EIGENVALUE SPECTRUM SUPPORT OF PAIRED RANDOM MATRICES WITH PSEUDO-INVERSE

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ABSTRACT

The Moore-Penrose pseudo-inverse $X^{\dagger}$, defined for rectangular matrices, naturally emerges in many areas of mathematics and science. For a pair of rectangular matrices X, Y where the corresponding entries are jointly Gaussian and i.i.d., we analyse the support of the eigenvalue spectrum of $XY^{\dagger}$.

Keywords Random Matrix Theory · Moore-Penrose pseudo-inverse · Eigenvalue Spectrum

1 Introduction

We are interested in pairs of rectangular random matrices of equal size where their corresponding elements are independent, identically distributed (i.i.d.) from a 2-D joint distribution.

Definition 1 (Real Paired Gaussian Matrices). *For $N, P \in \mathbb{N}$ and covariance matrix $\Sigma \in \mathbb{R}^{2\times 2}$, Real Paired Gaussian Matrices are a pair of real rectangular matrices of $X, Y \in \mathbb{R}^{N\times P}$ where corresponding entries $x = X_{i\mu}, y = Y_{i\mu}$ for any $i = 1 \dots N$, $\mu = 1 \dots P$, are jointly i.i.d. Gaussian $(x, y) \sim \mathcal{N}(0, \Sigma/N)$.*

Definition 2 (Complex Paired Gaussian Matrices with independent components). *For $N, P \in \mathbb{N}$, Σ as above, Complex Paired Gaussian Matrices with independent components are complex rectangular matrices $X, Y \in \mathbb{C}^{N\times P}$ such that $\mathrm{Re}(X), \mathrm{Re}(Y)$ are real paired Gaussian matrices with covariance Σ_{Re}, $\mathrm{Im}(X), \mathrm{Im}(Y)$ are real paired Gaussian matrices with covariance Σ_{Im}, and satisfy $\Sigma_{Re} + \Sigma_{Im} = \Sigma$ (i.e., the real and imaginary components are independent).*

Definition 3 (Complex Paired Gaussian Matrices). *For $N, P \in \mathbb{N}$, covariance matrix $\Gamma \in \mathbb{R}^{4\times 4}$, Complex Paired Gaussian Matrices are a pair of complex rectangular matrices of $X, Y \in \mathbb{C}^{N\times P}$ where corresponding entries $x = X_{i\mu}, y = Y_{i\mu}$ are jointly i.i.d. Gaussian $(\mathrm{Re}x, \mathrm{Im}, \mathrm{Re}y, \mathrm{Im}y) \sim \mathcal{N}(0, \Gamma/N)$.*

Note that Definition 2 generalises Definition 1 as it correspond to the case of $\Sigma_{\mathrm{Im}} = 0$. Without loss of generality,

$$\Sigma = \mathrm{Var}(x, y) = \begin{pmatrix} \sigma_x^2 & \tau\sigma_x\sigma_y \\ \bar{\tau}\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix}$$

for $\sigma_x, \sigma_y \in \mathbb{R}^+$ and $|\tau| \le 1$, where $\tau \in \mathbb{R}$ for the first two definitions and $\tau \in \mathbb{C}$ for the third definition.

Denoting the dimensions ratio $\alpha = P/N$, we consider a matrix $M \in \mathbb{C}^{N\times N}$ defined from paired Gaussian matrices X, Y using either a conjugate transpose $M = XY^*$ (a scenario previously discussed under the name "non-Hermitian Wishart ensemble" [1]) or a pseudo-inverse [2] $M = XY^{\dagger}$ and wish to calculate the support of the limiting spectral density of M in terms of $\sigma_x, \sigma_y, \tau, \alpha$, namely the set with positive density for $N \to \infty$.

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2 Results

We denote the empirical spectral density of a matrix $M_N \in \mathbb{C}^{N\times N}$ as $\mu_M^N(\omega) = \frac{1}{N}\sum_i^N \delta(\omega - \lambda_i(M_N))$. If a series of such measures converges weakly to a limiting spectral density, we denote it $\mu_M(\omega)$, and denote its support $\mathcal{S}_M = \{\omega : \mu_M(\omega) > 0\}$, with a slight abuse of notation, as both μ_M and $\mathcal{S}_M$ are not defined for a specific matrix M.

Theorem 1. *For complex paired Gaussian matrices with independent components X, Y, the support of the limiting spectral density of $M = XY^*$ is:*

$$\mathcal{S}_{XY^*} = \{0\}_{\alpha<1} \cup \left\{\lambda : \left(\frac{\mathrm{Re}\lambda - \sigma_x\sigma_y(1+\alpha)\mathrm{Re}(\tau)}{\sigma_x\sigma_y\sqrt{\alpha}\left(1+|\tau|^2\right)}\right)^2 + \left(\frac{\mathrm{Im}\lambda - \sigma_x\sigma_y(1+\alpha)\mathrm{Im}(\tau)}{\sigma_x\sigma_y\sqrt{\alpha}\left(1-|\tau|^2\right)}\right)^2 \leq 1\right\} \tag{1}$$

The support is an ellipsoid if $\alpha \geq 1$, or a union thereof with 0 if $\alpha < 1$. For the case discussed here, $\tau \in \mathbb{R}$ so that $\mathrm{Im}(\tau) = 0$, but this equation is valid in a more general case: for a complex τ the support is rotated by $\arg(\tau)$.

Theorem 2. *For complex paired Gaussian matrices X, Y, the support of the limiting spectral density of $M = XY^*$ is $e^{i\arg(\tau)}\mathcal{S}_{XY^*} = \left\{\lambda : e^{-i\arg(\tau)}\lambda \in \mathcal{S}_{XY^*}\right\}$.*

Theorem 3. *For paired Gaussian matrices X, Y with $\alpha \neq 1$, denoting $\beta = \max\{1/\alpha, \alpha\}$, the support of the limiting spectral density of $M = XY^\dagger$ is:*

$$\mathcal{S}_{XY^\dagger} = \{0\}_{\alpha<1} \cup \left\{\lambda : \left|\lambda - \frac{\sigma_x}{\sigma_y}\tau\right|^2 \leq \frac{\sigma_x^2}{\sigma_y^2}\frac{1-|\tau|^2}{\beta - 1}\right\} \tag{2}$$

The support is a circle if $\alpha > 1$, or a union thereof with 0 if $\alpha < 1$, and the case $\alpha = 1$ is not covered.

Conjecture 1. *For paired Gaussian matrices, the support of $\mu_{XY^*}^N$ converges to the support of the limiting μ_{XY^*}.*

This corresponds to the lack of isolated outliers for XY^*. For example, this property has been proven for the Ginibre ensemble, where it was further demonstrated that outliers can be created using bounded rank perturbations [3]. This might be proven by showing that the Brown measure is continuous with respect to the topology of convergence [4].

3 Proofs

Proof of Theorem 1 for $\alpha \geq 1$. This case is the main result of [1], where it is derived (using a different notation) for $\alpha \geq 1$ that $\mu_{XY^*}^N$ converges weakly to a limiting spectral density μ_{XY^*} with the specified support, with the additional assumptions that $\sigma_x = \sigma_y = 1$, and $\Sigma_{Re} = \Sigma_{Im}$. Because the resulting eigenvalues scale multiplicatively with $\sigma_x\sigma_y$, Eq. 1 is obtained from Eq. 1.7 in [1] by scaling λ into $\lambda/\sigma_x\sigma_y$. Furthermore, their result depends only on τ, the off-diagonal term of $\Sigma = \Sigma_{Re} + \Sigma_{Im}$, and thus generalises to any choice of Σ_{Re}, Σ_{Im}, as in our definitions. □

Proof of Theorem 1 for $\alpha < 1$. We note the characteristic polynomial of XY^* can be related to that of Y^*X by the Weinstein-Aronszajn identity $p_{XY^*}(x) = \det(xI - XY^*) = x^{N-P}\det(xI - Y^*X) = x^{N-P}p_{Y^*X}(x)$ so the eigenvalues of XY^* are $N-P$ zeros, and the P eigenvalues of Y^*X. This relation holds exactly for a finite N, $\mu_{XY^*}^N(\lambda) = (1-\alpha)\delta(\lambda) + \alpha\mu_{XY^*}^P(\lambda)$, and thus also for the limiting spectral density. The measure μ_{XY^*} is supported, according to the first half of the proof, at the following ellipsoid from Eq. 1 with dimensions ratio $1/\alpha > 1$, and additional scaling of $\sigma_x\sigma_y$ to $\frac{P}{N}\sigma_x\sigma_y$ due to correcting the scaling from Σ/N in Definition 1 into Σ/P. Those terms cancel, and the ellipsoid support from Eq. 1 is the same in both cases. □

This also provides the exact limiting spectral density of XY^* for $\alpha < 1$, in terms of the known result for $\alpha > 1$ [1].

Proof of Theorem 2. We note it is possible to diagonalise Γ, a 4×4 positive-definite matrix, using three rotation operations, one applied to components of x, one applied to the components of y, and one applied at the 2×2 block structure. The latter is equivalent to multiplication by a complex scalar c. The former are equivalent to multiplying the complex x (respectively y) by a constant c_x (respectively c_y) such that $c_x x$ (respectively $c_y y$) are complex Gaussian variables with independent real and imaginary components. As all the components of X, Y are identically distributed, $cc_x\bar{c_y}XY^*$ satisfy Definition 2 and hence their support is given by Eq. 1. Furthermore, this constant can be calculated as $\arg(\tau)$; the norms of the constants would not affect this normalised quantity. Finally, the effect of this multiplication is a rotation of the support around 0, so that the centre of the ellipsoid moves from $\sigma_x\sigma_y(1-\alpha)\tau + i0$ in the independent case to $\sigma_x\sigma_y(1-\alpha)(\mathrm{Re}(\tau) + i\mathrm{Im}(\tau))$ in the general case, as well as rotation of each λ into $e^{i\arg(\tau)}\lambda$. □

The following corollary can be drawn from Eq. 1, which we will use below. It was already noted in [1] for $\tau \in \mathbb{R}$.

Corollary 1. *For paired Gaussian matrices,* $0 \in \mathcal{S}_{XY^*}$ *iff* $|\tau|^2 \leq 1/\alpha$.

Proof. The rotation in Theorem 2 is around 0, so the condition is the same for complex paired Gaussian matrices with or without independent components. For $\alpha < 1$, the statement is trivial as both terms are true by definition. For $\alpha \geq 1$, the left term becomes $|\tau|(1+\alpha) \leq \sqrt{\alpha}\left(1+|\tau|^2\right)$ and denoting $g(x) = \frac{x}{1+x^2}$ yields $g(|\tau|) \leq g(1/\sqrt{\alpha})$. Thus, $|\tau| \leq 1/\sqrt{\alpha}$ from monotonicity of $g(x)$ for $x \in [0,1]$. □

We prove Theorem 3 by showing how the condition $\lambda \in \mathcal{S}_{XY^\dagger}$ can be reduced to the condition $0 \in Y^* Z_\lambda$, , for some matrix Z_λ, which we already understand from Corollary 1. The proof requires the yet unproven Conjecture 1.

Proof of Theorem 3 for $\alpha < 1$*, assuming Conjecture 1.* In this case $Y^\dagger = (Y^*Y)^{-1} Y^*$, and from the generalised matrix determinant lemma [5], the characteristic polynomial of $XY^\dagger$ would be $p_{XY^\dagger}(x) = \det\left(xI - X(Y^*Y)^{-1}Y^*\right) = \det\left(Y^*Y - \frac{1}{x}Y^*X\right)\frac{x^N}{\det(Y^*Y)}$ where $\det(Y^*Y)$ is a finite, strictly positive value for $\alpha < 1$ from Marchenko-Pastur [6]. We note that when $x \to 0$ the determinant is dominated by x^{-P} and the characteristic polynomial would have x^{N-P}. Thus, at least $N-P$ of the eigenvalues of $XY^\dagger$ are 0, and this value is included in the support. For $0 \neq \lambda \in \text{E.V.}\left(XY^\dagger\right)$ we have that it satisfies $0 = \det(Y^*Z)$ for $Z = Y - X/\lambda$. Now note that for a fixed $\lambda \neq 0$, we can consider a series of $P \times P$ matrices $M_P = Y^*Z$ where Y, Z are paired Gaussian matrices, with dimensions ratio $1/\alpha$. Assuming Conjecture 1, the support of the eigenvalue spectrum of M_P converges for $P \to \infty$ to the support of the limiting density Eq. 1, so except for a set whose measure vanishes, by Corollary 1 it is strictly positive for $|\tau_\lambda|^2 < \alpha$, and 0 otherwise, where $\tau_\lambda = \text{corrcoef}(y, y - x/\lambda)$. Using the joint distribution of x, y:

$$|\tau_\lambda|^2 = \frac{\langle \delta\bar{y}\delta(y - x/\lambda)\rangle \left\langle \delta y \delta\overline{y - x/\lambda}\right\rangle}{\langle \delta y \delta\bar{y}\rangle \left\langle \delta(y - x/\lambda)\delta\overline{y - x/\lambda}\right\rangle} = \frac{\sigma_y^2 - 2\sigma_x\sigma_y \text{Re}(\tau/\lambda) + |\tau|^2 \sigma_x^2/|\lambda|^2}{\sigma_y^2 - 2\sigma_x\sigma_y \text{Re}(\tau/\lambda) + \sigma_x^2/|\lambda|^2} \tag{3}$$

and substituting $\text{Re}(\tau/\lambda) = (\text{Re}\tau\text{Re}\lambda + \text{Im}\tau\text{Im}\lambda)/|\lambda|^2$ the condition on λ becomes:

$$(1-\alpha)|\lambda|^2 \sigma_y^2 - 2(1-\alpha)\sigma_x\sigma_y(\text{Re}\tau\text{Re}\lambda + \text{Im}\tau\text{Im}\lambda) + \left(|\tau|^2 - \alpha\right)\sigma_x^2 \leq 0 \tag{4}$$

which can be rewritten as a circular law $|\lambda - c|^2 \leq r^2$ with a centre $c = \tau\frac{\sigma_x}{\sigma_y}$ and square radius $r^2 = \frac{\sigma_x^2}{\sigma_y^2}\left(1-\tau^2\right)\frac{\alpha}{1-\alpha}$, so Eq. 2 follows for $\beta = 1/\alpha$. □

Proof of Theorem 3 for $\alpha > 1$*, assuming Conjecture 1.* In this case $Y^\dagger = Y^*(YY^*)^{-1}$, and from the generalised matrix determinant lemma [5] the characteristic polynomial is $p_{XY^\dagger}(x) = \det\left(xI - XY^*(YY^*)^{-1}\right) = \det\left(YY^* - \frac{1}{x}XY^*\right)\frac{x^N}{\det(YY^*)}$. It is not expected to have zeros at $x = 0$, as the determinant would contribute x^{-N} for $x \to 0$. For $0 \neq \lambda \in \text{E.V.}\left(XY^\dagger\right)$, we have that it satisfies $0 = \det(ZY^*)$ for $Z = Y - X/\lambda$, and the argument continues as in $\alpha < 1$. Here, for a fixed $\lambda \neq 0$, we can consider a series of $N \times N$ matrices $M_N = ZY^*$ where Y, Z are paired Gaussian matrices, with dimensions ratio α (instead of $1/\alpha$ in the $\alpha < 1$ case). Assuming Conjecture 1, the support of the eigenvalue spectrum of M_N converges for $N \to \infty$ to the support of the limiting density Eq. 1, so except for a set whose measure vanishes, by Corollary 1 it is strictly positive for $|\tau_\lambda|^2 < \alpha$, and 0 otherwise, so that Eq. 3 is unmodified and Eq. 4 has $1/\alpha$ terms instead of α terms. Eq. 2 follows with $\beta = \alpha$. □

We note that the above approach for Theorem 3 does not apply to $\alpha = 1$, as $Y^\dagger = Y^{-1}$ in this case.

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