Exponential tilting of subweibull distributions

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Abstract

The class of subweibull distributions has recently been shown to generalize the important properties of subexponential and subgaussian random variables. We describe alternative characterizations of subweibull distributions and detail the conditions under which their tail behavior is preserved after exponential tilting.

1 Introduction

Subexponential and subgaussian distributions are of fundamental importance in the application of high dimensional probability to machine learning (Vershynin, 2018; Wainwright, 2019). Recently it has been shown that the subweibull class unifies the subexponential and subgaussian families, while also incorporating distributions with heavier tails (Vladimirova et al., 2020; Kuchibhotla & Chakrabortty, 2022). Informally, a q-subweibull ($q > 0$) random variable has a survival function that decays at least as fast as $\exp(-\lambda x^q)$ for some $\lambda > 0$. For example, the exponential distribution is 1-subweibull and the Gaussian distribution is 2-subweibull. Here, we provide two alternative characterizations of the subweibull class and introduce a distinction between strictly and broadly subweibull distributions. As an example, the Poisson distribution is shown to be strictly subexponential ($q = 1$) but not subweibull for any $q > 1$. Finally, we detail the conditions under which the subweibull property is preserved after exponential tilting.

2 Laplace transforms

Definition 2.1. We define the two-sided Laplace transform of a random variable X with distribution function F as

$$\mathcal{L}_X(t) = \mathbb{E}[\exp(-tX)] = \int_{-\infty}^{\infty} \exp(-tx)dF(x)$$

We do not restrict X to be nonnegative or to have a density function. In the special case that $\mathcal{L}_X(t) < \infty$ for all t in an open interval around $t = 0$, then X has a moment generating function (MGF) which is $M_X(t) = \mathbb{E}[\exp(tX)] = \mathcal{L}_X(-t)$. The Laplace transform can characterize the distribution even if the MGF does not exist.

Lemma 2.0.1. If the Laplace transforms of random variables X and Y satisfy $\mathcal{L}_X(t) = \mathcal{L}_Y(t)$ for all t in any nonempty open interval $(a, b) \subset \mathbb{R}$, not necessarily containing zero, then $X \overset{d}{=} Y$.

For a proof refer to Mukherjea et al. (2006). A random variable X is considered subexponential iff the MGF exists (Vershynin, 2018). If $\mathcal{L}_X(t) = \infty$ for all $t > 0$ (respectively $t < 0$), X is said to have a heavy left (respectively, right) tail (Nair et al., 2022). If a tail is not heavy it is said to be light. It is well known that the one-sided Laplace transform characterizes nonnegative distributions (Feller, 1971). Lemma 2.0.1 shows that the two-sided Laplace transform characterizes any distribution with at least one light tail.

3 Subweibull random variables

Definition 3.1. A random variable X is *q-subweibull* if $\mathbb{E}[\exp(\lambda^q|X|^q)] < \infty$ for some $\lambda > 0$. X is *strictly q-subweibull* if the condition is satisfied for all $\lambda > 0$. If X is q-subweibull but not strictly so, we refer to it as *broadly q-subweibull*.

The first part of this definition was also proposed by Kuchibhotla & Chakrabortty (2022) and by Vladimirova et al. (2020) using a parameterization equivalent to $1/q$. Clearly X is (strictly) q-subweibull if and only if $|X|^q$ is (strictly) subexponential. As an example, the Laplace distribution is broadly 1-subweibull (ie broadly subexponential).

Definition 3.2. The *radius of convergence* of a q-subweibull random variable X is defined by

$$R_q = \sup\{\lambda > 0 : \mathbb{E}[\exp(\lambda^q |X|^q)] < \infty\}$$

and if no such $\lambda > 0$ exists we adopt the convention that $R_q = 0$.

In the case of strictly q-subweibull distributions, $R_q = \infty$. X has "heavy tails" (in the sense of Nair et al., 2022) iff it is not subexponential ($R_1 = 0$).

Lemma 3.0.1. Random variable X with $\Pr(X < 0) \notin \{0, 1\}$ is q-subweibull if and only if the nonnegative random variables $A = [-X \mid X < 0]$ and $B = [X \mid X \geq 0]$ are q-subweibull. Let R_{qx}, R_{qa}, and R_{qb} denote the radii of convergence for X, A, and B, respectively. Then $R_{qx} = \min\{R_{qa}, R_{qb}\}$.

Proposition 3.1. (Theorem 1 of Vladimirova et al. (2020)) The following are equivalent characterizations of a q-subweibull random variable X for $q > 0$.

1. Tail bound: $\exists\ K_{1a} > 0$ such that $\forall\ t \geq 0$,

$$\Pr(|X| > t) \leq 2\exp\big(-(t/K_{1a})^q\big)$$

2. Growth rate of absolute moments: $\exists\ K_2 > 0$ such that $\forall\ p \geq 1$,

$$\big(\mathbb{E}[|X|^p]\big)^{1/p} \leq K_2 p^{1/q}$$

3. MGF of $|X|^q$ finite in interval of zero: $\exists\ K_3 > 0$ such that $\forall\ 0 < \lambda \leq 1/K_3$

$$\mathbb{E}\left[\exp(\lambda^q |X|^q)\right] \leq \exp(K_3^q \lambda^q)$$

When $q \geq 1$, Condition 3 is equivalent to requiring the Orlicz norm $\|X\|_{\psi_q} < \infty$ where $\psi_q(x) = \exp(x^q) - 1$. The case of $q < 1$ (heavy tails) is further discussed in Kuchibhotla & Chakrabortty (2022). It is sometimes convenient to use the following asymptotic alternatives to Proposition 3.1 (1 and 2).

Corollary 3.0.1. A random variable X is q-subweibull if and only if either of the following hold

1. Tail bound: $\exists\ K_{1b} > 0$ such that

$$\limsup_{t\to\infty} \Pr(|X| > t)\exp\big((t/K_{1b})^q\big) < \infty$$

2. Growth rate of absolute moments:

$$\limsup_{p\to\infty} \frac{\big(\mathbb{E}[|X|^p]\big)^{1/p}}{p^{1/q}} < \infty$$

It was shown by Vladimirova et al. (2020) that a q-subweibull distribution is also r-subweibull for all $r < q$. We now show that this also implies it is strictly r-subweibull.

Corollary 3.0.2. If X is q-subweibull then it is strictly r-subweibull for all $r \in (0, q)$.

Corollary 3.0.3. Every bounded random variable is strictly q-subweibull for all $q > 0$.

Proof. If X is bounded then there exists $M \geq 0$ such that $|X| \leq M$. Then $\mathbb{E}[\exp(\lambda^q|X|^q)] \leq \exp(\lambda^q M^q) < \infty$ for all $\lambda > 0$ and $q > 0$. ☐

Corollary 3.0.4. If X is not strictly q-subweibull with $q \geq 1$ then it is not r-subweibull for any $r > q$.

3.1 Subweibull properties of the Poisson distribution

Corollaries 3.0.2 and 3.0.4 suggest a hierarchy of distributions based on the heaviness of the tails. Broadly q-subweibull distributions, which have a finite but nonzero radius of convergence (R_q), serve as "critical points" in the transition between the strictly r-subweibull regime ($r < q$), with $R_q = \infty$ and the not r-subweibull regime ($r > q$) with $R_q = 0$. However, the transition from strictly subweibull to not subweibull can be immediate, without passing through the stage of broadly subweibull. Here we provide a simple example: the Poisson tail is lighter than any exponential tail, but heavier than any weibull tail with $q > 1$.

Proposition 3.2. The Poisson distribution is strictly q-subweibull for $q \leq 1$ but not q-subweibull for any $q > 1$.

4 Exponential tilting

Definition 4.1. Let X be a random variable with distribution function F. If the Laplace transform satisfies $\mathcal{L}_X(-\theta) < \infty$ for some $\theta \neq 0$, then the *exponentially tilted distribution* is given by

$$F_\theta(x) = \int_{-\infty}^{x} \frac{\exp(\theta t)}{\mathcal{L}_X(-\theta)} dF(t)$$

We adopt the convention of using $-\theta$ instead of θ so that the interpretation of the tilting parameter is consistent with other works that assume X has an MGF, in which case one could equivalently require $M_X(\theta) < \infty$.

From the Radon-Nikodym theorem, F_θ is absolutely continuous with respect to F. Since the density function $e^{\theta x}/\mathcal{L}_X(-\theta)$ is also strictly positive, exponential tilting does not change the support. Generally speaking it is possible to produce a subexponential distribution by exponential tilting of any distribution with at least one light tail.

Proposition 4.1. If $X \sim F$ is a random variable having at least one light tail then exponential tilting is possible for all θ in some open interval $(-S, T)$ with $S, T \geq 0$ and $S + T > 0$. The resulting tilted distribution F_θ is subexponential with MGF $M_Z(t) = \mathcal{L}_X(-\theta - t)/\mathcal{L}_X(-\theta)$ finite for all $t \in (-S - \theta, T - \theta)$.

As an example, if $X \sim F$ is a nonnegative, heavy tailed random variable ($T = 0$), its left tail is strictly subexponential ($S = \infty$) so exponential tilting is possible for all $\theta < 0$. By Proposition 4.1 the resulting tilted distribution is subexponential and hence has lighter tails than the original distribution. On the other hand, if X is broadly subexponential, exponential tilting produces another broadly subexponential distribution, with a shifted interval of convergence.

While exponential tilting can alter the tail behavior of heavy tailed and broadly subexponential distributions, it does not affect the tail behavior of q-subweibull distributions with lighter than exponential tails (i.e., $q > 1$).

Lemma 4.0.1. Preservation of nonnegative subweibull tails under exponential tilting. Let θ be any real number. If $X \sim F$ is nonnegative and q-subweibull ($q > 1$), then the exponentially tilted variable $Z \sim F_\theta$ is also nonnegative and q-subweibull with the same radius of convergence.

1. $\mathbb{E}[\exp(\lambda^q X^q)] < \infty$ for all $\lambda \in [0, R_q)$ implies $\mathbb{E}[\exp(\lambda^q Z^q)] < \infty$ for all $\lambda \in [0, R_q)$.
2. $\mathbb{E}[\exp(\lambda^q X^q)] = \infty$ for all $\lambda > R_q$ implies $\mathbb{E}[\exp(\lambda^q Z^q)] = \infty$ for all $\lambda > R_q$.

We now extend Lemma 4.0.1 to general random variables.

Theorem 4.1. Preservation of subweibull tails under exponential tilting. Let θ be any real number.

1. If $X \sim F$ is q-subweibull ($q > 1$) with radius of convergence R_q, then the exponentially tilted variable $Z \sim F_\theta$ is also q-subweibull and has the same radius of convergence.

2. If $X \sim F$ is strictly q-subweibull ($q \geq 1$), the exponentially tilted variable $Z \sim F_\theta$ is also strictly q-subweibull.

3. If $X \sim F$ is not q-subweibull ($q > 1$), then $Z \sim F_\theta$ is also not q-subweibull.

5 Discussion

The theory of subexponential and subgaussian distributions is a key prerequisite to many results in nonparametric and nonasymptotic statistical inference such as concentration inequalities. A comprehensive overview with applications to high-dimensional machine learning problems is provided by Kuchibhotla & Chakrabortty (2022). Exponential tilting is used in a variety of statistical areas such as causal inference (McClean et al., 2024) and Monte Carlo sampling (Fuh & Wang, 2024). If F_θ is a tilted distribution, it is a natural exponential family with parameter θ. The exponential families are building blocks for generalized linear models (McCullagh & Nelder, 1989). For applications to machine learning see Li et al. (2023); Maity et al. (2023).

Here, we have provided a brief overview of subweibull distributions. We showed that the Poisson distribution is strictly 1-subweibull but not q-subweibull for any $q > 1$. Finally, we detailed the conditions under which the subweibull property is perserved under exponential tilting.

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A Proofs for Section 3 (Subweibull random variables)

Lemma 3.0.1. Random variable X with $\Pr(X<0)\notin\{0,1\}$ is q-subweibull if and only if the nonnegative random variables $A=[-X\mid X<0]$ and $B=[X\mid X\geq 0]$ are q-subweibull. Let R_{qx}, R_{qa}, and R_{qb} denote the radii of convergence for X, A, and B, respectively. Then $R_{qx}=\min\{R_{qa},R_{qb}\}$.

Proof. Let $p=\Pr(X<0)$ and define nonnegative random variables $A=[-X\mid X<0]$ and $B=[X\mid X\geq 0]$.

$$\begin{aligned}\mathbb{E}[\exp(\lambda^q|X|^q)]&=\mathbb{E}[\exp(\lambda^q(-X)^q)\mid X<0]p+\mathbb{E}[\exp(\lambda^q X^q)\mid X\geq 0](1-p)\\&=\mathbb{E}[\exp(\lambda^q A^q)]p+\mathbb{E}[\exp(\lambda^q B^q)](1-p)\end{aligned}$$

The left hand side is finite if and only if both terms on the right hand side are finite. If R_{qx} is the radius of convergence for X then $\mathbb{E}[\exp(\lambda^q|X|^q)]<\infty$ for all $\lambda\in[0,R_{qx})$. Clearly $\mathbb{E}[\exp(\lambda^q A^q)]<\infty$ and $\mathbb{E}[\exp(\lambda^q B^q)]<\infty$ for all $\lambda\in[0,R_{qx})$ also, implying $\min\{R_{qa},R_{qb}\}\geq R_{qx}$. However, if $\min\{R_{qa},R_{qb}\}>R_{qx}$ then there exists some $\lambda>R_{qx}$ such that $\mathbb{E}[\exp(\lambda^q|X|^q)]<\infty$, which by Definition 3.2 means R_{qx} is not the radius of convergence of X, a contradiction. Therefore, $\min\{R_{qa},R_{qb}\}=R_{qx}$. □

Corollary 3.0.1. A random variable X is q-subweibull if and only if either of the following hold

1. Tail bound: $\exists\, K_{1b}>0$ such that

$$\limsup_{t\to\infty}\Pr(|X|>t)\exp\left((t/K_{1b})^q\right)<\infty$$

2. Growth rate of absolute moments:

$$\limsup_{p\to\infty}\frac{\left(\mathbb{E}[|X|^p]\right)^{1/p}}{p^{1/q}}<\infty$$

Proof. Proposition 3.1 (1) $\implies$ (1) :

$$\limsup_{t\to\infty}\Pr(|X|>t)\exp\left((t/K_{1a})^q\right)\leq\sup_{t\geq 0}\Pr(|X|>t)\exp\left((t/K_{1a})^q\right)\leq 2<\infty$$

So we can simply set $K_{1b}=K_{1a}$.

(1) $\implies$ Proposition 3.1 (1): Assume

$$\limsup_{t\to\infty}\Pr(|X|>t)\exp\left((t/K_{1b})^q\right)=K$$

Then, for every $C>K$, there exists some T such that for all $t>T$,

$$\Pr(|X|>t)\exp\left((t/K_{1b})^q\right)<C$$

For all $t\in[0,T]$, $\Pr(|X|>t)\leq 1$ and $\exp\left((t/K_{1b})^q\right)\leq\exp\left((T/K_{1b})^q\right)$. Therefore

$$\sup_{t\geq 0}\Pr(|X|>t)\exp\left((t/K_{1b})^q\right)\leq\max\left\{C,\ \exp\left((T/K_{1b})^q\right)\right\}$$

Let $U=\max\left\{C,\ \exp\left((T/K_{1b})^q\right)\right\}$. If $U\leq 2$ this directly implies Proposition 3.1 (1) with $K_{1a}=K_{1b}$. In the case that $U>2$, set

$$K_{1a}=K_{1b}\left(\frac{\log U}{\log 2}\right)^{1/q}>K_{1b}$$

Let $f(t)=U\exp\left(-(t/K_{1b})^q\right)$, $g(t)=2\exp\left(-(t/K_{1a})^q\right)$, and $T^\star=K_{1b}(\log U)^{1/q}$. Since $f(T^\star)=g(T^\star)=1$ and $g(t)$ is a strictly decreasing function, this implies that $\Pr(|X|>t)\leq 1\leq g(t)$ for $t\in[0,T^\star]$. For $t\geq T^\star$, $g(t)\geq f(t)$ since $K_{1a}>K_{1b}$, and $f(t)\geq\Pr(|X|>t)$ by assumption therefore $2\exp\left(-(t/K_{1a})^q\right)\geq\Pr(|X|>t)$ for all $t\geq 0$.

Proposition 3.1 (2) $\implies$ (2):

$$\limsup_{p\to\infty} \frac{\left(\mathbb{E}[|X|^p]\right)^{1/p}}{p^{1/q}} \le \sup_{p\ge 1} \frac{\left(\mathbb{E}[|X|^p]\right)^{1/p}}{p^{1/q}} \le K_2 < \infty$$

(2) $\implies$ Proposition 3.1 (2): Assume

$$\limsup_{p\to\infty} \frac{\left(\mathbb{E}[|X|^p]\right)^{1/p}}{p^{1/q}} = K < \infty$$

Then for every $C > K$, there exists some $p^\star$ such that for all $p > p^\star$,

$$\frac{\left(\mathbb{E}[|X|^p]\right)^{1/p}}{p^{1/q}} < C$$

The L_p norm is increasing in p, so for $p \in [1, p^\star]$, $\left(\mathbb{E}[|X|^p]\right)^{1/p} \le \left(\mathbb{E}[|X|^{p^\star}]\right)^{1/p^\star}$ and $p^{1/q} \ge 1$, which establishes

$$\sup_{p\ge 1} \frac{\left(\mathbb{E}[|X|^p]\right)^{1/p}}{p^{1/q}} \le \max\left\{\left(\mathbb{E}[|X|^{p^\star}]\right)^{1/p^\star},\ C\right\}$$

□

Corollary 3.0.2. If X is q-subweibull then it is strictly r-subweibull for all $r \in (0, q)$.

Proof. If X is q-subweibull, by Proposition 3.1 we may assume there exists $K > 0$ such that $\forall\ p \ge 1$,

$$\left(\mathbb{E}[|X|^p]\right)^{1/p} \le K p^{1/q}$$

Let $r \in (0, q)$. The MGF of $|X|^r$ is given by

$$\begin{aligned}
\mathbb{E}\left[\exp(\lambda^r |X|^r)\right] &= 1 + \sum_{p=1}^{\infty} \frac{\lambda^{pr} E[|X|^{pr}]}{p!} \\
&\le 1 + \sum_{p=1}^{\infty} \frac{\lambda^{pr} K^{pr} (pr)^{pr/q}}{(p/e)^p} \\
&= 1 + \sum_{p=1}^{\infty} \left(\lambda^r K^r e r^{r/q}\right)^p p^{p(r/q-1)}
\end{aligned}$$

Apply the root test to the series to determine convergence.

$$R(p) = \lambda^r K^r e r^{r/q} p^{r/q-1}$$

Since $r < q$, then $\lim_{p\to\infty} R(p) = 0$ and the series converges regardless of the value of λ, which shows X is strictly r-subweibull. □

Corollary 3.0.4. If X is not strictly q-subweibull with $q \ge 1$ then it is not r-subweibull for any $r > q$.

Proof. From Corollary 3.0.1 we may assume $\exists\ \lambda > 0$ such that

$$\limsup_{t\to\infty} \Pr(|X| > t) \exp(\lambda t^q) = \infty$$

which implies there is an infinite sequence $t_n \to \infty$ such that

$$\lim_{n\to\infty} \Pr(|X| > t_n) \exp(\lambda t_n^q) = \infty$$

Now let $\rho > 0$ and $r > q$. Whenever $t \geq t^\star = (\lambda/\rho)^{1/(r-q)}$, we have $\exp(\rho t^r) \geq \exp(\lambda t^q)$. Let $\{t_m\}$ be the infinite subsequence of $\{t_n\}$ excluding the elements less than $t^\star$. Clearly $t_m \to \infty$ as well. Then

$$\lim_{m\to\infty} \Pr(|X| > t_m)\exp(\rho t_m^r) \geq \lim_{m\to\infty} \Pr(|X| > t_m)\exp(\lambda t_m^q) = \infty$$

which implies X cannot be r-subweibull. □

Proposition 3.2. The Poisson distribution is strictly q-subweibull for $q \leq 1$ but not q-subweibull for any $q > 1$.

Proof. Since the Poisson distribution has a finite MGF with infinite radius of convergence, it is strictly subexponential and by Corollary 3.0.2 strictly q-subweibull for all $q \leq 1$. Let $X \sim Poi(\mu)$. Without loss of generality assume $t > 1$ and let $n = \lfloor t \rfloor + 1$ with $t < n \leq t + 1$.

$$\Pr(X > t) = \sum_{j=n}^{\infty} \Pr(X = j) \geq \Pr(X = n) = \frac{\mu^n \exp(-\mu)}{n!} = \frac{\mu^n \exp(-\mu)}{n\Gamma(n)}$$

Since $t\Gamma(t)$ is increasing for $t \geq 1$, we have $n\Gamma(n) \leq (t+1)\Gamma(t+1)$. Also, $\Gamma(n) \leq n^n$ for $n \geq 1$. For the μ^n term, it is increasing for $\mu \geq 1$ and decreasing for $\mu < 1$, so $\mu^n \geq \min\{\mu^{t+1}, \mu^t\} = \mu^t \min\{\mu, 1\}$. Combining these we obtain

$$\Pr(X > t) \geq \frac{\mu^t \min\{\mu, 1\} e^{-\mu}}{(t+1)\Gamma(t+1)} = \frac{\mu^t \min\{\mu, 1\} e^{-\mu}}{(t+1)(t)\Gamma(t)} \geq \frac{\mu^t \min\{\mu, 1\} e^{-\mu}}{(t+1)(t)t^t}$$

To assess whether the tail follows a subweibull rate of decay, choose any $\lambda > 0$ and $q > 1$, then

$$\begin{aligned}
\limsup_{t\to\infty} \Pr(X > t)\exp(\lambda t^q) &\geq \min\{\mu, 1\} e^{-\mu} \lim_{t\to\infty} \frac{\mu^t}{(t+1)(t)t^t} \exp(\lambda t^q) \\
&= \min\{\mu, 1\} e^{-\mu} \exp\left[\lim_{t\to\infty} t\log\mu - \log(t+1) - \log t - t\log t + \lambda t^q\right]
\end{aligned}$$

The expression inside brackets is of the form $\infty - \infty$ so we rearrange terms and apply L'Hopital's rule. Define

$$\begin{aligned}
&\lim_{t\to\infty} t\log\mu - \log(t+1) - \log t - t\log t + \lambda t^q \\
&\quad = \lim_{t\to\infty} (t\log t)\left[\lim_{t\to\infty} \frac{\log\mu}{\log t} - \frac{\log(t+1)}{t\log t} - \frac{1}{t} + \frac{\lambda t^q}{t\log t}\right] \\
&\quad = \lim_{t\to\infty} (t\log t)\left[\lim_{t\to\infty} 0 - \frac{1/(t+1)}{1+\log t} - (0) + \frac{\lambda q t^{q-1}}{1+\log t}\right] \\
&\quad = \lim_{t\to\infty} (t\log t)\left[\lim_{t\to\infty} \frac{\lambda q(q-1)t^{q-2}}{1/t}\right] = \infty \cdot \infty = \infty
\end{aligned}$$

Therefore

$$\limsup_{t\to\infty} \Pr(X > t)\exp(\lambda t^q) = \infty$$

Since this holds for all $\lambda > 0$, X cannot satisfy Corollary 3.0.1 and therefore is not q-subweibull for any $q > 1$. □

B Proofs for Section 4 (Exponential tilting)

Proposition 4.1. If $X \sim F$ is a random variable having at least one light tail then exponential tilting is possible for all θ in some open interval $(-S, T)$ with $S, T \geq 0$ and $S + T > 0$. The resulting tilted distribution F_θ is subexponential with MGF $M_Z(t) = \mathcal{L}_X(-\theta - t)/\mathcal{L}_X(-\theta)$ finite for all $t \in (-S - \theta, T - \theta)$.

Proof. Without loss of generality assume the right tail is light so $\mathcal{L}_X(-\theta) < \infty$ for some $\theta > 0$. For all $\theta' \in [0, \theta)$,

$$\mathcal{L}_X(-\theta') = \mathbb{E}[\exp(\theta' X)] \leq \mathbb{E}[\exp(\theta X)] < \infty$$

Set $T = \sup\{\theta : \ \mathcal{L}_X(-\theta) < \infty\} > 0$. If X has a heavy left tail then $\mathcal{L}_X(-\theta) = \infty$ for all $\theta < 0$, so the interval of convergence is $(-S, T)$ with $S = 0$. If X has a light left tail then we can set $S = -\inf\{\theta : \mathcal{L}_X(-\theta) < \infty\} > 0$. This establishes the interval is $(-S, T)$ with $S, T \geq 0$ and $S + T > 0$. Let $Z \sim F_\theta$ follow the tilted distribution with $\theta \in (-S, T)$. Its Laplace transform is

$$\begin{aligned}\mathcal{L}_Z(t) = \mathbb{E}[\exp(-tZ)] &= \int_{-\infty}^{\infty} \exp(-tz) dF_\theta(z) = \int_{-\infty}^{\infty} \exp(-tx) \frac{\exp(\theta x)}{\mathcal{L}_X(-\theta)} dF(x) \\ &= \mathbb{E}[\exp(-(t - \theta)X)]/\mathcal{L}_X(-\theta) = \mathcal{L}_X(-(\theta - t))/\mathcal{L}_X(-\theta)\end{aligned}$$

This is finite when $\theta - t \in (-S, T)$ or equivalently $t \in (-T + \theta, S + \theta)$. Since $\theta \in (-S, T)$, the interval of convergence for $\mathcal{L}_Z(t)$ is an open interval containing zero, which proves Z is subexponential and has the MGF

$$M_Z(t) = \mathcal{L}_Z(-t) = \mathcal{L}_X(-\theta - t)/\mathcal{L}_X(-\theta)$$

which is finite on the interval $t \in (-S - \theta, T - \theta)$. □

Lemma 4.0.1. Preservation of nonnegative subweibull tails under exponential tilting. Let θ be any real number. If $X \sim F$ is nonnegative and q-subweibull ($q > 1$), then the exponentially tilted variable $Z \sim F_\theta$ is also nonnegative and q-subweibull with the same radius of convergence.

1. $\mathbb{E}[\exp(\lambda^q X^q)] < \infty$ for all $\lambda \in [0, R_q)$ implies $\mathbb{E}[\exp(\lambda^q Z^q)] < \infty$ for all $\lambda \in [0, R_q)$.
2. $\mathbb{E}[\exp(\lambda^q X^q)] = \infty$ for all $\lambda > R_q$ implies $\mathbb{E}[\exp(\lambda^q Z^q)] = \infty$ for all $\lambda > R_q$.

Proof. If X is q-subweibull with $q > 1$ then by Corollary 3.0.2 it is strictly subexponential and $\mathcal{L}_X(-\theta) < \infty$ for all $\theta \in \mathbb{R}$. Let $Z \sim F_\theta$. The MGF of Z^q is

$$\mathbb{E}[\exp(\lambda^q Z^q)] = \frac{\int \exp(\lambda^q x^q + \theta x) dF(x)}{\mathcal{L}_X(-\theta)}$$

(1) case of $\lambda < R_q$. If $\theta \leq 0$ then

$$\int \exp(\lambda^q x^q + \theta x) dF(x) \leq \int \exp(\lambda^q x^q + 0) dF(x) = \mathbb{E}[\exp(\lambda^q X^q)] < \infty$$

If $\theta > 0$, choose $\rho \in (\lambda, R_q)$ and define

$$x^\star = \left(\frac{\theta}{\rho^q - \lambda^q}\right)^{\frac{1}{q-1}}$$

Then for $x > x^\star$, $\lambda^q x^q + \theta x \leq \rho^q x^q$. Therefore,

$$\begin{aligned}\int \exp(\lambda^q x^q + \theta x) dF(x) &= \int_0^{x^\star} \exp(\lambda^q x^q + \theta x) dF(x) + \int_{x^\star}^{\infty} \exp(\lambda^q x^q + \theta x) dF(x) \\ &\leq \int_0^{x^\star} \exp\left(\theta x^\star + \lambda^q (x^\star)^q\right) dF(x) + \int_{x^\star}^{\infty} \exp(\rho^q x^q) dF(x) \\ &\leq \exp\left(\theta x^\star + \lambda^q (x^\star)^q\right) \Pr(X \leq x^\star) + \int_0^{\infty} \exp(\rho^q x^q) dF(x) \\ &< \infty\end{aligned}$$

(2) case of $\lambda > R_q$. If $\theta \geq 0$ then

$$\int \exp(\lambda^q x^q + \theta x) dF(x) \geq \int \exp(\lambda^q x^q + 0) dF(x) = \mathbb{E}[\exp(\lambda^q X^q)] = \infty$$

If $\theta < 0$. Choose $\rho \in (R_q, \lambda)$ and define

$$x^\star = \left(\frac{-\theta}{\lambda^q - \rho^q}\right)^{\frac{1}{q-1}}$$

Then for $x > x^\star$, $\lambda^q x^q + \theta x \geq \rho^q x^q$. Therefore,

$$\begin{aligned}\int \exp(\lambda^q x^q + \theta x) dF(x) &= \int_0^{x^\star} \exp(\lambda^q x^q + \theta x) dF(x) + \int_{x^\star}^{\infty} \exp(\lambda^q x^q + \theta x) dF(x) \\ &\geq \int_0^{x^\star} \exp(\lambda^q x^q + \theta x) dF(x) + \int_{x^\star}^{\infty} \exp(\rho^q x^q) dF(x)\end{aligned}$$

The first term is finite. We will show the second term is infinite. By assumption,

$$\begin{aligned}\int_0^{\infty} \exp(\rho^q x^q) dF(x) = \infty \\ &= \int_0^{x^\star} \exp(\rho^q x^q) dF(x) + \int_{x^\star}^{\infty} \exp(\rho^q x^q) dF(x)\end{aligned}$$

But

$$\int_0^{x^\star} \exp(\rho^q x^q) dF(x) \leq \exp\left(\rho^q (x^\star)^q\right) \Pr(X \leq x^\star) < \infty$$

Therefore

$$\int_{x^\star}^{\infty} \exp(\rho^q x^q) dF(x) = \infty$$

implying

$$\int \exp(\lambda^q x^q + \theta x) dF(x) = \infty$$

as well. □

Theorem 4.1. Preservation of subweibull tails under exponential tilting. Let θ be any real number.

1. If $X \sim F$ is q-subweibull ($q > 1$) with radius of convergence R_q, then the exponentially tilted variable $Z \sim F_\theta$ is also q-subweibull and has the same radius of convergence.

2. If $X \sim F$ is strictly q-subweibull ($q \geq 1$), the exponentially tilted variable $Z \sim F_\theta$ is also strictly q-subweibull.

3. If $X \sim F$ is not q-subweibull ($q > 1$), then $Z \sim F_\theta$ is also not q-subweibull.

Proof. (1) By Corollary 3.0.2, X is strictly subexponential so $\mathcal{L}_X(-\theta) < \infty$ for all $\theta \in \mathbb{R}$. Choose any arbitrary θ and set $M_1 = \mathcal{L}_X(-\theta)$. Define nonnegative random variables $A = [-X \mid X < 0]$ and $B = [X \mid X \geq 0]$ with distributions F^- and F^+, respectively. By Lemma 3.0.1 both A and B are q-subweibull and strictly subexponential. Let R_{qa} and R_{qb} be the radii of convergence of A and B, respectively. Let $p = \Pr(X < 0)$ and assume $p \notin \{0, 1\}$ (otherwise simply apply Lemma 4.0.1 to X or $-X$). Note that

$$M_1 = \mathcal{L}_A(\theta)p + \mathcal{L}_B(-\theta)(1-p) \tag{1}$$

We have

$$\begin{aligned}
\mathbb{E}[\exp(\lambda^q|Z|^q)] &= \int_{-\infty}^{\infty}\exp(\lambda^q|z|^q)dF_\theta(z) = \int_{-\infty}^{\infty}\exp(\lambda^q|x|^q)\frac{\exp(\theta x)}{M_1}dF(x)\\
&= \frac{\mathbb{E}[\exp(\lambda^q|X|^q+\theta X)]}{M_1}\\
&= \frac{p}{M_1}\mathbb{E}[\exp(\lambda^q A^q-\theta A)] + \frac{(1-p)}{M_1}\mathbb{E}[\exp(\lambda^q B^q+\theta B)]\\
&= \frac{p\mathcal{L}_A(\theta)}{M_1}\int_0^{\infty}\exp(\lambda^q x^q)\frac{\exp(-\theta x)}{\mathcal{L}_A(\theta)}dF^-(x)\dots\\
&\quad \dots + \frac{(1-p)\mathcal{L}_B(-\theta)}{\mathcal{L}_X(-\theta)}\int_0^{\infty}\exp(\lambda^q x^q)\frac{\exp(\theta x)}{\mathcal{L}_B(-\theta)}dF^+(x)\\
&= (\tilde{p})\int_0^{\infty}\exp(\lambda^q z^q)dF^-_{(-\theta)}(z) + (1-\tilde{p})\int_0^{\infty}\exp(\lambda^q z^q)dF^+_\theta(z)\\
&= \tilde{p}\mathbb{E}[\exp(\lambda^q U^q)] + (1-\tilde{p})\mathbb{E}[\exp(\lambda^q V^q)]
\end{aligned}$$

where (see Equation 1), $\tilde{p} = p\mathcal{L}_A(\theta)/M_1$ so that $\tilde{p}\in(0,1)$. The nonnegative random variable U is distributed as $F^-_{(-\theta)}$, which is the exponentially tilting of $A\sim F^-$ by $-\theta$ and $V\sim F^+_\theta$ is similarly defined as the exponential tilting of $B\sim F^+$. By Lemma 4.0.1, this implies U and V are q-subweibull with radii of convergence R_{qa} and R_{qb}, respectively. Let R_{qz} be the radius of convergence of Z. Note that

$$\begin{aligned}
\Pr(Z\geq 0) &= \int_0^{\infty}dF_\theta(z) = \int_0^{\infty}\frac{\exp(\theta x)}{\mathcal{L}_X(-\theta)}dF(x) = \frac{\mathbb{E}[\exp(\theta X)\mid X\geq 0]\Pr(X\geq 0)}{M_1}\\
&= \frac{\mathbb{E}[\exp(\theta B)](1-p)}{M_1} = \frac{\mathcal{L}_B(-\theta)(1-p)}{M_1}\\
&= 1-\tilde{p}
\end{aligned}$$

So $\Pr(Z<0)=\tilde{p}$. By Lemma 3.0.1 this implies Z is q-subweibull with radius of convergence $\min\{R_{qa},R_{qb}\}$, which is also the radius of convergence of X.

(2) For $q=1$ apply Proposition 4.1 with $S=\infty$ and $T=\infty$. For $q>1$, apply (1) with $R_q=\infty$.

(3) This is a direct corollary of (1) obtained in the case of $R_q=0$. □