Electronic Journal of Statistics
Vol. 11 (2017) 1235–1296
ISSN: 1935-7524
DOI: 10.1214/17-EJS1236

Efficient moment calculations for variance components in large unbalanced crossed random effects models*

Katelyn Gao† and Art Owen

Department of Statistics
Stanford University
Stanford, CA, 94305, USA
e-mail: kxgao@stanford.edu; owen@stanford.edu

Abstract: Large crossed data sets, often modeled by generalized linear mixed models, have become increasingly common and provide challenges for statistical analysis. At very large sizes it becomes desirable to have the computational costs of estimation, inference and prediction (both space and time) grow at most linearly with sample size.

Both traditional maximum likelihood estimation and numerous Markov chain Monte Carlo Bayesian algorithms take superlinear time in order to obtain good parameter estimates in the simple two-factor crossed random effects model. We propose moment based algorithms that, with at most linear cost, estimate variance components, measure the uncertainties of those estimates, and generate shrinkage based predictions for missing observations. When run on simulated normally distributed data, our algorithm performs competitively with maximum likelihood methods.

MSC 2010 subject classifications: Primary 62F10; secondary 62J10.
Keywords and phrases: Crossed random effects, variance components, big data.

Received January 2016.

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*This work was supported by US NSF under grant DMS-1407397.
†Supported by US NSF Graduate Research Fellowship under grant DGE-114747.

1. Introduction

Modern electronic activity generates enormous data sets with an unbalanced crossed random effects structure. The factors are customer IDs, URLs, product IDs, cookies, IP addresses, news stories, tweets, and query strings, among others. These variables could be treated as fixed effects, plain categorical variables that

just happen to have a large number of levels. But in many cases, the specific category levels are evanescent. Customers turn over at some rate, cookies get deleted at an even faster rate, products or news stories grow in popularity but then fade. In such cases it is more realistic to treat such variables as random effects. We want our inferences to apply to the population from which the future and observed levels of those variables are sampled. For realism we should also treat data in the same level of a factor as correlated.

The statistically efficient way to treat data sets with crossed random effects is through generalized linear mixed models (GLMMs), maximizing the likelihood with respect to both the parameters and the random effects. However, the cost of these computations is dominated by matrix algebra that takes time cubic in the number of distinct levels and space quadratic in that number; see [1] or [16]. Such costs are infeasible for big data.

It has been suggested to us that stochastic gradient descent (SGD) could provide an alternative way to maximize the likelihood. However, SGD approaches with theoretical guarantees have only been developed for data that can be split into independent subsets, which is not possible for data sets with crossed random effects.

With GLMMs infeasible, it is natural to consider the Gibbs sampler and other Markov Chain Monte Carlo (MCMC) methods. But, as shown in Section 2, those methods in the crossed random effects context have computational cost that is superlinear in the sample size. This is very different from the great success that MCMC has on hierarchical models for data with a nested structure. See for instance [5], [20] and [24].

With both likelihood and Bayesian methods running into difficulties, we turn to the method of moments. It seems ironic to use a 19th century method in this era of increased computer power. But data growth has been outpacing processing power for single-threaded computation, so it is appropriate to revisit methods from an earlier time when the data was large compared to the available computing power. A compelling advantage of the method of moments is that it is easily parallelizable. It also makes no distributional assumptions, has no tuning parameters, and does not require cumbersome diagnostics.

We are motivated by generalized linear mixed models with linear fixed effects but we focus the present paper on a very special case. We consider a setting with identity link, just two factors that are both random, and intercept only regression. In this paper, we assume that the data follow:

Model 1. *Two-factor crossed random effects,*

$$\begin{aligned} &Y_{ij} = \mu + a_i + b_j + e_{ij}, \quad i, j \in \mathbb{N} \quad \textit{where} \\ &a_i \overset{\text{iid}}{\sim} (0, \sigma_A^2), \quad b_j \overset{\text{iid}}{\sim} (0, \sigma_B^2), \quad e_{ij} \overset{\text{iid}}{\sim} (0, \sigma_E^2) \quad \textit{and} \\ &\mathbb{E}(a_i^4) < \infty, \quad \mathbb{E}(b_j^4) < \infty, \quad \mathbb{E}(e_{ij}^4) < \infty. \end{aligned} \tag{1}$$

For example, i might index a customer while j indexes a product and Y_{ij} is the most recent rating of product j by customer i. Then b_j represents product quality, a_i represents easy versus hard to please customers, and e_{ij} is noise.

More realistic models incorporate fixed effects and interactions and bilinear (SVD) terms. We choose this model because it is the simplest case that exhibits the intrinsic difficulty of the large unbalanced crossed random effects setting. Our goal is not to resolve the issue of analyzing massive crossed data sets via GLMMs in one go. Our contribution is a computationally affordable analogue of Henderson I (mentioned below) while analogues of Henderson II and Henderson III would offer richer modeling capabilities.

In the available data we only see N of the Y_{ij}, where $1 \leqslant N < \infty$, in R distinct rows (i's) and C distinct columns (j's). For example, R or C could be the number of distinct customers or products. We assume that observations are missing completely at random. See Section 7.1 for comments on informative missingness. Note that we do not make any distributional assumptions.

Let $\theta = (\sigma_A^2, \sigma_B^2, \sigma_E^2)^\mathsf{T}$ be the vector of variance components. Our first task is to get an unbiased estimate $\hat{\theta}$ of θ at computational cost $O(N)$ and using additional storage that is $O(R + C)$, which is often sublinear in N.

Our second and more challenging task is to find the variance of $\hat{\theta}$, $\mathrm{Var}(\hat{\theta} \mid \theta, \kappa)$. This variance depends on both θ and the vector of kurtoses of the random effects $\kappa = (\kappa_A, \kappa_B, \kappa_E)^\mathsf{T}$. We develop formulas $V(\theta, \kappa)$ approximating $\mathrm{Var}(\hat{\theta} \mid \theta, \kappa)$ that can be computed in $O(N)$ time and $O(R + C)$ storage, given values for θ and κ. After developing an estimate $\hat{\kappa}$ that can be computed in $O(N)$ time and $O(R+C)$ space, we let $\widehat{\mathrm{Var}}(\hat{\theta}) = V(\hat{\theta}, \hat{\kappa})$ be our plug-in estimate of the variance of $\hat{\theta}$.

Notice that in order to achieve the complexity bounds, we choose to overestimate $\mathrm{Var}(\hat{\theta})$. Specifically, we require the functions V to satisfy $\mathrm{diag}(V(\theta, \kappa)) \geqslant \mathrm{diag}(\mathrm{Var}(\hat{\theta} \mid \theta, \kappa))$. As we show, the overestimation is by a mild factor.

For large data sets we might suppose that $\mathrm{Var}(\hat{\theta})$ is necessarily very small and getting exact values is not important. While this may be true, it is wise to check. The effective sample size (as defined in [11]) in model (1) might be as small as R or C if the row or column effects dominate. Moreover, if the sampling frequencies of rows or columns are very unequal, then the effective sample size can be much smaller than R or C. For example, the Netflix data set [2] has $N \doteq 10^8$. But there are only about 18,000 movies and so for statistics dominated by the movie effect the effective sample size might be closer to 18,000. That the movies do not appear equally often would further reduce the effective sample size. Indeed, [13] shows that for some linear statistics the variance could be as much as 50,000 times larger than a formula based on IID sampling would yield. That factor is perhaps extreme but it would translate a nominal sample size of 10^8 into an effective sample size closer to 2,000.

An outline of this paper is as follows. Section 2 describes the difficulties with Gibbs sampling and other MCMC algorithms for crossed random effects, as suggested by theoretical results and shown through simulations. Section 3 introduces further notation and assumptions. Section 4 presents our linear-cost algorithm to estimate θ and conservatively approximate the variance of that estimate. Section 5 studies how knowledge of σ_A^2, σ_B^2, and σ_E^2 can be used to construct shrinkage predictions of unknown Y_{ij}. Section 6 illustrates the meth-

ods in Section 4 on both simulated Gaussian data and real world data. Section 7 concludes the paper and discusses informative missingness. Appendix A, Section 8, has a proof of convergence rates for MCMC methods and tables of their simulation results. Appendix B, Section 9, develops the variance formulas for our moment estimates and provides proofs of our theorems about prediction. We conclude this section with a few more pointers to the literature.

Our procedure to find variance component estimates is similar to those of Henderson [8] as described in Searle et al. [19, Chapter 5], but there are important differences in both cost and generality. Computing $\widehat{\mathrm{Var}}(\hat{\theta})$ for Henderson I requires terms k_{21} and k_{22} [19, p. 434] that cost $O(R^2 + C^2)$ space and $O(RC(R + C))$ time. That is over our budget. Henderson II addresses a flaw in Henderson I making it possible to incorporate fixed effects, but those fixed effects do not reduce the computational complexity. Henderson III addresses a flaw in Henderson II making it possible to incorporate interactions between fixed and random effects, but then the computation goes up to $O((R+C)^3)$. We get estimates and quantify their uncertainty within $O(R + C)$ space and $O(N)$ time budgets. As stated in [19], analyzing the variance of Henderson's estimators is difficult without the Gaussian assumption. Gaussian variables are not a reasonable assumption in our target applications. We use U-statistics that are weighted sums of within-row and within-column variances, while [8] uses statistics that estimate the variances of row means and column means. Following this approach we are able to find the variances of our U-statistics without any distributional assumptions. In our opinion, the quantities estimated by our U-statistics are easier to interpret than the ones Henderson uses.

For crossed random effects models with missing data [4] propose an alternating imputation-posterior (AIP) algorithm, which they show has good performance on fairly large data sets. It may be termed a 'pseudo-MCMC' method since it alternates between sampling the missing data from its distribution given the parameter estimates and sampling the parameters from a distribution centered on the maximum likelihood estimates. Because of this last step, we do not consider AIP to be scalable to Internet size problems.

In our model (1), for simplicity the variance components are homoscedastic. Alternatively, we could allow them to be heteroscedastic; see [13] or [14], who study bootstrap variance estimates for means and smooth functions of means. The latter paper also considers a more complex model in the sense that there are more than two factors as well as interactions among factors.

2. MCMC for large crossed data

In this section we consider some common MCMC methods to estimate the parameters σ_A^2, σ_B^2, and σ_E^2 of model (1). We show that numerous MCMC methods cannot scale. Some readers may prefer to skip directly to Section 3 describing our moments approach.

For this section we assume that a_i, b_j and e_{ij} are normally distributed in order to analyze MCMC. We also use normally distributed data in some sim-

ulations in Section 6, but otherwise no normality assumptions are used in this paper.

Balanced data is a fully sampled $R \times C$ matrix with Y_{ij} for rows $i = 1, \dots, R$ and columns $j = 1, \dots, C$. We present some analyses for balanced data with interspersed remarks on how the general unbalanced case behaves. The balanced case allows sharp formulas that we find useful and that case is the one we simulate. In particular, we can obtain convergence rates for some MCMC algorithms.

To estimate σ_A^2, σ_B^2, and σ_E^2 MCMC methods sample from the posterior distribution given the data: $\pi = p(\mu, a, b, \sigma_A^2, \sigma_B^2, \sigma_E^2 \mid Y)$ where a is the vector of a_i and b is the vector of b_j. Let

$$S^{(t)} = \begin{pmatrix} \mu^{(t)} & a^{(t)\mathsf{T}} & b^{(t)\mathsf{T}} & \sigma_A^{2(t)} & \sigma_B^{2(t)} & \sigma_E^{2(t)} \end{pmatrix}^{\mathsf{T}}, \quad \text{for } t \geqslant 1$$

denote the resulting chain. While MCMC is effective for hierarchical random effects models, it scales badly for crossed random effects models as we will see. Indeed, in limits where $R, C \to \infty$, the dimension of our chain $S^{(t)}$ approaches infinity. Convergence rates of many MCMC methods slow down as the dimension of the chain increases, making them ineffective for high dimensional parameter spaces.

The MCMC methods we consider go over the entire data set at each iteration. There are alternative samplers that save computation time by only looking at subsets of data at each iteration. However, so far those approaches are developed for IID data only.

2.1. Gibbs sampling

In each iteration of Gibbs sampling [6], we draw from the conditional posteriors of μ, a, b, σ_A^2, σ_B^2, and σ_E^2 in turn. To analyze the behavior of this sampler, let us consider the problem of Gibbs sampling from the 'smaller' distribution $\phi = p(a, b \mid \mu, \sigma_A^2, \sigma_B^2, \sigma_E^2, Y)$. At iteration $t+1$, we sample $a^{(t+1)} \sim p(a \mid b^{(t)}, \mu, \sigma_A^2, \sigma_B^2, \sigma_E^2, Y)$ and $b^{(t+1)} \sim p(b \mid a^{(t+1)}, \mu, \sigma_A^2, \sigma_B^2, \sigma_E^2, Y)$, which are normal distributions with diagonal covariance matrices. Let $X^{(t)}$ be the resulting chain.

Roberts and Sahu [18] give the following definition.

Definition 2.1. Let $\theta^{(t)}$, for integer $t \geqslant 0$ be a Markov chain with stationary distribution h. Its convergence rate is the minimum number ρ such that

$$\lim_{t \to \infty} \mathbb{E}_h \Big[(\mathbb{E}_h[f(\theta^{(t)}) \mid \theta^{(0)}] - \mathbb{E}_h[f(\theta)])^2 \Big] r^{-t} = 0$$

holds for all measurable functions f such that $\mathbb{E}_h(f(\theta)^2) < \infty$ and all $r > \rho$.

Theorem 2.1. *Let ρ be the convergence rate of $X^{(t)}$ to ϕ, as in Definition 2.1. Then,*

$$\rho = \frac{\sigma_B^2}{\sigma_B^2 + \sigma_E^2/R} \times \frac{\sigma_A^2}{\sigma_A^2 + \sigma_E^2/C}.$$

Proof. See Section 8.1. □

We see that $\rho \to 1$ as $R, C \to \infty$, outside of trivial cases with σ_A^2 or σ_B^2 equal to zero. If R and C grow proportionately then $\rho = 1 - \alpha/\sqrt{N} + O(1/N)$ for some $\alpha > 0$. We can therefore expect the Gibbs sampler to require at least some constant multiple of $\sqrt{N}$ iterations to approximate the target distribution sufficiently. When the data are not perfectly balanced, numerical computation of ρ shows that Gibbs still mixes increasingly slowly as $N \to \infty$ while the sampler requires $O(N)$ computation per iteration. In sum, Gibbs takes $O(N^{3/2})$ work to sample from ϕ, which is not scalable.

Because sampling from ϕ can be viewed as a subproblem of sampling from π, we believe that the Gibbs sampler that draws from π, which also requires $O(N)$ time per iteration, will exhibit the same slow convergence and hence require superlinear computation time.

2.2. Other MCMC algorithms

The Gibbs sampler is widely used for problems like this, where the full conditional distributions are tractable to sample from. But there are other MCMC algorithms that one could use. Here we consider random walk Metropolis (RWM), Langevin diffusion, and Metropolis adjusted Langevin (MALA). They also have difficulties scaling to large data sets.

At iteration $t+1$ of RWM, a Gaussian random walk proposal $S^{(t+1)} \sim \mathcal{N}(S^{(t)}, \sigma^2 I)$ for $\sigma^2 > 0$ is made and the step is taken with the Metropolis-Hastings acceptance probability. If the target distribution is a product distribution of dimension d, the chain $\tilde{S}^{(t)} \equiv S^{(dt)}$ (i.e. the chain formed by every dth state of the chain $S^{(t)}$) converges to a diffusion whose solution is the target distribution. We may interpret this as a convergence time for the algorithm that grows as $O(d)$ [17].

For our problem, evaluating the acceptance probability requires time at least $O(N)$, so the overall algorithm then takes $O(N(R+C))$ time. This is at best $O(N^{3/2})$, as we found for Gibbs sampling, and could be worse for sparse data where $N \ll RC$. Our target distribution is not of product form, but we have no reason to expect that RWM mixes orders of magnitude faster here than for a distribution of product form. Indeed, it seems more likely that mixing would be faster for product distributions than for distributions with more complicated dependence patterns such as ours.

At iteration $t+1$, Langevin diffusion steps $S^{(t+1)} \sim \mathcal{N}(S^{(t)} + (h/2)\nabla \log \pi(S^{(t)}), hI)$ for $h > 0$. As $h \to 0$, the stationary distribution for this process converges to π, as shown for general target distributions in [12]. Because $h \neq 0$ in practice, the Langevin algorithm is biased. To correct this, the MALA algorithm uses the Metropolis-Hastings algorithm with the Langevin proposal $S^{(t+1)}$. When the target distribution is a product distribution of dimension d, the chain $\tilde{S}^{(t)} \equiv S^{(d^{1/3}t)}$ converges to a diffusion with solution π; the convergence time grows as $O(d^{1/3})$ [17]. With similar reasoning as for RWM, the computation time is $O(N(R+C)^{1/3})$, which is at best $O(N^{1+1/6})$.

TABLE 1
Summary of simulation results for cases with $R = C = 1000$. The first row gives CPU time in seconds. The next four rows give median estimates of the 4 parameters. The next four rows give the number of lags required to get an autocorrelation below 0.5.

Method	Gibbs	Block	Reparam.	Lang.	MALA	Indp.	RWM	RWM Sub.	pCN
CPU sec.	3432	15046	4099	2302	4760	2513	2141	2635	1966
med μ	0.97	1.02	1.04	0.99	0.96	2.39	1.55	1.07	1.53
med σ_A^2	1.96	1.99	2.02	1.90	1.95	1.78	2.01	1.96	1.99
med σ_B^2	0.51	0.50	0.50	0.40	0.50	2.94	0.51	0.50	0.49
med σ_E^2	1.00	1.00	1.00	65.22	2.66	0.15	0	0.93	0
ACF(μ)	801	790	694	1	2501	5000+	1133	1656	1008
ACF(σ_A^2)	1	1	1	122	2656	5000+	1133	989	912
ACF (σ_B^2)	1	1	1	477	2514	5000+	1133	855	556
ACV(σ_E^2)	1	1	1	385	3062	5000+	1518	1724	621

2.3. Simulation results

We carried out simulations of the four algorithms described above, as well as five others: the block Gibbs sampler ('Block'), the reparameterized Gibbs sampler ('Reparam.'), the independence sampler ('Indp.'), RWM with subsampling ('RWM Sub.'), and the pCN algorithm of [7]. Descriptions of these five algorithms are given below with discussions of their simulation results. Every algorithm was implemented in MATLAB and run on a cluster using 4GB memory.

For each algorithm and a range of values of R and C, we generated balanced data from model (1) with $\mu = 1$, $\sigma_A^2 = 2$, $\sigma_B^2 = 0.5$, and $\sigma_E^2 = 1$. We ran 20,000 iterations of the algorithm, retaining the last 10,000 for analysis. We record the CPU time required, the median values of μ, σ_A^2, σ_B^2, and σ_E^2, and the number of lags needed for their sample auto-correlation functions (ACF) to go below 0.5.

The entire process is repeated in 10 independent runs. Table 1 presents median values of the recorded statistics over the 10 runs for the case $R = C = 1000$. Section 8 includes corresponding results at a range of (R, C) sizes.

Block Gibbs, which updates a and b together to try to improve mixing, has computation time superlinear in the number of observations. Also to improve mixing, reparameterized Gibbs scales the random effects to have equal variance. This gives an algorithm equivalent to the conditional augmentation of [21]. For all three Gibbs-type algorithms, the parameter estimates are good but μ mixes slower as R and C increase, while the variance components do not exhibit this behavior.

The computation times of Langevin diffusion ('Lang.') and MALA are approximately linear in the number of observations. However, σ_E^2 tends to explode for large data sets in Langevin diffusion, while the chain does not mix well in MALA.

The independent sampler is a Metropolis-Hastings algorithm where the proposal distribution is fixed. We propose $\mu \sim \mathcal{N}(1, 1)$, $a \sim \mathcal{N}(0, I_R)$, $b \sim \mathcal{N}(0, I_C)$, and $\sigma_A^2, \sigma_B^2, \sigma_E^2 \sim \text{InvGamma}(1, 1)$. The computation time grows linearly with the data size. The parameters do not mix well, and their estimates are not good. It is possible that better results would be obtained from a different proposal distribution, but it is not clear how best to choose one in practice.

RWM and RWM with subsampling, the latter of which updates a subset of parameters at each iteration, both have computation time linear in the number of observations. Neither algorithm mixed well, and for RWM σ_E^2 tended to go to zero in large data sets.

The pCN algorithm is a Metropolis-Hastings algorithm where the proposals are Gaussian random walk steps shrunken towards zero: $S^{(t+1)} \sim \mathcal{N}(\sqrt{1-\sigma^2}S^{(t)}, \sigma^2 I)$, for $\sigma^2 \leq 1$. Hairer, Stuart and Vollmer [7] show that under certain conditions on the target distribution, the convergence rate of this algorithm does not slow with the dimension of the distribution. We include it here, even though our π does not satisfy those conditions. The computation time grows linearly with the data size. However, the estimates for μ and σ_E^2 are not good, and those for σ_E^2 even get worse as the data size increases. None of the parameters seem to mix well.

In summary, for large data sets each algorithm mixes increasingly slowly or returns flawed estimates of μ and the variance components. We have also simulated some unbalanced data sets and slow mixing is once again the norm, with worse performance as R and C grow.

3. Further notation and assumptions

In this section, we go over pertinent notation and assumptions about the pattern of observations. Our data are realizations from model (1).

We refer to the first index of Y_{ij} as the 'row' and the second as the 'column'. We use integers i, i', r, r' to index rows and j, j', s, s' for columns. The actual indices may be URLs, customer IDs, or query strings and are not necessarily the integers we use here.

The variable Z_{ij} takes the value 1 if Y_{ij} is observed and 0 otherwise. We assume that there can be at most one observation in position (i, j).

The sample size is $N = \sum_{ij} Z_{ij} < \infty$. The number of observations in row i is $N_{i\bullet} = \sum_j Z_{ij}$ and the number in column j is $N_{\bullet j} = \sum_i Z_{ij}$. The number of distinct rows is $R = \sum_i 1_{N_{i\bullet}>0}$ and there are $C = \sum_j 1_{N_{\bullet j}>0}$ distinct columns. In the following, all of our sums over rows are only over rows i with $N_{i\bullet} > 0$, and similarly for sums over columns. We state this because there are a small number of expressions where omitting rows without data changes their values. Our convention corresponds to what happens when one makes a pass through the whole data set.

Let Z be the matrix containing Z_{ij}. Of interest are $(ZZ^{\mathsf{T}})_{ii'} = \sum_j Z_{ij}Z_{i'j}$, the number of columns for which we have data in both rows i and i', and $(Z^{\mathsf{T}}Z)_{jj'}$. Note that $(ZZ^{\mathsf{T}})_{ii'} \leqslant N_{i\bullet}$ and furthermore

$$\sum_{ir}(ZZ^{\mathsf{T}})_{ir} = \sum_{jir} Z_{ij}Z_{rj} = \sum_j N_{\bullet j}^2, \quad \text{and} \quad \sum_{js}(Z^{\mathsf{T}}Z)_{js} = \sum_i N_{i\bullet}^2.$$

Two other useful idioms are

$$T_{i\bullet} = \sum_j Z_{ij}N_{\bullet j} \quad \text{and} \quad T_{\bullet j} = \sum_i Z_{ij}N_{i\bullet}. \tag{2}$$

Here $T_{i\bullet}$ is the total number of observations in all of the columns j that are represented in row i.

Our notation allows for an arbitrary pattern of observations. Some special cases are as follows. A balanced crossed design can be described via $Z_{ij} = 1_{i\leqslant R}1_{j\leqslant C}$. If $\max_i N_{i\bullet} = 1$ but $\max_j N_{\bullet j} > 1$ then the data have a nested structure with rows nested in columns. If $\max_i N_{i\bullet} = \max_j N_{\bullet j} = 1$, then the observed Y_{ij} are IID.

Some patterns are difficult to handle. For example, if all the observations are in the same row or column, some of the variance components are not identifiable. We are motivated by problems that are not such worst cases.

Our main results are formulas that can be applied to finite samples. In some cases we use limits $N \to \infty$ to get insight into those formulas. The quantities

$$\epsilon_R = \max_i N_{i\bullet}/N, \quad \text{and} \quad \epsilon_C = \max_j N_{\bullet j}/N \tag{3}$$

measure the extent to which a single row or column dominates the data set. We expect that these are both small and in limiting arguments, where $N \to \infty$, we may assume that

$$\max(\epsilon_R, \epsilon_C) \to 0. \tag{4}$$

It is also often reasonable to suppose that $\max_i T_{i\bullet}/N$ and $\max_j T_{\bullet j}/N$ are both small.

In many data sets, the average row and column sizes are large, but much smaller than N. One way to measure the average row size is N/R. Another way to measure it is to randomly choose an observation and inspect its row size, obtaining an expected value of $(1/N)\sum_i N_{i\bullet}^2$. Similar formulas hold for the average column size. Therefore, we assume that as $N \to \infty$

$$\max(R/N, C/N) \to 0 \tag{5}$$

and

$$\begin{aligned} &\min\Big(\frac{1}{N}\sum_i N_{i\bullet}^2, \frac{1}{N}\sum_j N_{\bullet j}^2\Big) \to \infty, \quad \text{and} \\ &\max\Big(\frac{1}{N^2}\sum_i N_{i\bullet}^2, \frac{1}{N^2}\sum_j N_{\bullet j}^2\Big) \to 0. \end{aligned} \tag{6}$$

Notice that

$$\frac{1}{N^2}\sum_i N_{i\bullet}^2 \leqslant \frac{1}{N^2}\sum_i N_{i\bullet}(\epsilon_R N) \leqslant \epsilon_R, \quad \text{and} \quad \frac{1}{N^2}\sum_j N_{\bullet j}^2 \leqslant \epsilon_C \tag{7}$$

and so the second part of (6) follows from (3) and (4).

While the average row count may be large, many of the rows corresponding to newly seen entities can have $N_{i\bullet} = 1$. In our analysis, it is not necessary to assume that all of the rows and columns contain at least some minimum number of observations. Thus, we avoid losing information by the practice of iteratively removing all rows and columns with few observations.

To illustrate the appropriateness of our assumptions, the Netflix data has $N = 100{,}480{,}507$ ratings on $R = 17{,}770$ movies by $C = 480{,}189$ customers. Therefore $R/N \doteq 0.00018$ and $C/N \doteq 0.0047$. It is sparse with $N/(RC) \doteq 0.012$. It is not dominated by a single row or column because $\epsilon_R \doteq 0.0023$ and $\epsilon_C = 0.00018$ even though one customer has rated an astonishing 17,653 movies. Similarly

$$\frac{N}{\sum_i N_{i\bullet}^2} \doteq 1.78 \times 10^{-5}, \qquad \frac{\sum_i N_{i\bullet}^2}{N^2} \doteq 0.00056,$$
$$\frac{N}{\sum_j N_{\bullet j}^2} \doteq 0.0015, \quad \text{and} \qquad \frac{\sum_j N_{\bullet j}^2}{N^2} \doteq 6.43 \times 10^{-6}$$

so that the average row or column has size $\gg 1$ and $\ll N$.

There are various possible data storage models. We consider the log-file model with a collection of (i, j, Y_{ij}) triples, which for the purposes of this paper we assume are stored at the same location. A pass over the data proceeds via an iteration over all (i, j, Y_{ij}) triples in the data set. Such a pass may generate intermediate values that we assume can be retained for further computations.

4. Moment estimates of variance components

Here we develop a method of moments estimate $\hat{\theta}$ for $\theta = (\sigma_A^2, \sigma_B^2, \sigma_E^2)^{\mathsf{T}}$ that requires one pass over the data. We also find an expression for $\mathrm{Var}(\hat{\theta} \mid \theta, \kappa)$ and describe how to obtain an approximation of it after a second pass over the data.

Naturally, we would also want to estimate μ, and there are a number of ways to do so. The simplest is to let $\hat{\mu} = \bar{Y}_{\bullet\bullet}$, the sample mean. From [14],

$$\mathrm{Var}(\bar{Y}_{\bullet\bullet}) = \sigma_A^2 \frac{\sum_r N_{r\bullet}^2}{N^2} + \sigma_B^2 \frac{\sum_s N_{\bullet s}^2}{N^2} + \frac{\sigma_E^2}{N} \leqslant \epsilon_R \sigma_A^2 + \epsilon_C \sigma_B^2 + \frac{\sigma_E^2}{N}. \tag{8}$$

The upper bound in (8) is tight for balanced data, but otherwise it can be very conservative. The properties of this estimator have been well-studied in the literature, so in this paper we focus on estimating the variance components.

4.1. U-statistics for variance components

The usual unbiased sample variance estimate can be formulated as a U-statistic, which is more convenient to analyze. Thus, we use the following U-statistics as our method of moments estimators:

$$\begin{aligned} U_a &= \frac{1}{2} \sum_{ijj'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} (Y_{ij} - Y_{ij'})^2, \\ U_b &= \frac{1}{2} \sum_{jii'} N_{\bullet j}^{-1} Z_{ij} Z_{i'j} (Y_{ij} - Y_{i'j})^2, \quad \text{and} \\ U_e &= \frac{1}{2} \sum_{iji'j'} Z_{ij} Z_{i'j'} (Y_{ij} - Y_{i'j'})^2. \end{aligned} \tag{9}$$

To understand U_a note that for each row i, the quantities $Y_{ij} - \mu - a_i$ are IID with variance $\sigma_B^2 + \sigma_E^2$. Thus, U_a is a weighted sum of within-row unbiased estimates of $\sigma_B^2 + \sigma_E^2$. The explanation for U_b is similar, while U_e is proportional to the sample variance estimate of all N observations.

Lemma 4.1. *Let Y_{ij} follow the two-factor crossed random effects model (1) with the observation pattern Z_{ij} as described in Section 3. Then the U-statistics defined at (9) satisfy*

$$\begin{aligned}\mathbb{E}(U_a) &= (\sigma_B^2 + \sigma_E^2)(N - R)\\ \mathbb{E}(U_b) &= (\sigma_A^2 + \sigma_E^2)(N - C), \quad \textit{and}\\ \mathbb{E}(U_e) &= \sigma_A^2(N^2 - \sum_i N_{i\bullet}^2) + \sigma_B^2(N^2 - \sum_j N_{\bullet j}^2) + \sigma_E^2(N^2 - N).\end{aligned}$$

Proof. See Section 9.2.1. □

To obtain unbiased estimates $\hat\sigma_A^2$, $\hat\sigma_B^2$, and $\hat\sigma_E^2$ given values of the U-statistics, we solve the 3×3 system of equations

$$M\begin{pmatrix}\hat\sigma_A^2\\ \hat\sigma_B^2\\ \hat\sigma_E^2\end{pmatrix} = \begin{pmatrix}U_a\\ U_b\\ U_e\end{pmatrix} \text{ for } M = \begin{pmatrix}0 & N-R & N-R\\ N-C & 0 & N-C\\ N^2 - \sum_i N_{i\bullet}^2 & N^2 - \sum_j N_{\bullet j}^2 & N^2 - N\end{pmatrix} \tag{10}$$

For our method to return unique and meaningful estimates, the determinant of M

$$\begin{aligned}\det M &= (N-R)(N-C)\Big(N^2 - \sum_i N_{i\bullet}^2 - \sum_j N_{\bullet j}^2 + N\Big)\\ &\geqslant (N-R)(N-C)(N^2(1-\epsilon_R-\epsilon_C) + N)\end{aligned}$$

must be nonzero. This is true when no row or column has more than half of the data and at least one row and at least one column has more than one observation.

To compute the U-statistics, notice that $U_a = \sum_i S_{i\bullet}$, where $S_{i\bullet} = \sum_j Z_{ij}(Y_{ij} - \bar Y_{i\bullet})^2$ and $\bar Y_{i\bullet} = (1/N_{i\bullet})\sum_j Z_{ij}Y_{ij}$. In one pass over the data and time $O(N)$, we compute $N_{i\bullet}$, $\bar Y_{i\bullet}$, and $S_{i\bullet}$ for all R observed levels of i using the incremental algorithm described in the next paragraph. We can also compute N, R and C in such a pass if they are not known beforehand.

Chan, Golub and LeVeque [3] show how to compute both $Y_{i\bullet} = N_{i\bullet}\bar Y_{i\bullet}$ and $S_{i\bullet}$ in a numerically stable one pass algorithm. At the initial appearance of an observation in row i, with corresponding column $j = j(1)$, set $N_{i\bullet} = 1$, $Y_{i\bullet} = Y_{ij}$ and $S_{i\bullet} = 0$. After that, at the kth appearance of an observation in row i with corresponding column $j(k)$, $Y_{ij(k)}$,

$$N_{i\bullet} \leftarrow N_{i\bullet} + 1, \quad Y_{i\bullet} \leftarrow Y_{i\bullet} + Y_{ij(k)}, \quad \text{and} \quad S_{i\bullet} \leftarrow S_{i\bullet} + \frac{(k \times Y_{ij(k)} - Y_{i\bullet})^2}{k(k-1)}. \tag{11}$$

Chan, Golub and LeVeque [3] give a detailed analysis of roundoff error for update (11) as well as generalizations that update higher moments from groups of data values.

In that same pass over the data, U_e and the analogous quantities needed to compute U_b ($S_{\bullet j}$, $\bar{Y}_{\bullet j}$, $N_{\bullet j}$) are also computed using the incremental algorithm. Finally, in additional time $O(R+C)$, we calculate $\sum_i S_{i\bullet}$, $\sum_j S_{\bullet j}$, $\sum_i N_{i\bullet}^2$, and $\sum_j N_{\bullet j}^2$. Now, we have U_a, U_b, U_e, and all the entries of M.

Given U_a, U_b, U_e, and M we can calculate $\hat{\sigma}_A^2$, $\hat{\sigma}_B^2$, and $\hat{\sigma}_E^2$ in constant time. Therefore, finding our method of moments estimators takes $O(N)$ time overall. Note that in practice we may get negative estimates of the variance components, since the method of moments does not take into account any constraints. However, there is no consensus on how to deal with this situation. In this paper, we automatically set any negative variance component estimates to zero.

4.2. Variances of the estimators

Now we show how to estimate the covariance matrix of $\hat{\theta} = (\hat{\sigma}_A^2, \hat{\sigma}_B^2, \hat{\sigma}_E^2)^{\mathsf{T}}$.

4.2.1. True variance of $\hat{\theta}$

This section discusses the finite sample covariance matrix of $\hat{\theta}$. Theorem 4.1 below gives the exact variances and covariances of our U-statistics.

Theorem 4.1. *Let Y_{ij} follow the random effects model (1) with the observation pattern Z_{ij} as described in Section 3. Then the U-statistics defined at (9) have variances*

$$\begin{aligned}\operatorname{Var}(U_a) &= \sigma_B^4(\kappa_B+2)\sum_{ir}(ZZ^{\mathsf{T}})_{ir}(1-N_{i\bullet}^{-1})(1-N_{r\bullet}^{-1})\\ &\quad+2\sigma_B^4\sum_{ir}N_{i\bullet}^{-1}N_{r\bullet}^{-1}(ZZ^{\mathsf{T}})_{ir}[(ZZ^{\mathsf{T}})_{ir}-1]+4\sigma_B^2\sigma_E^2(N-R)\\ &\quad+\sigma_E^4(\kappa_E+2)\sum_i N_{i\bullet}(1-N_{i\bullet}^{-1})^2+2\sigma_E^4\sum_i(1-N_{i\bullet}^{-1}),\end{aligned}\tag{12}$$

and

$$\begin{aligned}\operatorname{Var}(U_b) &= \sigma_A^4(\kappa_A+2)\sum_{js}(Z^{\mathsf{T}}Z)_{js}(1-N_{\bullet j}^{-1})(1-N_{\bullet s}^{-1})\\ &\quad+2\sigma_A^4\sum_{js}N_{\bullet j}^{-1}N_{\bullet s}^{-1}(Z^{\mathsf{T}}Z)_{js}[(Z^{\mathsf{T}}Z)_{js}-1]+4\sigma_A^2\sigma_E^2(N-C)\\ &\quad+\sigma_E^4(\kappa_E+2)\sum_j N_{\bullet j}(1-N_{\bullet j}^{-1})^2+2\sigma_E^4\sum_j(1-N_{\bullet j}^{-1}),\end{aligned}\tag{13}$$

with $\operatorname{Var}(U_e)$ *equaling*

$$
\begin{aligned}
&2\sigma_A^4\Big[\Big(\textstyle\sum_i N_{i\bullet}^2\Big)^2 - \textstyle\sum_i N_{i\bullet}^4\Big] + 2\sigma_B^4\Big[\Big(\textstyle\sum_j N_{\bullet j}^2\Big)^2 - \displaystyle\sum_j N_{\bullet j}^4\Big]\\
&\quad + \sigma_A^4(\kappa_A+2)\Big(N^2\sum_i N_{i\bullet}^2 - 2N\sum_i N_{i\bullet}^3 + \sum_i N_{i\bullet}^4\Big)\\
&\quad + \sigma_B^4(\kappa_B+2)\Big(N^2\sum_j N_{\bullet j}^2 - 2N\sum_j N_{\bullet j}^3 + \sum_j N_{\bullet j}^4\Big)\\
&\quad + 2\sigma_E^4 N(N-1) + \sigma_E^4(\kappa_E+2)N(N-1)^2\\
&\quad + 4\sigma_A^2\sigma_B^2\Big(N^3 - 2N\sum_{ij} Z_{ij}N_{i\bullet}N_{\bullet j} + \sum_{ij} N_{i\bullet}^2 N_{\bullet j}^2\Big)\\
&\quad + 4\sigma_A^2\sigma_E^2\Big(N^3 - N\sum_i N_{i\bullet}^2\Big) + 4\sigma_B^2\sigma_E^2\Big(N^3 - N\sum_j N_{\bullet j}^2\Big).
\end{aligned}
\tag{14}
$$

Their covariances are

$$
\mathrm{Cov}(U_a, U_b) = \sigma_E^4(\kappa_E+2)\sum_{ij} Z_{ij}(1-N_{i\bullet}^{-1})(1-N_{\bullet j}^{-1}), \tag{15}
$$

$$
\begin{aligned}
\mathrm{Cov}(U_a, U_e) &= 2\sigma_B^4\Big(\sum_i N_{i\bullet}^{-1}T_{i\bullet}^2 - \sum_{ij} Z_{ij}N_{i\bullet}^{-1}N_{\bullet j}^2\Big)\\
&\quad + \sigma_B^4(\kappa_B+2)\sum_{ij} Z_{ij}(N-N_{\bullet j})N_{\bullet j}(1-N_{i\bullet}^{-1})\\
&\quad + 2\sigma_E^4(N-R) + \sigma_E^4(\kappa_E+2)(N-R)(N-1)\\
&\quad + 4\sigma_B^2\sigma_E^2 N(N-R), \quad \textit{and}
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
\mathrm{Cov}(U_b, U_e) &= 2\sigma_A^4\Big(\sum_j N_{\bullet j}^{-1}T_{\bullet j}^2 - \sum_{ij} Z_{ij}N_{\bullet j}^{-1}N_{i\bullet}^2\Big)\\
&\quad + \sigma_A^4(\kappa_A+2)\sum_{ij} Z_{ij}(N-N_{i\bullet})N_{i\bullet}(1-N_{\bullet j}^{-1})\\
&\quad + 2\sigma_E^4(N-C) + \sigma_E^4(\kappa_E+2)(N-C)(N-1)\\
&\quad + 4\sigma_A^2\sigma_E^2 N(N-C).
\end{aligned}
\tag{17}
$$

Proof. Equation (12) is proved in Section 9.4 and then equation (13) follows by exchanging indices. Equation (14) is proved in Section 9.5. Equation (15) is proved in Section 9.6. Equation (16) is proved in Section 9.7 and then equation (17) follows by exchanging indices. □

Now we consider $\mathrm{Var}(\hat\theta)$. From (10)

$$
\mathrm{Var}(\hat\theta) = M^{-1}\mathrm{Var}\begin{pmatrix} U_a\\ U_b\\ U_e\end{pmatrix}(M^{-1})^{\mathsf{T}}. \tag{18}
$$

We show in Section 4.2.2 that while $\mathrm{Var}(U_e)$ and the covariances of the U-statistics may be exactly computed in time $O(N)$, $\mathrm{Var}(U_a)$ and $\mathrm{Var}(U_b)$ cannot.

Therefore, we approximate $\mathrm{Var}(U_a)$ and $\mathrm{Var}(U_b)$ such that when we apply formula (18) we get conservative estimates of $\mathrm{Var}(\hat{\sigma}_A^2)$, $\mathrm{Var}(\hat{\sigma}_B^2)$, and $\mathrm{Var}(\hat{\sigma}_E^2)$ (the values of primary interest).

For intuition on what sort of approximation is needed, we give a linear expansion of $\mathrm{Var}(\hat{\theta})$ in terms of the variances and covariances of the U-statistics. Letting $\epsilon = \max(\epsilon_R, \epsilon_C, R/N, C/N)$ we have as $\epsilon \to 0$

$$M = \begin{pmatrix} N & & \\ & N & \\ & & N^2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} (1 + O(\epsilon))$$

and so

$$M^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} N^{-1} & & \\ & N^{-1} & \\ & & N^{-2} \end{pmatrix} (1 + O(\epsilon)).$$

It follows that

$$\begin{aligned} \hat{\sigma}_A^2 &= (U_e/N^2 - U_a/N)(1 + O(\epsilon)), \\ \hat{\sigma}_B^2 &= (U_e/N^2 - U_b/N)(1 + O(\epsilon)), \quad \text{and} \\ \hat{\sigma}_E^2 &= (U_a/N + U_b/N - U_e/N^2)(1 + O(\epsilon)). \end{aligned} \tag{19}$$

Disregarding the $O(\epsilon)$ terms,

$$\begin{aligned} \mathrm{Var}(\hat{\sigma}_A^2) &\doteq \mathrm{Var}(U_e)/N^4 + \mathrm{Var}(U_a)/N^2 - 2\mathrm{Cov}(U_a, U_e)/N^3, \\ \mathrm{Var}(\hat{\sigma}_B^2) &\doteq \mathrm{Var}(U_e)/N^4 + \mathrm{Var}(U_b)/N^2 - 2\mathrm{Cov}(U_b, U_e)/N^3, \quad \text{and} \\ \mathrm{Var}(\hat{\sigma}_E^2) &\doteq \mathrm{Var}(U_a)/N^2 + \mathrm{Var}(U_b)/N^2 + \mathrm{Var}(U_e)/N^4 \\ &\quad - 2\mathrm{Cov}(U_a, U_e)/N^3 - 2\mathrm{Cov}(U_b, U_e)/N^3 + 2\mathrm{Cov}(U_a, U_b)/N^2. \end{aligned} \tag{20}$$

In light of equation (20), to find computationally attractive but conservative approximations of $\mathrm{Var}(\hat{\theta})$ in finite samples, we use (slight) over-estimates of $\mathrm{Var}(U_a)$ and $\mathrm{Var}(U_b)$. We discuss how to do so in Section 4.2.2.

In practice, when obtaining $\widehat{\mathrm{Var}}(\hat{\theta})$, we plug in $\hat{\sigma}_A^2$, $\hat{\sigma}_B^2$, $\hat{\sigma}_E^2$, and estimates of the kurtoses into the covariance matrix of the U-statistics where $\mathrm{Var}(U_a)$ and $\mathrm{Var}(U_b)$ have been replaced by their over-estimates. Then we apply equation (18). We discuss estimating the kurtoses in Section 4.2.4.

4.2.2. Computable approximations of $\mathrm{Var}(U)$

First, we show how to obtain over-estimates of $\mathrm{Var}(U_a)$ in time $O(N)$; the case of $\mathrm{Var}(U_b)$ is similar. In addition to $N - R$, $\mathrm{Var}(U_a)$ contains the following quantities

$$\sum_{ir} (ZZ^{\mathsf{T}})_{ir}(1 - N_{i\bullet}^{-1})(1 - N_{r\bullet}^{-1}) \qquad \sum_{ir} N_{i\bullet}^{-1} N_{r\bullet}^{-1} (ZZ^{\mathsf{T}})_{ir}((ZZ^{\mathsf{T}})_{ir} - 1)$$

$$\sum_i N_{i\bullet}(1-N_{i\bullet}^{-1})^2, \quad \text{and} \qquad \sum_i (1-N_{i\bullet}^{-1}).$$

The third and fourth quantities above can be computed in $O(R)$ work after the first pass over the data.

The first quantity is a sum over i and r, and cannot be simplified any further. Computing it takes more than $O(N)$ work. Since its coefficient $\sigma_B^4(\kappa_B+2)$ is nonnegative, we must use an upper bound to obtain an over-estimate of $\mathrm{Var}(U_a)$. We have the bound

$$\begin{aligned}\sum_{ir}(ZZ^{\mathsf{T}})_{ir}(1-N_{i\bullet}^{-1})(1-N_{r\bullet}^{-1}) &\leqslant \sum_{ij}\sum_r Z_{ij}Z_{rj}(1-N_{i\bullet}^{-1})\\ &= \sum_j N_{\bullet j}^2 - \sum_{ij} Z_{ij}N_{\bullet j}N_{i\bullet}^{-1},\end{aligned}$$

which can be computed in $O(N)$ work in a second pass over the data. Other weaker bounds may be obtained without the second pass. An example is

$$\sum_{ir}(ZZ^{\mathsf{T}})_{ir}(1-N_{i\bullet}^{-1})(1-N_{r\bullet}^{-1}) \leqslant \sum_{ir}(ZZ^{\mathsf{T}})_{ir} = \sum_j N_{\bullet j}^2$$

which can be computed in $O(C)$ work.

For our motivating problems this over-estimation of variance is negligible. The true term is a weighted sum of $(ZZ^{\mathsf{T}})_{ir}$ and we use a weight of 1 instead of $1-N_{r\bullet}^{-1}$ and a typical $N_{r\bullet}$ will be large. Consider first a small row r, with $N_{r\bullet}=1$. Let $j(r)$ be the unique column with $Z_{rj(r)}=1$. Our bound replaces that row's contribution of 0 by

$$\sum_{ij} Z_{ij}Z_{rj}(1-N_{i\bullet}^{-1}) = \sum_i Z_{ij}Z_{rj(r)}(1-N_{i\bullet}^{-1}) = Z_{ij(r)}(1-N_{i\bullet}^{-1}) \leqslant 1$$

thereby adding at most 1 to the sum. The total from such terms is then at most the number of singleton rows which is in turn below $R \ll N \ll \sum_j N_{\bullet j}^2$. The latter quantity dominates that coefficient. When $N_{r\bullet}\geqslant 2$ our approximation at most doubles the contribution. A near doubling of one term under the extreme setting where most rows have only two observations, is acceptable.

For the same reason as the first quantity, the second quantity cannot be computed in time $O(N)$ and we upper bound it via $(ZZ^{\mathsf{T}})_{ir}\leqslant N_{r\bullet}$, getting

$$\begin{aligned}\sum_{ir} N_{i\bullet}^{-1}N_{r\bullet}^{-1}(ZZ^{\mathsf{T}})_{ir}((ZZ^{\mathsf{T}})_{ir}-1) &\leqslant \sum_{ir} N_{i\bullet}^{-1}N_{r\bullet}^{-1}(ZZ^{\mathsf{T}})_{ir}(N_{r\bullet}-1)\\ &= \sum_{ij} Z_{ij}N_{i\bullet}^{-1}N_{\bullet j} - \sum_{ir} N_{i\bullet}^{-1}N_{r\bullet}^{-1}(ZZ^{\mathsf{T}})_{ir}\\ &\leqslant \sum_{ij} Z_{ij}N_{i\bullet}^{-1}N_{\bullet j}\end{aligned}$$

which can be computed in $O(N)$ work on a second pass. Even this upper bound is a small part of the variance. It is a sum of column sizes divided by row sizes.

Another term in the variance is of order $\sum_{ij} Z_{ij} N_{\bullet j}$. When typical row sizes are large then $\sum_{ij} Z_{ij} N_{\bullet j} N_{i\bullet}^{-1} \ll \sum_{ij} Z_{ij} N_{\bullet j}$.

All but one expression in $\mathrm{Var}(U_e)$ (see (14)) can be computed in $O(R+C)$ time after the first pass over the data. The one expression is

$$\sum_{ij} Z_{ij} N_{i\bullet} N_{\bullet j}. \tag{21}$$

Equation (21) requires a second pass over the data in time $O(N)$, because it is the sum over i and j of a polynomial in Z_{ij}, $N_{i\bullet}$, and $N_{\bullet j}$. Hence computing $\mathrm{Var}(U_e)$ takes $O(N)$ time total.

With the same reasoning as for (21), we see that $\mathrm{Cov}(U_a, U_b)$ can be computed in a second pass over the data in time $O(N)$. This reasoning also shows that we can compute nearly every term in $\mathrm{Cov}(U_a, U_e)$ in a second pass over the data; the exception is

$$\sum_{i} N_{i\bullet}^{-1} T_{i\bullet}^2. \tag{22}$$

We compute $T_{i\bullet}$ for each i in a second pass over the data. But we must use additional time $O(R)$ to get (22). Nevertheless, the total computation time is still $O(N)$. Symmetrically $\mathrm{Cov}(U_b, U_e)$ can be computed in time $O(N)$ as well.

4.2.3. Asymptotic approximation of $\mathrm{Var}(\hat{\theta})$

Under asymptotic conditions, we may obtain simple, analytic approximate expressions for the covariance matrix of our method of moments estimators.

Theorem 4.2. *As described in Section 3, suppose that* $N_{i\bullet} \leqslant \delta N$,

$$N_{\bullet j} \leqslant \delta N, \quad R \leqslant \delta N, \quad C \leqslant \delta N, \quad N \leqslant \delta \sum_i N_{i\bullet}^2, \quad \textit{and} \quad N \leqslant \delta \sum_j N_{\bullet j}^2,$$

hold for the same small $\delta > 0$ *and that*

$$0 < \kappa_A + 2, \kappa_B + 2, \kappa_E + 2, \sigma_A^4, \sigma_B^4, \sigma_E^4 < \infty.$$

Suppose additionally that

$$\sum_{ij} Z_{ij} N_{i\bullet}^{-1} N_{\bullet j} \leqslant \delta \sum_i N_{i\bullet}^2, \quad \textit{and} \quad \sum_{ij} Z_{ij} N_{i\bullet} N_{\bullet j}^{-1} \leqslant \delta \sum_j N_{\bullet j}^2 \tag{23}$$

hold. Then

$$\mathrm{Var}(U_a) = \sigma_B^4 (\kappa_B + 2) \sum_j N_{\bullet j}^2 (1 + O(\delta))$$

$$\mathrm{Var}(U_b) = \sigma_A^4 (\kappa_A + 2) \sum_i N_{i\bullet}^2 (1 + O(\delta)), \quad \textit{and}$$

$$\mathrm{Var}(U_e) = \Big(\sigma_A^4(\kappa_A+2)N^2\sum_i N_{i\bullet}^2 + \sigma_B^4(\kappa_B+2)N^2\sum_j N_{\bullet j}^2\Big)(1+O(\delta)).$$

Similarly

$$\begin{aligned}
\mathrm{Cov}(U_a, U_b) &= \sigma_E^4(\kappa_E+2)N(1+O(\delta)),\\
\mathrm{Cov}(U_a, U_e) &= \sigma_B^4(\kappa_B+2)N\sum_j N_{\bullet j}^2(1+O(\delta)), \quad \textit{and}\\
\mathrm{Cov}(U_b, U_e) &= \sigma_A^4(\kappa_A+2)N\sum_i N_{i\bullet}^2(1+O(\delta)).
\end{aligned}$$

Finally $\hat\sigma_A^2$, $\hat\sigma_B^2$ *and* $\hat\sigma_E^2$ *are asymptotically uncorrelated as* $\delta\to 0$ *with*

$$\begin{aligned}
\mathrm{Var}(\hat\sigma_A^2) &= \sigma_A^4(\kappa_A+2)\frac{1}{N^2}\sum_j N_{i\bullet}^2(1+O(\delta))\\
\mathrm{Var}(\hat\sigma_B^2) &= \sigma_B^4(\kappa_B+2)\frac{1}{N^2}\sum_j N_{\bullet j}^2(1+O(\delta)), \quad \textit{and}\\
\mathrm{Var}(\hat\sigma_E^2) &= \sigma_E^4(\kappa_E+2)\frac{1}{N}(1+O(\delta)).
\end{aligned}$$

Proof. See Section 9.8. □

In an asymptotic setting with $\delta\to 0$ the three variance estimates become uncorrelated. Also each of them has the same variance it would have had if the other variance components had truly been zero.

4.2.4. Estimating kurtoses

Under a Gaussian assumption, $\kappa_A=\kappa_B=\kappa_E=0$. If however the data have heavier tails than this, a Gaussian assumption will lead to underestimates of $\mathrm{Var}(\hat\theta)$. Therefore, we will estimate the kurtoses using the method of moments on U-statistics.

Let $\mu_{A,4}=\mathbb{E}(a_i^4)=(\kappa_A+3)\sigma_A^4$, $\mu_{B,4}=\mathbb{E}(b_i^4)=(\kappa_B+3)\sigma_B^4$, and $\mu_{E,4}=\mathbb{E}(e_{ij}^4)=(\kappa_E+3)\sigma_E^4$. The fourth moment U-statistics we use are

$$\begin{aligned}
W_a &= \frac{1}{2}\sum_{ijj'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}(Y_{ij}-Y_{ij'})^4\\
W_b &= \frac{1}{2}\sum_{iji'} N_{\bullet j}^{-1} Z_{ij}Z_{i'j}(Y_{ij}-Y_{i'j})^4, \quad \text{and}\\
W_e &= \frac{1}{2}\sum_{iji'j'} Z_{ij}Z_{i'j'}(Y_{ij}-Y_{i'j'})^4.
\end{aligned} \tag{24}$$

Theorem 4.3. *Let* Y_{ij} *follow the random effects model* (1) *with the observation pattern* Z_{ij} *as described in Section 3. Then the statistics defined at* (24) *have means*

$$\begin{aligned}
\mathbb{E}(W_a) &= (\mu_{B,4} + 3\sigma_B^4 + 12\sigma_B^2\sigma_E^2 + \mu_{E,4} + 3\sigma_E^4)(N - R)\\
\mathbb{E}(W_b) &= (\mu_{A,4} + 3\sigma_A^4 + 12\sigma_A^2\sigma_E^2 + \mu_{E,4} + 3\sigma_E^4)(N - C), \quad \textit{and}\\
\mathbb{E}(W_e) &= (\mu_{A,4} + 3\sigma_A^4 + 12\sigma_A^2\sigma_E^2)(N^2 - \sum_i N_{i\bullet}^2)\\
&\quad + (\mu_{B,4} + 3\sigma_B^4 + 12\sigma_B^2\sigma_E^2)(N^2 - \sum_j N_{\bullet j}^2)\\
&\quad + (\mu_{E,4} + 3\sigma_E^4)(N^2 - N) + 12\sigma_A^2\sigma_B^2\Big(N^2 - \sum_i N_{i\bullet}^2 - \sum_j N_{\bullet j}^2 + N\Big).
\end{aligned}$$

Proof. See Section 9.9. □

Using Theorem 4.3, we compute estimates $\hat{\mu}_{A,4}$, $\hat{\mu}_{B,4}$, and $\hat{\mu}_{E,4}$, by solving the 3×3 system of equations

$$M \begin{pmatrix} \hat{\mu}_{A,4} \\ \hat{\mu}_{B,4} \\ \hat{\mu}_{E,4} \end{pmatrix} = \begin{pmatrix} W_a - m_a \\ W_b - m_b \\ W_e - m_e \end{pmatrix}, \tag{25}$$

where M is the same matrix that we used for the U-statistics in equation (10), with

$$\begin{aligned}
m_a &= (3\hat{\sigma}_B^4 + 12\hat{\sigma}_B^2\hat{\sigma}_E^2 + 3\hat{\sigma}_E^4)(N - R),\\
m_b &= (3\hat{\sigma}_A^4 + 12\hat{\sigma}_A^2\hat{\sigma}_E^2 + 3\hat{\sigma}_E^4)(N - C), \quad \text{and}\\
m_e &= (3\hat{\sigma}_A^4 + 12\hat{\sigma}_A^2\hat{\sigma}_E^2)(N^2 - \sum_i N_{i\bullet}^2) + (3\hat{\sigma}_B^4 + 12\hat{\sigma}_B^2\hat{\sigma}_E^2)(N^2 - \sum_j N_{\bullet j}^2)\\
&\quad + 3\hat{\sigma}_E^4(N^2 - N) + 12\hat{\sigma}_A^2\hat{\sigma}_B^2\Big(N^2 - \sum_i N_{i\bullet}^2 - \sum_j N_{\bullet j}^2 + N\Big).
\end{aligned}$$

Then, $\hat{\kappa}_A = \hat{\mu}_{A,4}/\hat{\sigma}_A^4 - 3$, $\hat{\kappa}_B = \hat{\mu}_{B,4}/\hat{\sigma}_B^4 - 3$, and $\hat{\kappa}_E = \hat{\mu}_{E,4}/\hat{\sigma}_E^4 - 3$.

We compute the statistics (24) via

$$\begin{aligned}
W_a &= \sum_i \Big(\sum_j Z_{ij}(Y_{ij} - \bar{Y}_{i\bullet})^4 + 3N_{i\bullet}^{-1}S_{i\bullet}^2\Big)\\
W_b &= \sum_j \Big(\sum_i Z_{ij}(Y_{ij} - \bar{Y}_{\bullet j})^4 + 3N_{\bullet j}^{-1}S_{\bullet j}^2\Big), \quad \text{and}\\
W_e &= N\sum_{ij} Z_{ij}(Y_{ij} - \bar{Y}_{\bullet\bullet})^4 + 3S_{\bullet\bullet}^2,
\end{aligned} \tag{26}$$

where $\bar{Y}_{\bullet\bullet} = N^{-1}\sum_{ij} Z_{ij}Y_{ij}$ and $S_{\bullet\bullet} = \sum_{ij} Z_{ij}(Y_{ij} - \bar{Y}_{\bullet\bullet})^2$.

Therefore, the kurtosis estimates $\hat{\kappa}$ requires $R + C + 1$ new quantities

$$\sum_j Z_{ij}(Y_{ij} - \bar{Y}_{i\bullet})^4, \quad \sum_i Z_{ij}(Y_{ij} - \bar{Y}_{\bullet j})^4, \quad \text{and} \quad \sum_{ij} Z_{ij}(Y_{ij} - \bar{Y}_{\bullet\bullet})^4 \tag{27}$$

beyond those used to compute $\hat{\theta}$. These can be computed in a second pass over the data after $\bar{Y}_{i\bullet}$, $\bar{Y}_{\bullet j}$ and $\bar{Y}_{\bullet\bullet}$ have been computed in the first pass. They can also be computed in the first pass using update formulas analogous to the second

moment formulas (11). Such formulas are given by [15], citing an unpublished paper by Terriberry.

Because the kurtosis estimates are used in formulas for $\widehat{\mathrm{Var}}(\hat{\theta})$ and those formulas already require a second pass over the data, it is more convenient to compute (27) and the sample fourth moments in a second pass. By a similar argument as in Section 4.1, obtaining $\hat{\kappa}_A$, $\hat{\kappa}_B$, and $\hat{\kappa}_E$ has space complexity $O(R+C)$ and time complexity $O(N)$, and is therefore scalable. As with the variance component estimates, in practice sometimes we get kurtosis estimates less than -2, outside the parameter space. In this paper, we simply threshold the kurtoses at -2, in line with the common practice of raising variance estimates to zero.

4.3. Algorithm summary

For clarity of exposition, here we gather all of the steps in our algorithm for estimating σ_A^2, σ_B^2, and σ_E^2 and the variances of those estimators. An outline is shown in Figure 1. We assume that all of the computations below can be done with large enough variable storage that overflow does not occur. This may require an extended precision representation beyond 64 bit floating point, such as that in the python package mpmath [9].

The first task is to compute $\hat{\theta}$. In a first pass over the data compute counts N, R, C, row values $N_{i\bullet}$, $\bar{Y}_{i\bullet}$, $S_{i\bullet}$ for all unique rows i in the data set, and column values $N_{\bullet j}$, $\bar{Y}_{\bullet j}$, $S_{\bullet j}$ for all unique columns j in the data set as well as $\bar{Y}_{\bullet\bullet}$ and $S_{\bullet\bullet}$. Incremental updates are used as described in (11).

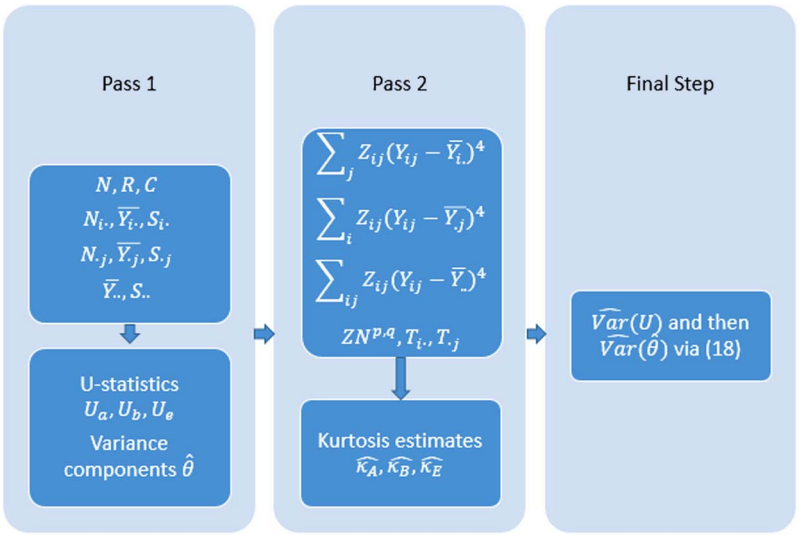


FIG 1. *Schematic of our algorithm. The expressions in the smallest boxes are the values computed at each step.*

Then compute

$$U_a = \sum_i S_{i\bullet}, \quad U_b = \sum_j S_{\bullet j}, \quad \text{and} \quad U_e = NS_{\bullet\bullet},$$

the matrix M from (10) and then $\hat{\theta} = (\hat{\sigma}_A^2, \hat{\sigma}_B^2, \hat{\sigma}_E^2)^{\mathsf{T}} = M^{-1}(U_a, U_b, U_e)^{\mathsf{T}}$ in time $O(R + C)$.

The second task is to compute approximately the variance of $\hat{\theta}$. First, we estimate the kurtoses. A second pass over the data computes the centered fourth moments in (27). Then one calculates the fourth order U-statistics of equation (26), solves (25) for the centered fourth moments, and converts them to kurtosis estimates, all in time $O(R + C)$.

In that second pass over the data, we also compute

$$\mathrm{ZN}^{p,q} \equiv \sum_{ij} Z_{ij} N_{i\bullet}^p N_{\bullet j}^q \tag{28}$$

for

$$\binom{p}{q} \in \left\{ \binom{-1}{-1}, \binom{1}{-1}, \binom{-1}{1}, \binom{1}{1}, \binom{-1}{2}, \binom{2}{-1} \right\}$$

as well as $T_{i\bullet}$ and $T_{\bullet j}$ of equation (2) for all i and j in the data.

Then, we estimate the variances of the U-statistics. Some of these next computations require even more bits per variable than are needed to avoid overflow, because they involve subtraction in a way that could lose precision. To estimate the variances of U_a and U_b, we apply the upper bounds discussed in Section 4.2.2 to (12) and (13) and plug in $\hat{\sigma}_A^2$, $\hat{\sigma}_B^2$, $\hat{\sigma}_E^2$, $\hat{\kappa}_A$, $\hat{\kappa}_B$, and $\hat{\kappa}_E$, calculating using time and space $O(R + C)$

$$\begin{aligned}\widehat{\mathrm{Var}}(U_a) = {} & \hat{\sigma}_B^4(\hat{\kappa}_B + 2)\Big(\sum_j N_{\bullet j}^2 - \mathrm{ZN}^{-1,1}\Big) + 2\hat{\sigma}_B^4\Big(\mathrm{ZN}^{-1,1} - R\sum_i N_{i\bullet}^{-1}\Big) \\ & + 4\hat{\sigma}_B^2\hat{\sigma}_E^2(N - R) + \hat{\sigma}_E^4(\hat{\kappa}_E + 2)\sum_i N_{i\bullet}(1 - N_{i\bullet}^{-1})^2 \\ & + 2\hat{\sigma}_E^4\sum_i (1 - N_{i\bullet}^{-1})\end{aligned}$$

and

$$\begin{aligned}\widehat{\mathrm{Var}}(U_b) = {} & \hat{\sigma}_A^4(\hat{\kappa}_A + 2)\Big(\sum_i N_{i\bullet}^2 - \mathrm{ZN}^{1,-1}\Big) + 2\hat{\sigma}_A^4\Big(\mathrm{ZN}^{1,-1} - C\sum_j N_{\bullet j}^{-1}\Big) \\ & + 4\hat{\sigma}_A^2\hat{\sigma}_E^2(N - C) + \hat{\sigma}_E^4(\hat{\kappa}_E + 2)\sum_j N_{\bullet j}(1 - N_{\bullet j}^{-1})^2 \\ & + 2\hat{\sigma}_E^4\sum_j (1 - N_{\bullet j}^{-1}).\end{aligned}$$

To estimate $\mathrm{Var}(U_e)$ and the covariances of the U-statistics, we again plug in the variance component and kurtosis estimates into Theorem 4.1 without

approximation. We get $\widehat{\mathrm{Var}}(U_e)$ from (14), using $\mathrm{ZN}^{1,1}$ from the second pass over the data. We get $\widehat{\mathrm{Cov}}(U_a, U_e)$ from (16) using $\mathrm{ZN}^{-1,1}$, $\mathrm{ZN}^{-1,2}$ and $T_{i\bullet}$, and $\widehat{\mathrm{Cov}}(U_b, U_e)$ from (17) using $\mathrm{ZN}^{1,-1}$, $\mathrm{ZN}^{2,-1}$ and $T_{\bullet j}$. We get $\widehat{\mathrm{Cov}}(U_a, U_b)$ from (15) using $\mathrm{ZN}^{-1,-1}$. It can be easily verified that these calculations also take time and space $O(R + C)$.

The final plug-in estimator of variance is

$$\widehat{\mathrm{Var}}\begin{pmatrix}\hat\sigma_A^2\\ \hat\sigma_B^2\\ \hat\sigma_E^2\end{pmatrix} = M^{-1}\begin{pmatrix}\widehat{\mathrm{Var}}(U_a) & \widehat{\mathrm{Cov}}(U_a,U_b) & \widehat{\mathrm{Cov}}(U_a,U_e)\\ \widehat{\mathrm{Cov}}(U_b,U_a) & \widehat{\mathrm{Var}}(U_b) & \widehat{\mathrm{Cov}}(U_b,U_e)\\ \widehat{\mathrm{Cov}}(U_e,U_a) & \widehat{\mathrm{Cov}}(U_e,U_b) & \widehat{\mathrm{Var}}(U_e)\end{pmatrix}(M^{-1})^{\mathsf{T}} \tag{29}$$

where M is the matrix in (10).

Aggregating the computation times and counting the number of intermediate values we must calculate, we see that our algorithm takes time $O(N)$ and space $O(R + C)$.

5. Predictions

Here we consider an application of variance component estimation to the prediction of a missing observation Y_{ij} at given values of i and j in model (1). An equivalent problem is predicting the expected value at those levels of the factors, $\mu + a_i + b_j = \mathbb{E}(Y_{ij} \mid a_i, b_j)$.

5.1. Best linear predictor

A gold standard is the best linear predictor (BLP), [19, Chapter 7.3], which minimizes the MSE over the class of predictors of the form $\hat Y_{ij}(\lambda) = \sum_{rs} \lambda_{rs} Z_{rs} Y_{rs}$, where λ is the vector of all λ_{rs}. In this section, we characterize the weights λ^*_{rs} of the BLP. We begin with the MSE

$$L(\lambda) = \mathbb{E}((\hat Y_{ij}(\lambda) - Y_{ij})^2) \tag{30}$$

Lemma 5.1. *The MSEs for the linear predictor $\sum_{rs} \lambda_{rs} Z_{rs} Y_{rs}$ are*

$$\begin{aligned} L(\lambda) = {}& \mu^2\Big(1 - \sum_{rs}\lambda_{rs}Z_{rs}\Big)^2 + \sigma_A^2 + \sigma_B^2 + \sigma_E^2\\ &+ \sigma_A^2\sum_{rss'}\lambda_{rs}\lambda_{rs'}Z_{rs}Z_{rs'} + \sigma_B^2\sum_{rsr'}\lambda_{rs}\lambda_{r's}Z_{rs}Z_{r's} + \sigma_E^2\sum_{rs}\lambda_{rs}^2Z_{rs}\\ &- 2\Big(\sigma_A^2\sum_s\lambda_{is}Z_{is} + \sigma_B^2\sum_r\lambda_{rj}Z_{rj} + \sigma_E^2\lambda_{ij}^2Z_{ij}\Big). \end{aligned} \tag{31}$$

Proof. See Section 9.10.1. □

The weights λ^*_{rs} of the BLP must satisfy the stationarity condition $\partial L(\lambda^*_{rs})/\partial\lambda = 0$. As shown in Section 9.10.2, when $Z_{rs} = 0$, the condition holds no matter the value of λ^*_{rs}. When $Z_{rs} = 1$, the condition becomes

$$\begin{aligned}\sigma_E^2\lambda^*_{rs} = \mu^2\Big(1 - \sum_{r's'}\lambda^*_{r's'}Z_{r's'}\Big) + \sigma_A^2\Big(1_{i=r} - \sum_{s'}\lambda^*_{rs'}Z_{rs'}\Big)\\ + \sigma_B^2\Big(1_{j=s} - \sum_{r'}\lambda^*_{r's}Z_{r's}\Big)\end{aligned} \tag{32}$$

We can compute λ^*_{rs} by solving an $N \times N$ system of equations but that ordinarily costs $O(N^3)$ time. Shortcuts are possible if there is a special pattern in the Z_{ij}, such as balanced data, but we don't know of any faster way to solve (32) for general Z. Therefore, we consider a smaller class of linear predictors called shrinkage predictors.

5.2. Shrinkage predictors

It is reasonable to suppose that the most important observations for predicting Y_{ij} are those in its row and column. Therefore we consider predicting Y_{ij} through a linear combination of the overall average, the average in row i, and the average in column j. We use estimators of the form

$$\hat{Y}_{ij}(\lambda) = \lambda_0\sum_{rs}Z_{rs}Y_{rs} + \lambda_a\sum_{s}Z_{is}Y_{is} + \lambda_b\sum_{r}Z_{rj}Y_{rj} \tag{33}$$

where $\lambda = \begin{pmatrix}\lambda_0 & \lambda_a & \lambda_b\end{pmatrix}^{\mathsf{T}}$. Then λ_0, λ_a, and λ_b are chosen to minimize $L(\lambda)$. By writing (33) in terms of row and column totals we avoid complicated treatments for the situation where row or column means are unavailable because $N_{i\bullet} = 0$ or $N_{\bullet j} = 0$ (or both). As an example, if $\min(N_{i\bullet}, N_{\bullet j}) > 0$, then the predictor $\hat{Y}_{ij} = \bar{Y}_{i\bullet} + \bar{Y}_{\bullet j} - \bar{Y}_{\bullet\bullet}$ (from Theorem 5.3 below) has $\lambda_0 = -1/N$, $\lambda_a = 1/N_{i\bullet}$ and $\lambda_b = 1/N_{\bullet j}$.

Lemma 5.2. *The MSEs for the linear predictor* (33) *are*

$$\begin{aligned}L(\lambda) = \mu^2\big(1 - \lambda_0 N - \lambda_a N_{i\bullet} - \lambda_b N_{\bullet j}\big)^2 + \lambda_0^2\Big(\sigma_A^2\sum_r N_{r\bullet}^2 + \sigma_B^2\sum_s N_{\bullet s}^2 + \sigma_E^2 N\Big)\\ + \lambda_a^2\Big(\sigma_A^2 N_{i\bullet}^2 + \sigma_B^2 N_{i\bullet} + \sigma_E^2 N_{i\bullet}\Big) + \lambda_b^2\Big(\sigma_A^2 N_{\bullet j} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}\Big)\\ + \sigma_A^2 + \sigma_B^2 + \sigma_E^2 - 2\lambda_0\Big(\sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j} + \sigma_E^2 Z_{ij}\Big)\\ - 2\lambda_a\Big(\sigma_A^2 N_{i\bullet} + \sigma_B^2 Z_{ij} + \sigma_E^2 Z_{ij}\Big) - 2\lambda_b\Big(\sigma_A^2 Z_{ij} + \sigma_B^2 N_{\bullet j} + \sigma_E^2 Z_{ij}\Big)\\ + 2\lambda_0\lambda_a\Big(\sigma_A^2 N_{i\bullet}^2 + \sigma_B^2\sum_s Z_{is}N_{\bullet s} + \sigma_E^2 N_{i\bullet}\Big)\\ + 2\lambda_0\lambda_b\Big(\sigma_A^2\sum_r Z_{rj}N_{r\bullet}\\ + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}\Big) + 2\lambda_a\lambda_b Z_{ij}\Big(\sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j} + \sigma_E^2\Big).\end{aligned}$$

Proof. See Section 9.10.3. □

Theorem 5.1. *The λ^* that minimizes the MSE $L = \mathbb{E}((\hat{Y}_{ij} - Y_{ij})^2)$ satisfies $H\lambda^* = c$, where*

$$c = \begin{pmatrix} N & N_{i\bullet} & N_{\bullet j} & Z_{ij} \\ N_{i\bullet} & N_{i\bullet} & Z_{ij} & Z_{ij} \\ N_{\bullet j} & Z_{ij} & N_{\bullet j} & Z_{ij} \end{pmatrix} \begin{pmatrix} \mu^2 \\ \sigma_A^2 \\ \sigma_B^2 \\ \sigma_E^2 \end{pmatrix}, \quad \textit{and} \quad H = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ * & H_{22} & H_{23} \\ * & * & H_{33} \end{pmatrix}$$

is a symmetric matrix with upper triangular elements

$$\begin{aligned} H_{11} &= \mu^2 N^2 + \sigma_A^2 \sum_r N_{r\bullet}^2 + \sigma_B^2 \sum_s N_{\bullet s}^2 + \sigma_E^2 N \\ H_{12} &= \mu^2 N N_{i\bullet} + \sigma_A^2 N_{i\bullet}^2 + \sigma_B^2 T_{i\bullet} + \sigma_E^2 N_{i\bullet} \\ H_{13} &= \mu^2 N N_{\bullet j} + \sigma_A^2 T_{\bullet j} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j} \\ H_{22} &= \mu^2 N_{i\bullet}^2 + \sigma_A^2 N_{i\bullet}^2 + \sigma_B^2 N_{i\bullet} + \sigma_E^2 N_{i\bullet} \\ H_{23} &= \mu^2 N_{i\bullet} N_{\bullet j} + \sigma_A^2 Z_{ij} N_{i\bullet} + \sigma_B^2 Z_{ij} N_{\bullet j} + \sigma_E^2 Z_{ij}, \quad \textit{and} \\ H_{33} &= \mu^2 N_{\bullet j}^2 + \sigma_A^2 N_{\bullet j} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}. \end{aligned}$$

Proof. See Section 9.10.4. □

Given estimates of μ and θ we can plug them in to get estimates of the optimal λ for prediction at (i, j). Assuming that the algorithm to compute $\hat{\theta}$ and its variance has been executed, all of c and most of H can be computed using quantities found in the first pass over the data. The rest are available after a second pass.

Therefore, since solving $H\lambda^* = c$ takes time $O(1)$, λ^* for predicting a given Y_{ij} can be found in time $O(N)$. If we wanted to find λ^* for k different sets of i and j, the computation cost is $O(N + k)$; we simply would have to store k different H's and c's.

Predicting a missing Y_{ij} using Theorem 5.1 is simple. Next we look at some special cases to understand how it performs.

Special case: Y_{ij} in new row and new column

In this case, $N_{rj} = N_{is} = 0$ for any r, s, and $N_{i\bullet} = N_{\bullet j} = 0$. The only nonzero entry of H is $H_{11} = \mu^2 N^2 + \sigma_A^2 \sum_r N_{r\bullet}^2 + \sigma_B^2 \sum_s N_{\bullet s}^2 + \sigma_E^2 N$, and the only nonzero entry of c is $c_1 = \mu^2 N$. Hence $\lambda_a^* = \lambda_b^* = 0$ and

$$\lambda_0^* = \frac{\mu^2 N}{\mu^2 N^2 + \sigma_A^2 \sum_r N_{r\bullet}^2 + \sigma_B^2 \sum_s N_{\bullet s}^2 + \sigma_E^2 N}.$$

The prediction $\hat{Y}_{ij}$ is then a shrinkage

$$\lambda_0^* Y_{\bullet\bullet} = N\lambda_0^* \bar{Y}_{\bullet\bullet} = \frac{\mu^2}{\mu^2 + \sigma_A^2 \sum_r N_{r\bullet}^2/N^2 + \sigma_B^2 \sum_s N_{\bullet s}^2/N^2 + \sigma_E^2/N} \bar{Y}_{\bullet\bullet}.$$

In practice we would plug in estimates of μ and the variance components. As we would expect, this estimate is very close to $\bar{Y}_{\bullet\bullet}$ for large N, when $\hat{\mu} \neq 0$ and

the limits (6) hold. In that case, the corresponding MSE is $L \doteq \sigma_A^2 + \sigma_B^2 + \sigma_E^2$, which can be verified to be approximately the same as the MSE of the BLP.

Special case: Y_{ij} in new row but old column

Suppose that $Z_{is} = 0$ for any s but $\exists r$ where $Z_{rj} = 1$, so $N_{i\bullet} = 0$ and $N_{\bullet j} > 0$. We would expect most of the weight to be on $\bar{Y}_{\bullet j}$, the average in the column containing Y_{ij}. This is indeed the case if $T_{\bullet j}$ is not large compared to N, that is, if the rows that are co-observed with column j do not comprise a large fraction of the data.

Let c_k denote the kth entry of c and $H_{k\ell}$ be the entry of H in row k and column ℓ. In this case, c_2 is zero as is the second row and second column of H. Therefore, without loss of generality we can take $\lambda_a^* = 0$ and $\tilde{\lambda}^* = \begin{pmatrix}\lambda_0^* & \lambda_b^*\end{pmatrix}^{\mathsf{T}}$ can be computed by solving the system $\tilde{H}\tilde{\lambda}^* = \tilde{c}$, where

$$\tilde{H} = \begin{pmatrix} H_{11} & H_{13} \\ H_{31} & H_{33} \end{pmatrix} \quad \text{and} \quad \tilde{c} = \begin{pmatrix} c_1 \\ c_3 \end{pmatrix}.$$

The following theorem describes the relative size of λ_0^* and λ_b^* in the big data limit.

Theorem 5.2. *Suppose that we are predicting Y_{ij} where $N_{i\bullet} = 0$ but $N_{\bullet j} > 0$. Assume that $0 < \mu^2, \sigma_A^2, \sigma_B^2, \sigma_E^2 < \infty$ and that $T_{\bullet j} \equiv \sum_r N_{r\bullet} Z_{rj} \leqslant \eta N$. Then*

$$\frac{\lambda_0^*}{\lambda_b^*} = \frac{1}{N}\frac{\sigma_A^2 + \sigma_E^2}{\sigma_B^2}(1 + O(\eta))$$

as $\eta \to 0$.

Proof. See Section 9.10.5. □

Note that λ_0^* is the coefficient of a sum of N observations, while λ_b^* is the coefficient of a sum of $N_{\bullet j}$ observations. Therefore, to more equitably compare the importances of the overall average and the column average for predicting Y_{ij}, we consider the ratio

$$\frac{N\lambda_0^*}{N_{\bullet j}\lambda_b^*} \approx \frac{\sigma_A^2 + \sigma_E^2}{\sigma_B^2 N_{\bullet j}}.$$

We may interpret this as the column j average being some multiple of $N_{\bullet j}$ times as important as the overall average. This makes sense because the more data we have in column j, the better estimate we would be able to get of $\mu + b_j$; the overall average only tells us about μ. Also, note that the larger σ_E^2 is relative to σ_B^2, the more weight we put on the overall average; we do not trust using only the column average.

Special case: Large $N_{i\bullet}$ and large $N_{\bullet j}$

Next we show that if both row i and column j have a very large number of observations, and the observation matrix Z is not too extreme, then $\hat{Y}_{ij}$ is

approximately $\bar{Y}_{i\bullet} + \bar{Y}_{\bullet j} - \bar{Y}_{\bullet\bullet}$ as we might expect. As a result, the customized weights in Theorem 5.1 are most useful for cases where one or both of $N_{i\bullet}$ and $N_{\bullet j}$ are not very large.

Theorem 5.3. *Suppose that* $1/\eta \leqslant N_{i\bullet} \leqslant \eta N$ *and* $1/\eta \leqslant N_{\bullet j} \leqslant \eta N$ *both hold for some* $\eta \in (0,1)$ *and that* $0 < \mu^2, \sigma_A^2, \sigma_B^2, \sigma_E^2 < \infty$. *Then*

$$\hat{Y}_{ij} = (\bar{Y}_{i\bullet} + \bar{Y}_{\bullet j} - \bar{Y}_{\bullet\bullet})(1 + O(\eta)), \quad \text{as } \eta \to 0.$$

Proof. See Section 9.10.6. □

6. Experimental results

6.1. Simulations

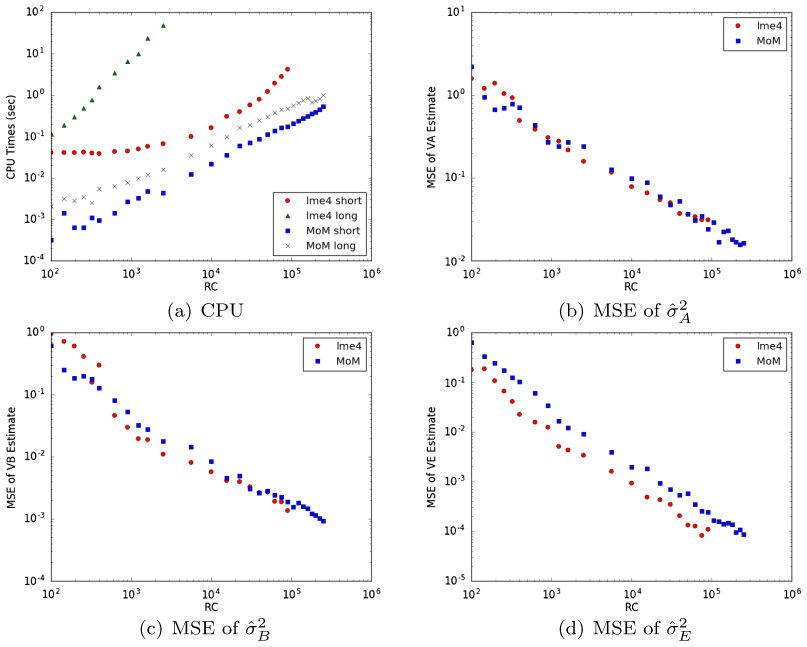
First, we compare the performance of our method of moments algorithm (MoM), described in Section 4.3, to maximum likelihood estimation as implemented in the commonly used R package for mixed models, lme4. We assume that the random effects are normally distributed. We do so because it is the assumption implicitly made by lme4, and so we expect that these are the conditions under which lme4 performs best.

For our algorithm, we consider a range of data sizes, with $R = C$ ranging from 10 to 500. At each fixed value of $R = C$, for 100 iterations, we generate data according to model (1) with $\sigma_A^2 = 2$, $\sigma_B^2 = 0.5$, and $\sigma_E^2 = 1$. Exactly 25 percent of the cells were randomly chosen to be observed. We measure the CPU time needed to obtain the variance component estimates $\hat{\sigma}_A^2$, $\hat{\sigma}_B^2$, and $\hat{\sigma}_E^2$ (labeled short) and the CPU time need to obtain the variance component estimates as well as conservative estimates of the variances of those estimates (labeled long). In addition, we measure the mean squared errors of the variance component estimates. At the end, those five measurements were averaged over the 100 iterations.

With regard to lme4, our simulation steps are nearly the same, with the following differences. Due to the slowness of lme4, we only consider data sizes with $R = C$ up to 300. In addition, because lme4 finds the maximum likelihood variance component estimates, the variances of those estimates were computed asymptotically using the inverse expected Fisher information matrix. The simulation results are shown in Figure 2.

Note that lme4 always takes more time than our algorithm. From Figure 2, we see that our method of moments algorithm takes time at most linear in the data size to compute both the variance component estimates and conservative estimates of the variances of those estimates. For lme4 the computation time is always superlinear in the data size, for data sets large enough that the startup cost of the package is no longer dominant.

The MSEs of $\hat{\sigma}_A^2$ for our algorithm and lme4 are comparable. Moreover, both decrease sublinearly with the data size. The same is true for the MSEs of $\hat{\sigma}_B^2$. However, the MSE of $\hat{\sigma}_E^2$ in lme4 is noticeably smaller than that of our algorithm;



(a) CPU (b) MSE of $\hat{\sigma}_A^2$

(c) MSE of $\hat{\sigma}_B^2$ (d) MSE of $\hat{\sigma}_E^2$

FIG 2. *Simulation results: log-log plots of the five recorded measurements against RC, which is proportional to the number of observations.*

this appears to be the price we pay for the decreased computation time. In both cases, though, the MSE of $\hat{\sigma}_E^2$ decreases approximately linearly with the data size. In sum, our algorithm provides estimates that are nearly as efficient as MLE and is scalable to huge data sets, unlike MLE.

We also considered data with other aspect ratios, e.g. $R = 2C$ and $C = 2R$. The CPU time required and the MSEs of the variance component estimates behave similarly to the case $R = C$. For the sake of brevity, we do not explicitly show the results of those simulations here.

Remark: In our simulations we investigated the conservativeness of the estimates of the variances of $\hat{\sigma}_A^2$, $\hat{\sigma}_B^2$, and $\hat{\sigma}_E^2$ due to the over-estimates described in Section 4.2.2. When $R = C$, it appears that they are conservative by a factor of at most four. The factors corresponding to σ_A^2 and σ_B^2 approached 1 as the data size increased. Therefore, we believe that the upper bounds are a reasonable approximation of the true variances of the variance components.

6.2. Real world data

We illustrate our algorithm, coded in Python, on three real world data sets that are too large for lme4 to handle in a timely manner.

The first, from [22], contains a random sample of ratings of movies by users, which are grades from A+ to F converted into a numeric scale. There are 211,231 ratings by 7,642 users on 11,916 movies, filtered with the condition that each user rates at least ten movies. Only 0.23 percent of the user-movie matrix is observed.

The estimated variances of the user random effect, the movie random effect, and the error are 2.57, 2.86, and 7.68. The estimated kurtoses are -2, -2, and 6.56. Conservative estimates of the variances of the estimated variance components are 0.0030, 0.0018, and 0.0060. Therefore, most of the variation in the moving rating comes from error or the interaction between movies and users; this is not surprising, since different people have different tastes.

The second data set, from [23], contains ratings of 1000 songs by 15,400 users, on a scale of 1 to 5. The first group of 10,000 users were randomly selected on the condition that they had rated at least 10 of the 1000 songs. The rest of the users were randomly selected from responders on a survey that asked them to rate a random subset of 10 of the 1000 songs. The songs were selected to have at least 500 ratings. Here, about 2 percent of the user-song pairs were observed.

The estimated variances of the user random effect, the song random effect, and the error are 0.97, 0.24, and 1.30. The estimated kurtoses are -2, -2, and 3.31. Conservative estimates of the variances of the estimated variance components are 4.5×10^{-5}, 10^{-5}, and 5.8×10^{-5}, yielding conservative standard errors of about 0.007, 0.003 and 0.008. In this case there is negligible sampling variability in the variance component estimates. For determining the rating, the user effect is dominant over the song effect, but as in the previous example, the greatest variation comes from error or interactions between song and user.

The third data set from [10] contains the numbers of times artists' songs are played by about $360,000$ users. Only the counts for the top k (for some k) artists for each user is recorded. The users are randomly selected. This data set is extremely sparse; only about 0.03 percent of user-artist pairs are observed.

The estimated variances of the user random effect, the artist random effect, and the error are 1.65, 0.22, and 0.27. The estimated kurtoses are 0.019, -2, and 23.14. Conservative estimates of the variances of the estimated variance components are 1.68×10^{-5}, 4.06×10^{-7}, and 1.37×10^{-6}, yielding small standard errors of about 0.004, 0.0006 and 0.001. The biggest source of variation in the number of plays is the user. The kurtosis of the row effect is nearly zero, indicating possible normality.

In all three data sets at least one of the estimated kurtoses was -2, which would be unexpected if the model is correctly specified. However, if model (1) does not fit the data well, such behavior may occur. We suspect that more realistic models incorporating fixed effects and SVD-like interactions would reduce the prevalence of such kurtosis estimates.

7. Conclusion

When traditional maximum likelihood or MCMC methods are used, with both theory and simulations, we have found that fitting large two-factor crossed un-

balanced random effects models has costs that are superlinear in the number of data points. With the method of moments it is possible to get, in linear time, parameter estimates and somewhat conservative estimates of their variance. The space requirements are proportional to the number of distinct levels of the factors; this will often be sublinear in N. We also developed shrinkage predictors of missing data that utilize our method of moments estimates.

Through simulations on normally distributed data, we show that our method of moments estimates are competitive with maximum likelihood estimates. We trade off a small increase in the MSE of one variance component for a dramatic decrease in computation time as N gets large. In the real data with large N, we saw negligible sampling uncertainty in our variance estimaes despite using over-estimates.

As stated in the introduction, the crossed random effects model we consider here is the simplest one for which we found no useful prior solution. We expect that richer models, which are the basis of our future work, will provide better fits to real world data.

7.1. Informative missingness

We have assumed throughout that the missingness pattern in Z_{ij} is not informative. But in many applications the observed values are likely to differ in some way from the missing values. For instance, in movie ratings data people may be more likely to watch and rate movies they believe they will like, and so missing values could be lower on average than observed ones. In general, the observed ratings may have both high and low values oversampled relative to middling values.

From observed values alone we cannot tell how different the missing values would be. To do so requires making untestable assumptions about the missingness mechanism. Even in cases where followup sampling can be made, e.g., giving some users incentives to make additional ratings, there will still be difficulties such as users refusing to make those ratings, or if forced, making inaccurate ratings. Methods to adjust for missingness have to be designed on a case by case basis, using whatever additional data and assumptions can be brought to bear. The uncertainties of the estimates from such methods can be quantified using, with further development, the techniques of this paper.

8. Appendix A

8.1. Proof of Theorem 2.1

In the balanced case we may assume that $i \in \{1, 2, \dots, R\}$ and $j \in \{1, 2, \dots, C\}$. The posterior distribution of the parameters is given by

$$p(\mu, a, b, \sigma_A^2, \sigma_B^2, \sigma_E^2 \mid Y) \propto \prod_{i=1}^{R} \frac{1}{\sqrt{2\pi\sigma_A^2}} \exp\Big(-\frac{a_i^2}{2\sigma_A^2}\Big) \prod_{j=1}^{C} \frac{1}{\sqrt{2\pi\sigma_B^2}} \exp\Big(-\frac{b_j^2}{2\sigma_B^2}\Big)$$

$$\times \prod_{i=1}^{R}\prod_{j=1}^{C} \frac{1}{\sqrt{2\pi\sigma_E^2}} \exp\Big(-\frac{(Y_{ij}-\mu-a_i-b_j)^2}{2\sigma_E^2}\Big)$$
$$\propto \sigma_A^{-R}\sigma_B^{-C}\sigma_E^{-RC} \exp\Big(-\frac{\sum_i a_i^2}{2\sigma_A^2} - \frac{\sum_j b_j^2}{2\sigma_B^2} - \frac{\sum_{ij}(Y_{ij}-\mu-a_i-b_j)^2}{2\sigma_E^2}\Big)$$

Then, $\phi \equiv p(a,b \mid \mu,\sigma_A^2,\sigma_B^2,\sigma_E^2,Y)$ is proportional to

$$\exp\Big(-\frac{\sum_i a_i^2}{2}\Big(\frac{1}{\sigma_A^2}+\frac{C}{\sigma_E^2}\Big) - \frac{\sum_j b_j^2}{2}\Big(\frac{1}{\sigma_B^2}+\frac{R}{\sigma_E^2}\Big) - \frac{\sum_{ij} a_i b_j}{\sigma_E^2}\Big).$$

Therefore, the posterior distribution of a and b is a joint normal with precision matrix

$$Q = \begin{pmatrix} \frac{\sigma_E^2+C\sigma_A^2}{\sigma_A^2\sigma_E^2} I_R & \frac{1}{\sigma_E^2} 1_R 1_C^{\mathsf{T}} \\ \frac{1}{\sigma_E^2} 1_C 1_R^{\mathsf{T}} & \frac{\sigma_E^2+R\sigma_B^2}{\sigma_B^2\sigma_E^2} I_C \end{pmatrix}.$$

From Theorem 1 of [18], for the Gibbs sampler described in Section 2.1, we have the following result. Let $A = I - \mathrm{diag}(Q_{11}^{-1}, Q_{22}^{-1})Q$, where Q_{11} denotes the upper left block of Q and Q_{22} denotes the lower right block. Let L be the block lower triangular part of A, and $U = A - L$. Then, the convergence rate ρ is given by the spectral radius of the matrix $B = (I-L)^{-1}U$. Now, we compute ρ. First

$$A = I - \begin{pmatrix} \frac{\sigma_A^2\sigma_E^2}{\sigma_E^2+C\sigma_A^2} I_R & 0 \\ 0 & \frac{\sigma_B^2\sigma_E^2}{\sigma_E^2+R\sigma_B^2} I_C \end{pmatrix} Q$$
$$= \begin{pmatrix} 0 & -\frac{\sigma_A^2}{\sigma_E^2+C\sigma_A^2} 1_R 1_C^{\mathsf{T}} \\ -\frac{\sigma_B^2}{\sigma_E^2+R\sigma_B^2} 1_C 1_R^{\mathsf{T}} & 0 \end{pmatrix}.$$

Next

$$L = \begin{pmatrix} 0 & 0 \\ -\frac{\sigma_B^2}{\sigma_E^2+R\sigma_B^2} 1_C 1_R^{\mathsf{T}} & 0 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 0 & -\frac{\sigma_A^2}{\sigma_E^2+C\sigma_A^2} 1_R 1_C^{\mathsf{T}} \\ 0 & 0 \end{pmatrix}$$

from which

$$B = \begin{pmatrix} I_R & 0 \\ \frac{\sigma_B^2}{\sigma_E^2+R\sigma_B^2} 1_C 1_R^{\mathsf{T}} & I_C \end{pmatrix}^{-1} U = \begin{pmatrix} I_R & 0 \\ -\frac{\sigma_B^2}{\sigma_E^2+R\sigma_B^2} 1_C 1_R^{\mathsf{T}} & I_C \end{pmatrix} U$$

TABLE 2
Median CPU time in seconds.

Method	Gibbs	Block	Reparam.	Lang.	MALA	Indp.	RWM	RWM Sub.	pCN
R=10 C=10	20	9	23	20	27	21	19	21	21
R=20 C=20	33	10	37	35	45	34	32	33	33
R=50 C=50	71	17	80	79	101	71	68	75	70
R=100 C=100	143	361	159	156	199	139	133	141	136
R=200 C=200	326	984	351	323	462	300	279	303	280
R=500 C=500	1157	2356	1205	955	1786	952	851	1019	817
R=1000 C=1000	3432	15046	4099	2302	4760	2513	2141	2635	1966
R=2000 C=2000	10348	88756	11434	6991	15836	7815	5712	9274	6006
R=50 C=100	105	287	121	112	151	103	101	107	102
R=10 C=200	138	316	167	139	200	138	137	142	138
R=100 C=1000	898	5148	964	807	1179	795	748	822	760

$$= \begin{pmatrix} 0 & -\dfrac{\sigma_A^2}{\sigma_E^2 + C\sigma_A^2} 1_R 1_C^{\mathsf{T}} \\ 0 & \dfrac{R\sigma_A^2\sigma_B^2}{(\sigma_E^2 + C\sigma_A^2)(\sigma_E^2 + R\sigma_B^2)} 1_C 1_C^{\mathsf{T}} \end{pmatrix}.$$

Clearly, B has rank one. Then, its spectral radius must be equal to its nonzero eigenvalue, which is also the trace of B. Hence,

$$\rho = \frac{RC\sigma_A^2\sigma_B^2}{(\sigma_E^2 + C\sigma_A^2)(\sigma_E^2 + R\sigma_B^2)}.$$

8.2. Simulation results

The results of our simulations described in Section 2 are presented here in Tables 2 through 6.

9. Appendix B

9.1. Partially observed random effects model

The random effects model is

$$Y_{ij} = \mu + a_i + b_j + e_{ij}, \quad i, j \in \mathbb{N} \tag{34}$$

for $a_i \overset{\text{iid}}{\sim} F_a$, $b_j \overset{\text{iid}}{\sim} F_b$ and $e_{ij} \overset{\text{iid}}{\sim} F_e$ independent of each other. These random variables have mean 0, variances σ_A^2, σ_B^2, σ_E^2 and kurtoses κ_A, κ_B, κ_E, respectively. We will not need their skewnesses.

TABLE 3
Median estimates of μ and number of lags after which $ACF(\hat{\mu}) \le 0.5$.

Method	Gibbs	Block	Reparam.	Lang.	MALA	Indp.	RWM	RWM Sub.	pCN
R=10	0.72	0.94	1.27	1.07	1.18	2.40	0.76	0.74	1.51
C=10	26	29	24	178	689	1604	1252	1522	1392
R=20	0.81	1.02	1.01	1.07	0.94	2.89	1.69	1.08	1.47
C=20	34	43	26	75	841	1019	1674	1720	1765
R=50	1.09	0.91	0.98	0.98	1.04	2.97	1.66	1.70	1.58
C=50	83	84	75	8	610	5000+	1158	1681	1104
R=100	0.98	1.02	1.13	0.99	0.85	2.73	1.57	1.61	1.49
C=100	123	185	144	2	398	5000+	1145	1713	1522
R=200	1.01	1.02	1.03	1.01	0.95	3.22	1.60	1.31	1.52
C=200	257	346	272	1	1	1278	1508	1692	807
R=500	0.99	1.01	0.99	0.99	1.00	2.26	1.58	1.15	1.55
C=500	536	617	576	9	4	1572	924	1687	1613
R=1000	0.97	1.02	1.04	0.99	0.96	2.39	1.55	1.07	1.53
C=1000	801	790	694	1	2501	5000+	1133	1656	1008
R=2000	0.98	1.01	1.00	1.01	1.00	2.57	1.55	1.03	1.55
C=2000	672	721	771	1	5000+	1086	1176	1716	799
R=50	0.89	1.03	0.95	1.01	1.06	2.70	1.57	1.61	1.45
C=100	144	155	118	7	1095	5000+	1219	1725	1371
R=10	0.86	1.08	0.84	0.94	0.80	2.40	1.41	1.36	1.23
C=200	329	244	299	120	944	3339	1518	1657	1437
R=100	1.06	1.06	1.02	1.01	1.03	2.73	1.57	1.11	1.55
C=1000	573	536	672	1	1	3330	1161	1681	3333

TABLE 4
Median estimates of σ_A^2 and number of lags after which $ACF(\hat{\sigma_A^2}) \le 0.5$.

Method	Gibbs	Block	Reparam.	Lang.	MALA	Indp.	RWM	RWM Sub.	pCN
R=10	2.76	2.49	2.05	2.07	2.45	2.39	1.88	2.05	1.38
C=10	1	1	1	898	768	1604	759	606	1232
R=20	2.00	2.06	1.65	1.89	2.32	1.48	1.96	1.76	2.00
C=20	1	1	1	930	829	850	873	822	1083
R=50	1.94	1.96	2.17	1.77	2.21	1.44	2.06	2.03	1.95
C=50	1	1	1	797	720	5000+	1035	1032	1079
R=100	2.21	2.14	2.23	1.88	1.87	1.11	2.19	1.92	1.95
C=100	1	1	1	649	398	5000+	994	917	1522
R=200	2.09	2.09	2.10	2.08	1.99	1.16	2.02	2.12	2.01
C=200	1	1	1	410	437	1281	1598	673	1135
R=500	1.97	2.12	1.99	1.64	1.96	1.07	2.02	2.01	1.97
C=500	1	1	1	407	197	1572	895	826	1599
R=1000	1.96	1.99	2.02	1.90	1.95	1.78	2.01	1.96	1.99
C=1000	1	1	1	122	2656	5000+	1133	989	912
R=2000	1.97	2.00	2.03	1.94	1.99	1.04	2.01	2.00	1.99
C=2000	1	1	1	69	5000+	1086	1181	1262	1161
R=50	2.22	2.29	2.05	2.24	1.98	1.10	2.00	1.96	2.09
C=100	1	1	1	948	672	5000+	1103	787	1005
R=10	2.34	1.74	3.05	2.70	2.72	0.88	1.89	1.43	1.16
C=200	1	1	1	891	1023	3309	1492	724	988
R=100	2.04	2.03	2.14	1.98	1.98	1.46	1.90	1.87	2.05
C=1000	1	1	1	512	450	3329	985	1086	3333

We use letters i, i', r, r' to index rows. Letters j, j', s, s' are used for columns. In internet applications, the actual indices may be people rating items, items being rated, cookies, URLs, IP addresses, query strings, image identifiers and so on. We simplify the index set to $\mathbb{N}$ for notational convenience. One feature of these variables is that we fully expect future data to bring hitherto unseen levels. That is why a countable index set is appropriate.

TABLE 5
Median estimates of σ_B^2 and number of lags after which $ACF(\hat{\sigma_B^2}) \leq 0.5$.

Method	Gibbs	Block	Reparam.	Lang.	MALA	Indp.	RWM	RWM Sub.	pCN
R=10	0.66	0.81	0.88	0.46	0.89	1.47	0.45	0.43	0.45
C=10	1	1	1	382	638	1604	1214	956	1297
R=20	0.54	0.45	0.44	0.43	0.44	1.55	0.49	0.46	0.57
C=20	1	1	1	261	410	978	937	1217	704
R=50	0.49	0.49	0.49	0.49	0.53	1.35	0.49	0.43	0.48
C=50	1	1	1	123	138	5000+	1308	786	1463
R=100	0.51	0.54	0.49	0.46	0.48	0.84	0.52	0.47	0.49
C=100	1	1	1	65	66	5000+	691	1169	1522
R=200	0.49	0.51	0.51	0.47	0.50	1.67	0.51	0.49	0.50
C=200	1	1	1	36	37	1266	1497	1241	831
R=500	0.51	0.49	0.50	0.28	0.47	1.56	0.50	0.48	0.47
C=500	1	1	1	770	16	1572	696	993	1619
R=1000	0.51	0.50	0.50	0.40	0.50	2.94	0.51	0.50	0.49
C=1000	1	1	1	477	2514	5000+	1133	855	556
R=2000	0.50	0.50	0.49	0.39	0.50	1.65	0.48	0.49	0.50
C=2000	1	1	1	224	5000+	1086	1220	830	1253
R=50	0.50	0.51	0.53	0.48	0.54	1.93	0.53	0.49	0.49
C=100	1	1	1	69	85	5000+	1378	910	1419
R=10	0.47	0.51	0.51	0.40	0.52	1.65	0.61	0.59	0.55
C=200	1	1	1	23	52	3332	1289	1004	1408
R=100	0.50	0.49	0.50	0.47	0.49	2.95	0.50	0.49	0.50
C=1000	1	1	1	6	8	3328	1345	962	3333

TABLE 6
Median estimates of σ_E^2 and number of lags after which $ACF(\hat{\sigma_E^2}) \leq 0.5$.

Method	Gibbs	Block	Reparam.	Lang.	MALA	Indp.	RWM	RWM Sub.	pCN
R=10	1.02	0.99	0.96	0.91	1.17	0.17	0.76	0.80	0.75
C=10	1	1	1	196	334	1604	1354	1329	1504
R=20	0.97	0.98	1.00	0.91	1.00	0.17	0.48	0.45	0.37
C=20	1	1	1	61	75	1218	1649	1614	1827
R=50	1.00	1.01	0.98	0.96	0.99	0.17	0	0.01	0
C=50	1	1	1	10	12	5000+	1107	1616	1466
R=100	1.00	1.00	1.00	0.98	1.00	0.16	0	0.38	0
C=100	1	1	1	3	3	5000+	1199	1714	1532
R=200	1.00	1.00	1.00	1.01	1.01	0.21	0	0.66	0
C=200	1	1	1	1	1	1266	1626	1691	636
R=500	1.00	1.00	1.00	118.45	52.70	0.14	0	0.87	0
C=500	1	1	1	545	138	1572	834	1702	1616
R=1000	1.00	1.00	1.00	65.22	2.66	0.15	0	0.93	0
C=1000	1	1	1	385	3062	5000+	1518	1724	621
R=2000	1.00	1.00	1.00	115.59	1.05	0.18	0	0.97	0
C=2000	1	1	1	10	5000+	1021	1194	1702	1014
R=50	1.01	0.99	1.00	0.98	1.01	0.15	0	0.19	0
C=100	1	1	1	5	6	5000+	1676	1774	1442
R=10	0.99	0.99	1.01	0.92	0.99	0.17	0	0.55	0
C=200	1	1	1	12	15	3309	1570	1678	1279
R=100	1.00	1.00	1.00	3.50	3.46	0.19	0	0.87	0
C=1000	1	1	1	3	3	3330	1454	1699	3333

We will want to estimate σ_A^2, σ_B^2, σ_E^2 and get a formula for the variance of those estimates. Many, perhaps most, of the Y_{ij} values are missing. Here we assume that the missingness is not informative. For further discussion see Section 7.1.

The variable $Z_{ij} \in \{0, 1\}$ takes the value 1 if Y_{ij} is available and 0 otherwise. The total sample size is $N = \sum_{ij} Z_{ij}$. We assume that $1 \leqslant N < \infty$. We also

need $N_{i\bullet} = \sum_j Z_{ij}$ and $N_{\bullet j} = \sum_i Z_{ij}$. The number of unique observed rows and columns are, respectively,

$$R \equiv \sum_i 1_{N_{i\bullet}>0}, \quad \text{and} \quad C \equiv \sum_j 1_{N_{\bullet j}>0}.$$

In the sum above, only finitely many summands are nonzero. When we sum over i, i', r, r', the sum is over the set $\{i \mid N_{i\bullet} > 0\}$. Similarly sums over column indices j, j', s, s' are over the set $\{j \mid N_{\bullet j} > 0\}$. These ranges are what one would naturally get in a pass over data logs showing all records.

We frequently need the number of columns jointly observed in two rows such as i and i'. This is $\sum_j Z_{ij}Z_{i'j} = (ZZ^{\mathsf{T}})_{ii'}$. Similarly, columns j and j' are jointly observed in $\sum_i Z_{ij}Z_{ij'} = (Z^{\mathsf{T}}Z)_{jj'}$ rows.

The matrix Z encodes several different measurement regimes as special cases. These include crossed designs, nested designs and IID sampling, as follows. A crossed design with an $R \times C$ matrix of completely observed data can be represented via $Z_{ij} = 1_{1\leqslant i\leqslant R}1_{1\leqslant j\leqslant C}$. If $\max_i N_{i\bullet} = 1$ and $\max_j N_{\bullet j} > 1$ then the data have a nested structure, with $N_{\bullet j}$ distinct rows in column j and $(Z^{\mathsf{T}}Z)_{jj'} = 0$ for $j \neq j'$. Similarly $\max_j N_{\bullet j} = 1$ with $\max_i N_{i\bullet} > 1$ yields columns nested in rows. If $\max_i N_{i\bullet} = \max_j N_{\bullet j} = 1$ then we have N IID observations.

We note some identities:

$$\sum_{ir}(ZZ^{\mathsf{T}})_{ir} = \sum_{ijr} Z_{ij}Z_{rj} = \sum_j N_{\bullet j}^2, \quad \text{and} \tag{35}$$

$$\sum_{ir} N_{i\bullet}^{-1}(ZZ^{\mathsf{T}})_{ir} = \sum_{ijr} N_{i\bullet}^{-1}Z_{ij}Z_{rj} = \sum_{ij} Z_{ij}N_{i\bullet}^{-1}N_{\bullet j}. \tag{36}$$

We need some notation for equality among index sets. The notation $1_{ij=rs}$ means $1_{i=r}1_{j=s}$. It is different from $1_{\{i,j\}=\{r,s\}}$ which we also use. Additionally, $1_{ij\neq rs}$ means $1 - 1_{ij=rs}$.

9.2. Weighted U statistics

We will work with weighted U-statistics

$$U_a = \frac{1}{2}\sum_{ijj'} u_i Z_{ij}Z_{ij'}(Y_{ij} - Y_{ij'})^2$$

$$U_b = \frac{1}{2}\sum_{iji'} v_j Z_{ij}Z_{i'j}(Y_{ij} - Y_{i'j})^2, \quad \text{and}$$

$$U_e = \frac{1}{2}\sum_{iji'j'} w_{ij} Z_{ij}Z_{i'j'}(Y_{ij} - Y_{i'j'})^2,$$

for weights u_i, v_j and w_{ij} chosen below.

We can write $U_a = \sum_i u_i N_{i\bullet}(N_{i\bullet} - 1)s_{i\bullet}^2$ where $s_{i\bullet}^2$ is an unbiased estimate of $\sigma_B^2 + \sigma_E^2$ from within any row i with $N_{i\bullet} \geqslant 2$. Under our model the values in row i are IID with mean $\mu + a_i$ and variance $\sigma_B^2 + \sigma_E^2$, and so

$$\mathrm{Var}(s^2_{i\bullet}) = (\sigma^2_B + \sigma^2_E)^2\Big(\frac{2}{N_{i\bullet}-1} + \frac{\kappa(b_j+e_{ij})}{N_{i\bullet}}\Big)$$

where $\kappa(b_j+e_{ij}) = (\kappa_B\sigma^4_B + \kappa_E\sigma^4_E)/(\sigma^2_B+\sigma^2_E)^2$ is the kurtosis of Y_{ij} for the given i and any j. Thus

$$\mathrm{Var}(s^2_{i\bullet}) = \frac{2(\sigma^2_B+\sigma^2_E)^2}{N_{i\bullet}-1} + \frac{\kappa_B\sigma^4_B}{N_{i\bullet}} + \frac{\kappa_E\sigma^4_E}{N_{i\bullet}}. \tag{37}$$

Inverse variance weighting then suggests that we weight $s^2_{i\bullet}$ proportionally to a value between $N_{i\bullet}$ and $N_{i\bullet}-1$. Weighting proportional to $N_{i\bullet}-1$ has the advantage of zeroing out rows with $N_{i\bullet}=1$. This consideration motivates us to take $u_i = 1/N_{i\bullet}$, and similarly $v_j = 1/N_{\bullet j}$.

If U_e is dominated by contributions from e_{ij} then the observations enter symmetrically and there is no reason to not take $w_{ij}=1$. Even if the e_{ij} do not dominate, the statistic U_e compares more data pairs than the others. It is unlikely to be the information limiting statistic. So $w_{ij}=1$ is a reasonable default.

If the data are IID then only U_e above is nonzero. This is appropriate as only the sum $\sigma^2_A+\sigma^2_B+\sigma^2_E$ can be identified in that case. For data that are nested but not IID, only two of the U-statistics above are nonzero and in that case only one of σ^2_A and σ^2_B can be identified separately from σ^2_E.

The U-statistics we use are then

$$\begin{aligned}
U_a &= \frac{1}{2}\sum_{ijj'} N^{-1}_{i\bullet} Z_{ij}Z_{ij'}(Y_{ij}-Y_{ij'})^2\\
U_b &= \frac{1}{2}\sum_{iji'} N^{-1}_{\bullet j} Z_{ij}Z_{i'j}(Y_{ij}-Y_{i'j})^2, \quad \text{and}\\
U_e &= \frac{1}{2}\sum_{iji'j'} Z_{ij}Z_{i'j'}(Y_{ij}-Y_{i'j'})^2.
\end{aligned} \tag{38}$$

Because we only sum over i with $N_{i\bullet}>0$ and j with $N_{\bullet j}>0$, our sums never include $0/0$.

9.2.1. Expected U-statistics

Here we find the expected values for our three U-statistics.

Lemma 9.1. *Under the random effects model* (34)*, the U-statistics in* (38) *satisfy*

$$\begin{pmatrix}\mathbb{E}(U_a)\\ \mathbb{E}(U_b)\\ \mathbb{E}(U_e)\end{pmatrix} = \begin{pmatrix} 0 & N-R & N-R\\ N-C & 0 & N-C\\ N^2-\sum_i N^2_{i\bullet} & N^2-\sum_j N^2_{\bullet j} & N^2-N\end{pmatrix}\begin{pmatrix}\sigma^2_A\\ \sigma^2_B\\ \sigma^2_E\end{pmatrix}. \tag{39}$$

Proof. First we note that

$$\begin{aligned}\mathbb{E}((a_i - a_{i'})^2) &= 2\sigma_A^2(1 - 1_{i=i'})\\ \mathbb{E}((b_j - b_{j'})^2) &= 2\sigma_B^2(1 - 1_{j=j'}), \quad \text{and}\\ \mathbb{E}((e_{ij} - e_{i'j'})^2) &= 2\sigma_E^2(1 - 1_{i=i'}1_{j=j'}).\end{aligned}$$

Now $Y_{ij} - Y_{ij'} = b_j - b_{j'} + e_{ij} - e_{ij'}$, and so

$$\begin{aligned}\mathbb{E}(U_a) &= \frac{1}{2}\sum_{ijj'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} \big(2\sigma_B^2(1 - 1_{j=j'}) + 2\sigma_E^2(1 - 1_{i=i}1_{j=j'})\big)\\ &= (\sigma_B^2 + \sigma_E^2)\sum_{ijj'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'}(1 - 1_{j=j'})\\ &= (\sigma_B^2 + \sigma_E^2)\sum_{ij'} Z_{ij'}(1 - 1_{j=j'})\\ &= (\sigma_B^2 + \sigma_E^2)\sum_{i}(N_{i\bullet} - 1)\\ &= (\sigma_B^2 + \sigma_E^2)(N - R).\end{aligned}$$

The same argument give $\mathbb{E}(U_b) = (\sigma_A^2 + \sigma_E^2)(N - C)$. □

The matrix in (39) is

$$M \equiv \begin{pmatrix} 0 & N - R & N - R\\ N - C & 0 & N - C\\ N^2 - \sum_i N_{i\bullet}^2 & N^2 - \sum_j N_{\bullet j}^2 & N^2 - N\end{pmatrix}. \tag{40}$$

Our moment based estimates are

$$\begin{pmatrix}\hat\sigma_A^2\\ \hat\sigma_B^2\\ \hat\sigma_E^2\end{pmatrix} = M^{-1}\begin{pmatrix}U_a\\ U_b\\ U_e\end{pmatrix}. \tag{41}$$

They are only well defined when M is nonsingular. The determinant of M is

$$\begin{aligned}&(N - R)\big[(N - C)(N^2 - \sum_j N_{\bullet j}^2)\big]\\ &\quad - (N - R)\big[(N - C)(N^2 - N) - (N - C)(N^2 - \sum_i N_{i\bullet}^2)\big]\\ &= (N - R)(N - C)\big[N^2 - \sum_i N_{i\bullet}^2 - \sum_j N_{\bullet j}^2 + N\big].\end{aligned}$$

The first factor is positive so long as $\max_i N_{i\bullet} > 1$, and the second factor requires $\max_j N_{\bullet j} > 1$. We already knew that we needed these conditions in order to have all three U-statistics depend on the Y_{ij}. It is still of interest to know when the third factor is positive. It is sufficient that no row or column has over half of the data.

9.3. The variance

From equation (41) we get

$$\operatorname{Var}\begin{pmatrix}\hat\sigma_A^2\\ \hat\sigma_B^2\\ \hat\sigma_E^2\end{pmatrix} = M^{-1}\operatorname{Var}\begin{pmatrix}U_a\\ U_b\\ U_e\end{pmatrix} M^{-1}$$

where M is given at (40). So we need the variances and covariances of the three U statistics.

To find variances, we will work out $\mathbb{E}(U^2)$ for our U-statistics. Those involve

$$\begin{aligned}
&\mathbb{E}((Y_{ij}-Y_{i'j'})^2(Y_{rs}-Y_{r's'})^2)\\
&=\mathbb{E}\big((a_i-a_{i'}+b_j-b_{j'}+e_{ij}-e_{i'j'})^2(a_r-a_{r'}+b_s-b_{s'}+e_{rs}-e_{r's'})^2\big)\\
&=\mathbb{E}\Big[\big((a_i-a_{i'})^2+(b_j-b_{j'})^2+(e_{ij}-e_{i'j'})^2\\
&\quad+2(a_i-a_{i'})(b_j-b_{j'})+2(a_i-a_{i'})(e_{ij}-e_{i'j'})+2(b_j-b_{j'})(e_{ij}-e_{i'j'})\big)\\
&\quad\times\big((a_r-a_{r'})^2+(b_s-b_{s'})^2+(e_{rs}-e_{r's'})^2\\
&\quad+2(a_r-a_{r'})(b_s-b_{s'})+2(a_r-a_{r'})(e_{rs}-e_{r's'})+2(b_s-b_{s'})(e_{rs}-e_{r's'})\big)\Big].
\end{aligned}$$

This expression involves 8 indices and it has 36 terms. Some of those terms simplify due to independence and some vanish due to zero means. To shorten some expressions we use

$$\begin{aligned}
\mathbb{B}_{A,ii',rr'} &\equiv \mathbb{E}((a_i-a_{i'})(a_r-a_{r'}))\\
\mathbb{D}_{A,ii'} &\equiv \mathbb{E}((a_i-a_{i'})^2),\quad \text{and,}\\
\mathbb{Q}_{A,ii',rr'} &\equiv \mathbb{E}((a_i-a_{i'})^2(a_r-a_{r'})^2)
\end{aligned}$$

with mnemonics bilinear, diagonal and quartic. There are similarly defined terms for component B. For the error term we have

$$\begin{aligned}
\mathbb{B}_{E,iji'j',rsr's'} &\equiv \mathbb{E}((e_{ij}-e_{i'j'})(e_{rs}-e_{r's'}))\\
\mathbb{D}_{E,ij,i'j'} &\equiv \mathbb{E}((e_{ij}-e_{i'j'})^2),\quad \text{and,}\\
\mathbb{Q}_{E,iji'j',rsr's'} &\equiv \mathbb{E}((e_{ij}-e_{i'j'})^2(e_{rs}-e_{r's'})^2).
\end{aligned}$$

The generic contribution $\mathbb{E}((Y_{ij}-Y_{i'j'})^2(Y_{rs}-Y_{r's'})^2)$ to the mean square of a U-statistic equals

$$\begin{aligned}
&\mathbb{Q}_{A,ii',rr'}+\mathbb{Q}_{B,jj',ss'}+\mathbb{Q}_{E,iji'j',rsr's'}+\mathbb{D}_{A,ii'}\mathbb{D}_{B,ss'}+\mathbb{D}_{A,ii'}\mathbb{D}_{E,rs,r's'}\\
&+\mathbb{D}_{B,jj'}\mathbb{D}_{A,rr'}+\mathbb{D}_{B,jj'}\mathbb{D}_{E,rs,r's'}+\mathbb{D}_{E,ij,i'j'}\mathbb{D}_{A,rr'}+\mathbb{D}_{E,ij,i'j'}\mathbb{D}_{B,ss'}\\
&+4\mathbb{B}_{A,ii',rr'}\mathbb{B}_{B,jj',ss'}+4\mathbb{B}_{A,ii',rr'}\mathbb{B}_{E,iji'j',rsr's'}+4\mathbb{B}_{B,jj',ss'}\mathbb{B}_{E,iji'j',rsr's'}.
\end{aligned}\tag{42}$$

The other 24 terms are zero.

9.3.1. Variance parts

Here we collect expressions for the quantities appearing in the generic term of our squared U-statistics.

Lemma 9.2. *In the random effects model* (34)*,*

$$\begin{aligned}\mathbb{B}_{A,ii',rr'} &= \sigma_A^2\big(1_{i=r} - 1_{i=r'} - 1_{i'=r} + 1_{i'=r'}\big),\\ \mathbb{B}_{B,jj',ss'} &= \sigma_B^2\big(1_{j=s} - 1_{j=s'} - 1_{j'=s} + 1_{j'=s'}\big), \quad \textit{and}\\ \mathbb{B}_{E,iji'j',rsr's'} &= \sigma_E^2\big(1_{ij=rs} - 1_{ij=r's'} - 1_{i'j'=rs} + 1_{i'j'=r's'}\big).\end{aligned}$$

Proof. The first one follows by expanding and using $\mathbb{E}(a_i a_r) = \sigma_A^2 1_{i=r}$, et cetera. The other two use the same argument. □

Lemma 9.3. *In the random effects model* (34)*,*

$$\begin{aligned}\mathbb{D}_{A,ii'} &= 2\sigma_A^2(1 - 1_{i=i'}),\\ \mathbb{D}_{B,jj'} &= 2\sigma_B^2(1 - 1_{j=j'}), \quad \textit{and}\\ \mathbb{D}_{E,ij,i'j'} &= 2\sigma_E^2(1 - 1_{ij=i'j'}).\end{aligned}$$

Proof. Take $i = r$ and $i' = r'$ in Lemma 9.2. □

Lemma 9.4. *In the random effects model* (34)*,*

$$\begin{aligned}\mathbb{Q}_{A,ii',rr'} &= 1_{i\neq i'}1_{r\neq r'}\sigma_A^4\Big(4 + (\kappa_A+2)(1_{i\in\{r,r'\}} + 1_{i'\in\{r,r'\}})\\ &\qquad + 4\times 1_{\{i,i'\}=\{r,r'\}}\Big),\\ \mathbb{Q}_{B,jj',ss'} &= 1_{j\neq j'}1_{s\neq s'}\sigma_B^4\Big(4 + (\kappa_B+2)(1_{j\in\{s,s'\}} + 1_{j'\in\{s,s'\}})\\ &\qquad + 4\times 1_{\{j,j'\}=\{s,s'\}}\Big), \quad \textit{and}\\ \mathbb{Q}_{E,iji'j',rsr's'} &= 1_{ij\neq i'j'}1_{rs\neq r's'}\sigma_E^4\Big(4 + (\kappa_E+2)(1_{ij\in\{rs,r's'\}} + 1_{i'j'\in\{rs,r's'\}})\\ &\qquad + 4\times 1_{\{ij,i'j'\}=\{rs,r's'\}}\Big).\end{aligned}$$

Proof. We prove the first one; the others are similar. This quantity is 0 if $i = i'$ or $r = r'$. When $i \neq i'$ and $r \neq r'$, there are 3 cases to consider: $|\{i,i'\}\cap\{r,r'\}| = 0$, $|\{i,i'\}\cap\{r,r'\}| = 1$ and $|\{i,i'\}\cap\{r,r'\}| = 2$. The kurtosis is defined via $\kappa_A = \mathbb{E}(a^4)/\sigma_A^4 - 3$, so $\mathbb{E}(a^4) = (\kappa_A+3)\sigma_A^4$.

For no overlap, we find

$$\mathbb{E}((a_1-a_2)^2(a_3-a_4)^2) = \mathbb{E}((a_1-a_2)^2)^2 = 4\sigma_A^4.$$

For a single overlap,

$$\begin{aligned}\mathbb{E}((a_1-a_2)^2(a_1-a_3)^2) &= \mathbb{E}((a_1^2 - 2a_1a_2 + a_2^2)(a_1^2 - 2a_1a_3 + a_3^2))\\ &= \mathbb{E}(a_1^4) + 3\sigma_A^4 = \sigma_A^4(\kappa_A + 6).\end{aligned}$$

For a double overlap,

$$\begin{aligned}\mathbb{E}((a_1-a_2)^4) &= \mathbb{E}(a_1^4 - 4a_1a_2^3 + 6a_1^2a_2^2 - 4a_1^3a_2 + a_2^4)\\ &= 2\mathbb{E}(a_1^4) + 6\sigma_A^4 = \sigma_A^4(2\kappa_A + 12).\end{aligned}$$

As a result,

$$\mathbb{E}((a_i-a_{i'})^2(a_r-a_{r'})^2) = \begin{cases} 4\sigma_A^4, & |\{i,i'\}\cap\{r,r'\}| = 0,\\ \sigma_A^4(\kappa_A+6), & |\{i,i'\}\cap\{r,r'\}| = 1,\\ \sigma_A^4(2\kappa_A+12), & |\{i,i'\}\cap\{r,r'\}| = 2,\end{cases}$$

and so $\mathbb{E}((a_i-a_{i'})^2(a_r-a_{r'})^2)$ equals

$$1_{i\neq i'}1_{r\neq r'}\sigma_A^4\Big(4 + (\kappa_A+2)(1_{i\in\{r,r'\}} + 1_{i'\in\{r,r'\}}) + 4\times 1_{\{i,i'\}=\{r,r'\}}\Big). \qquad \square$$

9.4. Variance of U_a

We will work out $\mathbb{E}(U_a^2)$ and then subtract $\mathbb{E}(U_a)^2$. First we write

$$U_a^2 = \frac{1}{4}\sum_{ijj'}\sum_{rss'} N_{i\bullet}^{-1}N_{r\bullet}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{rs'}(Y_{ij}-Y_{ij'})^2(Y_{rs}-Y_{rs'})^2.$$

For $\mathbb{E}(U_a^2)$ we use the special case $i=i'$ and $r=r'$ of (42),

$$\begin{aligned}\mathbb{E}(U_a^2) = \frac{1}{4}\sum_{ijj'}\sum_{rss'} N_{i\bullet}^{-1}N_{r\bullet}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{rs'}\Big[&\\ \mathbb{Q}_{A,ii,rr} + \mathbb{Q}_{B,jj',ss'} + \mathbb{Q}_{E,ijij',rsrs'} + \mathbb{D}_{A,ii}\mathbb{D}_{B,ss'} + \mathbb{D}_{A,ii}\mathbb{D}_{E,rs,rs'}&\\ + \mathbb{D}_{B,jj'}\mathbb{D}_{A,rr} + \mathbb{D}_{B,jj'}\mathbb{D}_{E,rs,rs'} + \mathbb{D}_{E,ij,ij'}\mathbb{D}_{A,rr} + \mathbb{D}_{E,ij,ij'}\mathbb{D}_{B,ss'}&\\ + 4\mathbb{B}_{A,ii,rr}\mathbb{B}_{B,jj',ss'} + 4\mathbb{B}_{A,ii,rr}\mathbb{B}_{E,ijij',rsrs'} + 4\mathbb{B}_{B,jj',ss'}\mathbb{B}_{E,ijij',rsrs'}\Big]&\\ = \frac{1}{4}\sum_{ijj'}\sum_{rss'} N_{i\bullet}^{-1}N_{r\bullet}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{rs'}\Big[\underbrace{\mathbb{Q}_{B,jj',ss'}}_{\text{Term 1}} + \underbrace{\mathbb{Q}_{E,ijij',rsrs'}}_{\text{Term 2}}&\\ + \underbrace{\mathbb{D}_{B,jj'}\mathbb{D}_{E,rs,rs'}}_{\text{Term 3}} + \underbrace{\mathbb{D}_{E,ij,ij'}\mathbb{D}_{B,ss'}}_{\text{Term 4}} + \underbrace{4\mathbb{B}_{B,jj',ss'}\mathbb{B}_{E,ijij',rsrs'}}_{\text{Term 5}}\Big]&\end{aligned}$$

after eliminating terms that are always 0. We handle these five sums in the next paragraphs.

Term 1

$$\begin{aligned}&\frac{1}{4}\sum_{ijj'}\sum_{rss'} N_{i\bullet}^{-1}N_{r\bullet}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{rs'}\mathbb{Q}_{B,jj',ss'}\\ &= \frac{\sigma_B^4}{4}\sum_{ijj'}\sum_{rss'} N_{i\bullet}^{-1}N_{r\bullet}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{rs'}(1-1_{j=j'})(1-1_{s=s'})\end{aligned}$$

$$\Big(4+(\kappa_B+2)(1_{j\in\{s,s'\}}+1_{j'\in\{s,s'\}})+4\times 1_{\{j,j'\}=\{s,s'\}}\Big)$$
$$=\sigma_B^4\Big((N-R)^2+(\kappa_B+2)\sum_{ir}(ZZ^{\mathsf{T}})_{ir}(1-N_{i\bullet}^{-1})(1-N_{r\bullet}^{-1})$$
$$+2\sum_{ir}N_{i\bullet}^{-1}N_{r\bullet}^{-1}(ZZ^{\mathsf{T}})_{ir}((ZZ^{\mathsf{T}})_{ir}-1)\Big).$$

Term 2

$$\frac{1}{4}\sum_{ijj'}\sum_{rss'}N_{i\bullet}^{-1}N_{r\bullet}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{rs'}\mathbb{Q}_{E,ijij',rsrs'}$$
$$=\frac{\sigma_E^4}{4}\sum_{ijj'}\sum_{rss'}N_{i\bullet}^{-1}N_{r\bullet}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{rs'}1_{j\neq j'}1_{s\neq s'}$$
$$\times\Big(4+(\kappa_E+2)1_{i=r}(1_{j\in\{s,s'\}}+1_{j'\in\{s,s'\}})+41_{i=r}1_{\{j,j'\}=\{s,s'\}}\Big)$$
$$=\sigma_E^4\Big((N-R)^2+(\kappa_E+2)\sum_i N_{i\bullet}(1-N_{i\bullet}^{-1})^2+2\sum_i(1-N_{i\bullet}^{-1})\Big).$$

Terms 3 and 4 These terms are equal by symmetry. We evaluate term 3.

$$\frac{1}{4}\sum_{ijj'}\sum_{rss'}N_{i\bullet}^{-1}N_{r\bullet}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{rs'}\mathbb{D}_{B,jj'}\mathbb{D}_{E,rs,rs'}$$
$$=\frac{1}{4}\Big(\sum_{ijj'}N_{i\bullet}^{-1}Z_{ij}Z_{ij'}\mathbb{D}_{B,jj'}\Big)\Big(\sum_{rss'}N_{r\bullet}^{-1}Z_{rs}Z_{rs'}\mathbb{D}_{E,rs,rs'}\Big).$$

Now

$$\sum_{ijj'}N_{i\bullet}^{-1}Z_{ij}Z_{ij'}\mathbb{D}_{B,jj'}=2\sigma_B^2\sum_{ijj'}N_{i\bullet}^{-1}Z_{ij}Z_{ij'}(1-1_{j=j'})=2\sigma_B^2(N-R)$$

and

$$\sum_{rss'}N_{r\bullet}^{-1}Z_{rs}Z_{rs'}\mathbb{D}_{E,rs,rs'}=2\sigma_E^2\sum_{rss'}N_{r\bullet}^{-1}Z_{rs}Z_{rs'}(1-1_{s=s'})=2\sigma_E^2(N-R)$$

by the same steps. Therefore term 3 of $\mathbb{E}(U_a^2)$ equals $\sigma_B^2\sigma_E^2(N-R)^2$ and the sum of terms 3 and 4 is $2\sigma_B^2\sigma_E^2(N-R)^2$.

Term 5 The term equals

$$\sum_{ijj'}\sum_{rss'}N_{i\bullet}^{-1}N_{r\bullet}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{rs'}\mathbb{B}_{B,jj',ss'}\mathbb{B}_{E,ijij',rsrs'}$$
$$=\sum_{ijj'}\sum_{ss'}N_{i\bullet}^{-1}Z_{ij}Z_{ij'}\mathbb{B}_{B,jj',ss'}\sum_r N_{r\bullet}^{-1}Z_{rs}Z_{rs'}\mathbb{B}_{E,ijij',rsrs'}.$$

Now

$$\sum_r N_{r\bullet}^{-1} Z_{rs} Z_{rs'} \mathbb{B}_{E,ijij',rsrs'} = \sigma_E^2 N_{i\bullet}^{-1} Z_{is} Z_{is'} \big(1_{j=s} - 1_{j=s'} - 1_{j'=s} + 1_{j'=s'}\big).$$

Term 5 is then

$$\begin{aligned}
&\sigma_E^2 \sum_{ijj'} \sum_{ss'} N_{i\bullet}^{-2} Z_{ij} Z_{ij'} Z_{is} Z_{is'} \big(1_{j=s} - 1_{j=s'} - 1_{j'=s} + 1_{j'=s'}\big) \mathbb{B}_{B,jj',ss'} \\
&= \sigma_E^2 \sigma_B^2 \sum_{ijj'} \sum_{ss'} N_{i\bullet}^{-2} Z_{ij} Z_{ij'} Z_{is} Z_{is'} 1_{j=s} \big(1_{j=s} - 1_{j=s'} - 1_{j'=s} + 1_{j'=s'}\big) \\
&\quad - \sigma_E^2 \sigma_B^2 \sum_{ijj'} \sum_{ss'} N_{i\bullet}^{-2} Z_{ij} Z_{ij'} Z_{is} Z_{is'} 1_{j=s'} \big(1_{j=s} - 1_{j=s'} - 1_{j'=s} + 1_{j'=s'}\big) \\
&\quad - \sigma_E^2 \sigma_B^2 \sum_{ijj'} \sum_{ss'} N_{i\bullet}^{-2} Z_{ij} Z_{ij'} Z_{is} Z_{is'} 1_{j'=s} \big(1_{j=s} - 1_{j=s'} - 1_{j'=s} + 1_{j'=s'}\big) \\
&\quad + \sigma_E^2 \sigma_B^2 \sum_{ijj'} \sum_{ss'} N_{i\bullet}^{-2} Z_{ij} Z_{ij'} Z_{is} Z_{is'} 1_{j'=s'} \big(1_{j=s} - 1_{j=s'} - 1_{j'=s} + 1_{j'=s'}\big) \\
&= 4\sigma_B^2 \sigma_E^2 (N - R)
\end{aligned}$$

Combination Combining the results of the previous sections, we have

$$\begin{aligned}
\mathbb{E}(U_a^2) = \sigma_B^4 \Big(&(N-R)^2 + (\kappa_B + 2) \sum_{ir} (ZZ^{\mathsf{T}})_{ir} (1 - N_{i\bullet}^{-1})(1 - N_{r\bullet}^{-1}) \\
&+ 2 \sum_{ir} N_{i\bullet}^{-1} N_{r\bullet}^{-1} (ZZ^{\mathsf{T}})_{ir} ((ZZ^{\mathsf{T}})_{ir} - 1) \Big) \\
&+ 2\sigma_B^2 \sigma_E^2 (N-R)^2 + 4\sigma_B^2 \sigma_E^2 (N-R) \\
&+ \sigma_E^4 \Big((N-R)^2 + (\kappa_E + 2) \sum_i N_{i\bullet} (1 - N_{i\bullet}^{-1})^2 + 2 \sum_i (1 - N_{i\bullet}^{-1}) \Big).
\end{aligned}$$

Subtracting $\mathbb{E}(U_a)^2 = (N-R)^2(\sigma_B^2 + \sigma_E^2)^2$ we find that $\operatorname{Var}(U_a)$ equals

$$\begin{aligned}
&4\sigma_B^2 \sigma_E^2 (N-R) + \sigma_E^4 \Big((\kappa_E + 2) \sum_i N_{i\bullet} (1 - N_{i\bullet}^{-1})^2 + 2 \sum_i (1 - N_{i\bullet}^{-1}) \Big) \\
&+ \sigma_B^4 \Big((\kappa_B + 2) \sum_{ir} (ZZ^{\mathsf{T}})_{ir} (1 - N_{i\bullet}^{-1})(1 - N_{r\bullet}^{-1}) \\
&\qquad + 2 \sum_{ir} N_{i\bullet}^{-1} N_{r\bullet}^{-1} (ZZ^{\mathsf{T}})_{ir} ((ZZ^{\mathsf{T}})_{ir} - 1) \Big).
\end{aligned} \tag{43}$$

9.4.1. Variance of U_b

This case is exactly symmetric to the one above with $\operatorname{Var}(U_a)$ given by (43). Therefore $\operatorname{Var}(U_b)$ equals

$$4\sigma_A^2 \sigma_E^2 (N - C) + \sigma_E^4 \Big((\kappa_E + 2) \sum_j N_{\bullet j} (1 - N_{\bullet j}^{-1})^2 + 2 \sum_j (1 - N_{\bullet j}^{-1}) \Big)$$

$$+\sigma_B^4\Big((\kappa_A+2)\sum_{js}(Z^{\mathsf T}Z)_{js}(1-N_{\bullet j}^{-1})(1-N_{\bullet s}^{-1}) \tag{44}$$
$$+2\sum_{js}N_{\bullet j}^{-1}N_{\bullet s}^{-1}(Z^{\mathsf T}Z)_{js}((Z^{\mathsf T}Z)_{js}-1)\Big).$$

9.5. *Variance of U_e*

As before, we find $\mathbb{E}(U_e^2)$ and then subtract $\mathbb{E}(U_e)^2$. Now

$$U_e^2=\frac14\sum_{ii'jj'}\sum_{rr'ss'}Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}(Y_{ij}-Y_{i'j'})^2(Y_{rs}-Y_{r's'})^2.$$

From (42),

$$\begin{aligned}\mathbb{E}(U_e^2)=\frac14\sum_{ii'jj'}\sum_{rr'ss'}Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}\Big[&\underbrace{\mathbb{Q}_{A,ii',rr'}}_{\text{Term 1}}+\underbrace{\mathbb{Q}_{B,jj',ss'}}_{\text{Term 2}}+\underbrace{\mathbb{Q}_{E,iji'j',rsr's'}}_{\text{Term 3}}\\
&+\underbrace{\mathbb{D}_{A,ii'}\mathbb{D}_{B,ss'}}_{\text{Term 4}}+\underbrace{\mathbb{D}_{A,ii'}\mathbb{D}_{E,rs,r's'}}_{\text{Term 5}}+\underbrace{\mathbb{D}_{B,jj'}\mathbb{D}_{A,rr'}}_{\text{Term 6}}+\underbrace{\mathbb{D}_{B,jj'}\mathbb{D}_{E,rs,r's'}}_{\text{Term 7}}\\
&+\underbrace{\mathbb{D}_{E,ij,i'j'}\mathbb{D}_{A,rr'}}_{\text{Term 8}}+\underbrace{\mathbb{D}_{E,ij,i'j'}\mathbb{D}_{B,ss'}}_{\text{Term 9}}+\underbrace{4\mathbb{B}_{A,ii',rr'}\mathbb{B}_{B,jj',ss'}}_{\text{Term 10}}\\
&+\underbrace{4\mathbb{B}_{A,ii',rr'}\mathbb{B}_{E,iji'j',rsr's'}}_{\text{Term 11}}+\underbrace{4\mathbb{B}_{B,jj',ss'}\mathbb{B}_{E,iji'j',rsr's'}}_{\text{Term 12}}\Big].\end{aligned}$$

We handle the twelve sums in the next paragraphs.

Terms 1 and 2 Term 1 is

$$\begin{aligned}&\frac14\sum_{ii'jj'}\sum_{rr'ss'}Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}\mathbb{Q}_{A,ii',rr'}\\
&=\frac14\sum_{ii'jj'}\sum_{rr'ss'}Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}1_{i\neq i'}1_{r\neq r'}\sigma_A^4\\
&\quad\Big(4+(\kappa_A+2)(1_{i\in\{r,r'\}}+1_{i'\in\{r,r'\}})+4\times1_{\{i,i'\}=\{r,r'\}}\Big)\\
&=\frac{\sigma_A^4}{4}\sum_{ii'jj'}\sum_{rr'ss'}Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}(1-1_{i=i'})(1-1_{r=r'})\\
&\quad\Big(4+(\kappa_A+2)(1_{i\in\{r,r'\}}+1_{i'\in\{r,r'\}})+4\times1_{\{i,i'\}=\{r,r'\}}\Big)\\
&=\sigma_A^4\Big(N^4-2N^2\sum_i N_{i\bullet}^2+3\Big(\sum_i N_{i\bullet}^2\Big)^2-2\sum_i N_{i\bullet}^4\Big)\\
&\quad+\sigma_A^4(\kappa_A+2)\Big(N^2\sum_i N_{i\bullet}^2-2N\sum_i N_{i\bullet}^3+\sum_i N_{i\bullet}^4\Big).\end{aligned}$$

We can use the symmetry of the roles of A and B and their indices. Therefore, term 2 is equal to

$$\sigma_B^4\Big(N^4 - 2N^2\sum_j N_{\bullet j}^2 + 3\Big(\sum_j N_{\bullet j}^2\Big)^2 - 2\sum_j N_{\bullet j}^4\Big)$$
$$+ \sigma_B^4(\kappa_B+2)\Big(N^2\sum_j N_{\bullet j}^2 - 2N\sum_j N_{\bullet j}^3 + \sum_j N_{\bullet j}^4\Big).$$

Term 3

$$\frac{1}{4}\sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}\mathbb{Q}_{E,iji'j',rsr's'}$$
$$= \frac{\sigma_E^4}{4}\sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}(1-1_{ij=i'j'})(1-1_{rs=r's'})$$
$$\Big(4+(\kappa_E+2)(1_{ij\in\{rs,r's'\}}+1_{i'j'\in\{rs,r's'\}}) + 4\times 1_{\{ij,i'j'\}=\{rs,r's'\}}\Big)$$
$$= \sigma_E^4 N(N-1)[N(N-1)+2] + \sigma_E^4(\kappa_E+2)N(N-1)^2.$$

Term 4

$$\frac{1}{4}\sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}\mathbb{D}_{A,ii'}\mathbb{D}_{B,ss'}$$
$$= \frac{1}{4}\Big(\sum_{ii'jj'} Z_{ij}Z_{i'j'}\mathbb{D}_{A,ii'}\Big)\Big(\sum_{rr'ss'} Z_{rs}Z_{r's'}\mathbb{D}_{B,ss'}\Big).$$

The first factor is

$$\sum_{ii'jj'} Z_{ij}Z_{i'j'}\mathbb{D}_{A,ii'} = 2\sigma_A^2\sum_{ii'jj'} Z_{ij}Z_{i'j'}(1-1_{i=i'}) = 2\sigma_A^2(N^2 - \sum_i N_{i\bullet}^2).$$

By the same argument, the second factor is

$$\sum_{rr'ss'} Z_{rs}Z_{r's'}\mathbb{D}_{B,ss'} = 2\sigma_B^2(N^2 - \sum_s N_{\bullet s}^2),$$

and so term 4 is

$$\sigma_A^2\sigma_B^2(N^2 - \sum_i N_{i\bullet}^2)(N^2 - \sum_j N_{\bullet j}^2).$$

Term 5

$$\frac{1}{4}\sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}\mathbb{D}_{A,ii'}\mathbb{D}_{E,rs,r's'}$$
$$= \frac{1}{4}\Big(\sum_{ii'jj'} Z_{ij}Z_{i'j'}\mathbb{D}_{A,ii'}\Big)\Big(\sum_{rr'ss'} Z_{rs}Z_{r's'}\mathbb{D}_{E,rs,r's'}\Big).$$

The first factor is computed in the previous section. The second factor is

$$\sum_{rr'ss'} Z_{rs}Z_{r's'}\mathbb{D}_{E,rs,r's'} = 2\sigma_E^2\sum_{rr'ss'} Z_{rs}Z_{r's'}(1-1_{rs=r's'}) = 2\sigma_E^2 N(N-1).$$

Thus, term 5 is

$$\sigma_A^2\sigma_E^2 N(N-1)(N^2-\sum_i N_{i\bullet}^2).$$

Terms 6–9 By symmetry of indices, term 6 is the same as term 4:

$$\sigma_A^2\sigma_B^2(N^2-\sum_i N_{i\bullet}^2)(N^2-\sum_j N_{\bullet j}^2).$$

Term 7 is like term 5 with factors A and B interchanged. Thus, term 7 is equal to

$$\sigma_B^2\sigma_E^2 N(N-1)(N^2-\sum_j N_{\bullet j}^2).$$

By symmetry of indices, term 8 is the same as term 5:

$$\sigma_A^2\sigma_E^2 N(N-1)(N^2-\sum_i N_{i\bullet}^2).$$

By symmetry of indices, term 9 is the same as term 7:

$$\sigma_B^2\sigma_E^2 N(N-1)(N^2-\sum_j N_{\bullet j}^2).$$

Term 10

$$\begin{aligned}
&\sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}\mathbb{B}_{A,ii',rr'}\mathbb{B}_{B,jj',ss'}\\
&=\sigma_A^2\sigma_B^2\sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}\big(1_{i=r}-1_{i=r'}-1_{i'=r}+1_{i'=r'}\big)\\
&\qquad\qquad\big(1_{j=s}-1_{j=s'}-1_{j'=s}+1_{j'=s'}\big)\\
&=4\sigma_A^2\sigma_B^2\Big(N^3-2N\sum_{ij}Z_{ij}N_{i\bullet}N_{\bullet j}+\sum_{ij}N_{i\bullet}^2N_{\bullet j}^2\Big).
\end{aligned}$$

Terms 11 and 12

$$\begin{aligned}
&\sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}\mathbb{B}_{A,ii',rr'}\mathbb{B}_{E,iji'j',rsr's'}\\
&=\sigma_A^2\sigma_E^2\sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}\big(1_{i=r}-1_{i=r'}-1_{i'=r}+1_{i'=r'}\big)\\
&\qquad\qquad\big(1_{ij=rs}-1_{ij=r's'}-1_{i'j'=rs}+1_{i'j'=r's'}\big)\\
&=\sigma_A^2\sigma_E^2\big(4N^3-4N\sum_i N_{i\bullet}^2\big).
\end{aligned}$$

For term 12, we can use the symmetry with term 11, interchanging rows columns. Thus, term 12 is

$$\sigma_B^2\sigma_E^2\big(4N^3-4N\sum_j N_{\bullet j}^2\big).$$

Combination Summing up the results of the previous twelve sections, we have that $\mathbb{E}(U_e^2)$ equals

$$\begin{aligned}
&\sigma_A^4 N^4 - 2\sigma_A^4 N^2 \sum_i N_{i\bullet}^2 + 3\sigma_A^4 \Big(\sum_i N_{i\bullet}^2\Big)^2 - 2\sigma_A^4 \sum_i N_{i\bullet}^4 + \sigma_B^4 N^4 - 2\sigma_B^4 \sum_j N_{\bullet j}^4 \\
&+ \sigma_A^4(\kappa_A+2)\Big(N^2\sum_i N_{i\bullet}^2 - 2N\sum_i N_{i\bullet}^3 + \sum_i N_{i\bullet}^4\Big) - 2\sigma_B^4 N^2 \sum_j N_{\bullet j}^2 \\
&+ 3\sigma_B^4\Big(\sum_j N_{\bullet j}^2\Big)^2 + \sigma_B^4(\kappa_B+2)\Big(N^2\sum_j N_{\bullet j}^2 - 2N\sum_j N_{\bullet j}^3 + \sum_j N_{\bullet j}^4\Big) \\
&+ \sigma_E^4\Big(N^4 - 2N^3 + 3N^2 - 2N\Big) + \sigma_E^4(\kappa_E+2)N(N-1)^2 \\
&+ \sigma_A^2\sigma_B^2(N^2 - \sum_i N_{i\bullet}^2)(N^2 - \sum_j N_{\bullet j}^2) + \sigma_A^2\sigma_E^2 N(N-1)(N^2 - \sum_i N_{i\bullet}^2) \\
&+ \sigma_A^2\sigma_B^2(N^2 - \sum_i N_{i\bullet}^2)(N^2 - \sum_j N_{\bullet j}^2) + \sigma_B^2\sigma_E^2 N(N-1)(N^2 - \sum_j N_{\bullet j}^2) \\
&+ \sigma_A^2\sigma_E^2 N(N-1)(N^2 - \sum_i N_{i\bullet}^2) + \sigma_B^2\sigma_E^2 N(N-1)(N^2 - \sum_j N_{\bullet j}^2) \\
&+ 4\sigma_A^2\sigma_B^2\Big(N^3 - 2N\sum_{ij} Z_{ij}N_{i\bullet}N_{\bullet j} + \sum_{ij} N_{i\bullet}^2 N_{\bullet j}^2\Big) + 4\sigma_A^2\sigma_E^2\big(N^3 - N\sum_i N_{i\bullet}^2\big) \\
&+ \sigma_B^2\sigma_E^2\big(4N^3 - 4N\sum_j N_{\bullet j}^2\big).
\end{aligned}$$

Then, we have, applying some simplifications,

$$\begin{aligned}
&\mathrm{Var}(U_e) = \mathbb{E}(U_e^2) - \mathbb{E}(U_e)^2 \\
&= 2\sigma_A^4\Big(\Big(\sum_i N_{i\bullet}^2\Big)^2 - \sum_i N_{i\bullet}^4\Big) + 2\sigma_B^4\Big(\Big(\sum_j N_{\bullet j}^2\Big)^2 - \sum_j N_{\bullet j}^4\Big) + 2\sigma_E^4 N(N-1) \\
&\quad + (\kappa_A+2)\sigma_A^4\sum_i N_{i\bullet}^2(N-N_{i\bullet})^2 + (\kappa_B+2)\sigma_B^4\sum_j N_{\bullet j}^2(N-N_{\bullet j})^2 \\
&\quad + (\kappa_E+2)\sigma_E^4 N(N-1)^2 + 4\sigma_A^2\sigma_B^2\sum_{ij}(N_{i\bullet}N_{\bullet j} - NZ_{ij})^2 \\
&\quad + 4\sigma_A^2\sigma_E^2 N\big(N^2 - \sum_i N_{i\bullet}^2\big) + 4\sigma_B^2\sigma_E^2 N\big(N^2 - \sum_j N_{\bullet j}^2\big). \qquad (45)
\end{aligned}$$

9.6. Covariance of U_a and U_b

We use the formula $\mathrm{Cov}(U_a, U_b) = \mathbb{E}(U_aU_b) - \mathbb{E}(U_a)\mathbb{E}(U_b)$, so we just need to compute $\mathbb{E}(U_aU_b)$. Using our preferred normalization,

$$U_aU_b = \frac{1}{4}\big(\sum_{ijj'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}(Y_{ij} - Y_{ij'})^2\big)\big(\sum_{rr's} N_{\bullet s}^{-1} Z_{rs}Z_{r's}(Y_{rs} - Y_{r's})^2\big)$$

$$= \frac{1}{4}\sum_{ijj'}\sum_{rr's} N_{i\bullet}^{-1}N_{\bullet s}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{r's}(Y_{ij}-Y_{ij'})^2(Y_{rs}-Y_{r's})^2.$$

Then,

$$\mathbb{E}(U_aU_b) = \frac{1}{4}\sum_{ijj'}\sum_{rr's} N_{i\bullet}^{-1}N_{\bullet s}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{r's}\big(\underbrace{\mathbb{Q}_{E,ijij',rsr's}}_{\text{Term 1}} + \underbrace{\mathbb{D}_{B,jj'}\mathbb{D}_{A,rr'}}_{\text{Term 2}} + \underbrace{\mathbb{D}_{B,jj'}\mathbb{D}_{E,rs,r's}}_{\text{Term 3}} + \underbrace{\mathbb{D}_{E,ij,ij'}\mathbb{D}_{A,rr'}}_{\text{Term 4}}\big).$$

We consider each term separately.

Term 1

$$\begin{aligned}
&\frac{1}{4}\sum_{ijj'}\sum_{rr's} N_{i\bullet}^{-1}N_{\bullet s}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{r's}\mathbb{Q}_{E,ijij',rsr's}\\
&= \frac{\sigma_E^4}{4}\sum_{ijj'}\sum_{rr's} N_{i\bullet}^{-1}N_{\bullet s}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{r's}1_{j\neq j'}1_{r\neq r'}\\
&\quad\Big(4+(\kappa_E+2)(1_{ij\in\{rs,r's\}}+1_{ij'\in\{rs,r's\}})+4\times 1_{\{ij,ij'\}=\{rs,r's\}}\Big)\\
&= \sigma_E^4\big(N-R\big)\big(N-C\big)+\sigma_E^4(\kappa_E+2)\sum_{ij}Z_{ij}(1-N_{i\bullet}^{-1})(1-N_{\bullet j}^{-1}).
\end{aligned}$$

Term 2

$$\begin{aligned}
&\frac{1}{4}\sum_{ijj'}\sum_{rr's} N_{i\bullet}^{-1}N_{\bullet s}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{r's}\mathbb{D}_{B,jj'}\mathbb{D}_{A,rr'}\\
&= \frac{1}{4}\sum_{ijj'}\sum_{rr's} N_{i\bullet}^{-1}N_{\bullet s}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{r's}2\sigma_B^2(1-1_{j=j'})2\sigma_A^2(1-1_{r=r'})\\
&= \sigma_A^2\sigma_B^2\big(N-R\big)\big(N-C\big).
\end{aligned}$$

Term 3

$$\begin{aligned}
&\frac{1}{4}\sum_{ijj'}\sum_{rr's} N_{i\bullet}^{-1}N_{\bullet s}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{r's}\mathbb{D}_{B,jj'}\mathbb{D}_{E,rs,r's}\\
&= \frac{1}{4}\sum_{ijj'}\sum_{rr's} N_{i\bullet}^{-1}N_{\bullet s}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{r's}2\sigma_B^2(1-1_{j=j'})2\sigma_E^2(1-1_{r=r'})\\
&= \sigma_B^2\sigma_E^2\big(N-R\big)\big(N-C\big).
\end{aligned}$$

Term 4

$$\frac{1}{4}\sum_{ijj'}\sum_{rr's} N_{i\bullet}^{-1}N_{\bullet s}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{r's}\mathbb{D}_{E,ij,ij'}\mathbb{D}_{A,rr'}$$

$$= \frac{1}{4}\sum_{ijj'}\sum_{rr's} N_{i\bullet}^{-1}N_{\bullet s}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{r's}2\sigma_E^2(1-1_{j=j'})2\sigma_A^2(1-1_{r=r'})$$
$$= \sigma_A^2\sigma_E^2\big(N-R\big)\big(N-C\big).$$

Combination Adding up the four terms, we have

$$\begin{aligned}\mathbb{E}(U_aU_b) &= \sigma_E^4\big(N-R\big)\big(N-C\big) + \sigma_E^4(\kappa_E+2)\sum_{ij}Z_{ij}(1-N_{i\bullet}^{-1})(1-N_{\bullet j}^{-1})\\ &\quad+\sigma_A^2\sigma_B^2\big(N-R\big)\big(N-C\big)+\sigma_B^2\sigma_E^2\big(N-R\big)\big(N-C\big)\\ &\quad+\sigma_A^2\sigma_E^2\big(N-R\big)\big(N-C\big),\end{aligned}$$

and so

$$\begin{aligned}\mathrm{Cov}(U_a,U_b) &= \mathbb{E}(U_aU_b)-\mathbb{E}(U_a)\mathbb{E}(U_b)\\ &= \mathbb{E}(U_aU_b)-(\sigma_B^2+\sigma_E^2)(\sigma_A^2+\sigma_E^2)(N-R)(N-C)\\ &= \sigma_E^4(\kappa_E+2)\sum_{ij}Z_{ij}(1-N_{i\bullet}^{-1})(1-N_{\bullet j}^{-1}).\end{aligned}$$

Notice that $\mathrm{Cov}(U_a,U_b)=0$ when $\sigma_E^2=0$. This can be verified by noting that when $\sigma_E^2=0$ then U_a is a function only of a_i while U_b is a function only of b_j. Therefore U_a and U_b are independent when $\sigma_E^2=0$.

9.7. Covariance of U_a and U_e

We use the formula $\mathrm{Cov}(U_a,U_e)=\mathbb{E}(U_aU_e)-\mathbb{E}(U_a)\mathbb{E}(U_e)$, so we just need to compute $\mathbb{E}(U_aU_e)$. First,

$$\begin{aligned}U_aU_e &= \frac{1}{4}\Big(\sum_{ijj'}N_{i\bullet}^{-1}Z_{ij}Z_{ij'}(Y_{ij}-Y_{ij'})^2\Big)\Big(\sum_{rr'ss'}Z_{rs}Z_{r's'}(Y_{rs}-Y_{r's'})^2\Big)\\ &= \frac{1}{4}\sum_{ijj'}\sum_{rr'ss'}N_{i\bullet}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{r's'}(Y_{ij}-Y_{ij'})^2(Y_{rs}-Y_{r's'})^2.\end{aligned}$$

Then,

$$\begin{aligned}\mathbb{E}(U_aU_e) = \frac{1}{4}\sum_{ijj'}\sum_{rr'ss'}N_{i\bullet}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{r's'}\big(&\underbrace{\mathbb{Q}_{B,jj',ss'}}_{\text{Term 1}}+\underbrace{\mathbb{Q}_{E,ijij',rsr's'}}_{\text{Term 2}}\\ &+\underbrace{\mathbb{D}_{B,jj'}\mathbb{D}_{A,rr'}}_{\text{Term 3}}+\underbrace{\mathbb{D}_{B,jj'}\mathbb{D}_{E,rs,r's'}}_{\text{Term 4}}+\underbrace{\mathbb{D}_{E,ij,ij'}\mathbb{D}_{A,rr'}}_{\text{Term 5}}+\underbrace{\mathbb{D}_{E,ij,ij'}\mathbb{D}_{B,ss'}}_{\text{Term 6}}\\ &+\underbrace{4\mathbb{B}_{B,jj',ss'}\mathbb{B}_{E,ijij',rsr's'}}_{\text{Term 7}}\big).\end{aligned}$$

We consider each term separately.

Term 1

$$\begin{aligned}
&\frac{1}{4}\sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{rs}Z_{r's'}\mathbb{Q}_{B,jj',ss'}\\
&= \frac{1}{4}\sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{rs}Z_{r's'}1_{j\neq j'}1_{s\neq s'}\sigma_B^4\\
&\quad\Big(4+(\kappa_B+2)(1_{j\in\{s,s'\}}+1_{j'\in\{s,s'\}})+4\times 1_{\{j,j'\}=\{s,s'\}}\Big)\\
&= 2\sigma_B^4\Big(\sum_i N_{i\bullet}^{-1}\Big(\sum_j Z_{ij}N_{\bullet j}\Big)^2-\sum_{ij}N_{i\bullet}^{-1}Z_{ij}N_{\bullet j}^2\Big)\\
&\quad+\sigma_B^4\big(N-R\big)\big(N^2-\sum_j N_{\bullet j}^2\big)+\sigma_B^4(\kappa_B+2)\sum_{ij}Z_{ij}(N-N_{\bullet j})N_{\bullet j}(1-N_{i\bullet}^{-1})
\end{aligned}$$

Term 2

$$\begin{aligned}
&\frac{1}{4}\sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{rs}Z_{r's'}\mathbb{Q}_{E,ijij',rsr's'}\\
&= \frac{1}{4}\sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{rs}Z_{r's'}1_{j\neq j'}1_{rs\neq r's'}\sigma_E^4\\
&\quad\Big(4+(\kappa_E+2)(1_{ij\in\{rs,r's'\}}+1_{ij'\in\{rs,r's'\}})+4\times 1_{\{ij,ij'\}=\{rs,r's'\}}\Big)\\
&= \sigma_E^4 N(N-1)(N-R)+2\sigma_E^4(N-R)+\sigma_E^4(\kappa_E+2)(N-R)(N-1)
\end{aligned}$$

Term 3

$$\begin{aligned}
&\frac{1}{4}\sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{rs}Z_{r's'}\mathbb{D}_{B,jj'}\mathbb{D}_{A,rr'}\\
&= \frac{1}{4}\sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{rs}Z_{r's'}2\sigma_B^2(1-1_{j=j'})2\sigma_A^2(1-1_{r=r'})\\
&= \sigma_A^2\sigma_B^2\big(N-R\big)\big(N^2-\sum_r N_{r\bullet}^2\big)
\end{aligned}$$

Term 4

$$\begin{aligned}
&\frac{1}{4}\sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{rs}Z_{r's'}\mathbb{D}_{B,jj'}\mathbb{D}_{E,rs,r's'}\\
&= \frac{1}{4}\sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{rs}Z_{r's'}2\sigma_B^2(1-1_{j=j'})2\sigma_E^2(1-1_{r=r'}1_{s=s'})\\
&= \sigma_B^2\sigma_E^2\big(N-R\big)\big(N^2-N\big)
\end{aligned}$$

Term 5

$$\frac{1}{4}\sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{rs}Z_{r's'}\mathbb{D}_{E,ij,ij'}\mathbb{D}_{A,rr'}$$

$$= \frac{1}{4}\sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{rs}Z_{r's'} 2\sigma_E^2(1-1_{j=j'})2\sigma_A^2(1-1_{r=r'})$$
$$= \sigma_A^2\sigma_E^2\big(N-R\big)\big(N^2-\sum_r N_{r\bullet}^2\big)$$

using the result for term 3.

Term 6

$$\frac{1}{4}\sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{rs}Z_{r's'}\mathbb{D}_{E,ij,ij'}\mathbb{D}_{B,ss'}$$
$$= \frac{1}{4}\sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{rs}Z_{r's'} 2\sigma_E^2(1-1_{j=j'})2\sigma_B^2(1-1_{s=s'})$$
$$= \sigma_B^2\sigma_E^2\big(N-R\big)\big(N^2-\sum_s N_{\bullet s}^2\big)$$

Term 7

$$\sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{rs}Z_{r's'}\mathbb{B}_{B,jj',ss'}\mathbb{B}_{E,ijij',rsr's'}$$
$$= \sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{rs}Z_{r's'}\sigma_B^2\big(1_{j=s}-1_{j=s'}-1_{j'=s}+1_{j'=s'}\big)$$
$$\times\,\sigma_E^2\big(1_{ij=rs}-1_{ij=r's'}-1_{ij'=rs}+1_{ij'=r's'}\big)$$
$$= 4\sigma_B^2\sigma_E^2 N(N-R)$$

Combination We add up the seven terms, replacing some $N_{r\bullet}$ and $N_{\bullet s}$ expressions by equivalents using $N_{i\bullet}$ and $N_{\bullet j}$, getting

$$\begin{aligned}\mathbb{E}(U_aU_e) = {}& \sigma_B^4\big(N-R\big)\big(N^2-\sum_j N_{\bullet j}^2\big)\\
&+ 2\sigma_B^4\Big(\sum_i N_{i\bullet}^{-1}\Big(\sum_j Z_{ij}N_{\bullet j}\Big)^2-\sum_{ij} N_{i\bullet}^{-1}Z_{ij}N_{\bullet j}^2\Big)\\
&+ \sigma_B^4(\kappa_B+2)\sum_{ij} Z_{ij}(N-N_{\bullet j})N_{\bullet j}(1-N_{i\bullet}^{-1}) + 2\sigma_E^4(N-R)\\
&+ \sigma_E^4 N(N-1)(N-R) + \sigma_E^4(\kappa_E+2)(N-R)(N-1)\\
&+ \sigma_A^2\sigma_B^2\big(N-R\big)\big(N^2-\sum_i N_{i\bullet}^2\big) + \sigma_B^2\sigma_E^2\big(N-R\big)\big(N^2-N\big)\\
&+ \sigma_A^2\sigma_E^2\big(N-R\big)\big(N^2-\sum_i N_{i\bullet}^2\big) + \sigma_B^2\sigma_E^2\big(N-R\big)\big(N^2-\sum_j N_{\bullet j}^2\big)\\
&+ 4\sigma_B^2\sigma_E^2 N(N-R).\end{aligned}$$

Now $\mathbb{E}(U_a)\mathbb{E}(U_e)$ equals

$$(N-R)(\sigma_B^2+\sigma_E^2)\Big(\sigma_A^2(N^2-\sum_i N_{i\bullet}^2)+\sigma_B^2(N^2-\sum_j N_{\bullet j}^2)+\sigma_E^2(N^2-N)\Big)$$

which contains terms equalling several of those in $\mathbb{E}(U_aU_e)$ above. Subtracting those term from $\mathbb{E}(U_aU_e)$ yields

$$\begin{aligned}\operatorname{Cov}(U_a,U_e) &= 2\sigma_B^4\Big(\sum_i N_{i\bullet}^{-1}\Big(\sum_j Z_{ij}N_{\bullet j}\Big)^2 - \sum_{ij} N_{i\bullet}^{-1}Z_{ij}N_{\bullet j}^2\Big)\\ &\quad+\sigma_B^4(\kappa_B+2)\sum_{ij}Z_{ij}(N-N_{\bullet j})N_{\bullet j}(1-N_{i\bullet}^{-1}) + 2\sigma_E^4(N-R)\\ &\quad+\sigma_E^4(\kappa_E+2)(N-R)(N-1) + 4\sigma_B^2\sigma_E^2N(N-R).\end{aligned}$$

9.7.1. Covariance of U_b and U_e

By interchanging the roles of the rows and columns in $\operatorname{Cov}(U_a,U_e)$, we find that

$$\begin{aligned}\operatorname{Cov}(U_b,U_e) &= 2\sigma_A^4\Big(\sum_j N_{\bullet j}^{-1}\Big(\sum_i Z_{ij}N_{i\bullet}\Big)^2 - \sum_{ij} N_{\bullet j}^{-1}Z_{ij}N_{i\bullet}^2\Big)\\ &\quad+\sigma_A^4(\kappa_A+2)\sum_{ij}Z_{ij}(N-N_{i\bullet})N_{i\bullet}(1-N_{\bullet j}^{-1})\\ &\quad+2\sigma_E^4(N-C)+\sigma_E^4(\kappa_E+2)(N-C)(N-1)\\ &\quad+4\sigma_A^2\sigma_E^2N(N-C).\end{aligned}$$

9.8. Asymptotic approximation: Proof of Theorem 4.2

We suppose that the following inequalities all hold

$$\begin{aligned}&N_{i\bullet}\leqslant\delta N, \quad N_{\bullet j}\leqslant\delta N, \quad R\leqslant\delta N, \quad C\leqslant\delta N,\\ &N\leqslant\delta\sum_i N_{i\bullet}^2, \quad N\leqslant\delta\sum_j N_{\bullet j}^2, \quad \sum_i N_{i\bullet}^2\leqslant\delta N^2, \quad\text{and}\quad \sum_j N_{\bullet j}^2\leqslant\delta N^2\end{aligned}$$

for the same small $\delta>0$. The first six inequalities are assumed in the theorem statement. The last two follow from the first two. We also assume that

$$0<\kappa_A+2,\kappa_B+2,\kappa_E+2,\sigma_A^4,\sigma_B^4,\sigma_E^4<\infty.$$

Note that we can bound $\sigma_A^2\sigma_B^2$, $\sigma_A^2\sigma_E^2$, and $\sigma_A^2\sigma_B^2$ away from 0 and ∞ uniformly with those other quantities.

We also suppose that

$$\sum_{ij}Z_{ij}N_{i\bullet}^{-1}N_{\bullet j}\leqslant\delta\sum_i N_{i\bullet}^2, \quad\text{and}\quad \sum_{ij}Z_{ij}N_{i\bullet}N_{\bullet j}^{-1}\leqslant\delta\sum_j N_{\bullet j}^2. \tag{46}$$

The bounds in (46) seem reasonable but it appears that they cannot be derived from the first eight bounds above.

We begin with the coefficient of $\sigma_B^4(\kappa_B+2)$ in $\operatorname{Var}(U_a)$ from equation (12). It is

$$\sum_{ir}(ZZ^{\mathsf T})_{ir}(1-N_{i\bullet}^{-1}-N_{r\bullet}^{-1}+N_{i\bullet}^{-1}N_{r\bullet}^{-1})$$

$$= \sum_j N_{\bullet j}^2 - 2\sum_{ij} Z_{ij} N_{i\bullet}^{-1} N_{\bullet j} + \sum_{ij} Z_{ij} N_{i\bullet}^{-1} N_{r\bullet}^{-1}$$
$$= \sum_j N_{\bullet j}^2 (1 + O(\delta)).$$

The third, fourth and fifth terms in $\mathrm{Var}(U_a)$ are all $O(\delta)$. The second term contains

$$\begin{aligned}\sum_{ir} N_{i\bullet}^{-1} N_{r\bullet}^{-1} (ZZ^{\mathsf{T}})_{ir}((ZZ^{\mathsf{T}})_{ir} - 1) &\leqslant \sum_{ir} N_{i\bullet}^{-1} (ZZ^{\mathsf{T}})_{ir} \\ &= \sum_{irj} N_{i\bullet}^{-1} Z_{ij} Z_{rj} \\ &= \sum_{ij} Z_{ij} N_{i\bullet}^{-1} N_{\bullet j} \\ &= O(\delta).\end{aligned}$$

It follows that $\mathrm{Var}(U_a) = \sigma_B^4(\kappa_B + 2)\sum_j N_{\bullet j}^2(1 + O(\delta))$. Similarly $\mathrm{Var}(U_b) = \sigma_A^4(\kappa_A + 2)\sum_i N_{i\bullet}^2(1 + O(\delta))$.

The expression for $\mathrm{Var}(U_e)$ contains terms $\sigma_A^4(\kappa_A + 2)N^2\sum_j N_{\bullet j}^2 + \sigma_B^4(\kappa_B + 2)N^2\sum_i N_{i\bullet}^2$. All other terms are $O(\delta)$ times these two, mostly through $N \ll \sum_i N_{i\bullet}^2, \sum_j N_{\bullet j}^2 \ll N^2$. The coefficient of $\sigma_A^2\sigma_B^2$ contains

$$N\sum_{ij} Z_{ij} N_{i\bullet} N_{\bullet j} \leqslant \delta N^2 \sum_{ij} Z_{ij} N_{i\bullet} = \delta N^2 \sum_i N_{i\bullet}^2$$

so it is of smaller order than the lead term, as well as

$$\sum_i N_{i\bullet}^2 \sum_j N_{\bullet j}^2 \leqslant \delta N^2 \sum_i N_{i\bullet}^2.$$

As a result

$$\mathrm{Var}(U_e) = \Big(\sigma_A^4(\kappa_A + 2)N^2 \sum_j N_{\bullet j}^2 + \sigma_B^4(\kappa_B + 2)N^2 \sum_i N_{i\bullet}^2\Big)(1 + O(\delta)).$$

Turning to the covariances

$$\begin{aligned}\mathrm{Cov}(U_a, U_b) &= \sigma_E^4(\kappa_E + 2)\sum_{ij} Z_{ij}(1 - N_{i\bullet}^{-1} - N_{\bullet j}^{-1} + N_{i\bullet}^{-1} N_{\bullet j}^{-1}) \\ &= \sigma_E^4(\kappa_E + 2)(N - R - C + O(R)) \\ &= \sigma_E^4(\kappa_E + 2)N(1 + O(\delta)).\end{aligned}$$

Next $\mathrm{Cov}(U_a, U_e)$ contains the term $\sigma_B^4(\kappa_B + 2)N\sum_{ij} Z_{ij} N_{\bullet j} = \sigma_B^4(\kappa_B + 2)N\sum_j N_{\bullet j}^2$. The terms appearing after that one are $O(N^2) = O(\delta N\sum_j N_{\bullet j}^2)$. The largest term preceding it is dominated by

$$\sum_i N_{i\bullet}^{-1}\Big(\sum_j Z_{ij} N_{\bullet j}\Big)^2 \leqslant \delta N \sum_i N_{i\bullet}^{-1}\Big(\sum_j Z_{ij} N_{\bullet j}\Big)\Big(\sum_j Z_{ij}\Big) = \delta N \sum_j N_{\bullet j}^2.$$

It follows that $\mathrm{Cov}(U_a, U_e) = \sigma_B^4(\kappa_B+2)N\sum_j N_{\bullet j}^2(1+O(\delta))$ and similarly, $\mathrm{Cov}(U_b, U_e) = \sigma_A^4(\kappa_A+2)N\sum_i N_{i\bullet}^2(1+O(\delta))$.

Next, using (19)

$$\begin{aligned}\mathrm{Var}(\hat\sigma_A^2) &= \Big(\frac{\mathrm{Var}(U_e)}{N^4} + \frac{\mathrm{Var}(U_a)}{N^2} - 2\frac{\mathrm{Cov}(U_a,U_e)}{N^3}\Big)(1+O(\delta))\\ &= \sigma_A^4(\kappa_A+2)\frac{1}{N^2}\sum_i N_{i\bullet}^2(1+O(\delta)), \quad \text{and similarly}\\ \mathrm{Var}(\hat\sigma_B^2) &= \sigma_B^4(\kappa_B+2)\frac{1}{N^2}\sum_j N_{\bullet j}^2(1+O(\delta)).\end{aligned}$$

The last variance is

$$\begin{aligned}\mathrm{Var}(\hat\sigma_E^2) &= \Big(\frac{\mathrm{Var}(U_a)}{N^2} + \frac{\mathrm{Var}(U_b)}{N^2} + \frac{\mathrm{Var}(U_e)}{N^4} - \frac{2}{N^3}\mathrm{Cov}(U_a,U_e)\\ &\qquad - \frac{2}{N^3}\mathrm{Cov}(U_b,U_e) + \frac{2}{N^2}\mathrm{Cov}(U_a,U_b)\Big)(1+O(\delta))\\ &= \sigma_E^4(\kappa_E+2)\frac{1}{N}(1+O(\delta)).\end{aligned}$$

Next we verify that these variance estimates are asymptotically uncorrelated. Ignoring the $1+O(\delta)$ factors we have

$$\begin{aligned}\mathrm{Cov}(\hat\sigma_A^2, \hat\sigma_B^2) &\doteq \frac{\mathrm{Var}(U_e)}{N^4} - \frac{\mathrm{Cov}(U_b,U_e)}{N^3} - \frac{\mathrm{Cov}(U_a,U_e)}{N^3} + \frac{\mathrm{Cov}(U_a,U_b)}{N^2}\\ &\doteq \frac{1}{N^2}\Big(\sigma_A^4(\kappa_A+2)\sum_i N_{i\bullet}^2 + \sigma_B^4(\kappa_B+2)\sum_j N_{\bullet j}^2)\Big)\\ &\quad - \frac{1}{N^2}\sigma_A^4(\kappa_A+2)\sum_i N_{i\bullet}^2 - \frac{1}{N^2}\sigma_B^4(\kappa_B+2)\sum_j N_{\bullet j}^2\\ &\quad + \frac{1}{N}\sigma_E^4(\kappa_E+2)\\ &= \frac{1}{N}\sigma_E^4(\kappa_E+2)\end{aligned}$$

which is $O(\delta)$ times $\mathrm{Var}(\hat\sigma_A^2)$ and $\mathrm{Var}(\hat\sigma_B^2)$. Likewise

$$\begin{aligned}\mathrm{Cov}(\hat\sigma_A^2, \hat\sigma_E^2) &\doteq \frac{1}{N^3}\mathrm{Cov}(U_a,U_e) + \frac{1}{N^3}\mathrm{Cov}(U_b,U_e) - \frac{1}{N^4}\mathrm{Var}(U_e) - \frac{1}{N^2}\mathrm{Var}(U_a)\\ &\quad - \frac{1}{N^2}\mathrm{Cov}(U_a,U_b) + \frac{1}{N^3}\mathrm{Cov}(U_a,U_e)\\ &\doteq \sigma_B^4(\kappa_B+2)\frac{2}{N^2}\sum_j N_{\bullet j}^2 + \sigma_A^4(\kappa_A+2)\frac{1}{N^2}\sum_i N_{i\bullet}^2\\ &\quad - \Big(\sigma_A^4(\kappa_A+2)\sum_i N_{i\bullet}^2 + \sigma_B^4(\kappa_B+2)\sum_j N_{\bullet j}^2\Big)\frac{1}{N^2}\\ &\quad - \sigma_B^4(\kappa_B+2)\frac{1}{N^2}\sum_j N_{\bullet j}^2 - \sigma_E^4(\kappa_E+2)\frac{1}{N}\end{aligned}$$

$$= -\sigma_E^4(\kappa_E + 2)\frac{1}{N}$$

which is much smaller than $\mathrm{Var}(\hat{\sigma}_A^2)$. Similarly $\mathrm{Cov}(\hat{\sigma}_B^2, \hat{\sigma}_E^2) \doteq -\sigma_E^4(\kappa_E+2)/N$, is much smaller than $\mathrm{Var}(\hat{\sigma}_B^2)$.

9.9. Estimating kurtoses

To estimate the kurtoses κ_A, κ_B and κ_E in our variance formulas, it suffices to estimate fourth central moments such as $\mu_{A,4} = \sigma_A^4(\kappa_A+3)$ and similarly defined $\mu_{B,4}$ and $\mu_{E,4}$. Given $\hat{\sigma}_A^2$, $\hat{\sigma}_B^2$, and $\hat{\sigma}_E^2$, we can do this via GMM. Consider the following estimating equations and their expectations,

$$\begin{aligned}
W_a &= \frac{1}{2}\sum_{ijj'} \frac{1}{N_{i\bullet}} Z_{ij}Z_{ij'}(Y_{ij} - Y_{ij'})^4 \\
W_b &= \frac{1}{2}\sum_{ii'j} \frac{1}{N_{\bullet j}} Z_{ij}Z_{i'j}(Y_{ij} - Y_{i'j})^4 \\
W_e &= \frac{1}{2}\sum_{ii'jj'} Z_{ij}Z_{i'j'}(Y_{ij} - Y_{i'j'})^4
\end{aligned}$$

Using previous results,

$$\begin{aligned}
\mathbb{E}(W_a) &= \frac{1}{2}\sum_{ijj'} \frac{1}{N_{i\bullet}} Z_{ij}Z_{ij'}\mathbb{E}\big((b_j - b_{j'} + e_{ij} - e_{ij'})^4\big) \\
&= \frac{1}{2}\sum_{ijj'} \frac{Z_{ij}Z_{ij'}}{N_{i\bullet}}\mathbb{E}\big((b_j - b_{j'})^4 + 6(b_j - b_{j'})^2(e_{ij} - e_{ij'})^2 + (e_{ij} - e_{ij'})^4\big) \\
&= (N - R)\big(\mu_{B,4} + 3\sigma_B^4 + 12\sigma_B^2\sigma_E^2 + \mu_{E,4} + 3\sigma_E^4\big).
\end{aligned}$$

By symmetry,

$$\mathbb{E}(W_b) = (N - C)\big(\mu_{A,4} + 3\sigma_A^4 + 12\sigma_A^2\sigma_E^2 + \mu_{E,4} + 3\sigma_E^4\big).$$

Next

$$\begin{aligned}
\mathbb{E}(W_e) &= \frac{1}{2}\sum_{ii'jj'} Z_{ij}Z_{i'j'}\mathbb{E}\big((Y_{ij} - Y_{i'j'})^4\big) \\
&= \frac{1}{2}\sum_{ii'jj'} Z_{ij}Z_{i'j'}\mathbb{E}\big((a_i - a_{i'} + b_j - b_{j'} + e_{ij} - e_{i'j'})^4\big) \\
&= \frac{1}{2}\sum_{ii'jj'} Z_{ij}Z_{i'j'}\mathbb{E}\big((a_i - a_{i'})^4 + 6(a_i - a_{i'})^2(b_j - b_{j'})^2 + (b_j - b_{j'})^4 \\
&\quad + 6(a_i - a_{i'})^2(e_{ij} - e_{i'j'})^2 + 6(b_j - b_{j'})^2(e_{ij} - e_{i'j'})^2 + (e_{ij} - e_{i'j'})^4\big) \\
&\quad + (\mu_{E,4} + 3\sigma_E^4)N(N-1) + 12\sigma_A^2\sigma_B^2(N^2 - \sum_i N_{i\bullet}^2 - \sum_j N_{\bullet j}^2 + N).
\end{aligned}$$

These expectations are all linear in the fourth moments. Therefore, given estimates of σ_A^2, σ_B^2, and σ_E^2, we can solve another three-by-three system of equations to get estimates of the fourth moments.

Letting M be the matrix in equation (40) we find that

$$\begin{pmatrix}\mathbb{E}(W_a)\\ \mathbb{E}(W_b)\\ \mathbb{E}(W_e)\end{pmatrix} = M\begin{pmatrix}\mu_{A,4}\\ \mu_{B,4}\\ \mu_{E,4}\end{pmatrix} + \begin{pmatrix}3(N-R)\sigma_B^4 + 12(N-R)\sigma_B^2\sigma_E^2 + 3(N-R)\sigma_E^4\\ 3(N-C)\sigma_A^4 + 12(N-C)\sigma_A^2\sigma_E^2 + 3(N-C)\sigma_E^4\\ H\end{pmatrix}$$

where

$$\begin{aligned}H = (3\sigma_A^4 + 12\sigma_A^2\sigma_E^2)(N^2 - \sum_i N_{i\bullet}^2) + (3\sigma_B^4 + 12\sigma_B^2\sigma_E^2)(N^2 - \sum_j N_{\bullet j}^2)\\ + 3\sigma_E^4 N(N-1) + 12\sigma_A^2\sigma_B^2(N^2 - \sum_i N_{i\bullet}^2 - \sum_j N_{\bullet j}^2 + N).\end{aligned}$$

For plug-in method of moment estimators we replace expected W-statistics by their sample quantities, replace the variance components by their estimates and solve the matrix equation getting $\hat{\mu}_{A,4}$ et cetera. Then $\hat{\kappa}_A = \hat{\mu}_{A,4}/\hat{\sigma}_A^4 - 3$ and so on.

9.10. Best linear predictor

Here we consider linear predicton of Y_{ij}. We begin with predictions of the form $\hat{Y}_{ij} = \hat{Y}_{ij}(\lambda) = \sum_{rs}\lambda_{rs}Z_{rs}Y_{rs}$. Then we consider predictions of a reduced form that consider only the totals in row i, in row j and in the whole data set.

9.10.1. Proof of Lemma 5.1

Let $\hat{Y}_{ij} = \sum_{rs} Z_{ij}\lambda_{ij}Y_{ij}$ and $L = \mathbb{E}((Y_{ij} - \hat{Y}_{ij})^2)$. Then

$$L = \mu^2\Big(1 - \sum_{rs}\lambda_{rs}Z_{rs}\Big)^2 + \mathrm{Var}(Y_{ij}) + \mathrm{Var}(\hat{Y}_{ij}) - 2\mathrm{Cov}(Y_{ij}, \hat{Y}_{ij}).$$

First $\mathrm{Var}(Y_{ij}) = \sigma_A^2 + \sigma_B^2 + \sigma_E^2$. Next

$$\begin{aligned}\mathrm{Cov}(Y_{ij}, \hat{Y}_{ij}) &= \sum_{rs}\lambda_{rs}Z_{rs}\big(\sigma_A^2 1_{i=r} + \sigma_B^2 1_{j=s} + \sigma_E^2 1_{i=r}1_{j=s}\big)\\ &= \sigma_A^2\sum_s \lambda_{is}Z_{is} + \sigma_B^2\sum_r \lambda_{rj}Z_{rj} + \sigma_E^2\lambda_{ij}^2 Z_{ij},\end{aligned}$$

and finally

$$\mathrm{Var}(\hat{Y}_{ij}) = \sum_{rs}\sum_{r's'}\lambda_{rs}\lambda_{r's'}Z_{rs}Z_{r's'}\big(\sigma_A^2 1_{r=r'} + \sigma_B^2 1_{s=s'} + \sigma_E^2 1_{r=r'}1_{s=s'}\big)$$

$$= \sigma_A^2 \sum_{rss'} \lambda_{rs}\lambda_{rs'} Z_{rs}Z_{rs'} + \sigma_B^2 \sum_{rsr'} \lambda_{rs}\lambda_{r's} Z_{rs}Z_{r's} + \sigma_E^2 \sum_{rs} \lambda_{rs}^2 Z_{rs}.$$

Thus

$$\begin{aligned} L = {} & \mu^2\Big(1 - \sum_{rs} \lambda_{rs} Z_{rs}\Big)^2 + \sigma_A^2 + \sigma_B^2 + \sigma_E^2 \\ & + \sigma_A^2 \sum_{rss'} \lambda_{rs}\lambda_{rs'} Z_{rs}Z_{rs'} + \sigma_B^2 \sum_{rsr'} \lambda_{rs}\lambda_{r's} Z_{rs}Z_{r's} + \sigma_E^2 \sum_{rs} \lambda_{rs}^2 Z_{rs} \\ & - 2\Big(\sigma_A^2 \sum_{s} \lambda_{is} Z_{is} + \sigma_B^2 \sum_{r} \lambda_{rj} Z_{rj} + \sigma_E^2 \lambda_{ij}^2 Z_{ij}\Big). \end{aligned}$$

9.10.2. Stationary conditions

The partial derivative of L with respect to $\lambda_{r''s''}$ is

$$\begin{aligned} & 2\mu^2\Big(1 - \sum_{rs} \lambda_{rs} Z_{rs}\Big)(-Z_{r''s''}) + 2\sigma_E^2 \lambda_{r''s''} Z_{r''s''} \\ & + \sigma_A^2 \sum_{rss'} Z_{rs}Z_{rs'}(\lambda_{rs'} 1_{rs=r''s''} + \lambda_{rs} 1_{rs'=r''s''}) - 2\sigma_A^2 \sum_{s} Z_{is} 1_{is=r''s''} \\ & + \sigma_B^2 \sum_{rsr'} Z_{rs}Z_{r's}(\lambda_{r's} 1_{rs=r''s''} + \lambda_{rs} 1_{r's=r''s''}) - 2\sigma_B^2 \sum_{r} Z_{rj} 1_{rj=r''s''}. \end{aligned}$$

After taking account of the indicator functions we get

$$\begin{aligned} 2Z_{r''s''}\Big(& \mu^2\Big(1 - \sum_{rs} \lambda_{rs} Z_{rs}\Big)(-1) + \sigma_E^2 \lambda_{r''s''} + \sigma_A^2 \sum_{s'} Z_{r''s'} \lambda_{r''s'} \\ & + \sigma_B^2 \sum_{r'} Z_{r's''} \lambda_{r's''} - \sigma_A^2 Z_{is''} 1_{i=r''} - \sigma_B^2 Z_{r''j} 1_{j=s''}\Big). \end{aligned}$$

We can replace $Z_{is''}1_{i=r''}$ by $1_{i=r''}$ because of the leading factor $Z_{r''s''}$. This and a corresponding change to the coefficient of σ_B^2 yield

$$\begin{aligned} 2Z_{rs}\Big(& \mu^2\Big(\sum_{r's'} \lambda_{r's'} Z_{r's'} - 1\Big) + \sigma_E^2 \lambda_{rs} + \sigma_A^2 \sum_{s'} Z_{rs'} \lambda_{rs'} + \sigma_B^2 \sum_{r'} Z_{r's} \lambda_{r's} \\ & - \sigma_A^2 1_{i=r} - \sigma_B^2 1_{j=s}\Big). \end{aligned}$$

9.10.3. Proof of Lemma 5.2

Here we consider

$$\hat{Y}_{ij} = \lambda_0 Y_{\bullet\bullet} + \lambda_a Y_{i\bullet} + \lambda_b Y_{\bullet j}$$

where Y_{rj}

$$Y_{\bullet\bullet} = \sum_{rs} Z_{rs} Y_{rs}, \quad Y_{i\bullet} = \sum_{s} Z_{is} Y_{is}, \quad \text{and} \quad Y_{\bullet j} = \sum_{r} Z_{rj} Y_{rj}.$$

The mean squared error is $L = \mathbb{E}((Y_{ij} - \hat{Y}_{ij})^2)$. Expanding it we get

$$L = \mu^2\big(1-(\lambda_0 N + \lambda_a N_{i\bullet} + \lambda_b N_{\bullet j})\big)^2 + \mathrm{Var}(Y_{ij}) + \lambda_0^2\mathrm{Var}(Y_{\bullet\bullet}) + \lambda_a^2\mathrm{Var}(Y_{i\bullet}) + \lambda_b^2\mathrm{Var}(Y_{\bullet j}) - 2\lambda_0\mathrm{Cov}(Y_{ij}, Y_{\bullet\bullet}) - 2\lambda_a\mathrm{Cov}(Y_{ij}, Y_{i\bullet}) - 2\lambda_b\mathrm{Cov}(Y_{ij}, Y_{\bullet j}) + 2\lambda_0\lambda_a\mathrm{Cov}(Y_{\bullet\bullet}, Y_{i\bullet}) + 2\lambda_0\lambda_b\mathrm{Cov}(Y_{\bullet\bullet}, Y_{\bullet j}) + 2\lambda_a\lambda_b\mathrm{Cov}(Y_{i\bullet}, Y_{\bullet j}).$$

As before $\mathrm{Var}(Y_{ij}) = \sigma_A^2 + \sigma_B^2 + \sigma_E^2$. We set about finding the other terms.

First

$$\begin{aligned}\mathrm{Var}(Y_{\bullet\bullet}) &= \sigma_A^2\sum_r N_{r\bullet}^2 + \sigma_B^2\sum_s N_{\bullet s}^2 + \sigma_E^2 N,\\ \mathrm{Var}(Y_{i\bullet}) &= \sigma_A^2 N_{i\bullet}^2 + \sigma_B^2 N_{i\bullet} + \sigma_E^2 N_{i\bullet}, \quad \text{and}\\ \mathrm{Var}(Y_{\bullet j}) &= \sigma_A^2 N_{\bullet j} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}.\end{aligned}$$

Second

$$\begin{aligned}\mathrm{Cov}(Y_{ij}, Y_{\bullet\bullet}) &= \sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j} + \sigma_E^2 Z_{ij},\\ \mathrm{Cov}(Y_{ij}, Y_{i\bullet}) &= \sigma_A^2 N_{i\bullet} + \sigma_B^2 Z_{ij} + \sigma_E^2 Z_{ij}, \quad \text{and}\\ \mathrm{Cov}(Y_{ij}, Y_{\bullet j}) &= \sigma_A^2 Z_{ij} + \sigma_B^2 N_{\bullet j} + \sigma_E^2 Z_{ij}.\end{aligned}$$

The remaining terms use somewhat longer arguments.

$$\begin{aligned}\mathrm{Cov}(Y_{i\bullet}, Y_{\bullet\bullet}) &= \sum_{rss'} Z_{rs}Z_{is'}\mathrm{Cov}(Y_{rs}, Y_{is'})\\ &= \sum_{rss'} Z_{rs}Z_{is'}\big(1_{i=r}\sigma_A^2 + 1_{s=s'}\sigma_B^2 + 1_{i=r}1_{s=s'}\sigma_E^2\big)\\ &= \sigma_A^2 N_{i\bullet}^2 + \sigma_B^2\sum_s Z_{is}N_{\bullet s} + \sigma_E^2 N_{i\bullet}, \quad \text{and then}\\ \mathrm{Cov}(Y_{\bullet j}, Y_{\bullet\bullet}) &= \sigma_A^2\sum_r Z_{rj}N_{r\bullet} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}\end{aligned}$$

by symmetry. Finally

$$\begin{aligned}\mathrm{Cov}(Y_{i\bullet}, Y_{\bullet j}) &= \sum_{rs} Z_{is}Z_{rj}\mathrm{Cov}(Y_{is}, Y_{rj})\\ &= \sum_{rs} Z_{is}Z_{rj}\big(\sigma_A^2 1_{i=r} + \sigma_B^2 1_{j=s} + \sigma_E^2 1_{i=r}1_{j=s}\big)\\ &= \sigma_A^2\sum_s Z_{is}Z_{ij} + \sigma_B^2\sum_r Z_{ij}Z_{rj} + \sigma_E^2 Z_{ij}\\ &= Z_{ij}\big(\sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j} + \sigma_E^2\big).\end{aligned}$$

Combining these pieces we find that

$$L = \mu^2\big(1-\lambda_0 N - \lambda_a N_{i\bullet} - \lambda_b N_{\bullet j}\big)^2 + \lambda_0^2\Big(\sigma_A^2\sum_r N_{r\bullet}^2 + \sigma_B^2\sum_s N_{\bullet s}^2 + \sigma_E^2 N\Big) + \sigma_A^2 + \sigma_B^2 + \sigma_E^2 + \lambda_a^2\Big(\sigma_A^2 N_{i\bullet}^2 + \sigma_B^2 N_{i\bullet} + \sigma_E^2 N_{i\bullet}\Big)$$

$$
\begin{aligned}
&+ \lambda_b^2\Big(\sigma_A^2 N_{\bullet j} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}\Big) - 2\lambda_0\Big(\sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j} + \sigma_E^2 Z_{ij}\Big)\\
&- 2\lambda_a\Big(\sigma_A^2 N_{i\bullet} + \sigma_B^2 Z_{ij} + \sigma_E^2 Z_{ij}\Big) - 2\lambda_b\Big(\sigma_A^2 Z_{ij} + \sigma_B^2 N_{\bullet j} + \sigma_E^2 Z_{ij}\Big)\\
&+ 2\lambda_0\lambda_a\Big(\sigma_A^2 N_{i\bullet}^2 + \sigma_B^2 \sum_s Z_{is} N_{\bullet s} + \sigma_E^2 N_{i\bullet}\Big)\\
&+ 2\lambda_0\lambda_b\Big(\sigma_A^2 \sum_r Z_{rj} N_{r\bullet} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}\Big)\\
&+ 2\lambda_a\lambda_b Z_{ij}\Big(\sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j} + \sigma_E^2\Big).
\end{aligned}
$$

9.10.4. Proof of Theorem 5.1

From the result of Lemma 5.2, we see that L is quadratic in λ. Since L is bounded below by 0, it follows that L attains its minimum on $\mathbb{R}^3$, which would be any solution of the stationarity condition $\nabla_\lambda L = 0$. We find the components of this gradient.

$$
\begin{aligned}
\frac{1}{2}\frac{\partial L}{\partial \lambda_0} &= N\mu^2(\lambda_0 N + \lambda_a N_{i\bullet} + \lambda_b N_{\bullet j} - 1) - \Big(\sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j}\Big)\\
&\quad + \lambda_0\Big(\sigma_A^2 \sum_r N_{r\bullet}^2 + \sigma_B^2 \sum_s N_{\bullet s}^2 + \sigma_E^2 N\Big)\\
&\quad + \lambda_a\Big(\sigma_A^2 N_{i\bullet}^2 + \sigma_B^2 \sum_s Z_{is} N_{\bullet s} + \sigma_E^2 N_{i\bullet}\Big)\\
&\quad + \lambda_b\Big(\sigma_A^2 \sum_r Z_{rj} N_{r\bullet} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}\Big)\\
\frac{1}{2}\frac{\partial L}{\partial \lambda_a} &= N_{i\bullet}\mu^2(\lambda_0 N + \lambda_a N_{i\bullet} + \lambda_b N_{\bullet j} - 1) + \lambda_a\Big(\sigma_A^2 N_{i\bullet}^2 + \sigma_B^2 N_{i\bullet} + \sigma_E^2 N_{i\bullet}\Big)\\
&\quad - \Big(\sigma_A^2 N_{i\bullet} + \sigma_B^2 Z_{ij}\Big) + \lambda_0\Big(\sigma_A^2 N_{i\bullet}^2 + \sigma_B^2 \sum_s Z_{is} N_{\bullet s} + \sigma_E^2 N_{i\bullet}\Big)\\
&\quad + \lambda_b Z_{ij}\Big(\sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j} + \sigma_E^2\Big), \quad \text{and}\\
\frac{1}{2}\frac{\partial L}{\partial \lambda_b} &= N_{\bullet j}\mu^2(\lambda_0 N + \lambda_a N_{i\bullet} + \lambda_b N_{\bullet j} - 1) + \lambda_b\Big(\sigma_A^2 N_{\bullet j} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}\Big)\\
&\quad - \Big(\sigma_A^2 Z_{ij} + \sigma_B^2 N_{\bullet j}\Big) + \lambda_0\Big(\sigma_A^2 \sum_r Z_{rj} N_{r\bullet} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}\Big)\\
&\quad + \lambda_a Z_{ij}\Big(\sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j} + \sigma_E^2\Big).
\end{aligned}
$$

We write this as

$$H\lambda^* = c$$

where

$$
c = \begin{pmatrix} N\mu^2 + \sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j} \\ N_{i\bullet}\mu^2 + \sigma_A^2 N_{i\bullet} + \sigma_B^2 Z_{ij} \\ N_{\bullet j}\mu^2 + \sigma_A^2 Z_{ij} + \sigma_B^2 N_{\bullet j} \end{pmatrix} = \begin{pmatrix} N & N_{i\bullet} & N_{\bullet j} \\ N_{i\bullet} & N_{i\bullet} & Z_{ij} \\ N_{\bullet j} & Z_{ij} & N_{\bullet j} \end{pmatrix} \begin{pmatrix} \mu^2 \\ \sigma_A^2 \\ \sigma_B^2 \end{pmatrix}
$$

and H is a symmetric matrix with upper triangle

$$H = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ * & H_{22} & H_{23} \\ * & * & H_{33} \end{pmatrix}$$

with elements

$$\begin{aligned}
H_{11} &= \mu^2 N^2 + \sigma_A^2 \sum_r N_{r\bullet}^2 + \sigma_B^2 \sum_s N_{\bullet s}^2 + \sigma_E^2 N \\
H_{12} &= \mu^2 N N_{i\bullet} + \sigma_A^2 N_{i\bullet}^2 + \sigma_B^2 \sum_s Z_{is} N_{\bullet s} + \sigma_E^2 N_{i\bullet} \\
H_{13} &= \mu^2 N N_{\bullet j} + \sigma_A^2 \sum_r Z_{rj} N_{r\bullet} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j} \\
H_{22} &= \mu^2 N_{i\bullet}^2 + \sigma_A^2 N_{i\bullet}^2 + \sigma_B^2 N_{i\bullet} + \sigma_E^2 N_{i\bullet} \\
H_{23} &= \mu^2 N_{i\bullet} N_{\bullet j} + \sigma_A^2 Z_{ij} N_{i\bullet} + \sigma_B^2 Z_{ij} N_{\bullet j} + \sigma_E^2 Z_{ij}, \quad \text{and} \\
H_{33} &= \mu^2 N_{\bullet j}^2 + \sigma_A^2 N_{\bullet j} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}.
\end{aligned}$$

Using $T_{i\bullet} \equiv \sum_s Z_{is} N_{\bullet s}$ and $T_{\bullet j} \equiv \sum_r Z_{rj} N_{r\bullet}$ some of these simplify:

$$\begin{aligned}
H_{12} &= \mu^2 N N_{i\bullet} + \sigma_A^2 N_{i\bullet}^2 + \sigma_B^2 T_{i\bullet} + \sigma_E^2 N_{i\bullet}, \quad \text{and} \\
H_{13} &= \mu^2 N N_{\bullet j} + \sigma_A^2 T_{\bullet j} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}.
\end{aligned}$$

9.10.5. Proof of Theorem 5.2

To begin with, we note that $N_{\bullet j} = \sum_r Z_{rj} \leqslant \sum_r N_{r\bullet} Z_{rj} \leqslant \eta N$. We write

$$\begin{pmatrix} \lambda_0^* \\ \lambda_b^* \end{pmatrix} = \frac{1}{\det \tilde{H}} \begin{pmatrix} H_{33} & -H_{13} \\ -H_{31} & H_{11} \end{pmatrix} \begin{pmatrix} c_1 \\ c_3 \end{pmatrix}.$$

Then

$$\begin{aligned}
\det \tilde{H} \lambda_0^* &= H_{33} c_1 - H_{13} c_3 \\
&= N_{\bullet j}(\mu^2 N_{\bullet j} + \sigma_A^2 + \sigma_B^2 N_{\bullet j} + \sigma_E^2)(N\mu^2 + N_{\bullet j}\sigma_B^2) \\
&\quad - (\mu^2 N N_{\bullet j} + \sigma_A^2 \sum_r Z_{rj} N_{r\bullet} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}) N_{\bullet j}(\mu^2 + \sigma_B^2) \\
&= \mu^2 \Big(\sigma_A^2 N N_{\bullet j} + \sigma_E^2 N N_{\bullet j} - \sigma_A^2 N_{\bullet j} \sum_r Z_{rj} N_{r\bullet} - \sigma_E^2 N_{\bullet j}^2 \Big) \\
&\quad + \sigma_B^2 \Big(\sigma_A^2 N_{\bullet j}^2 - \sigma_A^2 N_{\bullet j} \sum_r Z_{rj} N_{r\bullet} \Big) \\
&= \mu^2 (\sigma_A^2 + \sigma_E^2) N N_{\bullet j} (1 + O(\eta)),
\end{aligned}$$

and

$$\det \tilde{H} \lambda_b^* = H_{11} c_3 - H_{31} c_1$$

$$
\begin{aligned}
&= (\mu^2 N^2 + \sigma_A^2 \sum_r N_{r\bullet}^2 + \sigma_B^2 \sum_s N_{\bullet s}^2 + \sigma_E^2 N) N_{\bullet j}(\mu^2 + \sigma_B^2) \\
&\quad - (\mu^2 N N_{\bullet j} + \sigma_A^2 \sum_r Z_{rj} N_{r\bullet} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j})(N\mu^2 + N_{\bullet j}\sigma_B^2) \\
&= \mu^2 \Big(\sigma_A^2 N_{\bullet j} \sum_r N_{r\bullet}^2 + \sigma_B^2 N_{\bullet j} \sum_s N_{\bullet s}^2 - \sigma_A^2 N \sum_r Z_{rj} N_{r\bullet} - \sigma_B^2 N N_{\bullet j}^2 \Big) \\
&\quad + \sigma_B^2 \Big(\mu^2 N^2 N_{\bullet j} + \sigma_A^2 N_{\bullet j} \sum_r N_{r\bullet}^2 + \sigma_B^2 N_{\bullet j} \sum_s N_{\bullet s}^2 + \sigma_E^2 N N_{\bullet j} \\
&\quad\quad - \mu^2 N N_{\bullet j}^2 - \sigma_A^2 N_{\bullet j} \sum_r Z_{rj} N_{r\bullet} - \sigma_B^2 N_{\bullet j}^3 - \sigma_E^2 N_{\bullet j}^2 \Big) \\
&= \mu^2 \sigma_B^2 N^2 N_{\bullet j}(1 + O(\eta)).
\end{aligned}
$$

Thus

$$
\frac{\lambda_0^*}{\lambda_b^*} = \frac{\sigma_A^2 + \sigma_E^2}{\sigma_B^2 N}(1 + O(\eta)),
$$

and so

$$
\begin{aligned}
\det \tilde{H} &= H_{11} H_{33} - H_{13}^2 \\
&= \Big(\mu^2 N^2 + \sigma_A^2 \sum_r N_{r\bullet}^2 + \sigma_B^2 \sum_s N_{\bullet s}^2 + \sigma_E^2 N \Big) \\
&\quad \Big(\mu^2 N_{\bullet j}^2 + \sigma_B^2 N_{\bullet j}^2 + \sigma_A^2 N_{\bullet j} + \sigma_E^2 N_{\bullet j} \Big) \\
&\quad - \Big(\mu^2 N N_{\bullet j} + \sigma_A^2 \sum_r N_{r\bullet} Z_{rj} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j} \Big)^2 \\
&\approx \mu^2 N^2 N_{\bullet j}^2 (\mu^2 + \sigma_B^2) - (\mu^2 N N_{\bullet j})^2 = \mu^2 N^2 N_{\bullet j}^2 \sigma_B^2.
\end{aligned}
$$

As a result the prediction for a new row in a large column is essentially that column average plus $O(1/N_{\bullet j})$ times the global average.

9.10.6. Asymptotic weights: Proof of Theorem 5.3

Here we have

$$
\begin{aligned}
&1 \leqslant N_{i\bullet} \leqslant \eta N, \qquad 1 \leqslant N_{\bullet j} \leqslant \eta N, \qquad N_{i\bullet} \leqslant \eta N_{i\bullet}^2, \\
&N_{\bullet j} \leqslant \eta N_{\bullet j}^2, \qquad N \leqslant \eta N^2, \qquad \sum_r N_{r\bullet}^2 \leqslant \eta N^2, \\
&\sum_s N_{\bullet s}^2 \leqslant \eta N^2, \qquad \sum_r N_{r\bullet} Z_{rj} \leqslant \eta N N_{\bullet j}, \quad \text{and} \quad \sum_s N_{\bullet s} Z_{is} \leqslant \eta N N_{i\bullet}.
\end{aligned}
$$

The first five follow easily from $1 < 1/\eta \leqslant N_{i\bullet}, N_{\bullet j} \leqslant \eta N$. The last four follow from the others. For instance $\sum_r N_{r\bullet}^2 \leqslant \sum_r N_{r\bullet}(\eta N) = \eta N^2$, and $\sum_r N_{r\bullet} Z_{rj} \leqslant \sum_r Z_{rj}(\eta N) = \eta N_{\bullet j} N$. We also have $0 < \mu^2, \sigma_A^2, \sigma_B^2, \sigma_E^2 < \infty$.

Then

$$H = \begin{pmatrix} \mu^2 N^2 & \mu^2 N N_{i\bullet} & \mu^2 N N_{\bullet j} \\ \mu^2 N N_{i\bullet} & (\mu^2+\sigma_A^2) N_{i\bullet}^2 & \mu^2 N_{i\bullet} N_{\bullet j} \\ \mu^2 N N_{\bullet j} & \mu^2 N_{i\bullet} N_{\bullet j} & (\mu^2+\sigma_B^2) N_{\bullet j}^2 \end{pmatrix} (1+O(\eta))$$

and using symbolic computation (via Wolfram|Alpha, September 6, 2015)

$$H^{-1} = \begin{pmatrix} \dfrac{\mu^2(\sigma_A^2+\sigma_B^2)+\sigma_A^2\sigma_B^2}{\sigma_A^2\sigma_B^2\mu^2N^2} & \dfrac{-1}{\sigma_A^2 N_{i\bullet} N} & \dfrac{-1}{\sigma_B^2 N_{\bullet j} N} \\ \dfrac{-1}{\sigma_A^2 N_{i\bullet} N} & \dfrac{1}{\sigma_A^2 N_{i\bullet}^2} & 0 \\ \dfrac{-1}{\sigma_B^2 N_{\bullet j} N} & 0 & \dfrac{1}{\sigma_B^2 N_{\bullet j}^2} \end{pmatrix} (1+O(\eta)).$$

The determinant of H^{-1} is $(\sigma_A^2\sigma_B^2\mu^2 N_{i\bullet}^2 N_{\bullet j}^2 N^2)^{-1}(1+O(\eta))$, so we need $N_{i\bullet} \geqslant 1$ and $N_{\bullet j} \geqslant 1$ to make matrix inversion a continuous operation. Similarly

$$c = \begin{pmatrix} N\mu^2 \\ N_{i\bullet}(\mu^2+\sigma_A^2) \\ N_{\bullet j}(\mu^2+\sigma_B^2) \end{pmatrix} (1+O(\eta)).$$

Thus ignoring the $O(\eta)$ terms

$$\begin{aligned} \lambda_0^* &\doteq \Big(\frac{\mu^2(\sigma_A^2+\sigma_B^2)+\sigma_A^2\sigma_B^2}{\sigma_A^2\sigma_B^2\mu^2N^2}\Big)N\mu^2 - \Big(\frac{1}{\sigma_A^2 N_{i\bullet} N}\Big)N_{i\bullet}(\mu^2+\sigma_A^2) \\ &\quad - \Big(\frac{1}{\sigma_B^2 N_{\bullet j} N}\Big)N_{\bullet j}(\mu^2+\sigma_B^2) \\ &= \frac{\mu^2(\sigma_A^2+\sigma_B^2)+\sigma_A^2\sigma_B^2}{\sigma_A^2\sigma_B^2 N} - \frac{\mu^2+\sigma_A^2}{\sigma_A^2 N} - \frac{\mu^2+\sigma_B^2}{\sigma_B^2 N} \\ &= \frac{\mu^2(\sigma_A^2+\sigma_B^2)+\sigma_A^2\sigma_B^2}{\sigma_A^2\sigma_B^2 N} - \frac{\mu^2\sigma_B^2+\sigma_A^2\sigma_B^2}{\sigma_A^2\sigma_B^2 N} - \frac{\mu^2\sigma_A^2+\sigma_B^2\sigma_B^2}{\sigma_A^2\sigma_B^2 N} \\ &= -\frac{1}{N}. \end{aligned}$$

The end result $-1/N$ is of the same order of magnitude as the original terms. Therefore $\lambda_0^* = (-1/N)(1+O(\eta))$. Similarly

$$\lambda_a^* \doteq -\frac{1}{\sigma_A^2 N_{i\bullet} N} N\mu^2 + \frac{1}{\sigma_A^2 N_{i\bullet}^2} N_{i\bullet}(\mu^2+\sigma_A^2) = -\frac{\mu^2}{\sigma_A^2 N_{i\bullet}} + \frac{\mu^2+\sigma_A^2}{\sigma_A^2 N_{i\bullet}} = \frac{1}{N_{i\bullet}}$$

and

$$\lambda_b^* \doteq \frac{1}{N_{\bullet j}},$$

and both of these approximations involve multiplication by $1+O(\eta)$. In this limit then

$$\hat{Y}_{ij} = \bar{Y}_{i\bullet}(1+O(\eta)) + \bar{Y}_{\bullet j}(1+O(\eta)) - \bar{Y}_{\bullet\bullet}(1+O(\eta))$$

which make intuitive sense as $(\hat{\mu}+\hat{a}_i)+(\hat{\mu}+\hat{b}_j)-\hat{\mu}$.

Acknowledgments

This work was supported by US NSF under grant DMS-1407397. KG was supported by US NSF Graduate Research Fellowship under grant DGE-114747. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

We would like to thank Brad Klingenberg for his motivation and encouragement during this project. We would also like to thank Rob Tibshirani for his suggestions about our experiments, and Lester Mackey and Norm Matloff for some helpful discussions. Finally, two anonymous reviewers and the editors made very helpful comments that led us to improve the paper.

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