
Privacy Amplification by Mixing and Diffusion Mechanisms

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Abstract

A fundamental result in differential privacy states that the privacy guarantees of a mechanism are preserved by any post-processing of its output. In this paper we investigate under what conditions stochastic post-processing can amplify the privacy of a mechanism. By interpreting post-processing as the application of a Markov operator, we first give a series of amplification results in terms of uniform mixing properties of the Markov process defined by said operator. Next we provide amplification bounds in terms of coupling arguments which can be applied in cases where uniform mixing is not available. Finally, we introduce a new family of mechanisms based on diffusion processes which are closed under post-processing, and analyze their privacy via a novel heat flow argument. On the applied side, we generalize the analysis of “privacy amplification by iteration” in Noisy SGD and show it admits an exponential improvement in the strongly convex case, and study a mechanism based on the Ornstein–Uhlenbeck diffusion process which contains the Gaussian mechanism with optimal post-processing on bounded inputs as a special case.

1 Introduction

Differential privacy (DP) [15] has arisen in the last decade into a strong de-facto standard for privacy-preserving computation in the context of statistical analysis. The success of DP is based, at least in part, on the availability of robust building blocks (e.g., the Laplace, exponential and Gaussian mechanisms) together with relatively simple rules for analyzing complex mechanisms built out of these blocks (e.g., composition and robustness to post-processing). The inherent tension between privacy and utility in practical applications has sparked a renewed interest into the development of further rules leading to tighter privacy bounds. A trend in this direction is to find ways to measure the privacy introduced by sources of randomness that are not accounted for by standard composition rules. Generally speaking, these are referred to as *privacy amplification* rules, with prominent examples being amplification by *subsampling* [9, 18, 20, 6, 5, 8, 2, 27], *shuffling* [16, 10, 3] and *iteration* [17].

Motivated by these considerations, in this paper we initiate a systematic study of privacy amplification by *stochastic post-processing*. Specifically, given a DP mechanism M producing (probabilistic) outputs in $\mathbb{X}$ and a Markov operator K defining a stochastic transition between $\mathbb{X}$ and $\mathbb{Y}$, we are interested in measuring the privacy of the post-processed mechanism $K \circ M$ producing outputs in $\mathbb{Y}$. The standard post-processing property of DP states that $K \circ M$ is at least as private as M . Our goal is to understand under what conditions the post-processed mechanism $K \circ M$ is *strictly* more private than M . Roughly speaking, this amplification should be non-trivial when the operator K “forgets” information about the distribution of its input $M(D)$. Our main insight is that, at least when $\mathbb{Y} = \mathbb{X}$,

the forgetfulness of K from the point of view of DP can be measured using similar tools to the ones developed to analyze the speed of convergence, i.e. *mixing*, of the Markov process associated with K .

In this setting, we provide three types of results, each associated with a standard method used in the study of convergence for Markov processes. In the first place, Section 3 provides DP amplification results for the case where the operator K satisfies a uniform mixing condition. These include standard conditions used in the analysis of Markov chains on discrete spaces, including the well-known Dobrushin coefficient and Doeblin’s minorization condition [19]. Although in principle uniform mixing conditions can also be defined in more general non-discrete spaces [12], most Markov operators of interest in $\mathbb{R}^d$ do not exhibit uniform mixing since the speed of convergence depends on how far apart the initial inputs are. Convergence analyses in this case rely on more sophisticated tools, including Lyapunov functions [22], coupling methods [21] and functional inequalities [1].

Following these ideas, Section 4 investigates the use of coupling methods to quantify privacy amplification by post-processing under Rényi DP [23]. These methods apply to operators given by, e.g., Gaussian and Laplace distributions, for which uniform mixing does not hold. Results in this section are intimately related to the privacy amplification by iteration phenomenon studied in [17] and can be interpreted as extensions of their main results to more general settings. In particular, our analysis unpacks the *shifted* Rényi divergence used in the proofs from [17] and allows us to easily track the effect of iterating arbitrary noisy Lipschitz maps. As a consequence, we show an exponential improvement on the privacy amplification by iteration of Noisy SGD in the strongly convex case which follows from applying this generalized analysis to *strict* contractions.

Our last set of results concerns the case where K is replaced by a family of operators $(P_t)_{t \geq 0}$ forming a *Markov semigroup* [1]. This is the natural setting for *continuous-time* Markov processes, and includes *diffusion* processes defined in terms of stochastic differential equations [25]. In Section 5 we associate (a collection of) diffusion mechanisms $(M_t)_{t \geq 0}$ to a diffusion semigroup. Interestingly, these mechanisms are, by construction, closed under post-processing in the sense that $P_s \circ M_t = M_{s+t}$. We show the Gaussian mechanism falls into this family – since Gaussian noise is closed under addition – and also present a new mechanism based on the Ornstein-Uhlenbeck process which has better mean squared error than the standard Gaussian mechanism (and matches the error of the optimally post-processed Gaussian mechanism with bounded inputs). Our main result on diffusion mechanisms provides a generic Rényi DP guarantee based on an intrinsic notion of sensitivity derived from the geometry induced by the semigroup. The proof relies on a heat flow argument reminiscent of the analysis of mixing in diffusion processes based on functional inequalities [1].

2 Background

We start by introducing notation and concepts that will be used throughout the paper. We write $[n] = \{1, \dots, n\}$, $a \wedge b = \min\{a, b\}$ and $[a]_+ = \max\{a, 0\}$.

Probability. Let $\mathbb{X} = (\mathbb{X}, \Sigma, \lambda)$ be a measurable space with sigma-algebra Σ and base measure λ . We write $\mathcal{P}(\mathbb{X})$ to denote the set of probability distributions on $\mathbb{X}$. Given a probability distribution $\mu \in \mathcal{P}(\mathbb{X})$ and a measurable event $E \subseteq \mathbb{X}$ we write $\mu(E) = \mathbb{P}[X \in E]$ for a random variable $X \sim \mu$, denote its expectation under $f : \mathbb{X} \rightarrow \mathbb{R}^d$ by $\mathbb{E}[f(X)]$, and can get back its distribution as $\mu = \text{Law}(X)$. Given two distributions μ, ν (or, in general, arbitrary measures) we write $\mu \ll \nu$ to denote that μ is absolutely continuous with respect to ν , in which case there exists a Radon-Nikodym derivative $\frac{d\mu}{d\nu}$. We shall reserve the notation $p_\mu = \frac{d\mu}{d\lambda}$ to denote the density of μ with respect to the base measure. We also write $\mathcal{C}(\mu, \nu)$ to denote the set of couplings between μ and ν ; i.e. $\pi \in \mathcal{C}(\mu, \nu)$ is a distribution on $\mathcal{P}(\mathbb{X} \times \mathbb{X})$ with marginals μ and ν . The support of a distribution is $\text{supp}(\mu)$.

Markov Operators. We will use $\mathcal{K}(\mathbb{X}, \mathbb{Y})$ to denote the set of Markov operators $K : \mathbb{X} \rightarrow \mathcal{P}(\mathbb{Y})$ defining a stochastic transition map between $\mathbb{X}$ and $\mathbb{Y}$ and satisfying that $x \mapsto K(x)(E)$ is measurable for every measurable $E \subseteq \mathbb{Y}$. Markov operators act on distributions $\mu \in \mathcal{P}(\mathbb{X})$ on the left through $(\mu K)(E) = \int K(x)(E) \mu(dx)$, and on functions $f : \mathbb{Y} \rightarrow \mathbb{R}$ on the right through $(Kf)(x) = \int f(y) K(x, dy)$, which can also be written as $(Kf)(x) = \mathbb{E}[f(X)]$ with $X \sim K(x)$. The kernel of a Markov operator K (with respect to λ) is the function $k(x, \cdot) = \frac{dK(x)}{d\lambda}$ associating with x the density of $K(x)$ with respect to a fixed measure.

Divergences. A popular way to measure dissimilarity between distributions is to use Csiszár divergences $D_\phi(\mu \| \nu) = \int \phi(\frac{d\mu}{d\nu}) d\nu$, where $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}$ is convex with $\phi(1) = 0$. Taking

$\phi(u) = \frac{1}{2}|u - 1|$ yields the total variation distance $\text{TV}(\mu, \nu)$, and the choice $\phi(u) = [u - e^\varepsilon]_+$ with $\varepsilon \geq 0$ gives the hockey-stick divergence D_{e^ε} , which satisfies

$$D_{e^\varepsilon}(\mu\|\nu) = \int \left[\frac{d\mu}{d\nu} - e^\varepsilon \right]_+ d\nu = \int [p_\mu - e^\varepsilon p_\nu]_+ d\lambda = \sup_{E \subseteq \mathbb{X}} (\mu(E) - e^\varepsilon \nu(E)) .$$

It is easy to check that $\varepsilon \mapsto D_{e^\varepsilon}(\mu\|\nu)$ is monotonically decreasing and $D_1 = \text{TV}$. All Csiszár divergences satisfy joint convexity $D((1-\gamma)\mu_1 + \gamma\mu_2\|(1-\gamma)\nu_1 + \gamma\nu_2) \leq (1-\gamma)D(\mu_1\|\nu_1) + \gamma D(\mu_2\|\nu_2)$ and the data processing inequality $D(\mu K\|\nu K) \leq D(\mu\|\nu)$ for any Markov operator K . Rényi divergences¹ are another way to compare distributions. For $\alpha > 1$ the Rényi divergence of order α is defined as $R_\alpha(\mu\|\nu) = \frac{1}{\alpha-1} \log \int (\frac{d\mu}{d\nu})^\alpha d\nu$, and also satisfies the data processing inequality. Finally, to measure similarity between $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ we sometimes use the ∞ -Wasserstein distance:

$$W_\infty(\mu, \nu) = \inf_{\pi \in \mathcal{C}(\mu, \nu)} \inf \{w \geq 0 : \|X - Y\| \leq w \text{ holds almost surely for } (X, Y) \sim \pi\} .$$

Differential Privacy. A mechanism $M : \mathbb{D}^n \rightarrow \mathcal{P}(\mathbb{X})$ is a randomized function that takes a dataset $D \in \mathbb{D}^n$ over some universe of records $\mathbb{D}$ and returns a (sample from) distribution $M(D)$. We write $D \simeq D'$ to denote two databases differing in a single record. We say that M satisfies² (ε, δ) -DP if $\sup_{D \simeq D'} D_{e^\varepsilon}(M(D)\|M(D')) \leq \delta$ [15]. Furthermore, we say that M satisfies (α, ϵ) -RDP if $\sup_{D \simeq D'} R_\alpha(M(D)\|M(D')) \leq \epsilon$ [23].

3 Amplification From Uniform Mixing

We start our analysis of privacy amplification by stochastic post-processing by considering settings where the Markov operator K satisfies one of the following uniform mixing conditions.

Definition 1. Let $K \in \mathcal{K}(\mathbb{X}, \mathbb{Y})$ be a Markov operator, $\gamma \in [0, 1]$ and $\varepsilon \geq 0$. We say that K is:

- (1) γ -Dobrushin if $\sup_{x, x'} \text{TV}(K(x), K(x')) \leq \gamma$,
- (2) (γ, ε) -Dobrushin if $\sup_{x, x'} D_{e^\varepsilon}(K(x)\|K(x')) \leq \gamma$,
- (3) γ -Doebelin if there exists a distribution $\omega \in \mathcal{P}(\mathbb{Y})$ such that $K(x) \geq (1-\gamma)\omega$ for all $x \in \mathbb{X}$,
- (4) γ -ultra-mixing if for all $x, x' \in \mathbb{X}$ we have $K(x) \ll K(x')$ and $\frac{dK(x)}{dK(x')} \geq 1-\gamma$.

Most of these conditions arise in the context of mixing analyses in Markov chains. In particular, the Dobrushin condition can be tracked back to [13], while Doebelin's condition was introduced earlier [14] (see also [24]). Ultra-mixing is a strengthening of Doebelin's condition used in [12]. The (γ, ε) -Dobrushin is, on the other hand, new and is designed to be a generalization of Dobrushin tailored for amplification under the hockey-stick divergence.

It is not hard to see that Dobrushin's is the weakest among these conditions, and in fact we have the implications summarized in Figure 1 (see Lemma 9). This explains why the amplification bounds in the following result are increasingly stronger, and in particular why the first two only provide amplification in δ , while the last two also amplify the ε parameter.

Theorem 1. Let M be an (ε, δ) -DP mechanism. For a given Markov operator K , the post-processed mechanism $K \circ M$ satisfies:

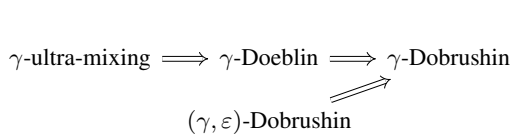
- (1) (ε, δ') -DP with $\delta' = \gamma\delta$ if K is γ -Dobrushin,
- (2) (ε, δ') -DP with $\delta' = \gamma\delta$ if K is $(\gamma, \tilde{\varepsilon})$ -Dobrushin with³ $\tilde{\varepsilon} = \log(1 + \frac{e^\varepsilon - 1}{\delta})$,
- (3) (ε', δ') -DP with $\varepsilon' = \log(1 + \gamma(e^\varepsilon - 1))$ and $\delta' = \gamma(1 - e^{\varepsilon' - \varepsilon}(1 - \delta))$ if K is γ -Doebelin,
- (4) (ε', δ') -DP with $\varepsilon' = \log(1 + \gamma(e^\varepsilon - 1))$ and $\delta' = \gamma\delta e^{\varepsilon' - \varepsilon}$ if K is γ -ultra-mixing.

A few remarks about this result are in order. First we note that (2) is stronger than (1) since the monotonicity of hockey-stick divergences implies $\text{TV} = D_1 \geq D_{e^\varepsilon}$. Also note how in the results above we always have $\varepsilon' \leq \varepsilon$, and in fact the form of ε' is the same as obtained under amplification

¹Rényi divergences do not belong to the family of Csiszár divergences.

²This divergence characterization of DP is due to [4].

³We take the convention $\tilde{\varepsilon} = \infty$ whenever $\delta = 0$, in which case the (γ, ∞) -Dobrushin condition is obtained with respect to the divergence $D_\infty(\mu\|\nu) = \mu(\text{supp}(\mu) \setminus \text{supp}(\nu))$.



Mixing Condition	Local DP Condition
γ -Dobrushin	$(0, \gamma)$ -LDP
(γ, ε) -Dobrushin	(ε, γ) -LDP
γ -Doebelin	Blanket condition ⁴
γ -ultra-mixing	$(\log \frac{1}{1-\gamma}, 0)$ -LDP

take $\omega = \mathcal{N}(w, \sigma_1^2 I)$ for some w to be determined later, and couple ω and μ through a translation $\tau = u - w$, yielding a coupling π with $p_\pi(x, y) \propto \exp(-\frac{\|x-w\|^2}{2\sigma_1^2})\mathbb{I}[y = x + \tau]$ and a transport operator H_π with kernel $h_\pi(x, y) = \mathbb{I}[y = x + \tau]$. Plugging these into (1) we get

$$R_\alpha(\mu K \| \nu K) \leq \frac{\alpha \|w - v\|^2}{2\sigma_1^2} + \sup_{x \in \mathbb{R}^d} R_\alpha(K(x + \tau) \| K(x)) = \frac{\alpha}{2} \left(\frac{\|w - v\|^2}{\sigma_1^2} + \frac{\|u - w\|^2}{\sigma_2^2} \right).$$

Finally, taking $w = \theta u + (1 - \theta)v$ with $\theta = (1 + \frac{\sigma_2^2}{\sigma_1^2})^{-1}$ yields $R_\alpha(\mu K \| \nu K) \leq \frac{\alpha \Delta^2}{2(\sigma_1^2 + \sigma_2^2)}$.

Example 2 (Iterated Laplace). To illustrate the flexibility of this technique, we also apply it to get an amplification result for iterated Laplace noise, in which Laplace noise is added to the output of a Laplace mechanism. We begin by noting a negative result that there is no amplification in the $(\varepsilon, 0)$ -DP regime.

Lemma 3. *Let $M(D) = \text{Lap}(f(D), \lambda_1)$ for some function $f : \mathbb{D} \rightarrow \mathbb{R}$ with global L_1 -sensitivity Δ and let the Markov operator K be given by $K(x) = \text{Lap}(x, \lambda_2)$. The post-processed mechanism $(K \circ M)$ does not achieve $(\varepsilon, 0)$ -DP for any $\varepsilon < \frac{\Delta}{\max\{\lambda_1, \lambda_2\}}$. Note that M achieves $(\frac{\Delta}{\lambda_1}, 0)$ -DP and $K(f(D))$ achieves $(\frac{\Delta}{\lambda_2}, 0)$ -DP.*

However, the iterated Laplace mechanism $K \circ M$ above still offers additional privacy in the relaxed RDP setting. An application of (1) allows us to identify some of this improvement. Recall from [23, Corollary 2] that M satisfies $(\alpha, \frac{1}{\alpha-1} \log g_\alpha(\frac{\Delta}{\lambda_1}))$ -RDP with $g_\alpha(z) = \frac{\alpha}{2\alpha-1} \exp(z(\alpha-1)) + \frac{\alpha-1}{2\alpha-1} \exp(-z\alpha)$. As in Example 1, we take $\omega = \text{Lap}(w, \lambda_1)$ for some w to be determined later, and couple ω and μ through a translation $\tau = u - w$. Through (1) we obtain

$$\begin{aligned} R_\alpha(\mu K \| \nu K) &\leq \frac{1}{\alpha-1} \log \left(g_\alpha \left(\frac{|w-v|}{\lambda_1} \right) \right) + \sup_{x \in \mathbb{R}} R_\alpha(K(x + \tau) \| K(x)) \\ &= \frac{1}{\alpha-1} \log \left(g_\alpha \left(\frac{|w-v|}{\lambda_1} \right) g_\alpha \left(\frac{|u-w|}{\lambda_2} \right) \right). \end{aligned}$$

In the simple case where $\lambda_1 = \lambda_2$, an amplification result is observed from the log-convexity of g_α , since $g_\alpha(a)g_\alpha(b) \leq g_\alpha(a+b)$. When $\lambda_1 \neq \lambda_2$, certain values of w still result in amplification, but they depend nontrivially on α . However, we also observe that this improvement vanishes as $\alpha \rightarrow \infty$, since the necessary convexity also vanishes. In the limit, the lowest upper bound offered by (1) for R_∞ (which reduces to $(\varepsilon, 0)$ -DP) matches the $\frac{\Delta}{\max\{\lambda_1, \lambda_2\}}$ result of Lemma 3.

Example 3 (Lipschitz Kernel). As a warm-up for the results in Section 4.1, we now re-work Example 1 with a slightly more complex Markov operator. Suppose ψ is an L -Lipschitz map⁶ and let $K(x) = \mathcal{N}(\psi(x), \sigma_2^2 I)$. Taking M to be the Gaussian mechanism from Example 1, we will show that the post-processed mechanism $K \circ M$ satisfies $(\alpha, \frac{\alpha \Delta^2}{2\sigma_2^2})$ -RDP with $\sigma_*^2 = \sigma_1^2 + \frac{\sigma_2^2}{L^2}$. To prove this bound, we instantiate the notation from Example 1, and use the same coupling strategy to obtain

$$R_\alpha(\mu K \| \nu K) \leq \frac{\alpha}{2} \left(\frac{\|w - v\|^2}{\sigma_1^2} + \sup_{x \in \mathbb{R}^d} \frac{\|\psi(x + \tau) - \psi(x)\|^2}{\sigma_2^2} \right) \leq \frac{\alpha}{2} \left(\frac{\|w - v\|^2}{\sigma_1^2} + \frac{L^2 \|u - w\|^2}{\sigma_2^2} \right),$$

where the second inequality uses the Lipschitz property. As before, the result follows from taking $w = \theta u + (1 - \theta)v$ with $\theta = (1 + \frac{\sigma_2^2}{L^2 \sigma_1^2})^{-1}$. This example shows that we get amplification (i.e. $\sigma_*^2 > \sigma_1^2$) for any $L < \infty$ and $\sigma_2 > 0$, although the amount of amplification decreases as L grows. On the other hand, for $L < 1$ the amplification is stronger than just adding Gaussian noise (Example 1).

4.1 Amplification by Iteration in Noisy Projected SGD with Strongly Convex Losses

Now we use Theorem 2 and the computations above to show that the proof of privacy amplification by iteration [17, Theorem 22] can be extended to explicitly track the Lipschitz coefficients in a “noisy iteration” algorithm. In particular, this allows us to show an exponential improvement on the rate of privacy amplification by iteration in noisy SGD when the loss is strongly convex. To obtain this result we first provide an *iterated* version of Theorem 2 in $\mathbb{R}^d$ with Lipschitz Gaussian

⁶That is, $\|\psi(x) - \psi(y)\| \leq L\|x - y\|$ for any pair x, y .

kernels. This version of the analysis introduces an explicit dependence on the W_∞ distances along an “interpolating” path between the initial distributions $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ which could later be optimized for different applications. In our view, this helps to clarify the intuition behind the previous analysis of amplification by iteration.

Theorem 4. *Let $\alpha \geq 1$, $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ and let $\mathbb{K} \subseteq \mathbb{R}^d$ be a convex set. Suppose $K_1, \dots, K_r \in \mathcal{K}(\mathbb{R}^d, \mathbb{R}^d)$ are Markov operators where $Y_i \sim K_i(x)$ is obtained as⁷ $Y_i = \Pi_{\mathbb{K}}(\psi_i(x) + Z_i)$ with $Z_i \sim \mathcal{N}(0, \sigma^2 I)$, where the maps $\psi_i : \mathbb{K} \rightarrow \mathbb{R}^d$ are L -Lipschitz for all $i \in [r]$. For any $\mu_0, \mu_1, \dots, \mu_r \in \mathcal{P}(\mathbb{R}^d)$ with $\mu_0 = \mu$ and $\mu_r = \nu$ we have*

$$R_\alpha(\mu K_1 \cdots K_r \| \nu K_1 \cdots K_r) \leq \frac{\alpha L^2}{2\sigma^2} \sum_{i=1}^r L^{2(r-i)} W_\infty(\mu_i, \mu_{i-1})^2. \quad (2)$$

Furthermore, if $L \leq 1$ and $W_\infty(\mu, \nu) = \Delta$, then

$$R_\alpha(\mu K_1 \cdots K_r \| \nu K_1 \cdots K_r) \leq \frac{\alpha \Delta^2 L^{r+1}}{2r\sigma^2}. \quad (3)$$

Note how taking $L = 1$ in the bound above we obtain $\frac{\alpha \Delta^2}{2r\sigma^2} = O(1/r)$, which matches [17, Theorem 1]. On the other hand, for L strictly smaller than 1, the analysis above shows that the amplification rate is $O(L^{r+1}/r)$ as a consequence of the maps ψ_i being strict contractions, i.e. $\|\psi_i(x) - \psi_i(y)\| < \|x - y\|$. For $L > 1$ this result is not useful since the sum will diverge; however, the proof could easily be adapted to handle the case where each ψ_i is L_i -Lipschitz with some $L_i > 1$ and some $L_i < 1$. We now apply this result to improve the per-person privacy guarantees of noisy projected SGD (Algorithm 1) in the case where the loss function is smooth and strongly convex.

Algorithm 1: Noisy Projected Stochastic Gradient Descent — NoisyProjSGD($D, \ell, \eta, \sigma, \xi_0$)

Input: Dataset $D = (z_1, \dots, z_n)$, loss function $\ell : \mathbb{K} \times \mathbb{D} \rightarrow \mathbb{R}$, learning rate η , noise parameter σ , initial distribution $\xi_0 \in \mathcal{P}(\mathbb{K})$

Sample $x_0 \sim \xi_0$

for $i \in [n]$ **do**

$x_i \leftarrow \Pi_{\mathbb{K}}(x_{i-1} - \eta(\nabla_x \ell(x_{i-1}, z_i) + Z))$ with $Z \sim \mathcal{N}(0, \sigma^2 I)$

return x_n

A function $f : \mathbb{K} \subseteq \mathbb{R}^d \rightarrow \mathbb{R}$ defined on a convex set is β -smooth if it is continuously differentiable and ∇f is β -Lipschitz, i.e., $\|\nabla f(x) - \nabla f(y)\| \leq \beta \|x - y\|$, and is ρ -strongly convex if the function $g(x) = f(x) - \frac{\rho}{2} \|x\|^2$ is convex. When we say that a loss function $\ell : \mathbb{K} \times \mathbb{D} \rightarrow \mathbb{R}$ satisfies a property (e.g. smoothness) we mean the property is satisfied by $\ell(\cdot, z)$ for all $z \in \mathbb{D}$. Furthermore, we recall from [17] that a mechanism $M : \mathbb{D}^n \rightarrow \mathbb{X}$ satisfies (α, ϵ) -RDP at index i if $R_\alpha(M(D) \| M(D')) \leq \epsilon$ holds for any pair of databases D and D' differing on the i th coordinate.

Theorem 5. *Let $\ell : \mathbb{K} \times \mathbb{D} \rightarrow \mathbb{R}$ be a C -Lipschitz, β -smooth, ρ -strongly convex loss function. If $\eta \leq \frac{2}{\beta + \rho}$, then NoisyProjSGD($D, \ell, \eta, \sigma, \xi_0$) satisfies $(\alpha, \alpha \epsilon_i)$ -RDP at index i , where $\epsilon_n = \frac{2C^2}{\sigma^2}$ and $\epsilon_i = \frac{2C^2}{(n-i)\sigma^2} (1 - \frac{2\eta\beta\rho}{\beta + \rho})^{\frac{n-i+1}{2}}$ for $1 \leq i \leq n-1$.*

Since [17, Theorem 23] shows that for smooth Lipschitz loss functions the guarantee at index i of NoisyProjSGD is given by $\epsilon_i = O(\frac{C^2}{(n-i)\sigma^2})$, our result provides an exponential improvement in the strongly convex case. This implies, for example, that using the technique in [17, Corollary 31] one can show that, in the strongly convex setting, running $\Theta(\log(d))$ additional iterations of NoisyProjSGD on *public* data is enough to attain (up to constant factors) the same optimization error as non-private SGD while providing privacy for all individuals.

5 Diffusion Mechanisms

Now we go beyond the analysis from previous sections and simultaneously consider a family of Markov operators $\mathbf{P} = (P_t)_{t \geq 0}$ indexed by a continuous parameter t and satisfying the semigroup

⁷Here $\Pi_{\mathbb{K}}(x) = \arg \min_{y \in \mathbb{K}} \|x - y\|$ denotes the projection operator onto the convex set $\mathbb{K} \subseteq \mathbb{R}^d$.

property $P_t P_s = P_{t+s}$. Such $\mathbf{P}$ is called a *Markov semigroup* and can be used to define a family of output perturbation mechanisms $M_t^f(D) = P_t(f(D))$ which are closed under post-processing by $\mathbf{P}$ in the sense that $P_s \circ M_t^f = M_{t+s}^f$. The semigroup property greatly simplifies the analysis of privacy amplification by post-processing, since, for example, if we show that M_t^f satisfies $(\alpha, \epsilon(t))$ -RDP, then this immediately provides RDP guarantees for any post-processing of M_t by any number of operators in $\mathbf{P}$. The main result of this section provides such privacy analysis for mechanisms arising from symmetric diffusion Markov semigroups in Euclidean space. We will show this class includes the well-known Gaussian mechanism, and also identify another interesting mechanism in this class arising from the Ornstein-Uhlenbeck diffusion process.

Roughly speaking, a *diffusion* Markov semigroup $\mathbf{P} = (P_t)_{t \geq 0}$ on $\mathbb{R}^d$ corresponds to the case where $X_t \sim P_t(x)$ defines a Markov process $(X_t)_{t \geq 0}$ arising from a (time-homogeneous Itô) *stochastic differential equation* (SDE) of the form $X_0 = x$ and $dX_t = u(X_t)dt + v(X_t)dW_t$, where W_t is a standard d -dimensional Wiener process, and the *drift* $u : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and *diffusion* $v : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ coefficients satisfy appropriate regularity assumptions.⁸ In this paper, however, we shall follow [1] and take a more abstract approach to Markov diffusion semigroups. We synthesize this approach by making a number of hypotheses on $\mathbf{P}$ that we discuss after introducing two core concepts from the theory of Markov semigroups.

In the context of a Markov semigroup $\mathbf{P}$, the action of the Markov operators P_t on functions can be used to define the *generator* L of the semigroup as the operator given by $Lf = \frac{d}{dt}(P_t f)|_{t=0}$. In particular, for a diffusion semigroup arising from the SDE $dX_t = u(X_t)dt + v(X_t)dW_t$ it is well-known that one can write the generator as $Lf = \langle u, \nabla f \rangle + \frac{1}{2} \langle vv^\top, H(f) \rangle$, where $H(f)$ is the Hessian of f and the second term is a Frobenius inner product. Using the generator one also defines the so-called *carré du champ* operator $\Gamma(f, g) = \frac{1}{2}(L(fg) - fLg - gLf)$. This operator is bilinear and non-negative in the sense that $\Gamma(f) \triangleq \Gamma(f, f) \geq 0$. The *carré du champ* operator can be interpreted as a device to measure how far L is from being a first-order differential operator, since, e.g., if $L = \sum_i a_i \frac{\partial}{\partial x_i}$ then $L(fg) = fLg + gLf$ and therefore $\Gamma(f, g) = 0$. The operator Γ can also be related to notions of curvature/contractivity of the underlying semigroup [1]. Below we illustrate these concepts with the example of Brownian motion; but first we formally state our assumptions on the semigroup.

Assumption 1. Suppose the Markov semigroup $\mathbf{P} = (P_t)_{t \geq 0} \subset \mathcal{K}(\mathbb{R}^d, \mathbb{R}^d)$ satisfies the following:

- (1) There exists a unique non-negative invariant measure λ ; that is, $\lambda P_t = \lambda$ for all $t \geq 0$. When the invariant measure is finite we normalize it to be a probability measure.
- (2) The operators P_t admit a symmetric kernel $p_t(x, y) = p_t(y, x)$ with respect to the invariant measure. Equivalently, the invariant measure λ is reversible for the Markov process X_t .
- (3) The generator L satisfies the diffusion property $L\phi(f) = \phi'(f)Lf + \phi''(f)\Gamma(f)$ for any differentiable $\phi : \mathbb{R} \rightarrow \mathbb{R}$. This is a chain rule property saying that L is a second-order differential operator without constant terms.

Example 4 (Brownian Motion). The simplest diffusion process is the Brownian motion given by the simple SDE $dX_t = \sqrt{2}dW_t$, which corresponds to the semigroup $\mathbf{P}$ given by $P_t(x) = \mathcal{N}(x, 2t)$. In this case, the mechanism $M_t^f(D) = P_t(f(D))$ is a Gaussian mechanism with variance $\sigma^2 = 2t$ and therefore satisfies $(\alpha, \frac{\alpha \Delta^2}{4t})$ -RDP, where Δ is the global L_2 -sensitivity of f . A direct substitution with $u = 0$ and $v = \sqrt{2}I$ shows that the semigroup's generator is the standard Laplacian in $\mathbb{R}^d$, $L = \nabla^2 = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$, and a simple calculation yields the expression $\Gamma(f, g) = \langle \nabla f, \nabla g \rangle$ for the carré du champ operator. Now we check that $\mathbf{P}$ satisfies the conditions in Assumption 1. First, we recall that Brownian motion has the Lebesgue measure λ on $\mathbb{R}^d$ as its unique invariant measure; this happens to be a non-finite measure. With respect to λ , the semigroup has kernel $p_t(x, y) \propto \exp(-\frac{\|x-y\|^2}{4t})$ which is clearly symmetric. Finally, we use the chain rule for the gradient to verify that

$$Lf = \nabla^2 \phi(f) = \nabla(\phi'(f)\nabla f) = \phi''(f)\langle \nabla f, \nabla f \rangle + \phi'(f)\nabla^2 f = \phi''(f)\Gamma(f) + \phi'(f)Lf.$$

Now we turn to the main result of this section, which provides a privacy analysis for the diffusion mechanism M_t^f associated with an arbitrary symmetric diffusion Markov semigroup. The key insight

⁸The details are not relevant here since we work directly with semigroups satisfying Assumption 1. We refer to [25] for details.

behind this result is that the carré du champ operator of the semigroup provides a measure $\Lambda(t)$ of *intrinsic sensitivity* for the mechanism M_t^f defined as:

$$\Lambda(t) = \sup_{D \simeq D'} \int_t^\infty \kappa_{f(D), f(D')}(s) ds, \quad \text{where} \quad \kappa_{x, x'}(t) = \sup_{y \in \mathbb{R}^d} \Gamma \left(\log \frac{p_t(x, y)}{p_t(x', y)} \right).$$

Theorem 6. *Let $f : \mathbb{D}^n \rightarrow \mathbb{R}^d$ and let $\mathbf{P} = (P_t)_{t \geq 0}$ be a Markov semigroup on $\mathbb{R}^d$ satisfying Assumption 1. If the mechanism $M_t^f(D) = P_t(f(D))$ has intrinsic sensitivity $\Lambda(t)$, then it satisfies $(\alpha, \alpha\Lambda(t))$ -RDP for any $\alpha > 1$ and $t > 0$.*

Example 5 (Brownian Motion, Continued). To illustrate the use of Theorem 6 we show how it can be used to recover the privacy guarantees of the Gaussian mechanism through its connection with Brownian motion. We let $\mathbf{P}$ be the semigroup from Example 4 and start by using $\Gamma(f) = \|\nabla f\|^2$ to compute $\kappa_{x, x'}(t)$ as follows:

$$\Gamma \left(\log \frac{p_t(x, y)}{p_t(x', y)} \right) = \left\| \nabla_y \left(\frac{\|x' - y\|^2 - \|x - y\|^2}{4t} \right) \right\|^2 = \frac{\|x - x'\|^2}{4t^2}.$$

Now we use $\int_t^\infty \frac{1}{s^2} ds = \frac{1}{t}$ and $\Delta^2 = \sup_{D \simeq D'} \|f(D) - f(D')\|^2$ to see that the mechanism associated with $\mathbf{P}$ has intrinsic sensitivity $\Lambda(t) = \frac{\Delta^2}{4t}$, yielding the privacy guarantee from Example 4.

5.1 The Ornstein-Uhlenbeck Mechanism

Beyond Brownian motion, another well-known diffusion process is the *Ornstein-Uhlenbeck* process with parameters $\theta, \rho > 0$ given by the SDE $dX_t = -\theta X_t dt + \sqrt{2\rho} dW_t$. This diffusion process is associated with the semigroup $\mathbf{P} = (P_t)_{t \geq 0}$ given by $P_t(x) = \mathcal{N}(e^{-\theta t}x, \frac{\rho^2}{\theta}(1 - e^{-2\theta t})I)$. One interpretation of this diffusion process is to think of X_t as a Brownian motion with variance ρ^2 applied to a mean reverting flow that pulls a particle towards the origin at a rate θ . In particular, the mechanism $M_t^f(D)$ is given by releasing $e^{-\theta t}f(D) + \mathcal{N}(0, \frac{\rho^2}{\theta}(1 - e^{-2\theta t}))$.

Taking the limit $t \rightarrow \infty$ one sees that the (unique) invariant measure of $\mathbf{P}$ is the Gaussian distribution $\lambda = \mathcal{N}(0, \frac{\rho^2}{\theta}I)$. From the SDE characterization of this process it is easy to check that its generator is $Lf = \rho^2 \nabla^2 f - \theta \langle x, \nabla f \rangle$ and the associated carré du champ operator is $\Gamma(f, g) = \rho^2 \langle \nabla f, \nabla g \rangle$. Thus, $\mathbf{P}$ satisfies conditions (1) and (3) in Assumption 1. To check the symmetry condition we apply a change of measure to the Gaussian density $\tilde{p}_t(x, y)$ of P_t with respect to the Lebesgue measure to get its density w.r.t. λ :

$$p_t(x, y) = \frac{\tilde{p}_t(x, y)}{\tilde{p}_\lambda(y)} \propto \frac{\exp \left(-\frac{\theta \|y - e^{-\theta t}x\|^2}{2\rho^2(1 - e^{-2\theta t})} \right)}{\exp \left(-\frac{\theta \|y\|^2}{2\rho^2} \right)} = \exp \left(-\theta \frac{\|x\|^2 - 2e^{\theta t} \langle x, y \rangle + \|y\|^2}{2\rho^2(e^{2\theta t} - 1)} \right),$$

where $\tilde{p}_\lambda$ is the density of λ w.r.t. the Lebesgue measure. Thus, Theorem 6 yields the following.

Corollary 7. *Let $f : \mathbb{D}^n \rightarrow \mathbb{R}^d$ have global L_2 -sensitivity Δ and $\mathbf{P} = (P_t)_{t \geq 0}$ be the Ornstein-Uhlenbeck semigroup with parameters θ, ρ . For any $\alpha > 1$ and $t > 0$ the mechanism $M_t^f(D) = P_t(f(D))$ satisfies $(\alpha, \alpha\Lambda(t))$ -RDP with $\Lambda(t) = \frac{\theta \Delta^2}{2\rho^2(e^{2\theta t} - 1)}$.*

The Ornstein-Uhlenbeck mechanism is not an unbiased mechanism since $\mathbb{E}[M_t^f(D)] = e^{-\theta t}f(D)$. This bias is the reason why the privacy guarantee in Corollary 7 exhibits a rate $O(e^{-2\theta t})$, while, for example, the Brownian motion mechanism only exhibits a rate $O(t^{-1})$. In particular, the Ornstein-Uhlenbeck mechanism achieves its privacy not only by introducing noise, but also by shrinking $f(D)$ towards a data-independent point (the origin in this case); this effectively corresponds to reducing the sensitivity of f from Δ to $e^{-\theta t}\Delta$. This provides a way to trade-off variance and bias in the mean-squared error (MSE) incurred by privately releasing $f(D)$ in a similar way that can be achieved by post-processing the Gaussian mechanism when $f(D)$ is known to be bounded.

To formalize this result we define the *mean squared error* $\mathcal{E}_{\text{OU}}(\theta, \rho, t)$ of the Ornstein-Uhlenbeck mechanism with parameters θ, ρ at time t , which is given by:

$$\mathcal{E}_{\text{OU}}(\theta, \rho, t) \triangleq \mathbb{E}[\|f(D) - M_t^f(D)\|^2] = (1 - e^{-\theta t})^2 \|f(D)\|^2 + \frac{d\rho^2}{\theta}(1 - e^{-2\theta t}). \quad (4)$$

Similarly, we define $\mathcal{E}_{\text{GM}}(\theta, \rho, t)$ as the mean squared error of a Gaussian mechanism with the same privacy guarantees as M_t^f with parameters θ, ρ . In particular, we have $\mathcal{E}_{\text{GM}}(\theta, \rho, t) = d\tilde{\sigma}^2$, where $\tilde{\sigma}^2 \triangleq \frac{\rho^2(e^{2\theta t}-1)}{\theta}$ (cf. Corollary 7). We also note the post-processed Gaussian mechanism (PGM) $D \mapsto \beta(f(D) + \mathcal{N}(0, \tilde{\sigma}^2 I))$ which multiplies the output by a scalar β optimized to minimize the MSE under the condition $\|f(D)\| \leq R$ yields error $\mathcal{E}_{\text{PGM}}(\theta, \rho, t) \leq \mathcal{E}_{\text{GM}}(\theta, \rho, t)(1 + \frac{d\tilde{\sigma}^2}{R^2})^{-1}$.

Theorem 8. Suppose $f : \mathbb{D}^n \rightarrow \mathbb{R}^d$ has global L_2 -sensitivity Δ and satisfies $\sup_D \|f(D)\| \leq R$. If $\theta R^2 \leq 4d\rho^2$ then we have $\frac{\mathcal{E}_{\text{OU}}(\theta, \rho, t)}{\mathcal{E}_{\text{GM}}(\theta, \rho, t)} \leq 1$ for all $t \geq 0$ and $\lim_{t \rightarrow \infty} \frac{\mathcal{E}_{\text{OU}}(\theta, \rho, t)}{\mathcal{E}_{\text{GM}}(\theta, \rho, t)} = 0$. In particular, taking $\theta = \log\left(1 + \frac{d\Delta^2}{2\epsilon R^2}\right)$ and $\rho^2 = \frac{\theta\Delta^2}{2\epsilon(e^{2\theta}-1)}$ with $\epsilon > 0$, the mechanism M_t^f satisfies $(\alpha, \alpha\epsilon)$ -RDP at time $t = 1$ and we have $\frac{\mathcal{E}_{\text{OU}}(\theta, \rho, 1)}{\mathcal{E}_{\text{GM}}(\theta, \rho, 1)} \leq \left(1 + \frac{d\Delta^2}{2\epsilon R^2}\right)^{-1}$.

This result not only shows that the Ornstein-Uhlenbeck mechanism is uniformly better than the Gaussian mechanism for any level of privacy, but also shows that in this mechanism the error always stays bounded and can attain the same level of error as the Gaussian mechanism with optimal post-processing. To see this note that with the choices of parameters made in the second statement give $\mathcal{E}_{\text{GM}}(\theta, \rho, 1) = \frac{d\Delta^2}{2\epsilon}$ and therefore $\mathcal{E}_{\text{OU}}(\theta, \rho, 1) \leq \frac{d\Delta^2 R^2}{2\epsilon R^2 + d\Delta^2}$, which behaves like $O(R^2)$ with Δ constant and either $\epsilon \rightarrow 0$ or $d \rightarrow \infty$.

6 Conclusion

We have undertaken a systematic study of amplification by post-processing. Our results yield improvements over recent work on amplification by iteration, and introduce a new Ornstein-Uhlenbeck mechanism which is more accurate than the Gaussian mechanism. In the future it would be interesting to study applications of amplification by post-processing. One promising application is *Hierarchical Differential Privacy*, where information is released under increasingly strong privacy constraints (e.g. to a restricted group within a company, globally within a company, and finally to outside parties).

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