Abstract

In many real-world scenarios it is crucial to be able to reliably and efficiently reason under uncertainty while capturing complex relationships in data. Probabilistic circuits (PCs), a prominent family of tractable probabilistic models, offer a remedy to this challenge by composing simple, tractable distributions into a high-dimensional probability distribution. However, learning PCs on heterogeneous data is challenging and densities of some parametric distributions are not available in closed form, limiting their potential use. We introduce characteristic circuits (CCs), a family of tractable probabilistic models providing a unified formalization of distributions over heterogeneous data in the spectral domain. The one-to-one relationship between characteristic functions and probability measures enables us to learn high-dimensional distributions on heterogeneous data domains and facilitates efficient probabilistic inference even when no closed-form density function is available. We show that the structure and parameters of CCs can be learned efficiently from the data and find that CCs outperform state-of-the-art density estimators for heterogeneous data domains on common benchmark data sets.

1 INTRODUCTION

Probabilistic circuits (PCs) have gained increasing attention in the machine learning community as a promising modelling family that renders many probabilistic inferences tractable with little compromise in their expressivity. Their beneficial properties have prompted many successful applications in density estimation [e.g., Peharz et al., 2020, Di Mauro et al., 2021, Dang et al., 2022, Correia et al., 2023] and in areas where probabilistic reasoning is key, for example, neuro-symbolic reasoning [Ahmed et al., 2022], certified fairness [Selvam et al., 2023], or causality [Zečević et al., 2021]. Moreover, recent works have explored ways of specifying PCs for more complex modelling scenarios, such as time-series [e.g., Trapp et al., 2020, Yu et al., 2021b,a] or tractable representation of graphs [Wang et al., 2022].

Density estimation is at the very core of many machine learning techniques, e.g., approximate Bayesian inference [Murphy, 2012], and a fundamental tool in statistics to identify characteristics of the data such as \( n \)th order moments or multimodality [Silverman, 2018]. However, even in the case of parametric families, densities are sometimes not available in closed-form, e.g., only special cases of \( \alpha \)-stable distributions provide closed-form densities [Nolan, 2013]. Fortunately, there exists a one-to-one correspondence between probability measures and characteristic functions (CFs) [Sasvári, 2013], which can be understood as the Fourier-Stieltjes transform of the probability measures. Thus, enabling the characterisation of any probability measure through its CF. Henceforth, the CF of probability measures has found wide applicability in statistics, ranging from its use as a non-parametric estimator through the empirical characteristic function (ECF) [Feuerverger and Mureika, 1977] to estimate heavy-tailed data e.g., through the family of \( \alpha \)-stable distributions [Nolan, 2013]. However, even though the CF
has many advantageous properties and provides greater flex-

in this work, we bridge between the CF of probability mea-
sures and PCs, by examining PCs from a more general

Instead of defining the circuit over density functions, we propose to form the circuit over the characteristic function of the respective probability measures, illustrated in Fig. 1. Motivated by the property of characteristic function that the moments can be easily obtained by differentiating, we notice learning a circuit in the spectral domain has many interesting benefits: (i) characteristic functions as the base enables a unified view for discrete and continuous random variables (RVs), (ii) directly representing the characteristic function allows to learn distributions that do not have closed-form expressions for their density.

In summary, our contributions are: (1) We propose characteristic circuits, a novel deep probabilistic model representing the joint of discrete and continuous RVs through a unifying view in the spectral domain. (2) Moreover, we show that characteristic circuits retain tractability of PCs despite the change of domain and enable efficient computation of densities, marginals, and conditionals. (3) Lastly, we derive parameter and structure learning for characteristic circuits and find that characteristic circuits outperform SOTA density estimators in the majority of tested benchmarks.

2 PRELIMINARIES ON PROBABILISTIC CIRCUITS AND CHARACTERISTIC FUNCTIONS

Before introducing characteristic circuit, let us recap probabilistic circuits and characteristic functions.

Probaabilistic Circuits. PCs are tractable probabilistic models, structured as rooted directed acyclic graphs, where each leaf node $L$ represents a probability distribution over a univariate RV, each sum node $S$ models a mixture of its children, and each product node $P$ models a product distribution (assuming independence) of their children. A PC over a set of RVs $X$ can be viewed as a computational graph $G$ representing a tractable probability distribution over $X$, and the value obtained at the root node is the probability computed by the circuit. We refer to Choi et al. [2020] for more details.

Each node in $G$ is associated with a subset of $X$ called the scope of a node $N$ and is denoted as $\psi(N)$. The scope of an inner node is the union of the scope of its children. Sum nodes compute a weighted sum of their children $S = \sum_{N \in \psi(N)} w_{N,S}N$, and product nodes compute the product of their children $P = \prod_{N \in \psi(P)} N$, where $\psi(N)$ denotes the children of a node. The weights $w_{N,S}$ are generally assumed to be non-negative and normalized (sum up to one) at each sum node. We also assume the PC to be smooth (complete) and decomposable [Darwiche, 2003], where smooth requires all children of a sum node having the same scope, and decomposable means all children of a product node having pairwise disjoint scopes.

Characteristic Functions. CFs provide a unified view for discrete and continuous RVs through the Fourier–Stieltjes transform of their probability measures. Let $X$ be a random vector, the CF of $X$ for $t \in \mathbb{R}^d$ is given as:

$$\varphi_X(t) = \mathbb{E}[\exp(i t^\top X)] = \int_{x \in \mathbb{R}^d} \exp(i t^\top x) \mu_X(dx),$$

where $\mu_X$ is the distribution/probability measure of $X$. CFs have certain useful properties. We will briefly review those that are relevant for the remaining discussion: (i) $\varphi_X(0) = 1$ and $|\varphi_X(t)| \leq 1$; (ii) for any two RVs $X_1, X_2$, both have the same distribution iff $\varphi_{X_1} = \varphi_{X_2}$; (iii) if $X$ has $k$ moments, then $\varphi_X$ is $k$-times differentiable; and (iv) two RVs $X_1, X_2$ are independent iff $\varphi_{X_1, X_2}(s,t) = \varphi_{X_1}(s)\varphi_{X_2}(t)$. We refer to Sasvári [2013] for more details of CFs.

Theorem 2.1 (Lévy’s inversion theorem). Let $X$ be a real-valued random variable, $\mu_X$ its probability measure, and $\varphi_X : \mathbb{R} \to \mathbb{C}$ its characteristic function. Then for any $a, b \in \mathbb{R}$, $a < b$ we have that

$$\lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{\exp(-it a) - \exp(-it b)}{it} \varphi_X(t) dt = \mu_X([a, b]) + \frac{1}{2}(\mu_X(a) + \mu_X(b))$$

(1)

and, hence, $\varphi_X$ uniquely determines $\mu_X$.

Corollary. If $\int_{\mathbb{R}} |\varphi_X(t)| dt < \infty$, then $X$ has a continuous probability density function $f_X$ given by

$$f_X(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp(-itx) \varphi_X(t) dt.$$ (2)

Note that not every probability measure admits an analytical solution to Eq. (2), e.g., only special cases of $\alpha$-stable distributions have a closed-form density function [Nolan, 2013], and numerical integration might be needed.

3 CHARACTERISTIC CIRCUIT

Now we have everything at hand to derive characteristic circuit. We first give a recursive definition of CC, followed
by devising each type of node in a CC. We then show CCs feature efficient computation of densities, and in the end introduce how to learn a CC.

**Definition 3.1 (Characteristic Circuit).** Let $X = \{X_1, \ldots, X_d\}$ be a set of random variables. A characteristic circuit denoted as $C$ is a tuple consisting of a rooted directed acyclic graph $G$, a scope function $\psi: V(G) \to \mathcal{P}(X)$, parameterized by a set of graph parameters $\theta_G$. Nodes in $G$ are either sum ($S$), product ($P$), or leaf ($L$).

We define the characteristic circuit in the following recursive way: (1) a CF for a scalar random variable is a characteristic circuit, (2) a product of characteristic circuits is a characteristic circuit, (3) a convex combination of characteristic circuits is a characteristic circuit.

Denote $\varphi_C(t)$ the output of $C$ computed at the root of $C$ which represents the estimation of CF given argument of the CF $t \in \mathbb{R}^d$. Further, we denote the number of RVs in the scope of $N$ as $p_n := |\psi(N)|$ and use $\varphi_N(t)$ for the CF of a node. Throughout the paper we assume the CC to be smooth and decomposable.

**Product Nodes.** A product node in CC encodes the independence of its children. Let $X$ and $Y$ be two RVs. Following property (iv) of CFs, the CF of $X, Y$ is given as $\varphi_{XY}(t, s) = \varphi_X(t)\varphi_Y(s)$, if and only if $X$ and $Y$ are independent. Therefore, by definition, with $t = \bigcup_{N \in \text{ch}(P)} t\psi(N)$, the characteristic function of product nodes is given as:

$$\varphi_P(t) = \prod_{N \in \text{ch}(P)} \varphi_N(t\psi(N)).$$

**Sum Nodes.** A sum node in CC encodes the mixture of its children. Let the parameters of $S$ be given as $\sum_{N \in \text{ch}(S)} w_{S,N} = 1$ and $w_{S,N} \geq 0, \forall S, N$. Then the sum node in a CC is given as:

$$\varphi_S(t) = \int_{\mathbb{R}^d} \exp(it^T x) \left[ \sum_{N \in \text{ch}(S)} w_{S,N} \mu_N(dx) \right].$$

**Leaf Nodes.** The leaf node of a CC models the characteristic function of a univariate RV. For discrete RVs, we utilize categorical distributions and for continuous RVs, we use either normal distribution or $\alpha$-stable distributions. Besides, ECF leaf $\varphi_{\text{ECF}}(t) = \frac{1}{n} \sum_{j=1}^{n} \exp(itx_j)$ is also employed as a non-parametric leaf. A more detailed discussion on leaf types can be found in Appendix A.

### 3.1 THEORETIC PROPERTIES

With everything at hand, we can derive the marginal density.

**Efficient Computation of Densities.** Through their recursive nature, CCs enable efficient computation of densities in high-dimensional settings even if the density function is not available in closed form. For this, we present an extension of Theorem 2.1 for CCs, formulated using the notion of induced trees $T$ [Zhao et al., 2016].

**Lemma 3.2 (Inversion).** Let $C = \langle G, \psi, \theta_G \rangle$ be a CC on RVs $X = \{X_j\}_{j=1}^{d}$ with univariate leaf nodes. If $\int_{\mathbb{R}} \varphi_C(t) dt < \infty$ for every $L \in V(G)$, then $X$ has a continuous probability density function $f_x$ given by $f_x(x) = \frac{1}{(2\pi)^d} \sum_{i=1}^{T} \prod_{(S,N) \in E(T_i)} w_{S,N} \prod_{L \in V(T_i)} \int_{\mathbb{R}} \exp(-itx\psi(L))\varphi_L(t) dt$.

**Computation of Marginals.** Similar to PCs over distribution functions, CCs allow efficient computation of arbitrary marginals. Given a CC on RVs $Z = X \cup Y$, we can obtain the marginal CC over $X$ as follows. Let $n = |X|$, $m = |Y|$, $t = tx \cup ty \in \mathbb{R}^{n+m}$ and let the characteristic function of the circuit be given by $\varphi_C(t_x, t_y) = \int_{\mathbb{R}^{n+m}} \exp(it^T x)\mu_S(dx)$, where $\mu_S$ denotes the distribution of the root. Then following property (i) of CFs, the marginal CC over $X$ is given by setting $t_y = 0$.

### 3.2 LEARNING CHARACTERISTIC CIRCUIT

Inspired by Gens and Pedro [2013], we propose a structure learning algorithm to learn the structure of the CC, depicted in Algorithm 1. Structure learning recursively splits the data slice and creates sum and product nodes of the CC. Univariate CF leaves are created to model the local data.

On the other hand, given a random circuit structure, parameter learning can be employed to learn a CC. Instead of maximising the likelihood, which is not guaranteed to be tractable, we minimise the approximated squared characteristic function distance (CFD, c.f. Appendix C) between the

<table>
<thead>
<tr>
<th>Algorithm 1 CC Structure Learning</th>
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<tbody>
<tr>
<td><strong>Input:</strong> training data $D$, RVs $X$, threshold $\text{min}_k$, number of splits $k_S$ and $k_P$</td>
</tr>
<tr>
<td><strong>Output:</strong> $C$</td>
</tr>
<tr>
<td><strong>Function</strong> buildCF($D, X$)</td>
</tr>
<tr>
<td>$\text{L}$ return $L \leftarrow \text{univariate CF}_{D, X}(t)$</td>
</tr>
<tr>
<td><strong>Function</strong> buildSumNode($D, X$)</td>
</tr>
<tr>
<td>if $</td>
</tr>
<tr>
<td>$S \leftarrow \text{buildCF}(D, X)$</td>
</tr>
<tr>
<td>else</td>
</tr>
<tr>
<td>if $</td>
</tr>
<tr>
<td>Partition $D$ into $</td>
</tr>
<tr>
<td>$S \leftarrow \prod_{i=1}^{</td>
</tr>
<tr>
<td>else</td>
</tr>
<tr>
<td>Partition $D$ into $k_S$ clusters $D_j$</td>
</tr>
<tr>
<td>$S \leftarrow \sum_{i=1}^{k_S} \prod_{D_j} \text{buildProdNode}(D_j, X_j)$</td>
</tr>
<tr>
<td>return $S$</td>
</tr>
<tr>
<td><strong>Function</strong> buildProdNode($D, X$)</td>
</tr>
<tr>
<td>Partition $D$ into $k_P$ independent subsets $D_j$</td>
</tr>
<tr>
<td>return $P \leftarrow \prod_{j=1}^{k_P} \text{buildSumNode}(D_j, X_j)$</td>
</tr>
<tr>
<td>$C \leftarrow \text{buildSumNode}(D, X)$</td>
</tr>
</tbody>
</table>
We first learn CCs applying structure learning with various world tabular data usually contain both discrete and real-valued elements, and thus are in most cases heterogeneous data. Therefore, we also conduct experiments on the UCI data distribution better than ECF, which can be justified the CFD between the CC and the ECF by providing a smaller CFD. We visualize the CFD Figure 2: CC approximates the true distribution better than the ECF by providing a smaller CFD. We visualize the CFD for CC with CC-P ( ), CC-E ( ), CC-N ( ) and a single ECF ( ) learned from MN (Left) and EN (Right).

Table 1: Average test log-likelihoods from CC and SOTA algorithms on heterogeneous data. CC with α-stable distribution leaves (CC-A) wins 7 out of 12 data sets.

<table>
<thead>
<tr>
<th>Data Set</th>
<th>MSPN</th>
<th>ABDA</th>
<th>BSPN</th>
<th>CC-P</th>
<th>CC-A</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abalone</td>
<td>9.73</td>
<td>2.22</td>
<td>3.92</td>
<td>4.27</td>
<td>3.24</td>
</tr>
<tr>
<td>Adult</td>
<td>-44.07</td>
<td>-5.91</td>
<td>-4.62</td>
<td>-31.37</td>
<td>-8.65</td>
</tr>
<tr>
<td>Autism</td>
<td>-39.20</td>
<td>-27.93</td>
<td>-4.07</td>
<td>-34.70</td>
<td>-13.56</td>
</tr>
<tr>
<td>Breast</td>
<td>-28.01</td>
<td>-25.48</td>
<td>-25.02</td>
<td>-54.75</td>
<td>-13.22</td>
</tr>
<tr>
<td>Chess</td>
<td>-13.01</td>
<td>-12.30</td>
<td>-11.54</td>
<td>-13.03</td>
<td>-13.03</td>
</tr>
<tr>
<td>Crx</td>
<td>-36.26</td>
<td>-12.82</td>
<td>-19.38</td>
<td>-32.62</td>
<td>-3.80</td>
</tr>
<tr>
<td>Diabetes</td>
<td>-31.22</td>
<td>-17.48</td>
<td>-21.21</td>
<td>-30.33</td>
<td>-24.72</td>
</tr>
<tr>
<td>German</td>
<td>-26.05</td>
<td>-25.83</td>
<td>-26.76</td>
<td>-27.29</td>
<td>-14.96</td>
</tr>
<tr>
<td>Student</td>
<td>-30.18</td>
<td>-28.73</td>
<td>-29.51</td>
<td>-31.59</td>
<td>-27.79</td>
</tr>
<tr>
<td>Wine</td>
<td>-0.13</td>
<td>-10.12</td>
<td>-8.62</td>
<td>-6.91</td>
<td>3.49</td>
</tr>
</tbody>
</table>

| # best | 1 | 0 | 4 | 0 | 7 |

4 EXPERIMENTAL EVALUATION

Here, we evaluate the performance of characteristic circuit on synthetic data sets and the UCI data sets, which are composed of heterogeneous data.

Better Approximation of Characteristic Functions. We begin by describing and evaluating the performance of CC on two synthetic data sets: a mixture of multivariate distributions (denoted as MN): \( p(x) = \sum_{i=1}^{K} \frac{1}{n} \exp [ (t_j \cdot x_i) - \varphi_C (t_j) ]^2 \), where \( t_j \sim \omega (t, \eta) \), in this paper \( \mathcal{N} (0, \text{diag}(\eta^2)) \). Furthermore, we provide an analytical solution to the CFD between two CCs in Appendix D.

We first learn CCs applying structure learning with various leaf types: CC with ECF as leaves (CC-E), CC with normal distribution for all leaf nodes (CC-N). The trained CCs are evaluated with the CFD between the CC and the ground truth CF. Following Chwialkowski et al. [2015] and Ansari et al. [2020], we illustrate both the CFD with varying scale \( \eta \) in \( \omega (t; \eta) \) and also optimising \( \eta \) for the largest CFD, shown in Fig. 2. The CFDs are averaged for 5 runs and the standard deviations are visualized. Fig. 2 shows that both CC-E and CC-P have almost equally lower CFD values and also lower maximum CFD values compared to the ECF, which indicates the CC structure better encodes the data distribution than the ECF. Therefore, CC estimates the data distribution better than ECF, which can be justified by the smaller CFD from CC-E compared with ECF.

Better Density Estimator on Heterogeneous Data. Real-world tabular data usually contain both discrete and real-valued elements, and thus are in most cases heterogeneous data. Therefore, we also conduct experiments on the UCI heterogeneous data sets [Molina et al., 2018] and compare against SOTA PC methods, including MSPN [Molina et al., 2018], ABDA [Vergari et al., 2019] and BSPN [Trapp et al., 2019], for density estimation. We employ structure learning and all the continuous RVs are modelled with either normal distribution (CC-P) or α-stable distribution (CC-A).

The test log-likelihoods are presented in Table 1. CC-P does not win on all the data sets but performs as runner-up on Abalone, and is also competitive with MSPN and ABDA on most of the data sets. CC-A outperforms the other methods on 7 out of 12 data sets, marked as bold. This implies that CC is a competitive density estimator compared with SOTA PCs. Furthermore, α-stable distribution leaf, which is not available in current PCs, is a more suitable choice for CC on heterogeneous tabular data.

5 CONCLUSION

We introduced characteristic circuit, the first circuit based characteristic function estimator that leverages an arithmetic circuit with univariate characteristic function leaves for modelling the joint of heterogeneous data distributions. Compared to PCs, CC models the CF of data distribution in the continuous spectral domain, providing a unified view for discrete and continuous RVs, and can further model distributions that do not have closed-form probability density functions. We show that both joint and marginal probability densities can be calculated exactly and efficiently via CC. Finally, we empirically show that CC approximates data distribution better than ECF, and can also perform as a competitive density estimator on heterogeneous data sets.

The circuit structure of CC generated by structure learning has a high impact on the performance of the CC, thus an inappropriate structure can limit the modelling power of CC. Therefore, one interesting direction of future work is to apply parameter learning of CC on more advanced circuit structures [Peharz et al., 2020] or in combinations with flows.
Acknowledgements

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References


Supplementary Material:
Characteristic Circuit

A LEAF TYPES OF CHARACTERISTIC CIRCUIT

Here we describe the leaf types that are used in the characteristic circuit.

ECF leaf. In many cases, a parametric form of the data distribution is not available and one needs to use a non-parametric estimator. The ECF [Feuerverger and Mureika, 1977, Cramér, 1999] is an unbiased and consistent non-parametric estimator of the population characteristic function. Given data \( \{x_j\}_{j=1}^n \) the ECF is given by \( \hat{\varphi}_p(t) = \frac{1}{n} \sum_{j=1}^n \exp(i t^\top x_j) \). Thus, the most straightforward way for modelling the leaf node is to directly employ the empirical characteristic function for the local data at each leaf, defined as \( \varphi_{\text{ECF}}(t) = \frac{1}{n} \sum_{j=1}^n \exp(it x_j) \), where \( n \) is the number of instances at leaf \( L \), and \( x_j \) is the \( j^{th} \) instance. The ECF leaf is non-parametric and is determined by the \( n \) instances \( x_j \) at \( L \).

Parametric leaf for continuous RVs. Motivated by existing SPN literature, we can assume that the RV at a leaf node follows a parametric continuous distribution, e.g. normal distribution. With this, the leaf node is equipped with the CF of normal distribution \( \varphi_{\text{normal}}(t) = \exp(it \mu - \frac{1}{2} \sigma^2 t^2) \), where parameters \( \mu \) and \( \sigma^2 \) are the mean and variance.

Parametric leaf for discrete RVs. For discrete RVs, if it is assumed to follow categorical distribution \( P(X = j) = p_j \), then the CF at the leaf node can be defined as \( \varphi_{\text{Categorical}}(t) = \sum_j p_j \exp(it x_j) \). Other discrete distributions which are widely used in probabilistic circuits can also be employed as leaf nodes in CCS, e.g. Bernoulli, Poisson and geometric distributions.

\( \alpha \)-stable leaf. In the case of financial data or data distributed with heavy tails, the \( \alpha \)-stable distribution is frequently employed. \( \alpha \)-stable distributions are more flexible in modelling e.g. data with skewed centered distributions. The characteristic function of an \( \alpha \)-stable distribution is \( \varphi_{\text{\alpha-stable}}(t) = \exp(it \mu - |ct|\alpha(1 - i\beta \text{sgn}(t) \Phi)) \), where \( \text{sgn}(t) \) takes the sign of \( t \) and \( \Phi = \begin{cases} \frac{\tan(\pi \alpha/2)}{\pi} & \alpha \neq 1 \\ -2/|\log |t|| & \alpha = 1 \end{cases} \). The parameters in \( \alpha \)-stable distribution are the stability parameter \( \alpha \), the skewness parameter \( \beta \), the scale parameter \( c \) and the location parameter \( \mu \). Despite its modelling power, \( \alpha \)-stable distribution is never employed in PCs, as it is represented analytically by its CF and in most cases does not have a closed-form probability density function.

B THEORETIC PROPERTIES

In this section, we provide a detailed proof of Lemma 3.2.

Proof. Let \( C = \langle G, \psi, \theta_G \rangle \) be a characteristic circuit on RVs \( X = \{X_j\}_{j=1}^d \) with univariate leave nodes and \( p_N \) the number of RVs in the scope of \( N \). In order to calculate the density function of \( C \), we need to integrate over the \( d \)-dimensional real space \( \mathbb{R}^d \), i.e.,

\[
\hat{f}_C(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp(-i t^\top x) \varphi_C(t) \lambda_d(dt),
\]

where \( \varphi_C(t) \) denotes the CF defined by the root of the characteristic circuit and \( \lambda_d \) is the Lebesgue measure on \( (\mathbb{R}^d, B(\mathbb{R}^d)) \). We can examine the computation of Eq. (5) recursively for every node.

Leaf Nodes. If \( N \) is a leaf node \( L \), we obtain \( \hat{f}_L(\cdot) \) by calculating:

\[
\hat{f}_L(x) = 2\pi f_L(x) = \int_{\mathbb{R}} \exp(-itx) \varphi_X(t) \lambda(dt),
\]

which follows from Theorem 2.1.
Where we used that

Therefore, computing the inverse for $S$ reduces to inversion at its children.

**Product Nodes.** If $N$ is a product node $P$, then:

\[
\hat{f}_P(x) = \int_{t \in \mathbb{R}^p} \exp(-i t^T x) \varphi_P(t) \lambda_{pp}(dt) = \prod_{N \in \text{ch}(P)} \int_{t \in \mathbb{R}^{p_N}} \exp(-i s^T x_{|\{N\}}) \varphi_{N}(s) \lambda_{pn}(ds),
\]

where we used that $\lambda_{pp} = \otimes_{N \in \text{ch}(P)} \lambda_{pn}$ is a product measure on a product space, applied Fubini’s theorem, and used the additivity property of exponential functions. Consequently, computing the inverse for $P$ reduces to inversion at its children.

Through the recursive application of Eq. (7) and Eq. (8), we obtain that Eq. (5) reduces to integration at the leaves and, therefore, can be solved either analytically or efficiently through one-dimensional numerical integration.

\[
\boxed{	ext{C CHARACTERISTIC FUNCTION DISTANCE}}
\]

To measure the distance between two distributions represented by their characteristic functions, the squared characteristic function distance (CFD) can be employed. The CFD between two distributions $P$ and $Q$ is defined as:

\[
\text{CFD}_2^2(P, Q) = \int_{\mathbb{R}^d} |\varphi_P(t) - \varphi_Q(t)|^2 \omega(t; \eta) dt,
\]

where $\omega(t; \eta)$ is a weighting function parameterized by $\eta$ and guarantees the integral in Eq. (9) converge. When $\omega(t; \eta)$ is a probability density function, Eq. (9) can be rewritten as:

\[
\text{CFD}_2^2(P, Q) = \mathbb{E}_{t \sim \omega(t; \eta)} \left[|\varphi_P(t) - \varphi_Q(t)|^2 \right].
\]

Sriperumbudur et al. [2010] showed that using the uniqueness theorem of CFs, $\text{CFD}_2^2(P, Q) = 0$ iff $P = Q$. Computing Eq. (10) is generally intractable, therefore, we use Monte-Carlo integration to approximate the expectation, resulting in $\text{CFD}_2^2(P, Q) \approx \frac{1}{K} \sum_{k=1}^{K} |\varphi_P(t_k) - \varphi_Q(t_k)|^2$, where $\{t_1, \cdots, t_k\} \sim \omega(t; \eta)$. We refer to Ansari et al. [2020] for a detailed discussion.

\[
\boxed{D \text{ ANALYTICAL SOLUTION OF THE CFD}}
\]

The squared characteristic function distance (CFD)

\[
\text{CFD}_2^2(P, Q) = \int_{\mathbb{R}^d} |\varphi_P(t) - \varphi_Q(t)|^2 \omega(t; \eta) dt
\]

can not only be estimated with MC methods by sampling from $\omega(t; \eta)$, but also be calculated through the characteristic circuit with an analytical solution, if $\varphi_P(t)$ and $\varphi_Q(t)$ are compatible characteristic circuits.

Eq. (11) can be rewritten as

\[
\text{CFD}_2^2(P, Q) = \int_{\mathbb{R}^d} (\varphi_P(t) - \varphi_Q(t)) \left(\frac{\varphi_P(t) - \varphi_Q(t)}{\sqrt[2]{\varphi_P(t) - \varphi_Q(t)}}\right) \omega(t; \eta) dt
\]

\[
= \int_{\mathbb{R}^d} (\varphi_P(t)^2 - \varphi_P(t)\varphi_Q(t) - \varphi_Q(t)\varphi_P(t) + \varphi_Q(t)^2) \omega(t; \eta) dt,
\]

\[
\text{CFD}_2^2(P, Q) = \int_{\mathbb{R}^d} \varphi_P(t)^2 \omega(t; \eta) dt - \int_{\mathbb{R}^d} \varphi_P(t)\varphi_Q(t) \omega(t; \eta) dt - \int_{\mathbb{R}^d} \varphi_Q(t)\varphi_P(t) \omega(t; \eta) dt + \int_{\mathbb{R}^d} \varphi_Q(t)^2 \omega(t; \eta) dt.
\]

\[
\boxed{8}
\]
where \( \bar{z} \) denotes the conjugate of the complex number \( z \). Without loss of generality, let us derive the analytical solution of \( \int_{R_d} \varphi_P(t) \varphi_Q(t) \omega(t; \eta) dt \), since the derivation can be directly applied to the other terms in Eq. (12). In the following, we omit the term \( \omega(t; \eta) \) at sum and product nodes for simplicity. At sum nodes \( S \) and \( S' \),

\[
\int S(t) S'(t) dt = \int \left( \sum_{N \in \text{ch}(S)} w_{S,N} N(t) \right) \left( \sum_{N' \in \text{ch}(S')} w_{S',N'} N'(t) \right) dt
\]

\[
= \int \sum_{N \in \text{ch}(S)} \sum_{N' \in \text{ch}(S')} w_{S,N} w_{S',N} N(t) N'(t) dt
\]

\[
= \sum_{N \in \text{ch}(S)} \sum_{N' \in \text{ch}(S')} w_{S,N} w_{S',N'} \int N(t) N'(t) dt. \tag{13}
\]

At product nodes \( P \) and \( P' \),

\[
\int P(t) P'(t) dt = \int \left( \prod_{N \in \text{ch}(P)} N(t_{[\psi(N)]}) \right) \left( \prod_{N' \in \text{ch}(P')} N'(t_{[\psi(N)]}) \right) dt
\]

\[
= \prod_{(N,N') \in \Delta_{P \times P'}} \int_{R^{|P| \times |P'|}} N(t) N'(t) dt. \tag{compatibility} \tag{14}
\]

where \( \Delta_{P \times P'} \) denotes the diagonal of the Cartesian product of the children of \( P \) and \( P' \), i.e., \( \text{diag} (\text{ch}(P) \times \text{ch}(P')) \), compatibility ensures that both product nodes apply the same partition of the scope \( \psi(P) = \psi(P') \) with parts in the same order, and \( t_{[\psi(N)]} \) is the projection of \( t \) to the scope of \( N \). Therefore,

\[
= \prod_{(N,N') \in \Delta_{P \times P'}} \int_{R^{|P| \times |P'|}} N(t) N'(t) dt. \tag{compatibility} \tag{15}
\]

At univariate leaf nodes \( L \) and \( L' \), assuming both leaf nodes model univariate normal distribution with parameters \( (\mu, \sigma) \) and \( (\mu', \sigma') \), and \( \omega(t; \eta) = \frac{1}{\eta \sqrt{2\pi}} \exp(-\frac{t^2}{2\eta^2}) \), then

\[
\int L(t) L'(t) \omega(t; \eta) dt = \int \exp(i t \mu - \frac{1}{2} \sigma^2 t^2) \exp(i t \mu' - \frac{1}{2} \sigma'^2 t^2) \frac{1}{\eta \sqrt{2\pi}} \exp(-\frac{t^2}{2\eta^2}) dt
\]

\[
= \frac{1}{\eta \sqrt{2\pi}} \int \exp(i t (\mu - \mu') - \frac{1}{2} (\sigma^2 + \sigma'^2) t^2) dt
\]

\[
= \frac{1}{\eta \sqrt{2\pi}} \exp(-\frac{\bar{\mu}^2}{2\sigma^2}), \quad \text{(integral of a Gaussian function)} \tag{16}
\]

where \( \bar{\mu} = \mu - \mu' \) and \( \bar{\sigma} = \sqrt{\sigma^2 + \sigma'^2 + 1/\eta^2} \). Therefore, at univariate leaf nodes, it can be solved either analytically or with Monte-Carlo integration:

\[
\int L(t) L'(t) \omega(t; \eta) dt = \frac{1}{k} \sum_{j=1}^{k} \varphi_L(t_j) \varphi_L'(t_j), \tag{17}
\]

where \( \{t_1, \ldots, t_k\} \overset{\text{iid}}{\sim} \omega(t; \eta) \). With the above properties, the CFD between two compatible CCs can be calculated from bottom-up analytically and efficiently.