right)^2 + p_a. \tag{20}$$

Here $v_i = 2b_{scale}(\frac{v_i^{b-1}(1-v_i)^{b-1}}{B(b,b)} - \frac{1}{2})$ represents the unobserved covariate of the i-th user and $B(b,b) = \frac{\Gamma(b)\Gamma(b)}{\Gamma(2b)}$. The prior b controls the clustering of the generated network. b_{scale} is a pre-defined scalar. Under the Beta distribution $B(b,b)$, the generated network is similar to a complete random graph when $b > 10$. The prior p_b represents the average probability that users are to connect with others. The p_a is a corresponding minimum likelihood. That two parameters, especially the p_b, control the density of the generated network. Moreover, the random variable θ_i, $i = 1, ..., n$, represents that compared to the population connect density p_b, how easily the i-th user is likely to connect with others. Note that v_i and θ_i can imply certain unobserved covariates that affect the connection generation—for example, users' age, education, etc.

C. Code and Data

The R codes and real data used for the numerical studies are available at https://github.com/jialush/AB-Testing-in-Network-Data-with-Covariate-Adaptive-Randomization.git.