Provable Robust Watermarking for AI-Generated Text

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Abstract

As AI-generated text increasingly resembles human-written content, the ability to detect machine-generated text becomes crucial. To address this challenge, we present GPTWatermark, a robust and high-quality solution designed to ascertain whether a piece of text originates from a specific model. Our approach extends existing watermarking strategies and employs a fixed group design to enhance robustness against editing and paraphrasing attacks. We show that our watermarked language model enjoys strong provable guarantees on generation quality, correctness in detection, and security against evasion attacks. Experimental results on various large language models (LLMs) and diverse datasets demonstrate that our method achieves superior detection accuracy and comparable generation quality in perplexity, thus promoting the responsible use of LLMs. Code is available at `https://github.com/XuandongZhao/GPTWatermark`.

1. Introduction

Generative Artificial Intelligence (AI) (Brown et al., 2020; Ramesh et al., 2022; Saharia et al., 2022; OpenAI, 2023a) has achieved significant progress in recent years, spanning from computer vision (CV) to natural language processing (NLP). Large language models (LLMs) such as ChatGPT (OpenAI, 2022) can generate coherent and contextually relevant long-form text in response to user-specified prompts. However, the ease of using LLMs has raised concerns about their potential misuse (Zellers et al., 2019; Weidinger et al., 2021; Stokel-Walker, 2022). For example, LLMs could be used to generate fake news, contaminate web content, or assist in academic dishonesty. Additionally, the proliferation of synthetic data from LLMs poses challenges for training new models, as synthetic data needs to be detected and excluded before model training (Radford et al., 2022; Carlini et al., 2023).

There are two main camps of existing attempts to address these challenges. One camp, inspired by Turing (1950), aims at generically distinguishing machine-generated text from that of the humans (Gehrmann et al., 2019; Mitchell et al., 2023; Hovy, 2016; Zellers et al., 2019; OpenAI, 2023b). These works primarily leverage hand-crafted or learned "statistical patterns" of generated text, thus their performance is not robust to distribution changes (e.g., by prompting / conditioning), prone to biases (Liang et al., 2023), and vulnerable to adversarial attacks. Moreover, recent research (Sadasivan et al., 2023) presents an impossibility result arguing that, as language models improve over time, AI-generated text increasingly resembles human-generated text, hence rendering any classifiers ineffective.

The other camp advocates active intervention by injecting carefully-designed watermarks to machine-generated text (Kirchenbauer et al., 2023; Zhao et al., 2023). The watermarking approach does not search for statistical patterns (which could be hit-or-miss), but rather deliberately *plant* subtle but distinctive patterns within the content to enable downstream detection. Compared to the passive detection approaches, the watermarking methods aim at determining whether the text is coming from a *specific* language model rather than solving the Turing test generically. As a result, watermarking approaches are robust to distribution-shift and can essentially *prove* — rather than *predict* — the origin of the suspect text.

The most notable challenge for the watermarking approach is that the planted patterns could be post-processed away. As an example, Kirchenbauer et al. (2023)'s soft watermarking method divides the vocabulary into a "green list" and a "red list" based on the prefix token, and subtly increases the probability of choosing from the green list. If the watermarked sentence is edited by changing every other token into its synonym, then it is no longer possible to determine the green/red lists for each candidate token, thus ruining the detector. One could also simply paraphrase the sentence as a whole using another off-the-shelf LLM.

In this paper, we take a first stab at formally defining robustness in the context of watermarking LLMs. Our contribu-

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tions are fourfold.

1. We devise a rigorous theoretical framework for quantifying the performance drop, the correctness of detection, and the security property against post-processing.
2. We propose to simplify the scheme of Kirchenbauer et al. (2023) by using a fixed Green-Red split consistently and show that the new watermark, named GPTWatermark, is *twice as robust* to edits as the baseline, provably.
3. We prove that the watermarked LLM is close to the original LLM (in all Renyi divergences) and show that the Type I/Type II errors of the detection algorithm decay exponentially as the suspect text length gets longer and more diverse.
4. We conduct experiments utilizing various large language models on diverse datasets. The results indicate that our method achieves superior detection accuracy and improved robustness against different attacks, thus promoting the responsible use of LLMs.

To the best of our knowledge, we are the first to formulate the LLM watermarking as a cryptographic problem and to obtain provably robust guarantees for watermarks for LLMs against arbitrary edits.

2. Problem definition

We start with an overview of the language model watermarking problem. The definitions and notations introduced in this section will be used throughout the paper.

Symbols and mathematical notations. We use $\mathbb{P}[\cdot]$, $\mathbb{E}[\cdot]$, $\mathbb{P}[\cdot|\cdot]$ and $\mathbb{E}[\cdot|\cdot]$ to denote the probability, expectation operator, conditional probability and conditional expectation respectively. Whenever there is ambiguity on which distribution the random variables are drawn from, we explicitly state them, e.g., $\mathbb{P}_{(X,Y)\sim\mathcal{D}}[X<3|Y=y]$, or equivalently $\mathbb{P}[X<3|Y=y\ ;\ (X,Y)\sim\mathcal{D}]$. To avoid clutter, we do not distinguish between random variables and constants as the distinctions are clear from the context. Boldface symbols denote a vector, e.g., a probability mass function $\mathbf{p}$ or a sequence of tokens $\boldsymbol{y}$. $\|\cdot\|_2$, $\|\cdot\|_\infty$ denotes the standard ℓ_2 and ℓ_∞-norms of a vector. In addition, $[n]$ is a shorthand for $\{1,2,...,n\}$. Other symbols and their meanings will be defined as we encounter them.

Language models. A language model (LM) $\mathcal{M}$ is a statistical model that describes the probability of a sequence of words occurring in a sentence. Common neural language models (e.g., GPT-2/3 (Radford et al., 2019; Brown et al., 2020)) are designed for next-word prediction which typically uses a transformer neural network (Vaswani et al., 2017). The LM has a "vocabulary" $\mathcal{V}$ with $N := |\mathcal{V}| = 50,000$ tokens or more (Radford et al., 2019; Liu et al., 2019). Let $\boldsymbol{x}$ be an input prompt. $\boldsymbol{y} := [y_1,\ldots,y_n]$ are n tokens generated by $\mathcal{M}$. During inference, $\mathcal{M}$ receives the input prompt $\boldsymbol{x}$ as the prefix of generation. It iteratively produces $|\mathcal{V}|$ logit scores for every next token. A soft-(arg)max function converts these scores into a probability distribution over $\mathcal{V}$ for the next token. The generic procedure for an LM $\tilde{\mathcal{M}}$ to generate text is described in Algorithm 1.

Algorithm 1 Text generation from a language model

```
Input: prompt $\boldsymbol{x}$, language model $\tilde{\mathcal{M}}$.
for $t = 1, 2, \cdots$ do
    Apply $\tilde{\mathcal{M}}$ to prior tokens $[\boldsymbol{x}, \boldsymbol{y}_{1:t-1}]$ to obtain the logits $\tilde{\boldsymbol{\ell}}_t$.
    Sample $y_t \sim \tilde{\mathbf{p}}_t$ where
        $$\tilde{\mathbf{p}}_t[v] = \frac{\exp\left(\tilde{\boldsymbol{\ell}}_t[v]\right)}{\sum_{i\in\mathcal{V}}\exp\left(\tilde{\boldsymbol{\ell}}_t[i]\right)} \text{ for all } v\in\mathcal{V}. \quad (1)$$
end for
Output: Sequence $\boldsymbol{y} \leftarrow [y_1, ..., y_n]$.
```

2.1. Definition of language model watermarking

In the language model watermarking problem, the objective for the model owner is to embed a secret message known as "watermark" within the generated sequence $\boldsymbol{y}$ for a given prompt $\boldsymbol{x}$. There are two desired requirements for watermarking. First, the quality of the watermarked model should be comparable to the quality of the original, un-watermarked model. Second, an adversary needs to modify sufficiently many AI-generated text in order to evade detection.

Definition 2.1 (Edit distance)**.** The edit distance, denoted as $\mathsf{ED}(\boldsymbol{y},\boldsymbol{z})$, quantifies the number of basic operations required to transform a sequence $\boldsymbol{y}$ into another sequence $\boldsymbol{z}$. These operations include "insertion", "deletion", and "replacement" of tokens.

Definition 2.2 (Language model watermarking)**.** A **language model watermarking scheme** consists of two probabilistic polynomial-time algorithms $(\mathsf{Watermark}, \mathsf{Detect})$:

- $\mathsf{Watermark}(\mathcal{M})$: Let $\mathcal{M}$ be a language model and let $\mathbf{p}_t := \mathbb{P}_{\mathcal{M}(\boldsymbol{x})}[y_t = \cdot|\boldsymbol{y}_{1:t-1}]$ be the *conditional* probability distribution of t-th token on $\mathcal{V}$ generated by $\mathcal{M}$. This algorithm produces a new model $\hat{\mathcal{M}}$ with a new conditional distribution $\hat{\mathbf{p}}_t := \mathbb{P}_{\hat{\mathcal{M}}(\boldsymbol{x})}[y_t = \cdot|\boldsymbol{y}_{1:t-1}]$ on $\mathcal{V}$. Additionally, it outputs a detection key k associated with $\hat{\mathcal{M}}$. The watermark could contain certain randomness.
- $\mathsf{Detect}(\mathsf{k},\boldsymbol{y})$: This algorithm takes input detection key k and sequence $\boldsymbol{y}$, then outputs 1 (indicating it was generated by $\hat{\mathcal{M}}$) or 0 (indicating it was *not* generated by $\hat{\mathcal{M}}$).

We require the following **three correctness properties** to hold:

- ω-Quality of watermarked output, for $\omega \in \mathbb{R}$: Assume the original language model $\mathcal{M}$ generates a probability vector $\mathbf{p}_t$ for the token at position t. The watermarked model $\hat{\mathcal{M}}$ predicts the token at position t using the modified probability vector $\hat{\mathbf{p}}_t$. It is required that the distance between the two probability distributions satisfies: $D\left(\hat{\mathbf{p}}_t \| \mathbf{p}_t\right) \leq \omega$ for any fixed prompts and prefixes.
- $\alpha_{(\boldsymbol{x},\mathcal{M})}$-Type I error ("No false positives"): for any algorithm $\mathcal{A}$ that takes $\boldsymbol{x}$ and certain auxiliary information aux as input
$$\mathbb{P}\left[\mathsf{Detect}(\mathsf{k},\boldsymbol{y}) = 1\ ;\ \substack{(\hat{\mathcal{M}},\mathsf{k})\sim\mathsf{Watermark}(\mathcal{M})\\ \boldsymbol{y}\sim\mathcal{A}(\boldsymbol{x},\mathsf{aux})}\right] \leq \alpha_{(\boldsymbol{x},\mathcal{M})}.$$
- $\beta_{(\boldsymbol{x},\mathcal{M})}$-Type II error ("No false negatives"):
$$\mathbb{P}\left[\mathsf{Detect}(\mathsf{k},\boldsymbol{y}) = 0\ ;\ \substack{(\hat{\mathcal{M}},\mathsf{k})\sim\mathsf{Watermark}(\mathcal{M})\\ \boldsymbol{y}\sim\hat{\mathcal{M}}(\boldsymbol{x})}\right] \leq \beta_{(\boldsymbol{x},\mathcal{M})}.$$

We also require the following **security property** (parameterized by $\epsilon \geq 0$ and $\eta(\boldsymbol{y},\mathsf{k},\epsilon)$):

- For any adversary $\mathcal{A}$ that postprocesses $\boldsymbol{y}$ with auxiliary information aux and any prompt $\boldsymbol{x} \in \mathcal{V}^*$
$$\mathbb{P}\left[\mathsf{Detect}(\mathsf{k},\boldsymbol{y}_{\mathcal{A}}) = 1 \text{ or } \mathsf{ED}(\boldsymbol{y},\boldsymbol{y}_{\mathcal{A}}) \geq \eta(\mathsf{k},\boldsymbol{y},\epsilon) \middle| \substack{\boldsymbol{y},\mathsf{k},\\ \mathsf{Detect}(\mathsf{k},\boldsymbol{y})=0}\ ;\ \substack{(\hat{\mathcal{M}},\mathsf{k})\sim\mathsf{Watermark}(\mathcal{M})\\ \boldsymbol{y}\sim\hat{\mathcal{M}}(\boldsymbol{x})\\ \boldsymbol{y}_{\mathcal{A}}\sim\mathcal{A}(\boldsymbol{y},\mathsf{aux})}\right] \geq 1-\epsilon.$$

Remark 2.3 (Discussion on Definition 2.2). Informally, our definition allows us to formally quantify the essential properties of a language model watermarking scheme including its generation quality relative to the input LM, the accuracy of detection in terms of both false positives and false negatives, as well as the robustness to attacks.

The security property, in particular, states the following: suppose a malicious adversary intends to evade the detection algorithm, then the adversarial answer, to some input prompt $\boldsymbol{x}$, should be far away (in edit distance) from any AI-generated answer. In other words, the optimal strategy to evade the detection algorithm would necessitate executing a minimum number of insert/delete/replacement operations, captured by the function $\eta(\cdot)$ in Definition 2.2. This conceptually suggests that the adversary must exert considerable effort to successfully elude detection.

On the other hand, our definition does not capture the following attacks:

- Adversary with embedded response: If the adversary possesses the response $\boldsymbol{y}_{\mathcal{A}}$ embedded within it as supplementary data, they are not exerting genuine effort to circumvent detection. Nevertheless, there are scenarios where it is reasonable to assume that the adversary is not initially aware of an answer to the prompt $\boldsymbol{x}$, such as when $\boldsymbol{x}$ is drawn from a distribution.
- Paraphrasing attacks: One potential approach the adversary can undertake to evade detection is to paraphrase the AI-generated answer using an un-watermarked LLM. But it is plausible, at least in some scenarios, that the adversary does not have access to an un-watermarked LLM. In general, paraphrasing attacks can be problematic when reliably detecting AI-generated text and further research is needed to come up with formal definitions capturing these attacks.

2.2. Threat models

Adversary's capabilities. We consider an adversary with black-box input-output access to the language model. This adversary has the capacity to modify the sequence within a *bounded edit distance*. Given an input prompt $\boldsymbol{x}$, the watermarked language model generates a text output $\boldsymbol{y} \leftarrow \hat{\mathcal{M}}(\boldsymbol{x})$. The adversary, equipped with arbitrary side-information and computational resources, can then produce a modified output $\boldsymbol{y}_{\mathcal{A}}$ such that the edit distance between the original and modified output, $\mathsf{ED}(\boldsymbol{y},\boldsymbol{y}_{\mathcal{A}})$, is bounded, i.e. $\mathsf{ED}(\boldsymbol{y},\boldsymbol{y}_{\mathcal{A}}) < \eta$.

Adversary's objective. The primary objective of the adversary is to render the watermark detection algorithm ineffective. Specifically, the adversary aims to produce a $\boldsymbol{y}_{\mathcal{A}}$ such that $\mathsf{Detect}(\mathsf{k},\boldsymbol{y}_{\mathcal{A}}) = 0$ while at the same time, $\boldsymbol{y}_{\mathcal{A}}$ is a minor modification of an AI-generated text $\boldsymbol{y}$.

3. Method

Algorithm 2 GPTWatermark: Watermark

```
Input: random number generator $F$, green list size $\gamma \in (0,1)$, watermark strength $\delta$.
Randomly generate a watermark key k using $F$.
Use watermark key to partition the vocabulary of $\mathcal{M}$ into a "green list" $G \subset \mathcal{V}$ of size $\gamma|\mathcal{V}|$, and a "red list" $R = G^c$.
Define a new language model $\hat{\mathcal{M}}$ where for $t$ and any prefix $[\boldsymbol{x}, \boldsymbol{y}_{1:t-1}]$, the resulting logits satisfy
    $\hat{\boldsymbol{\ell}}_t[v] := \boldsymbol{\ell}_t[v] + \delta\mathbf{1}(v \in G)$,
    where $\mathbf{1}(\cdot)$ is the indicator function and the logit vector $\boldsymbol{\ell}_t \in \mathbb{R}^{|\mathcal{V}|}$ is obtained by the passing the same prefix to $\mathcal{M}$.
Output: watermark key k, watermarked language model $\hat{\mathcal{M}}$.
```

We present our watermarking scheme GPTWatermark based on Definition 2.2. The outline of our watermarking scheme

Prompt	Q: what is codependent and why is it bad? \n A:
Human PPL: 7.87 z-score: 0.14	Had a psychology teacher explain it this way and it always stuck with me. this is a **codependent** relationship. Each line represents one person. Each relies on the other for support and stability. If you take either person out, the other will fall. this is a **dependent** relationship. One person is a self sufficient individual (represented by the vertical line). They can stand on their own with or without the other person. The leaning line is dependent on the other person. [continues...]
LLaMA-7B (un-watermarked) PPL: 12.19 z-score: 1.96	Imagine yourself in a relationship with a spouse, friend, family member, or co-worker who is like a drug addict. They go through cycles of behavior that can make you crazy. You think they are in the right, so you defend them. You feel so badly for them that you don't really tell them the truth about how their behaviors are hurting you, your kids, your family, or even other people. [continues...]
LLaMA-7B (watermarked) PPL: 9.47 z-score: 9.58	To define codependency as it relates to mental health, one has to understand what it means to be emotionally dependent. While a person can be dependent on another person for fulfillment, as evidenced through feelings of low self-esteem and fear of loss, a codependent person will often keep another person in their life despite their poor and sometimes abusive behavior. [continues...]

Table 1: Comparison of human-written text and machine-generated text examples for a given prompt question. Our method, utilizing the LLaMA-7B model, produces watermarked text of comparable quality to the un-watermarked text, without any noticeable degradation. However, there exists a significant disparity in the z-scores for watermark detection.

Algorithm 3 GPTWatermark: Detect

```
Input: suspect text $\boldsymbol{y}$, watermark detection key $\mathsf{k}$, threshold $\tau$.
Output: 1 or 0 (whether the text is watermarked).
Use the watermark detection key $\mathsf{k}$ to find the "green list" $G$.
Calculate the number of green list tokens $|\boldsymbol{y}|_G = \sum_{t=1}^{n} \mathbf{1}(y_t \in G)$ in $[y_1, \dots, y_n]$.
Compute the $z$-statistic:
    $z_{\boldsymbol{y}} = (|\boldsymbol{y}|_G - \gamma n) \,/ \sqrt{n\gamma(1-\gamma)}$.   (2)
if $z_{\boldsymbol{y}} > \tau$ then return 1, i.e., "The suspect text is watermarked."
else return 0, i.e., "The suspect text is not watermarked."
```

is provided in Algorithm 2 and 3. In Algorithm 2, we utilize a randomly generated watermark key to partition the vocabulary of the language model into two distinct sets: the green list and the red list. The logits of the language model for the green list tokens are increased by δ while the logits remaining tokens in the red list remain unchanged. This is the same procedure from (Kirchenbauer et al., 2023) with the objective of adaptively increasing the probability of generating tokens from the green list relative to those from the red list. Generation from the watermarked language model follows simply by passing the returned $\hat{\mathcal{M}}$ to Algorithm 1.

The watermarking procedure is parameterized by two *watermark strength parameters* γ, δ. γ determines the fraction of the vocabulary included in the green list. We typically set γ to be a constant, e.g., $1/3$ or 0.5. δ specifies the increase in the logits associated with the green list tokens. The larger δ is, the lower the quality of the watermarked LM, but the easier it is to detect.

The critical difference from (Kirchenbauer et al., 2023) is that our algorithm employs a fixed partition, as opposed to using the hash of previously generated tokens as a random seed in their soft watermarking algorithm. As a result, unlike (Kirchenbauer et al., 2023), our work guarantees information-theoretic security in the so-called plain model without relying upon cryptographic tools. In addition, we provide a formal definition of watermarking as well as a proof of the properties, which has not been done in (Kirchenbauer et al., 2023).

Our detection procedure is outlined in Algorithm 3. It calculates the total number of tokens from the suspect sequence $\boldsymbol{y}$ that fall within the green list: $\sum_{t=1}^{n} \mathbf{1}(y_t \in G)$. We start by presuming the null hypothesis H_0*: The text sequence is produced without regard to the green list rule*, and then compute a z-statistic using Equation 2. If the z-score exceeds a predetermined threshold, we reject the null hypothesis and identify the watermark. We show the examples of real prompts and watermarked outputs in Table 1.

It is worth noting that the detection procedure can be computed in linear time using the green list G and the suspect sequence $\boldsymbol{y}$ alone. It does not require access to either the prompt $\boldsymbol{x}$ or the language model $\mathcal{M}$. This feature makes it widely applicable to common situations where users of a language model often omit the prompt and may use only one chunk of the generated text. No costly computation of the logits is needed for the detection.

Overall, the proposed watermarking scheme requires almost no overhead in its implementation, is extremely simple, and is easy to maintain. The big question is:

```
How well does this watermark scheme
              work?
```

We provide answers to this question with provable guarantees on the properties from Definition 2.2 and extensive experiments. For a deeper understanding, we refer the reader to the Appendix section which contains all detailed explanations.

4. Limitations and conclusion

Conclusion. In this paper, we have addressed the concerns surrounding the potential misuse of large language models and proposed an effective watermarking approach, GPTWatermark, for detecting machine-generated text from a *specific* language model. Our contributions include the development of a rigorous theoretical framework, designing a provable effective and robust watermarking scheme under this framework, as well as conducting extensive experiments to demonstrate the effectiveness and robustness of our method in practice. We anticipate that our work will inspire future research to develop more resilient watermarking methods capable of withstanding a broader range of attacks.

Limitations. While our watermarking method, GPTWatermark, demonstrates improved robustness against edits, its reliance on a fixed Green-Red split may not be universally optimal. The performance and robustness of watermarking methods can vary depending on the specific characteristics of the LLM and the generated text. Additionally, although our method enhances detection capabilities, it is not immune to all possible attacks.

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A. Related work

Watermarking natural languages. The concept of watermarking, which involves hiding identifying information within data, has a long history. However, watermarking digital text has been challenging due to its discrete nature (Stefan et al., 2000). Early approaches relied on techniques such as synonym substitution (Topkara et al., 2006), syntactic structure restructuring (Atallah et al., 2001), or paraphrasing (Atallah et al., 2002). Later, advancements in modern neural language models led to improved methods that move away from rule-based approaches. Different approaches have been proposed, such as encoding messages by context-aware lexical substitution (Yang et al., 2022) or using mask-infilling models for editing text (Ueoka et al., 2021). Recent studies (Zhao et al., 2023; Kirchenbauer et al., 2023) explore modifying the logits of language models during token generation and embedding invisible watermarks in the decoding process. Our objective is to develop a robust watermarking technique for natural language models that maintains high text quality while effectively concealing identifying information.

Post-hoc detection. Rather than watermarking, an alternative approach involves developing detection models for post-hoc analysis of machine-generated text. Some detection methods use statistical outlier detection techniques without requiring additional training. For example, GLTR (Gehrmann et al., 2019) assesses the expected probability of individual tokens and applies thresholding to identify AI-generated content. DetectGPT (Mitchell et al., 2023) suggests that AI-generated passages tend to reside in the negative curvature of the log probability of texts. Another set of methods relies on classifiers that are fine-tuned to distinguish between human-written and machine-generated text. Initial efforts in this domain focus on detecting fake reviews (Hovy, 2016) and fake news (Zellers et al., 2019). More recently, OpenAI releases a web interface that uses a finetuned GPT model for this discrimination task (OpenAI, 2023b). However, as language models improve, AI-generated text is becoming increasingly similar to human-generated text, making it more challenging to detect. (Gambini et al., 2022) find that existing detection strategies designed for GPT-2 struggle with GPT-3. Moreover, known detectors are found to be fragile to adversarial attacks (Wolff, 2020) and biased towards non-native English writers (Liang et al., 2023).

Impossibility results. Sadasivan et al. (2023) pose the question of whether detecting machine-generated text is possible and argue that as the human distribution and LLM distribution of texts get closer, any classifier will have to either have a large Type I error or a large Type II error. The authors also argue that (in Corollary 2) if the watermarking scheme can be learned then paraphrasing attacks either evade the detector or also classify humans with a similar distribution as false positives. This does not invalidate our results as we made no theoretical claim about paraphrasing. Also, the learnability of the watermarking scheme is questionable too since the green-red lists are generated randomly — these can be seen as injecting a very special “style” to an LLM. The style being randomly generated makes sure that it is extremely unlikely for any human to develop the same style by chance.

Language model watermarks with provable guarantees. Concurrent to our work, Christ et al. (2023) consider the problem of formally defining watermarking language models and propose a construction with provable guarantees. The main differences between their work and ours are:

- In Christ et al. (2023), the watermarked distribution is computationally indistinguishable (i.e., indistinguishable against probabilistic polynomial-time algorithms) from the un-watermarked distribution whereas in our case, we insist that the watermarked distribution is statistically close to the un-watermarked distribution (of each token). The Type-I/Type-II error guarantees and the security properties are qualitatively different in both works.

- We both use different approaches to achieve our definitions. The advantage of our construction is that it satisfies robustness to edits property whereas they have no such guarantees. On the other hand, our construction uses a very different set of assumptions (e.g., high entropy) on the language model and prompt that appears to be incompatible with theirs.

- Finally, we implement our construction and conduct a thorough empirical evaluation to demonstrate its practicality while they don’t provide any implementation of their construction.

B. Theoretical guarantees

In this section, we present the quality, correctness, and security property of GPTWatermark as described in Definition 2.2. We will start with the security property which ensures the resilience of the watermark to a wide family of evasion attacks.

B.1. Security property of GPTWatermark

We demonstrate the robustness of our watermarking scheme against editing attempts through Theorem B.1. As a baseline of comparison, we also obtain new robustness guarantees for the soft watermarking method proposed in (Kirchenbauer et al., 2023). The detailed proof is deferred to the Appendix.

Theorem B.1 (Robustness to editing)**.** *Let $\boldsymbol{y} = [y_1, \dots, y_n]$ represent the watermarked sequence. Suppose the adversary $\mathcal{A}$ follows Definition 2.2 and outputs a modified text $\boldsymbol{u} = [u_1, \dots, u_m]$. Following Equation 2, we calculate z-score $z_{\boldsymbol{y}}$ and $z_{\boldsymbol{u}}$. Assume edit distance between $\boldsymbol{y}$ and $\boldsymbol{u}$ (denoted as η) satisfies $\eta < n$. Then we have*

$$z_{\boldsymbol{u}} \geq z_{\boldsymbol{y}} - \max\{\frac{(1+\gamma/2)\eta}{\sqrt{n}}, \frac{(1-\gamma/2)\eta}{\sqrt{n-\eta}}\}.$$

In particular, when $\eta \leq \frac{2\gamma n}{(1+\gamma/2)^2}$, we can drop the second term in the max.

Remark B.2. This theorem highlights the relationship between the z-scores in the modified sequence and the watermarked sequence, considering the maximum allowable edit distance η. It demonstrates the robustness of our algorithm against editing attacks, illustrating that the z-score of the modified sequence remains bounded when compared to the z-score in the watermarked sequence.

Remark B.3. Note that the green list can be completely revealed. This may initially seem counter-intuitive since, if the green list is known, one could potentially post-process the text using synonym replacement to balance the occurrence of words from the green and red lists. However, under the edit-distance metric (Definition 2.1), effective balancing cannot be achieved unless the number of edits exceeds η.

Corollary B.4. *Algorithm 3 with threshold τ satisfies the* **security property** *from Definition 2.2 with $\epsilon = 0$ and*

$$\eta(\boldsymbol{y}, \mathsf{k}, \epsilon) = \frac{\sqrt{n}(z_{\boldsymbol{y}} - \tau)}{1+\gamma/2} \mathbf{1}\left(z_{\boldsymbol{y}} - \tau < \frac{\gamma\sqrt{n}}{1+\gamma/2}\right).$$

In comparison, the best bound on the security property parameter one can obtain for the scheme of Kirchenbauer et al. (2023) is (a formal statement and proof are included in Appendix F.2)

$$\eta(\boldsymbol{y}, \mathsf{k}, \epsilon) = \frac{\sqrt{n}(z_{\boldsymbol{y}} - \tau)}{2+\gamma/2} \mathbf{1}\left(z_{\boldsymbol{y}} - \tau < \frac{\gamma\sqrt{n}}{2+\gamma/2}\right).$$

To say it differently, our method, GPTWatermark, utilizing a fixed Green-Red split, achieves *twice the robustness* to edits compared to Kirchenbauer et al. (2023)'s baseline approach.

B.2. Quality guarantee of GPTWatermark

The following theorems demonstrate that the distance between the original probability vector $\mathbf{p}_t$ and the watermarked probability vector $\hat{\mathbf{p}}_t$ are very close to each other in almost all popular metrics of probabilistic distances (and divergence).

Theorem B.5. *Consider $\boldsymbol{h}$ as the input to the language model at step t, denoted as $\boldsymbol{h} = [\boldsymbol{x}, \boldsymbol{y}_{1:t-1}]$. Fix green list G. Let δ represent the watermark strength. For any $\boldsymbol{h}$, the α-th order Renyi-divergence between the watermarked probability distribution $\hat{\mathbf{p}}_t = \hat{\mathbf{p}}_t(\cdot|\boldsymbol{h})$ at time step t and the original probability distribution $\mathbf{p}_t = \mathbf{p}_t(\cdot|\boldsymbol{h})$ satisfies:*

$$\forall \boldsymbol{h}, \max\left(D_\alpha\big(\hat{\mathbf{p}}_t \| \mathbf{p}_t\big), D_\alpha\big(\mathbf{p}_t \| \hat{\mathbf{p}}_t\big)\right) \leq \min\{\delta, \alpha\delta^2/8\}.$$

The proof, deferred to the appendix, leverages a surprising connection to modern techniques in the differential privacy literature (Dwork et al., 2006; Dong et al., 2020).

Remark B.6 (KL-divergence and other probability distance metrics). Renyi-divergence is very general. Kullback-Leibler-divergence and chi-square divergence are directly implied by the α-Renyi divergence bound of $\min\{\delta, \alpha\delta^2/8\}$ by choosing $\alpha = 1$ and $\alpha = 2$ respectively and swap $\hat{\mathbf{p}}$ and $\mathbf{p}$. Hellinger distance can be obtained by choosing $\alpha = 0.5$. By Pinsker's inequality, we get a Total Variation distance bound of $\min\{\sqrt{\delta/2}, \delta/4\}$. Moreover, by choosing $\alpha \to \infty$, we obtain an upper bound of δ for a very strong multiplicative guarantee known as max-divergence. The resulting two distributions $\hat{\mathbf{p}}$ and $\mathbf{p}$ are referred to by cryptographers as $(\delta, 0)$-*indistinguishable*, which says that for any measurable event S, the log-odds ratio satisfies

$$-\delta \leq \log \frac{\hat{\mathbf{p}}_t(y_t \in S|\boldsymbol{h})}{\mathbf{p}_t(y_t \in S|\boldsymbol{h})} \leq \delta.$$

To summarize, our result shows that Algorithm 2 produces $\hat{\mathcal{M}}$ that satisfies ω-quality of watermarked output with ω (as a function of δ) for almost all commonly used probability distance D.

B.3. Type I error of GPTWatermark

Theorem B.7 (No false positives)**.** *Consider $\boldsymbol{y} = \boldsymbol{y}_{1:n}$ as any* fixed *suspect text. Let $N =: |\mathcal{V}|$ and $G \subset |\mathcal{V}|$ satisfying $|G| = \gamma N$. G is selected through Algorithm 2, using a uniform random choice. Let $|\boldsymbol{y}|_G$ denote the number of tokens in G and $z_{\boldsymbol{y}} := \frac{|\boldsymbol{y}|_G - \gamma n}{\sqrt{n\gamma(1-\gamma)}}$ as in Algorithm 3. Then the following statements hold true:*

1. *Assume $n \geq 1$, then*
$$\mathbb{E}[|\boldsymbol{y}|_G | \boldsymbol{y}] = \gamma n \quad \textit{and} \quad \mathbb{E}[z_{\boldsymbol{y}} | \boldsymbol{y}] = 0.$$

2. *Define $C_{\max}(\boldsymbol{y}) := \max_{i \in [N]} \sum_{j=1}^{n} \mathbf{1}(y_j = i)$ and $V(\boldsymbol{y}) := \frac{1}{n} \sum_{i=1}^{N} (\sum_{j=1}^{n} \mathbf{1}(y_j = i))^2$, then with probability $1 - \alpha$ (over only the randomness of G),*
$$\mathbb{P}\left[|\boldsymbol{y}|_G \geq \gamma n + \sqrt{64\gamma n V \log(9/\alpha)} + C_{\max} \log(9/\alpha) \middle| \boldsymbol{y}\right] \leq \alpha$$
or equivalently (when $n \geq 1$)
$$\mathbb{P}\left[z_{\boldsymbol{y}} \geq \sqrt{\frac{64V \log(9/\alpha)}{c(1-\gamma)}} + \frac{C_{\max} \log(9/\alpha)}{\sqrt{n\gamma(1-\gamma)}} \middle| \boldsymbol{y}\right] \leq \alpha.$$

Remark B.8 (Wide applicability). Note that the theorem does not impose assumptions on how $\boldsymbol{y}$ is generated. It covers any procedure (including human generation) that produces $\boldsymbol{y}$ in a manner *independently* of the secret partition G. In cases where $\boldsymbol{y}$ is generated by a language model, it could be the output of greedy search from $\mathbf{p}(y_t | \boldsymbol{x}, \boldsymbol{y}_{1:t-1})$, nucleus sampling, beam search, or any other decoding methods.

Remark B.9 (Diversity parameters). The V and $C_{\max}$ parameters in Theorem B.7 measure the *diversity* of the suspect text $\boldsymbol{y}$ and are necessary for the high-probability bound. As an example, if the prompt says "`Repeat "Goal" for a hundred thousand times like a soccer commentator.`" Then the resulting generated sequence will be "`Goal goal goal ...`", and has either n green tokens or 0 green tokens. No meaningful Type I error bound can be obtained.

Remark B.10 (Controlling false positive rate). The theorem implies that if we choose $\tau > \sqrt{\frac{64V \log(9/\alpha)}{c(1-\gamma)}} + \frac{C_{\max} \log(9/\alpha)}{\sqrt{n\gamma(1-\gamma)}}$, then the false-positive rate is smaller than α. Note that V and $C_{\max}$ can be computed directly from $\boldsymbol{y}$, allowing us to choose an input-dependent τ as a function of $V, C_{\max}$ that achieves a α-Type I error guarantee with a fixed α for all inputs. In particular, the Type I error α decreases exponentially as we increase the threshold τ.

Remark B.11 (Robustness to edits). When combined with Theorem B.1, it implies that if we choose $\tau > \sqrt{\frac{64V \log(9/\alpha)}{c(1-\gamma)}} + \frac{C_{\max} \log(9/\alpha)}{\sqrt{n\gamma(1-\gamma)}} + \max\{\frac{(2+\gamma/2)\eta}{\sqrt{n}}, \frac{(2-\gamma/2)\eta}{\sqrt{n-\eta}}\}$, it guarantees that the false-positive rate is smaller than α for any adversary that edits the sentence arbitrarily by at most η.

B.4. Type II error of GPTWatermark

To bound the Type II error, i.e., false negative rates, we need to make certain assumptions about $\mathbf{p}$ of the language model and the prompt $\boldsymbol{x}$. These assumptions include a "on-average high entropy" assumption and a "homophily" condition. We will provide a detailed definition and discussion of these assumptions in Appendix E.4.1 and Appendix E.4.2, but let us first explain them informally with examples.

On-average high entropy. The "on-average high entropy" assumption requires the probability of the roll-out to be "sufficiently diverse" on average. Specifically, it requires a subset of $\sum_t \|\mathbf{p}_t\|_2^2$, $\|\sum_t \mathbf{p}_t\|_2$, $\sum_t \|\mathbf{p}_t\|_\infty^2$ or $\|\sum_t \mathbf{p}_t\|_\infty$ to be small either in expectation or with high probability. These bounds can be viewed as requiring a lower bound on the average of certain Tsallis entropy — a generalization of the standard Shannon entropy. They are related but different from the "spike entropy" assumption used by Kirchenbauer et al. (2023).

The high-entropy assumption is necessary to ensure that the increases in the logits have an effect on the generated outcome. For example, if the prompt writes

“Generate the English alphabet in capital letters for 200 times please.”

Then the language model would generate

“ABC...XYZ, ABC...XYZ, ...”.

Despite that the generated sequence is very long, i.e., n is as large as $5,200$, the added watermark does not change the distribution very much. To see this, if $\mathbf{p}(y_3 = \text{“C”}|\boldsymbol{x}, \boldsymbol{h}) \geq 1 - \epsilon$ for a tiny ϵ, and then by our quality guarantee, $\hat{\mathbf{p}}(y_3 = \text{“C”}\ |\boldsymbol{x}, \boldsymbol{h}) \geq 1 - \epsilon e^{\delta}$.

Homophily. We also require a “homophily” assumption about the distribution induced by the state-transitions of the language model $\mathcal{M}$ which essentially implies that if we roll in with $\hat{\mathcal{M}}$ to step t instead of $\mathcal{M}$, then it will not make the probabilities of seeing the green-list words less likely.

This “homophily” assumption is needed to rule out the unnatural situation where increasing the green list tokens initially ends up reducing the number of green list tokens in the long run. To illustrate this, consider the following example utilizing the prompt:

$\boldsymbol{x} =$ “Randomly select a color, state what it is. Then write a short poem about it without naming this color at all.”

The generated text from a commercial language model is

“Color choice: green
Emerald whispers in the meadow’s sway, Life’s verdant rhythm in ceaseless play.
It cradles the world in a leafy embrace, A silent serenade to nature’s grace.”

Notice that if the token “green” $\in G$, it increases the probability of the language model generating “green” at the beginning. However, regardless of the text’s length, the subsequent portion of the generated text will not contain the word “green”, as instructed by the prompt. This decreases the expected number of times the token “green” appears.

Theorem B.12 (Only true detection)**.** *For a fixed language model $\mathcal{M}$ and a prompt $\boldsymbol{x}$. The sentence $\boldsymbol{y}_{1:n}$ generated from $\hat{\mathcal{M}}(\boldsymbol{x})$ where $\hat{\mathcal{M}}$ is an output of our watermarking scheme* $\mathsf{Watermark}_{\delta,\gamma}(\mathcal{M})$ *with parameter δ, γ. Then the following statements are true.*

1. *Assume homophily (Assumption E.8), then*

$$\mathbb{E}[|\boldsymbol{y}|_G] \geq \frac{n\gamma e^{\delta}}{1 + (e^{\delta} - 1)\gamma} - \gamma(1-\gamma)e^{\delta} \sum_{t=1}^{n} \mathop{\mathbb{E}}_{\boldsymbol{y}_{1:t-1} \sim \mathbf{p}(\cdot|x)} \|\mathbf{p}_t\|^2.$$

In particular, if Assumption E.5 condition is true with parameter $\xi \leq (1-\kappa)\frac{e^{\delta}-1}{(1+(e^{\delta}-1)\gamma)e^{\delta}}$ for a parameter $0 < \kappa < 1$, then

$$\mathbb{E}[|\boldsymbol{y}|_G] \geq n\gamma \left(1 + \kappa \frac{(e^{\delta}-1)(1-\gamma)}{1 + (e^{\delta}-1)\gamma}\right) \text{ or equivalently } \mathbb{E}[z_{\boldsymbol{y}}] \geq \frac{\kappa(e^{\delta}-1)\sqrt{n\gamma(1-\gamma)}}{1 + (e^{\delta}-1)\gamma}.$$

2. *Assume high-probability version of homophily (Assumption E.9). There exists a parameter $C_{\delta,\gamma}$ that depends only δ, γ such that with probability at least $1 - \beta$ for any $\beta > 0$ (over both G and $\boldsymbol{y} \sim \hat{\mathbf{p}}(\cdot|\boldsymbol{x}, G)$),*

$$\begin{aligned} \|\boldsymbol{y}\|_G \geq & \frac{n\gamma e^{\delta}}{1 + (e^{\delta}-1)\gamma} - \sqrt{2n\log(6/\beta)} \\ & - C_{\delta,\gamma} \log^2 \frac{27(n+1)}{\beta} \left(\|\sum_{t=1}^{n} \mathbf{p}_t\| + \sum_{t=1}^{n} \|\mathbf{p}_t\|^2 + \|\sum_{t=1}^{n} \mathbf{p}_t\|_{\infty} + \sum_{t=1}^{n} \|\mathbf{p}_t\|_{\infty}^2 \right). \end{aligned}$$

In particular, if for a parameter $0 < \kappa < 1$,

$$n \geq \frac{8\log(6/\beta)(1-\gamma+e^{\delta}\gamma)^2}{(1-\kappa)^2\gamma^2(1-\gamma)^2(e^{\delta}-1)^2} = \tilde{\Omega}(1/\delta^2) \quad (3)$$

and Assumption E.6 condition is true with parameter $(\xi, \beta/3)$ *where*

$$\xi \leq \frac{(1-\kappa)\gamma(1-\gamma)(e^{\delta}-1)}{8C_{\delta,\gamma}(1-\gamma+e^{\delta}\gamma)\log^2\left(\frac{27(n+1)}{\beta}\right)} = \tilde{O}(\delta), \tag{4}$$

then

$$\mathbb{P}\left[\|\boldsymbol{y}\|_G < n\gamma(1+\kappa\frac{(e^{\delta}-1)(1-\gamma)}{1-\gamma+\gamma e^{\delta}})\right] = \mathbb{P}\left[z_{\boldsymbol{y}} < \frac{\kappa(e^{\delta}-1)\sqrt{n\gamma(1-\gamma)}}{1+(e^{\delta}-1)\gamma}\right] \leq \beta.$$

Remark B.13 (Exponentially small Type I and Type II error guarantees). Recall that according to Theorem B.7, in order to have a false positive rate controlled at level α, we need to set the threshold $\tau \gtrsim \sqrt{\log(1/\alpha)}$ for sufficiently high-entropy sequences. Theorem B.12 says that if we want the false negative rate to be smaller than β, we only need the threshold $\tau \lesssim \kappa\delta n$ under similar (slightly different) high-entropy sequences for $n \gtrsim \log(1/\beta)/\delta^2$. Observe that there is a wide range of valid choices of τ for us to have a detection algorithm that does not make Type I or Type II error with high probability. These observations together suggest that we can afford to choose $\delta \asymp 1/\sqrt{n}$ if the sequence is sufficiently high-entropy.

Remark B.14 (Information-theoretic optimality). The sample complexity of $n \gtrsim 1/\delta^2$ is information-theoretically optimal (up to a logarithmic factor) in δ because, our accuracy guarantee (together with the composition theorem) indicates that the KL-divergence between a sequence of length n generated from $\mathbf{p}$ and that generated from $\hat{\mathbf{p}}$ is $n\delta^2$ indistinguishable, i.e., $n > 1/\delta^2$ for any classifier — even the uniform most-powerful Neyman-Pearson likelihood-ratio test (which requires additional information, e.g., $\boldsymbol{x}$ and $\mathbf{p}$ which we do not have) — to make no mistakes with a constant probability.

C. Experiment

In this section, we aim to conduct experiments to demonstrate the performance of watermark detection, the quality of watermarked text, and the robustness against various attacking schemes, as compared to the baseline method.

C.1. Experiment setting

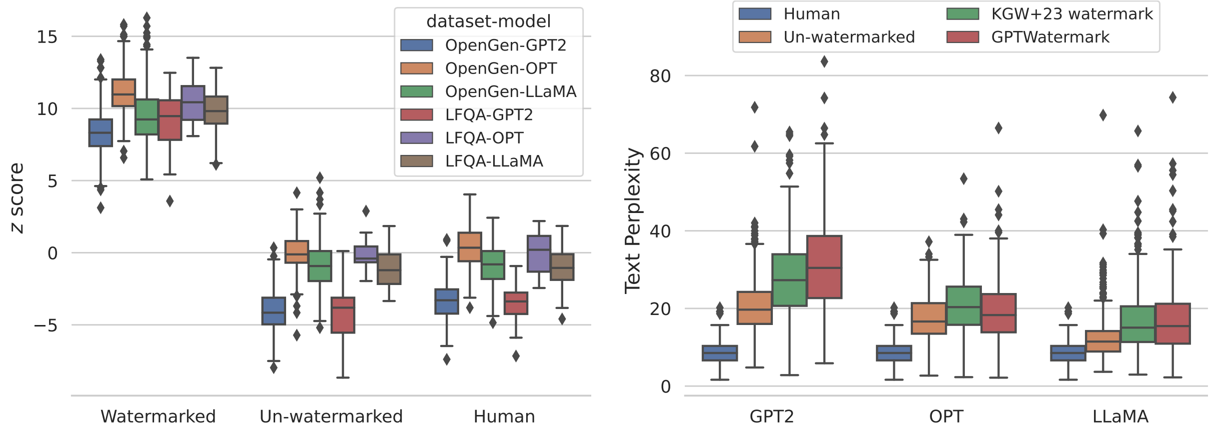
Datasets and prompts. We utilize two long-form text datasets: OpenGen and LFQA. OpenGen, collected by (Krishna et al., 2023), consists of 3K two-sentence chunks sampled from the validation split of WikiText-103 (Merity et al., 2017). The subsequent 300 tokens serve as the human-written continuation. LFQA is a long-form question-answering dataset created by (Krishna et al., 2023) by scraping questions from Reddit, posted between July and December 2021, across six domains. (Krishna et al., 2023) randomly select 500 questions from each domain and pair them with their corresponding longest human-written answers, resulting in 3K QA pairs. In our experiments, we use the questions as prompts and the corresponding answers as human-written text.

Language models. We conduct experiments using three state-of-the-art public language models of varying sizes from different model families: GPT2-XL with 1.5B parameters (Radford et al., 2019), OPT-1.3B (Zhang et al., 2022), and LLaMA-7B (Touvron et al., 2023). Nucleus Sampling (Holtzman et al., 2020) is employed as the default decoding algorithm to introduce randomness while maintaining human-like text output. The models are loaded from the Huggingface library (Wolf et al., 2019), and the `generate` API function is used to adjust the logits distribution of the language model.

Evaluation methods. Maintaining a low false positive rate is crucial to prevent misclassifying un-watermarked text as watermarked. To ensure this, we set the false positive rates at 1% and 10% for all detection algorithms and adjust the detection threshold accordingly. We report the true positive rate (TPR), F1 score, and present ROC curves. GPT3 (`text-davinci-003`), with 175 billion parameters and reinforcement learning from human feedback (Ouyang et al., 2022), is used as the oracle model for perplexity evaluation. The experiments are conducted on Nvidia A100 GPUs.

C.2. Watermarking results

We use a watermark strength of $\delta = 2.0$ and a green list ratio of $\gamma = 0.5$. We also use different watermark keys k for different models. Stronger watermarks can be achieved for shorter sequences for a smaller γ and a larger δ. From the two datasets, we generate 500 watermarked sentences and 500 un-watermarked sentences using three different models (GPT2-XL, OPT-1.3B, and LLaMA-7B). We label them as “watermarked” and “un-watermarked” respectively. We also have corresponding human-written text for each prompt, referred to as "human". All sentences are cropped to a length of 200 tokens. z-scores are calculated for hypothesis testing as shown in Algorithm 3 between different sentence groups.



(a) z-scores of watermarked and un-watermarked machine-generated text, along with the z-score of human-generated text. The watermarked text z-score surpasses the empirical threshold of $z = 6.0$.

(b) Text perplexity comparison (evaluated by GPT-3) between human-generated text and text generated by various models on the OpenGen dataset.

Figure 1: z-score comparison and text perplexity comparison.

The results (Figure 1a) indicate a clear distinction between watermarked and non-watermarked text. A default threshold of z-score $= 6.0$ can be used to determine if a text is watermarked. For a fair comparison with (Kirchenbauer et al., 2023), we also set $\delta = 2.0$ and $\gamma = 0.5$ for their method.

Figure 1b demonstrates the text perplexity of human, un-watermarked machine-generated, and two watermarking-generated texts, evaluated on the OpenGen dataset. The perplexity of human text is significantly lower, likely due to the expertise contributed in the Wikipedia-based dataset used to train GPT3. We observe that the perplexity of the watermarked text is comparable to that of human-generated text, especially with the use of the largest model LLaMA-7B. This finding further supports the effectiveness of our method in preserving linguistic characteristics and coherence, ensuring seamless integration of watermarks without compromising overall text quality. One example of the prompt questions, human answers, and machine-generated answers can be found in Table 1.

C.3. Robustness results

One of the key advantages of our method is its robustness. To provide comprehensive evidence of its resilience, we conduct experiments to test its resilience against various attacking methods.

Paraphrasing attack. In the soft watermarking scheme proposed by (Kirchenbauer et al., 2023), the selection of an output token from the language model's green list relies on the token's prefix. However, this approach is vulnerable to paraphrase attacks that aim to remove the watermark signature. To demonstrate the superior robustness of our method, supported by our theorem, we devise experiments to compare its performance against (Kirchenbauer et al., 2023). We employ different paraphrase attack techniques targeting the removal of the watermark. Firstly, we utilized two versions of the DIPPER model(Krishna et al., 2023), we denote them as "DIPPER-1" and "DIPPER-2". DIPPER-2 has greater diversity than DIPPER-1. Additionally, we leverage the ChatGPT API, generating paraphrased text by providing prompts such as *Rewrite the following paragraph:*. Furthermore, we employ BART (Lewis et al., 2019) (`bart-large-cnn`, a large-sized model fine-tuned on the CNN Daily Mail dataset (Hermann et al., 2015)) for text summarization as another type of paraphrasing attack. The results of our experiments are shown in Figure 2 and Table 2. We also show the true positive rate, F1 score for false positive rates at 1% and 10%. The results illustrate the substantial improvement in robustness achieved by our method compared to (Kirchenbauer et al., 2023). Notably, our method achieves an accuracy rate of over 85% with a false positive rate of 10%.

Editing attack. To further evaluate the robustness of our method against edit attacks, we examine its performance when subjected to synonym replacement, random deletion, and random swapping. These edit attack scenarios represent common techniques used to manipulate text and potentially remove watermarks. We conduct various editing attacks for the watermarked text of GPTWatermark and KGW+23. The results are shown in Figure 2. In each scenario, our

Setting	Method	OpenGen				LFQA			
		1% FPR		10% FPR		1% FPR		10% FPR	
		TPR	F1	TPR	F1	TPR	F1	TPR	F1
No attack	KGW+23	1.000	0.995	1.000	0.952	1.000	0.995	1.000	0.952
	GPTWatermark	1.000	0.995	1.000	0.952	1.000	0.995	1.000	0.952
ChatGPT	KGW+23	0.565	0.704	0.853	0.747	0.327	0.453	0.673	0.490
	GPTWatermark	0.866	0.910	0.961	0.818	0.442	0.568	0.865	0.584
DIPPER-1	KGW+23	0.386	0.546	0.738	0.720	0.372	0.534	0.740	0.767
	GPTWatermark	0.729	0.830	0.922	0.837	0.639	0.770	0.909	0.865
DIPPER-2	KGW+23	0.490	0.646	0.810	0.769	0.432	0.595	0.845	0.839
	GPTWatermark	0.777	0.862	0.941	0.852	0.693	0.810	0.948	0.894
BART	KGW+23	0.342	0.505	0.667	0.759	0.457	0.617	0.783	0.836
	GPTWatermark	0.590	0.730	0.861	0.857	0.656	0.784	0.885	0.897

Table 2: Performance comparison of our method (GPTWatermark) and the soft watermarking method proposed in (Kirchenbauer et al., 2023) (denoted as KGW+23). Both methods employ LLaMA-7B with nucleus sampling, utilizing $\delta = 2.0$ and $\gamma = 0.5$. We use ChatGPT, DIPPER, and BART for paraphrasing the watermarked text as paraphrasing attacks. True positive rate and F1 score are presented for fixing the false positive rates at 1% and 10%. When there is no attack, both methods exhibit perfect watermark detection. Nevertheless, when subjected to paraphrasing attacks, GPTWatermark consistently outperforms KGW+23.

method consistently outperforms (Kirchenbauer et al., 2023) watermarking scheme, showcasing its enhanced resilience and effectiveness in protecting the integrity of the embedded watermarks.

C.4. Distinguishing human-written text

An interesting observation emphasized by (Liang et al., 2023) is the misclassification of non-native English writing samples as AI-generated by existing AI content detectors. In light of this, our method has the unique capability to effectively establish the origin of suspicious text and maintain its robustness against distribution shifts. We evaluate the effectiveness of our watermark in distinguishing human-written text on a dataset of human-written TOEFL essays collected by (Liang et al., 2023) (more details are deferred to the Appendix). Our method demonstrates a remarkable ability to accurately classify human-written text, as evidenced by significantly lower z-scores compared to the empirical threshold of $z = 6.0$. This outcome underscores the effectiveness of our watermark in discerning text generated by human authors, further enhancing its practical utility and reliability.

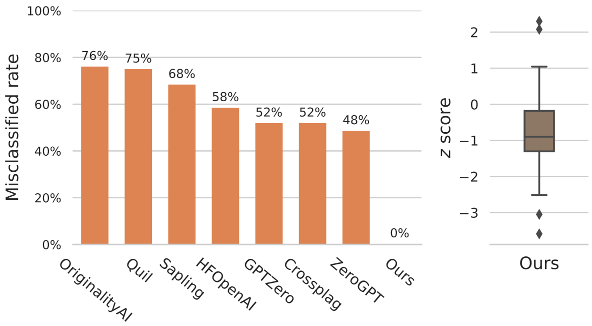
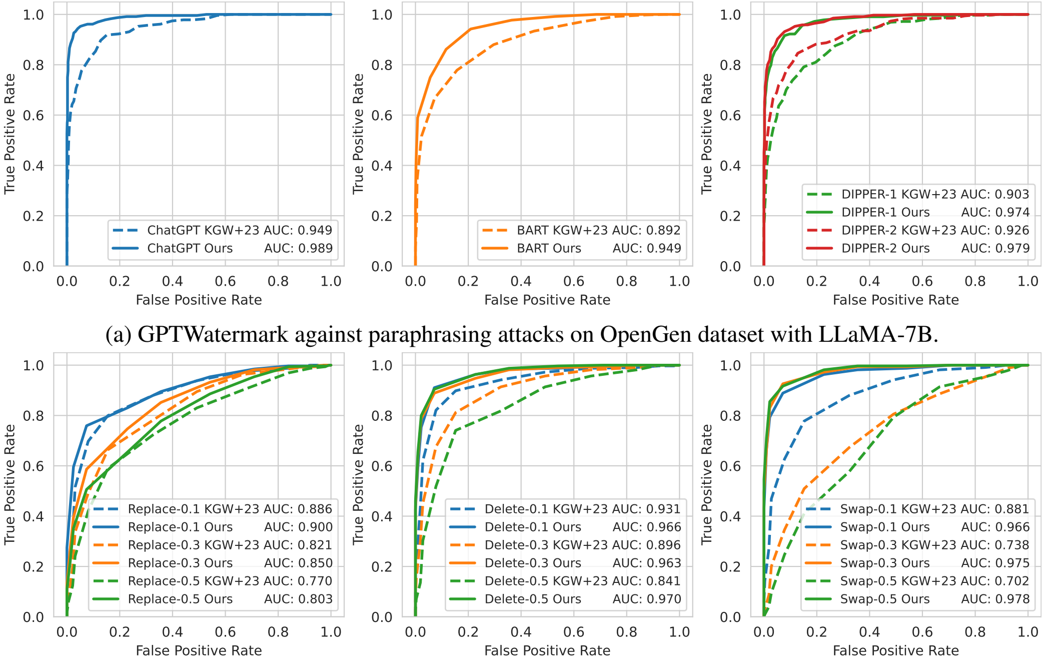


Figure 3: Distinguishing human-written text on TOEFL dataset.



(a) GPTWatermark against paraphrasing attacks on OpenGen dataset with LLaMA-7B.

(b) GPTWatermark against editing attacks on LFQA dataset with LLaMA-7B. We vary the rates of synonym replacement, random deletion, and random swapping (0.1, 0.3, 0.5) to demonstrate different attack scenarios.

Figure 2: ROC curves with corresponding AUC values for watermark detection against various attack methods. Complete results can be found in the Appendix. Our method (GPTWatermark) exhibits superior robustness compared to the baseline (KGW+23) across both datasets and all attack scenarios.

D. Additional experiment results

D.1. Empirical error rates

We perform experiments on two datasets (OpenGen and LFQA) using three different models (GPT2-XL, OPT-1.3B, and LLaMA-7B). Table 3 presents the error rates, showcasing the sensitivity of the resulting hypothesis test based on observed z-scores. The results demonstrate that there are no Type-I (false positive) errors for all models, with true positive rates exceeding 0.94 for a threshold of $z = 6.0$.

		$z = 6.0$				$z = 7.0$			
Dataset	Model	FPR	TNR	TPR	FNR	FPR	TNR	TPR	FNR
OpenGen	GPT2-XL	0.0	1.0	0.943	0.057	0.0	1.0	0.832	0.168
	OPT-1.3B	0.0	1.0	0.998	0.002	0.0	1.0	0.996	0.004
	LLaMA-7B	0.0	1.0	0.974	0.026	0.0	1.0	0.911	0.089
LFQA	GPT2-XL	0.0	1.0	0.948	0.052	0.0	1.0	0.889	0.111
	OPT-1.3B	0.0	1.0	1.000	0.000	0.0	1.0	0.997	0.003
	LLaMA-7B	0.0	1.0	0.976	0.024	0.0	1.0	0.942	0.058

Table 3: Empirical error rates for watermark detection using different models on two datasets. All models employ nucleus sampling with $\delta = 2.0$ and $\gamma = 0.5$. No Type-I (false positive) errors are observed across all models.

D.2. Different watermark parameters

We conduct an analysis to understand the impact of changing watermark strength (δ), green list size (γ), and sampling methods on two datasets. The results are summarized in Table 4. When using nucleus sampling with a fixed $\gamma = 0.5$, increasing the watermark strength resulted in higher true positive rates (TPR), but it also led to an increase in perplexity (lower quality). Furthermore, for the same watermark strength δ, varying the green list ratio from 0.25 to 0.5 and 0.75 showed improved detection results with smaller γ. Additionally, we explore different decoding methods, transitioning from nucleus sampling to multinomial sampling and beam search. Remarkably, watermark detection performed effectively with all decoding methods. It is worth noting that the perplexity score for beam search is significantly lower than that of nucleus sampling. However, beam search tends to generate shorter sequences with repeated words.

					$z = 6.0$				$z = 7.0$			
Dataset	decoding	δ	γ	PPL	FPR	TNR	TPR	FNR	FPR	TNR	TPR	FNR
OpenGen	nucleus	1.0	0.5	$18.37_{6.45}$	0.0	1.0	0.576	0.424	0.0	1.0	0.310	0.690
	nucleus	2.0	0.5	$19.42_{8.78}$	0.0	1.0	0.998	0.002	0.0	1.0	0.996	0.004
	nucleus	5.0	0.5	$19.44_{15.02}$	0.0	1.0	1.000	0.000	0.0	1.0	1.000	0.000
	nucleus	10.0	0.5	$19.20_{18.01}$	0.0	1.0	1.000	0.000	0.0	1.0	1.000	0.000
	nucleus	2.0	0.25	$17.96_{9.54}$	0.0	1.0	1.000	0.000	0.0	1.0	1.000	0.000
	nucleus	2.0	0.75	$20.03_{7.67}$	0.0	1.0	0.820	0.180	0.0	1.0	0.485	0.515
	m-nom.	2.0	0.5	$1.75_{0.59}$	0.0	1.0	0.951	0.049	0.0	1.0	0.924	0.076
	4-beams	2.0	0.5	$1.83_{0.97}$	0.0	1.0	0.992	0.008	0.0	1.0	0.982	0.018
	6-beams	2.0	0.5	$1.89_{1.10}$	0.0	1.0	0.984	0.016	0.0	1.0	0.982	0.018
	8-beams	2.0	0.5	$1.96_{1.23}$	0.0	1.0	0.986	0.014	0.0	1.0	0.984	0.016
LFQA	nucleus	1.0	0.5	$18.63_{7.19}$	0.0	1.0	0.455	0.545	0.0	1.0	0.199	0.801
	nucleus	2.0	0.5	$19.14_{11.11}$	0.0	1.0	1.000	0.000	0.0	1.0	0.997	0.003
	nucleus	5.0	0.5	$16.37_{15.39}$	0.0	1.0	1.000	0.000	0.0	1.0	1.000	0.000
	nucleus	10.0	0.5	$16.07_{14.25}$	0.0	1.0	0.998	0.002	0.0	1.0	0.998	0.002
	nucleus	2.0	0.25	$15.27_{10.00}$	0.0	1.0	1.000	0.000	0.0	1.0	1.000	0.000
	nucleus	2.0	0.75	$19.44_{8.20}$	0.0	1.0	0.893	0.107	0.0	1.0	0.582	0.418
	m-nom.	2.0	0.5	$3.17_{2.39}$	0.0	1.0	0.934	0.066	0.0	1.0	0.914	0.086
	4-beams	2.0	0.5	$3.24_{2.85}$	0.0	1.0	0.990	0.010	0.0	1.0	0.986	0.014
	6-beams	2.0	0.5	$3.20_{2.52}$	0.0	1.0	0.994	0.006	0.0	1.0	0.994	0.006
	8-beams	2.0	0.5	$3.13_{2.37}$	0.0	1.0	0.994	0.006	0.0	1.0	0.992	0.008

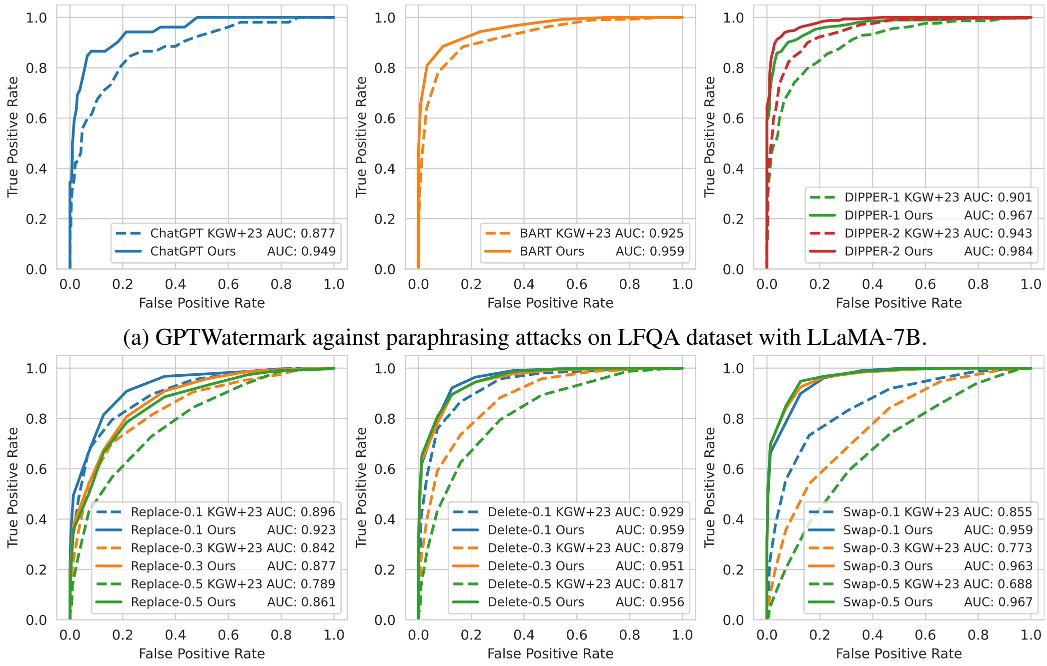
Table 4: Comparison of empirical error rates for watermark detection using nucleus sampling, multinomial decoding, and beam search. Each row represents the average of 500 sequences. While sequences generated with beam search exhibit lower perplexity, they tend to favor shorter outputs, potentially resulting in less diverse text.

D.3. Additional robustness results

In addition to the previously discussed robustness evaluations, we provide further analysis of our method's resilience against paraphrasing attacks and editing attacks. The results are presented in Figure 4. Notably, our proposed method (GPTWatermark) consistently outperforms the baseline approach (KGW+23) across various datasets and attack scenarios. This demonstrates the superior robustness of our method in accurately detecting watermarked text.

E. Proofs of the theoretical results

In this section, we provide the full proof details for the guarantees for GPTWatermark which certifies the required quality, correctness, and security properties of a language model watermarking scheme from Definition 2.2.



(a) GPTWatermark against paraphrasing attacks on LFQA dataset with LLaMA-7B.

(b) GPTWatermark against editing attacks on OpenGen dataset with LLaMA-7B. We vary the rates of synonym replacement, random deletion, and random swapping (0.1, 0.3, 0.5) to demonstrate different attack scenarios.

Figure 4: ROC curves with corresponding AUC values for watermark detection against various attack methods.

E.1. Quality guarantees

We start by providing a strong utility analysis of the watermarked language model than the "perplexity" bound from (Kirchenbauer et al., 2023). Our results work for the entire family of Rényi-divergence and imply guarantees in Kullback-Leibler (KL) divergence and Total Variation-distance.

The Renyi-divergence of two distributions P, Q is defined as

$$D_\alpha(P\|Q) = \frac{1}{\alpha-1}\log \mathop{\mathbb{E}}_{x\sim Q}\left[(\frac{dP}{dQ})^\alpha\right]$$

where $\frac{dP}{dQ}$ is the Radon–Nikodym derivative. When $\alpha \to 1$, the Renyi divergence converges to the KL-divergence. Additionally, when $\alpha = 0.5$, it serves as an upper bound for the TV-distance.

On the technical level, we leverage a surprising connection to a modern machinery developed in the differential privacy literature known as "bounded range" analysis (Dong et al., 2020) of the classical exponential mechanism (McSherry and Talwar, 2007).

Theorem E.1 (Restatement of Theorem B.5)**.** *Consider $\boldsymbol{h}$ as the input to the language model at step t, denoted as $\boldsymbol{h} = [\boldsymbol{x}, \boldsymbol{y}_{1:t-1}]$. Fix green list G. Let δ represent the watermark strength. For any $\boldsymbol{h}$, the α-th order Renyi-divergence between the watermarked probability distribution $\hat{\mathbf{p}}_t = \hat{\mathbf{p}}_t(\cdot|\boldsymbol{h})$ at time step t and the original probability distribution $\mathbf{p}_t = \mathbf{p}_t(\cdot|\boldsymbol{h})$ satisfies:*

$$\forall \boldsymbol{h}, \max\left(D_\alpha(\hat{\mathbf{p}}_t\|\mathbf{p}_t), D_\alpha(\mathbf{p}_t\|\hat{\mathbf{p}}_t)\right) \le \min\{\delta, \alpha\delta^2/8\}.$$

Proof. We define $\delta_v = 0$ when $v \in R$ and $\delta_v = \delta$ when $v \in G$. Using this definition, we have:

$$\hat{\mathbf{p}}(v|\boldsymbol{h}) = \frac{\exp(\boldsymbol{\ell}_v + \delta_v)}{\sum_w \exp(\boldsymbol{\ell}_w + \delta_w)} \le \frac{\exp(\delta)\exp(\boldsymbol{\ell}_v)}{\exp(-\delta)\sum_w \exp(\boldsymbol{\ell}_w)} = e^{2\delta}\mathbf{p}(v|\boldsymbol{h})$$

Similarly, $\hat{\mathbf{p}}(v|\boldsymbol{h}) \ge e^{-2\delta}\mathbf{p}(v|\boldsymbol{h})$.

Consequently, $\hat{\mathbf{p}}$ and $\mathbf{p}$ are 2δ-close in terms of max-divergence, which can be interpreted as $(\epsilon, \tilde{\delta})$-indistinguishable, similar to the concept of Differential Privacy (Dwork et al., 2006) with $\tilde{\delta} = 0$ and $\epsilon = 2\delta$.

Additionally, $\hat{\mathbf{p}}(v|\boldsymbol{h})$ and $\mathbf{p}(v|\boldsymbol{h})$ satisfy δ-BoundedRange (Proposition 1 in (Dong et al., 2020)) with parameter δ, since the changes to $\boldsymbol{\ell}_v$ is monotonic. Lemma 3.2 in (Cesar and Rogers, 2021) shows that δ-Bounded Range implies $\delta^2/8$-concentrated differential privacy, which says that $D_\alpha(\hat{\mathbf{p}}\|\mathbf{p}) \leq \frac{\delta^2\alpha}{8}$ for all $\alpha \geq 1$ (where D_α represents Rényi Divergence of order α). Specifically, when $\alpha = 1$, the KL-divergence satisfies $D_{\mathrm{KL}}(\hat{\mathbf{p}}\|\mathbf{p}) \leq \frac{\delta^2}{8}$.

Furthermore, δ-BoundedRange implies δ-DP (or rather $(\delta, 0)$-indistinguishability, since we are dealing with just two distributions rather than a family of neighbor distributions). It follows from the that

$$D_{\mathrm{KL}}(\hat{\mathbf{p}}\|\mathbf{p}) \leq D_\infty(\hat{\mathbf{p}}\|\mathbf{p}) \leq \delta$$

□

Corollary E.2. *For any prompt $\boldsymbol{x}$, the KL-divergence between the probability distribution of the watermarked sequence and the original sequence satisfies:*

$$\forall \boldsymbol{x}, \max\{D_{\mathrm{KL}}\big(\hat{\mathbf{p}}(\boldsymbol{y}_{1:n}|\boldsymbol{x})\|\mathbf{p}(\boldsymbol{y}_{1:n}|\boldsymbol{x})\big), D_{\mathrm{KL}}\big(\mathbf{p}(\boldsymbol{y}_{1:n}|\boldsymbol{x})\|\hat{\mathbf{p}}(\boldsymbol{y}_{1:n}|\boldsymbol{x})\big)\} \leq \alpha \min\{n\delta, n\delta^2/8\}$$

Proof. The proof follows from the adaptive composition theorem for Renyi-divergence, and max-divergence (from the DP literature) for the autoregressive decomposition of $\hat{\mathbf{p}}(\boldsymbol{y}_{1:n}|\boldsymbol{x})$ and $\mathbf{p}(\boldsymbol{y}_{1:n}|\boldsymbol{x})$ and then invoke Theorem B.5 for each factor. □

E.2. Robustness / Security guarantees

In this section, we provide the proof for Theorems B.1, F.1, and B.5 to ensure completeness and precision. We begin by restating the theorems and providing the corresponding proofs with necessary modifications.

Theorem E.3 (Robustness to editing (Restatement of Theorem B.1))**.** *Let $\boldsymbol{y} = [y_1, \ldots, y_n]$ represent the watermarked sequence. Suppose the adversary $\mathcal{A}$ follows Definition 2.2 and outputs a modified text $\boldsymbol{u} = [u_1, \ldots, u_m]$. Following Equation 2, we calculate z-score $z_{\boldsymbol{y}}$ and $z_{\boldsymbol{u}}$. Assume edit distance between $\boldsymbol{y}$ and $\boldsymbol{u}$ (denoted as η) satisfies $\eta < n$. Then we have*

$$z_{\boldsymbol{u}} \geq z_{\boldsymbol{y}} - \max\{\frac{(1+\gamma/2)\eta}{\sqrt{n}}, \frac{(1-\gamma/2)\eta}{\sqrt{n-\eta}}\}.$$

In particular, when $\eta \leq \frac{2\gamma n}{(1+\gamma/2)^2}$, we can drop the second term in the max.

Proof. Define bivariate function $f(x,y) = \frac{x-\gamma y}{\sqrt{y}}$. By Taylor's theorem

$$f(x-k_x, y-k_y) = f(x,y) + \begin{bmatrix} \partial_x f(x-\tilde{k}_x y-\tilde{k}_y) \\ \partial_y f(x-\tilde{k}_x y-\tilde{k}_y) \end{bmatrix}^T \begin{bmatrix} -k_x \\ -k_y \end{bmatrix} = f(x,y) - \left(\frac{k_x}{\sqrt{y-\tilde{k}_y}} - \frac{\gamma k_y}{2\sqrt{y-\tilde{k}_y}}\right)$$

where $\tilde{k}_x$ is between 0 and k_x and $\tilde{k}_y$ is between 0 and k_y. We also know that $|k_x| \leq k$ and $|k_y| \leq k$.

A lower bound of the above can be obtained by finding an upper bound to

$$\frac{k_x}{\sqrt{y-\tilde{k}_y}} - \frac{\gamma k_y}{2\sqrt{y-\tilde{k}_y}} = \frac{k_x - \frac{\gamma}{2}k_y}{\sqrt{y-\tilde{k}_y}}$$

First observe that we can always choose $k_x = k$. Next we discuss two possibilities of k_y. If k_y is negative, then choosing $k_y = -k$ and $\tilde{k} = 0$ maximizes the bound, which gives $\frac{(1+\gamma/2)k}{\sqrt{y}}$.

If k_y is positive, then we should always choose $\tilde{k}_y = k_y$ to maximize the expression, which gives us an upper bound of

$$\frac{k - \frac{\gamma}{2}k_y}{\sqrt{y-k_y}} = \frac{k + \frac{\gamma}{2}(y-k_y) - \frac{\gamma}{2}y}{\sqrt{y-k_y}} = \frac{k - \frac{\gamma}{2}y}{\sqrt{y-k_y}} + \frac{\gamma\sqrt{y-k_y}}{2}.$$

We will discuss two cases again, the first case is when $k - \gamma y/2 \leq 0$. In this case, the function $g(u) = a/u + bu$ with $a \leq 0$ has a derivative of $-a/u^2 + b \geq 0$, thus g is monotonically increasing. Thus we should choose $k_y = 0$. The second case is when $k - \gamma y/2 > 0$, in this case the $a > 0$ in the above $g(u)$ and $g(u)$ is convex, thus $\max_{u_{\min} \leq u \leq u_{\max}} g(u) = \max\{g(u_{\max}), g(u_{\min})\}$. Thus we should just compare the two cases when $k_y = 0$ and $k_y = k$, i.e., $\max\{\frac{k}{\sqrt{y}}, \frac{(1-\gamma/2)k}{\sqrt{y-k}}\}$.

Collect everything together, we get an upper bound o

$$\max\{\frac{(1+\gamma/2)k}{\sqrt{y}}, \frac{k}{\sqrt{y}}, \frac{(1-\gamma/2)k}{\sqrt{y-k}}\} = \max\left\{\frac{(1+\gamma/2)k}{\sqrt{y}}, \frac{(1-\gamma/2)k}{\sqrt{y-k}}\right\}$$

i.e.,

$$f(x-k_x, y-k_y) - f(x,y) \geq -\max\left\{\frac{(1+\gamma/2)k}{\sqrt{y}}, \frac{(1-\gamma/2)k}{\sqrt{y-k}}\right\}.$$

Now notice that our z-score has the same form as the $f(x,y)$ function. We can take $y = n$ and $x = |\boldsymbol{y}|_G$. Instantiate k be the maximum number of edits η. Observe that given that the adversary has a bounded edit distance, each operation of "insertion", "deletion", or "edit" can, at most, alter one token from the green list to the red list. They also can only alter the length by the number of edits. The above result translates into

$$z_{\boldsymbol{u}} \geq z_{\boldsymbol{y}} - \max\{\frac{(1+\gamma/2)\eta}{\sqrt{n}}, \frac{(1-\gamma/2)\eta}{\sqrt{n-\eta}}\},$$

where η denotes the edit distance between y and u. □

The robustness theorem above implies the security guarantees as we discussed in Corollary B.4.

E.3. No false positive (Type I error guarantees)

Theorem E.4 (No false positives (Restatement of Theorem B.7))**.** *Consider $\boldsymbol{y} = \boldsymbol{y}_{1:n}$ as any* fixed *suspect text. Let $N =: |\mathcal{V}|$ and $G \subset |\mathcal{V}|$ satisfying $|G| = \gamma N$. G is selected through Algorithm 2, using a uniform random choice. Let $|\boldsymbol{y}|_G$ denote the number of tokens in G and $z_{\boldsymbol{y}} := \frac{|\boldsymbol{y}|_G - \gamma n}{\sqrt{n\gamma(1-\gamma)}}$ as in Algorithm 3. Then the following statements hold true:*

1. *Assume $n \geq 1$, then*
$$\mathbb{E}[|\boldsymbol{y}|_G|\boldsymbol{y}] = \gamma n \quad \text{and} \quad \mathbb{E}[z_{\boldsymbol{y}}|\boldsymbol{y}] = 0.$$

2. *Define $C_{\max}(\boldsymbol{y}) := \max_{i\in[N]} \sum_{j=1}^n \mathbf{1}(y_j = i)$ and $V(\boldsymbol{y}) := \frac{1}{n}\sum_{i=1}^N (\sum_{j=1}^n \mathbf{1}(y_j = i))^2$, then with probability $1-\alpha$ (over only the randomness of G),*
$$\mathbb{P}\left[|\boldsymbol{y}|_G \geq \gamma n + \sqrt{64\gamma n V \log(9/\alpha)} + C_{\max}\log(9/\alpha)\middle|\boldsymbol{y}\right] \leq \alpha$$
or equivalently (when $n \geq 1$)
$$\mathbb{P}\left[z_{\boldsymbol{y}} \geq \sqrt{\frac{64V\log(9/\alpha)}{c(1-\gamma)}} + \frac{C_{\max}\log(9/\alpha)}{\sqrt{n\gamma(1-\gamma)}}\middle|\boldsymbol{y}\right] \leq \alpha.$$

Proof. To prove the first statement, observe that any fixed token has a probability γ to be included in the green list, thus by the linearity of the expectation and the independence of $\boldsymbol{y}$ in G.

$$\mathbb{E}[|\boldsymbol{y}|_G|\boldsymbol{y}] = \sum_{i=1}^n \mathbb{E}[\mathbf{1}(y_i \in G)|\boldsymbol{y}] = \sum_{i=1}^n \gamma = \gamma n.$$

Next, we will prove the second statement by applying Lemma G.1 to obtain the result stated in the third statement. Let $a_{i,j} = \mathbf{1}(j \leq \gamma N)\sum_{\ell=1}^n \mathbf{1}(y_\ell = i)$. By our assumption $0 \leq a_{i,j} \leq C_{\max}$ for all i, j. Observe that $\sum_{i=1}^N a_{i,\Pi_N(i)}$ is *identically distributed* with $|\boldsymbol{y}|_G$.

By Lemma G.1 with $t = 16\log(8e^{1/16}/\alpha)$, we get that with probability $1-\alpha$,

$$||\boldsymbol{y}|_G - \gamma n| < 2\sqrt{\frac{16\log(9/\alpha)}{N}N\gamma nV} + 16C_{\max}\log(9/\alpha)$$

where we used that $8e^{1/16} \leq 9$ and the fact that only γN columns of the $a_{i,j}$ matrix $a_{i,j}$ is nonzero, and for each non-zero column L2-norm of the column is bounded by $\sqrt{nV}$ by our definition of V. The result for the z-score follows trivially. □

E.4. Only true detection (Type II error guarantees)

For bounding the Type II error, i.e., false negative rates, we will work with our proposed method that generates $\boldsymbol{y}$ from the language model, i.e., sampling from the watermarked distribution $\hat{\mathbf{p}}$ recursively one token at a time.

Let's first recall a few notations. $\boldsymbol{h}$ is the input to the language model at step t, i.e., $\boldsymbol{h} = [\boldsymbol{x}, \boldsymbol{y}_{1:t-1}]$. Let δ represent the watermark strength from Equation 1. The green list $G \subset [N]$ is a random index set of the vocabulary of size γN. The watermarked probability distribution $\hat{\mathbf{p}}_t = \hat{\mathbf{p}}_t(\cdot|\boldsymbol{h})$ at time step t. The process of generating the sentence $y_1, y_2, \ldots, y_n$ involves recursively sampling from $\hat{\mathbf{p}}_t$, which we refer to as a "roll-out" procedure.

We need to make a few assumptions about the language model's probability distribution $\mathbf{p}$ and the prompt $\boldsymbol{x}$. We will first state them and then explain why these are natural and arguably needed for the Type II error to be small.

E.4.1. On-average high entropy assumption

The first such assumption requires the probability of the roll-out to be "sufficiently diverse" on average. We will introduce the notation $\|\mathbf{p}\|_2 := \sqrt{\sum_{i=1}^N \mathbf{p}[i]^2}$.

Assumption E.5 (On-average-high-entropy)**.** We say a language model's probability distribution $\mathbf{p}$ with a prompt $\boldsymbol{x}$ satisfies ξ-on-average-high-entropy if

$$\frac{1}{n}\sum_{t=1}^{n} \mathop{\mathbb{E}}_{\boldsymbol{y}_{1:t-1}\sim\mathbf{p}(\cdot|\boldsymbol{x})}[\|\mathbf{p}_t\|^2] \leq \xi.$$

This assumption requires the distribution of the roll-out to be sufficiently diffuse on average (either in expectation or with high probability).

The purpose of these assumptions is to rule out the cases when $\boldsymbol{y}_{1:n}$ is almost deterministic under $\mathbf{p}$ and perturbing the logits by δ does not change the distribution much at all.

For example, if the prompt writes

"`Generate the English alphabet in capital letters for 200 times please.`"

Then the language model would generate

"`ABC...XYZ, ABC...XYZ, ...`".

Despite that the generated sequence is very long, i.e., n is as large as $5,200$, the added watermark does not change the distribution very much at all. To see this, if $\mathbf{p}(y_3 = \text{"C"}|\boldsymbol{x}, \boldsymbol{h}) \geq 1-\epsilon$ for a tiny ϵ, and then by our quality guarantee, $\hat{\mathbf{p}}(y_3 = \text{"C"}\ |\boldsymbol{x}, \boldsymbol{h}) \geq 1 - \epsilon e^{\delta}$.

Quantitatively, for nearly uniform $\mathbf{p}_t$, $\xi = O(1/N)$, if $\mathbf{p}_t$ concentrates on a single token for all t, e.g., when a football commentator exclaims "`Goal goal goal goal ....`", then we cannot obtain a better bound than the trivial $\xi \leq 1$. In the alphabet example above $\xi \leq 1/26$.

Why is it called entropy? Assumption E.5 is related to the "high-entropy" assumption in Kirchenbauer et al. (2023) but for a slightly different kind of entropy. In a more formal sense, the quantity $\|\mathbf{p}_t\|^2$ is connected to the Tsallis entropy of order 2, defined as $S_2(\mathbf{p}_t) = k_B(1 - \|\mathbf{p}_t\|^2)$ where k_B is known as the Boltzmann constant. Our assumption requires the expected Tsallis entropy of the conditional distribution $\mathbf{p}_t$ over the roll-out of $\mathbf{p}$ to be larger than $k_B(1-\xi)$ on average among $t = 1, ..., n$.

For a high-probability result, we also need a stronger version.

Assumption E.6 (On-average-high-entropy (high probability))**.** We say that a language model's probability distribution $\mathbf{p}$ with a prompt $\boldsymbol{x}$ satisfies (ξ, β)-on-average-high-entropy if with probability at least $1-\beta$ over the generated sequence $\boldsymbol{y}_{1:n}$,

$$\frac{1}{n}\max\left\{\left\|\sum_{t=1}^{n}\mathbf{p}_t\right\|, \sum_{t=1}^{n}\|\mathbf{p}_t\|^2, \left\|\sum_{t=1}^{n}\mathbf{p}_t\right\|_\infty, \sum_{t=1}^{n}\|\mathbf{p}_t\|_\infty^2\right\} \leq \xi.$$

The behavior is similar to that of the expectation version of the assumption. When $\mathbf{p}_t$ is nearly uniform, $\mathbf{p}_t[i] = O(1/N)$, then $\xi = O(1/\sqrt{N})$. When $\mathbf{p}_t$ is supported only on one token, then $\xi = 1$. In practice, ξ is a small constant. As we will present in the main theorem, as long as $\xi \asymp \delta$, the number of green list tokens is guaranteed to grow faster γn as n gets larger.

One may also ask whether it is necessary to make entropy assumptions on the conditional probabilities instead of the marginal probabilities induced by $\mathbf{p}$ or $\hat{\mathbf{p}}$, but this is unfortunately not sufficient as illustrated in the following example.

Example E.7 (Marginal high entropy is insufficient)**.** Let the prompt $\boldsymbol{x}$ be

“Generate the first token uniformly at random, then repeat the token you generated for the remaining $n-1$ tokens”.

In this case, a good language model that follows the instruction will have $\mathbb{P}_{\mathbf{p}}(y_t = i) = 1/N$ for all i and all $t = 1, ..., n$ marginally, which implies that the entropy is the maximum and for any green list G, $\mathbb{P}_{\mathbf{p}}(y_t \in G) = \gamma$. On the other hand, with probability γ, $|\boldsymbol{y}|_G = n$ and with probability $1-\gamma$, $|\boldsymbol{y}|_G = 0$. There isn't any concentration around γn possible. Moreover, check that if we apply watermark, then $\mathbb{P}_{\hat{\mathbf{p}}}(y_t \in G) = \frac{\gamma e^\delta}{\gamma e^\delta + (1-\gamma)}$ for all t and all G. This changes the probability of seeing $|\boldsymbol{y}|_G = n$ slightly but the two world remains indistinguishable.

E.4.2. A “HOMOPHILY” ASSUMPTION

The second assumption that we need to make is called “homophily”, which says that increasing the probability of a group of tokens by adding the watermarks will not decrease the probability of generating the same group of tokens in the future as the language model rolls out.

Assumption E.8 (“Homophily”)**.** We say a language model's probability distribution $\mathbf{p}$ and prompt $\boldsymbol{x}$ satisfy “homophily” if for any G, the corresponding watermarked $\hat{\mathbf{p}}$ satisfies that

$$\underset{\boldsymbol{h}\sim\hat{\mathbf{p}}(\cdot|\boldsymbol{x})}{\mathbb{E}}\left[\underset{y\sim\hat{\mathbf{p}}(\cdot|\boldsymbol{h},\boldsymbol{x})}{\mathbb{P}}(y \in G)\right] \geq \underset{\boldsymbol{h}\sim\mathbf{p}(\cdot|\boldsymbol{x})}{\mathbb{E}}\left[\underset{y\sim\hat{\mathbf{p}}(\cdot|\boldsymbol{h},\boldsymbol{x})}{\mathbb{P}}(y \in G)\right]$$

where $\boldsymbol{h}$ denotes the generated sequence before y.

This assumption says that by increasing the probability of tokens in G, the induced distribution of the prefix $\boldsymbol{h}$ cannot counter-intuitively reduce the probability of tokens in G in the future on average.

The assumption is not unreasonable, because we expect a language model to be more likely to refer to text it has generated in the prefix than those that did not appear in the prefix.

This assumption is needed to rule out the unnatural situation where increasing the green list tokens initially ends up reducing the number of green list tokens in the long run. We gave an example in Section 3 to demonstrate the type of prompts that may lead to such a counter-intuitive sequence distribution.

To hammer it home, consider the following more quantitative construction of that works no matter which random green list G realizes.

$\boldsymbol{x} =$ “Choose the first k token by random sampling without replacement. Then sample from all but the token you choose uniformly for n-k rounds.”

It's easy to calculate that the expected number of times any token appears in a language model that perfectly follows the instruction will be n/N. However, the watermarked language model, let's say we use a very large δ such that the first k tokens are from the green list, then the expected number of times a green-list token appears is $\frac{k}{\gamma N} + \frac{\gamma N - k}{\gamma N}\frac{(n-k)(\gamma N-k)}{N-k}$ which is bounded by 1 if $k = \gamma N$ instead of growing linearly in n as in the original language model.

To obtain a concentration bound, we also need a stronger version of the homophily assumption as follows.

Assumption E.9 (High probability on-average homophily)**.** There exists a coupling – a joint distribution of $\boldsymbol{y}_{1:n}$ and $\hat{\boldsymbol{y}}_{1:n}$ where marginally $\boldsymbol{y}_{1:n} \sim \mathbf{p}(\cdot|\boldsymbol{x})$, $\hat{\boldsymbol{y}}_{1:n} \sim \hat{\mathbf{p}}(\cdot|\boldsymbol{x})$ – such that for any G, with probability $1-\beta$ over the joint distribution,

$$\frac{1}{n}\sum_{t=1}^{n} \hat{\mathbf{p}}_t(G|\hat{\boldsymbol{y}}_{1:t-1})) \geq \frac{1}{n}\sum_{t=1}^{n} \hat{\mathbf{p}}_t(G|\boldsymbol{y}_{1:t-1})).$$

The reason for defining the existence of a coupling is for technical reasons, but the purpose of the assumption is identical to that of the in-expectation version.

E.5. Theorem statement on "Only true detection"

Now we are ready to state the main theorem.

Theorem E.10 (Only true detection; Restating Theorem B.12)**.** *For a fixed language model $\mathcal{M}$ and a prompt $\boldsymbol{x}$. The sentence $\boldsymbol{y}_{1:n}$ generated from $\hat{\mathcal{M}}(\boldsymbol{x})$ where $\hat{\mathcal{M}}$ is an output of our watermarking scheme* $\mathsf{Watermark}_{\delta,\gamma}(\mathcal{M})$ *with parameter δ, γ. Then the following statements are true.*

1. *Assume homophily (Assumption E.8), then*

$$\mathbb{E}[|\boldsymbol{y}|_G] \geq \frac{n\gamma e^{\delta}}{1+(e^{\delta}-1)\gamma} - \gamma(1-\gamma)e^{\delta}\sum_{t=1}^{n} \mathop{\mathbb{E}}_{\boldsymbol{y}_{1:t-1}\sim\mathbf{p}(\cdot|\boldsymbol{x})} \|\mathbf{p}_t\|^2.$$

In particular, if Assumption E.5 condition is true with parameter $\xi \leq (1-\kappa)\frac{e^{\delta}-1}{(1+(e^{\delta}-1)\gamma)e^{\delta}}$ for a parameter $0<\kappa<1$, then

$$\mathbb{E}[|\boldsymbol{y}|_G] \geq n\gamma\left(1+\kappa\frac{(e^{\delta}-1)(1-\gamma)}{1+(e^{\delta}-1)\gamma}\right) \text{ or equivalently } \mathbb{E}[z_{\boldsymbol{y}}] \geq \frac{\kappa(e^{\delta}-1)\sqrt{n\gamma(1-\gamma)}}{1+(e^{\delta}-1)\gamma}.$$

2. *Assume high-probability version of homophily (Assumption E.9). There exists a parameter $C_{\delta,\gamma}$ that depends only δ, γ such that with probability at least $1-\beta$ for any $\beta>0$ (over both G and $\boldsymbol{y} \sim \hat{\mathbf{p}}(\cdot|\boldsymbol{x}, G)$),*

$$\begin{aligned}\|\boldsymbol{y}\|_G \geq & \frac{n\gamma e^{\delta}}{1+(e^{\delta}-1)\gamma} - \sqrt{2n\log(6/\beta)} \\ & - C_{\delta,\gamma}\log^2\frac{27(n+1)}{\beta}\left(\|\sum_{t=1}^{n}\mathbf{p}_t\| + \sum_{t=1}^{n}\|\mathbf{p}_t\|^2 + \|\sum_{t=1}^{n}\mathbf{p}_t\|_\infty + \sum_{t=1}^{n}\|\mathbf{p}_t\|_\infty^2\right).\end{aligned}$$

In particular, if for a parameter $0<\kappa<1$,

$$n \geq \frac{8\log(6/\beta)(1-\gamma+e^{\delta}\gamma)^2}{(1-\kappa)^2\gamma^2(1-\gamma)^2(e^{\delta}-1)^2} = \tilde{\Omega}(1/\delta^2) \tag{5}$$

and Assumption E.6 condition is true with parameter $(\xi, \beta/3)$ where

$$\xi \leq \frac{(1-\kappa)\gamma(1-\gamma)(e^{\delta}-1)}{8C_{\delta,\gamma}(1-\gamma+e^{\delta}\gamma)\log^2\left(\frac{27(n+1)}{\beta}\right)} = \tilde{O}(\delta), \tag{6}$$

then

$$\mathbb{P}\left[\|\boldsymbol{y}\|_G < n\gamma(1+\kappa\frac{(e^{\delta}-1)(1-\gamma)}{1-\gamma+\gamma e^{\delta}})\right] = \mathbb{P}\left[z_{\boldsymbol{y}} < \frac{\kappa(e^{\delta}-1)\sqrt{n\gamma(1-\gamma)}}{1+(e^{\delta}-1)\gamma}\right] \leq \beta.$$

E.6. Proof of Theorem B.12

In the false negative error cases, $\boldsymbol{y}$ is drawn from the watermarked language model $\hat{\mathcal{M}}$. To be explicit, let us write $\boldsymbol{y} = [\hat{y}_1, ..., \hat{y}_n] = \hat{\boldsymbol{y}}_{1:n}$. Now let's also define a hypothetical (possibly coupled) sequence $\boldsymbol{y}_{1:n}$ which is drawn from the original (un-watermarked) language model $\mathcal{M}$.

For convenience, we define the following shorthand $\mathbf{p}(G) := \mathbb{P}_{y\sim\mathbf{p}}[y \in G]$. for a probability mass function $\mathbf{p}$ defined on the vocabulary $\mathcal{V}$. Specifically, $\hat{\mathbf{p}}_t(G|\hat{\boldsymbol{y}}_{1:t-1})$ means $\mathbb{P}_{y\sim\hat{\mathbf{p}}_t(\cdot|\boldsymbol{x},\hat{\boldsymbol{y}}_{1:t-1})}[y \in G]$, parameterized by a fixed green list G. Similarly, $\mathbf{p}_t(G|\boldsymbol{y}_{1:t-1})$ denotes $\mathbb{P}_{y\sim\mathbf{p}_t(\cdot|\boldsymbol{x},\boldsymbol{y}_{1:t-1})}[y \in G]$.

The proof of Theorem B.12 considers the following decomposition

$$
\begin{aligned}
|\boldsymbol{y}|_G = & |\boldsymbol{y}|_G - \sum_t \hat{\mathbf{p}}_t(G|\hat{\boldsymbol{y}}_{1:t-1}) && (7)\\
& + \sum_t \hat{\mathbf{p}}_t(G|\hat{\boldsymbol{y}}_{1:t-1}) - \sum_t \hat{\mathbf{p}}_t(G|\boldsymbol{y}_{1:t-1}) && (8)\\
& + \sum_t \hat{\mathbf{p}}_t(G|\boldsymbol{y}_{1:t-1}) && (9)
\end{aligned}
$$

steps to prove a lower bound to each of the three terms. We will start with the high probability bound (the second statement in Theorem B.12) then deal with the expectation.

E.6.1. Many green list tokens with high probability

To obtain a high-probability lower bound, it requires us to obtain concentration for each of the three terms. Specifically,

1. To bound Term (7), we use Lemma E.11 which invokes Martingale concentration over the randomness in $\boldsymbol{y}$ to show $|\boldsymbol{y}|_G$ is close to $\sum_t \hat{\mathbf{p}}_t(G|\hat{\boldsymbol{y}}_{1:t-1})$.

2. We will show Term (8) is non-negative with high probability by using the homophily assumption (Assumption E.9). This allows us to study the roll-out $\hat{\boldsymbol{y}}_{1:t-1}$ under $\hat{\mathcal{M}}(\boldsymbol{x})$ (or $\hat{\mathbf{p}}$) by studying a hypothetical alternative roll-out $\boldsymbol{y}_{1:t-1}$ sampled under $\mathcal{M}(\boldsymbol{x})$ (or $\mathbf{p}$).

3. Then we control Term (9) by first Taylor expanding it into quantities involving $\mathbf{p}_t(G|\boldsymbol{y}_{1:t-1})$ instead of $\hat{\mathbf{p}}(G|\boldsymbol{y}_{1:t-1})$, then apply concentration inequalities for each expanded terms over the randomness of G (while fixing $\boldsymbol{y}_{1:t-1}$) to obtain a high probability lower bound. Proposition E.14 gives the results.

We start by tackling (7) via Martingale concentration.

Lemma E.11. *For any green list G and prompt $\boldsymbol{x}$.*

$$
\mathbb{E}\left[|\boldsymbol{y}|_G - \sum_{t=1}^{n} \underset{y_t\sim\hat{\mathbf{p}}(\cdot|\boldsymbol{x},\boldsymbol{y}_{1:t-1})}{\mathbb{P}}[y_t \in G]\right] = 0.
$$

Moreover, with probability at least $1-\beta$ over the roll-out

$$
|\boldsymbol{y}|_G \geq \sum_{t=1}^{n} \underset{y_t\sim\hat{\mathbf{p}}(\cdot|\boldsymbol{x},\boldsymbol{y}_{1:t-1})}{\mathbb{P}}[y_t \in G] - \sqrt{2n\log(2/\beta)}.
$$

Proof. We fix G and construct a martingale sequence $X_1, X_2, ..., X_n$ where $X_0 = 0$ and:

$$
X_t = X_{t-1} + \mathbf{1}(y_t \in G) - \underset{y_t\sim\hat{\mathbf{p}}(\cdot|\boldsymbol{x},\boldsymbol{y}_{1:t-1})}{\mathbb{P}}[y_t \in G].
$$

Check that $\mathbb{E}[X_t|\boldsymbol{y}_{1:t-1}] = X_{t-1}$. The underlying filtration is the sigma-field generated by $y_{1:t}$.

The claim about the expectation follows from that $X_0 = 0$ and an inductive argument following the tower property of conditional probabilities.

By the fact that $|X_t - X_{t-1}| \leq 1$ we can apply Azuma-Hoeffding's inequality and get

$$
\mathbb{P}\left[|X_n - \mathbb{E}[X_n]| \geq u\right] \leq 2e^{-\frac{u^2}{2n}}.
$$

Check that by an inductive argument $\mathbb{E}[X_n] = 0$. So we get that with probability at least $1 - \delta$

$$|X_n| = \left| \sum_{t=1}^{n} \mathbf{1}(y_t \in G) - \sum_{t=1}^{n} \underset{y_t \sim \hat{\mathbf{p}}(\cdot|\boldsymbol{x}, \boldsymbol{y}_{1:t-1})}{\mathbb{P}} [y_t \in G] \right| \leq \sqrt{2n \log(2/\delta)}.$$

□

To handle (8), we apply Assumption E.9 with parameter $\beta/3$, which says that with probability $1 - \beta/3$ (8)≥ 0. This converts a roll-out from $\hat{y} \sim \hat{\mathbf{p}}$ to a roll-out from the original p.

Before we deal with (9), let us write a lemma that rewrites $\hat{\mathbf{p}}_t(G|\boldsymbol{y}_{1:t-1})$ into a more convenient form.

Lemma E.12. *For any t, $\boldsymbol{h}_t$. Fix G. Denote short hands $\hat{\mathbf{p}}(G) := \mathbb{P}_{y_t \sim \hat{\mathbf{p}}_t(\cdot|\boldsymbol{x}, \boldsymbol{h}_t)}[y_t \in G]$ and $\mathbf{p}(G) := \mathbb{P}_{y_t \sim \mathbf{p}_t(\cdot|\boldsymbol{x}, \boldsymbol{h}_t)}[y_t \in G]$.*

$$\hat{\mathbf{p}}(G) = \frac{e^{\delta} \mathbf{p}(G)}{1 + (e^{\delta} - 1)\mathbf{p}(G)} = \left(1 + \frac{(e^{\delta} - 1)(1 - \mathbf{p}(G))}{1 + (e^{\delta} - 1)\mathbf{p}(G)}\right) \mathbf{p}(G).$$

Proof. By definition,

$$\begin{aligned} \hat{\mathbf{p}}(G) &= \frac{\sum_{y \in G} e^{\ell_y + \delta}}{\sum_{y \in G} e^{\ell_y + \delta} + \sum_{y \notin G} e^{\ell_y}} \\ &= \frac{e^{\delta} \mathbf{p}(G)}{e^{\delta} \mathbf{p}(G) + 1 - \mathbf{p}(G)} = \frac{e^{\delta}}{1 + (e^{\delta} - 1)\mathbf{p}(G)} \mathbf{p}(G) \\ &= \left(1 + \frac{(e^{\delta} - 1)(1 - \mathbf{p}(G))}{1 + (e^{\delta} - 1)\mathbf{p}(G)}\right) \mathbf{p}(G). \end{aligned}$$

□

The lemma implies that $\hat{\mathbf{p}}(G) \geq \mathbf{p}(G)$ and that if $\mathbf{p}(G)$ is bounded away from 1, $\hat{\mathbf{p}}(G) \geq (1 + O(\delta))\mathbf{p}(G)$.

Lemma E.13. *For any t, $\boldsymbol{h}_t$. Fix G.*

$$\hat{\mathbf{p}}(G) \geq \frac{e^{\delta} \gamma}{1 + (e^{\delta} - 1)\gamma} + \frac{e^{\delta}}{(1 + (e^{\delta} - 1)\gamma)^2} (\mathbf{p}(G) - \gamma) - e^{\delta} (\mathbf{p}(G) - \gamma)^2$$

Proof. By the second-order Taylor's theorem

$$\frac{e^{\delta} x}{1 + (e^{\delta} - 1)x} = \frac{e^{\delta} \gamma}{1 + (e^{\delta} - 1)\gamma} + \frac{e^{\delta}}{(1 + (e^{\delta} - 1)\gamma)^2} (x - \gamma) - \frac{e^{\delta}}{(1 + (e^{\delta} - 1)\tilde{x})^3} (x - \gamma)^2$$

where $\tilde{x} \in [x, \gamma]$ is a function of x. By relaxing $\tilde{x}$ to 0 we obtain the lower bound as claimed. □

Now we are ready to handle (9) with high probability in the following proposition.

Proposition E.14 (Concentration)**.** *For any fixed sequence $\boldsymbol{y}_{1:n}$, and the corresponding language model's probability distribution $\mathbf{p}$ that gives conditional distributions $\mathbf{p}_1, ..., \mathbf{p}_n$. There exists a parameter $C_{\delta,\gamma}$ that depends only δ, γ. Then with probability at least $1 - \beta$ for any $\beta > 0$ (over G),*

$$\begin{aligned} \sum_{t=1}^{n} \underset{y_t \sim \mathbf{p}(\cdot|\boldsymbol{x}, \boldsymbol{y}_{1:t-1})}{\mathbb{P}} [y_t \in G] &\geq \frac{n \gamma e^{\delta}}{1 + (e^{\delta} - 1)\gamma} \\ &\quad - C_{\delta,\gamma} \log^2 \frac{9(n+1)}{\beta} \left(\| \sum_{t=1}^{n} \mathbf{p}_t[\cdot] \| + \sum_{t=1}^{n} \|\mathbf{p}_t[\cdot]\|^2 + \| \sum_{t=1}^{n} \mathbf{p}_t[\cdot] \|_{\infty} + \sum_{t=1}^{n} \|\mathbf{p}_t[\cdot]\|_{\infty}^2 \right). \end{aligned}$$

Proof. By Lemma E.12 and E.13

$$
\begin{aligned}
&\sum_{t=1}^{n} \underset{y_t \sim \hat{\mathbf{p}}(\cdot|\boldsymbol{x}, \boldsymbol{y}_{1:t-1})}{\mathbb{P}} [y_t \in G] \\
=&\sum_t \frac{e^\delta \mathbf{p}_t(G)}{1 + (e^\delta - 1)\mathbf{p}_t(G)} \\
\geq&\sum_t \frac{e^\delta \gamma}{1 + (e^\delta - 1)\gamma} + \frac{e^\delta(\mathbf{p}_t(G) - \gamma)}{(1 + (e^\delta - 1)\gamma)^2} - e^\delta(\mathbf{p}_t(G) - \gamma)^2 \\
=&\frac{n\gamma e^\delta}{1 + (e^\delta - 1)\gamma} + \frac{e^\delta}{(1 + (e^\delta - 1)\gamma)^2} \left(\underbrace{\sum_t \sum_{i=1}^{N\gamma} \mathbf{p}_t[\pi[i]] - n\gamma}_{(*)} \right) - e^\delta \sum_t \left(\underbrace{\sum_{i=1}^{N\gamma} \mathbf{p}_t[\pi[i]] - \gamma}_{(**)} \right)^2
\end{aligned}
$$

where π is a random permutation of the index set $\{1, ..., N\}$.

We will now apply Lemma G.1 to lowerbound $(*)$ with high probability and to bound the absolute value of $(**)$ with high probability.

Remark E.15. The reason why we can apply these lemmas even after we condition on $\boldsymbol{y}_{1:t-1}$ is due to the "high-probability homophily" assumption which allows us to use the fact that $\boldsymbol{y}_{1:t-1}$ is independent to G, i.e., the distribution of the green list remains uniform at random after we condition on each qualifying $\boldsymbol{y}_{1:t-1}$ separately.

Using a similar argument from the proof of Theorem B.7, we can apply Lemma G.1 and get that with probability $1 - \beta$,

$$
(*) \geq -\sqrt{64\gamma \| \sum_{t=1}^{n} \mathbf{p}_t(\cdot) \|^2 \log(9/\beta)} - \| \sum_{t=1}^{n} \mathbf{p}_t(\cdot) \|_\infty \log(9/\beta).
$$

Similarly by Lemma G.1 again to bound $(**) = \sum_{i=1}^{N\gamma} \mathbf{p}_t[\pi[i]] - \gamma$ w.h.p for each t.

$$
\big|(**)\big| \leq \sqrt{64\gamma \|\mathbf{p}_t(\cdot)\|^2 \log(9/\beta)} + \|\mathbf{p}_t(\cdot)\|_\infty \log(9/\beta).
$$

To put things together, with probability $1 - (n+1)\beta$,

$$
\begin{aligned}
&\sum_{t=1}^{n} \underset{y_t \sim \mathbf{p}(\cdot|\boldsymbol{x}, \boldsymbol{y}_{1:t-1})}{\mathbb{P}} [y_t \in G] \\
\geq&\frac{n\gamma e^\delta}{1 + (e^\delta - 1)\gamma} - \frac{e^\delta}{(1 + (e^\delta - 1)\gamma)^2} \left(\sqrt{64\gamma \| \sum_{t=1}^{n} \mathbf{p}_t[\cdot] \|^2 \log(9/\beta)} + \| \sum_{t=1}^{n} \mathbf{p}_t[\cdot] \|_\infty \log(9/\beta) \right) \\
&\quad - e^\delta \gamma(1 - \gamma) \sum_t \|\mathbf{p}_t[\cdot]\|^2 - 2e^\delta \left(64\gamma \sum_{t=1}^{n} \|\mathbf{p}_t[\cdot]\|_2^2 \log(9/\beta) + \sum_{t=1}^{n} \|\mathbf{p}_t[\cdot]\|_\infty^2 \log^2(9/\beta) \right) \\
\geq&\frac{n\gamma e^\delta}{1 + (e^\delta - 1)\gamma} - C_{\delta,\gamma} \log(9/\beta)^2 \left(\| \sum_{t=1}^{n} \mathbf{p}_t[\cdot] \| + \sum_{t=1}^{n} \|\mathbf{p}_t[\cdot]\|^2 + \| \sum_{t=1}^{n} \mathbf{p}_t[\cdot] \|_\infty + \sum_{t=1}^{n} \|\mathbf{p}_t[\cdot]\|_\infty^2 \right)
\end{aligned}
$$

for a constant $C_{\delta,\gamma}$ that depends only in δ, γ. The proof is complete by defining $\tilde{\beta} = 9(n+1)\beta$, and get the same result under probability $1 - \tilde{\beta}$. □

E.6.2. Many Green List Tokens in Expectation

To obtain the lower bound in expectation, we just need to bound the expectation of (7), (8) and (9).

1. Observe that $\mathbb{E}[\text{Term (7)}|G] = 0$ (from Lemma E.11)
2. Also, observe that (8) ≥ 0 under the *homophily* assumption (Assumption E.8).
3. Term (9) can be further lower bounded by a second-order Taylor expansion argument (Lemma E.13) and a variance calculation for sampling without replacement (Lemma E.16), which ends up depending on the *on-average high-entropy* parameter from Definition E.5. The formal result is stated in Proposition E.17.

Lemma E.16. *Fix* $\mathbf{p}_t$

$$\mathop{\mathbb{E}}_{G}[(\mathbf{p}_t(G) - \gamma)^2] \leq \gamma(1-\gamma)\|\mathbf{p}_t[\cdot]\|^2.$$

Proof. First observe that $\mathbb{E}_G[\mathbf{p}_t(G)] = \gamma$ because every token has γ probability to be included. By the variance formula for sampling without replacement (N choose $N\gamma$),

$$\mathrm{Var}_G[\mathbf{p}_t(G)|\boldsymbol{y}_{1:t-1}] = \gamma N \frac{1}{N}\sum_{i=1}^{N}(\mathbf{p}_t[i]^2 - N^{-2})(1 - \frac{\gamma N - 1}{N-1}) \leq \gamma(1-\gamma)\sum_{i=1}^{N}\mathbf{p}_t[i]^2.$$

□

Proposition E.17. *Assume homophily, then*

$$\mathbb{E}\left[\sum_{t=1}^{n} \mathop{\mathbb{P}}_{y_t\sim\hat{\mathbf{p}}(\cdot|\boldsymbol{x},\boldsymbol{y}_{1:t-1})}[y_t \in G]\right] \geq n\gamma\left(\frac{e^{\delta}}{1+(e^{\delta}-1)\gamma} - \frac{(1-\gamma)e^{\delta}}{n}\sum_{t=1}^{n}\mathop{\mathbb{E}}_{\boldsymbol{y}_{1:t-1}\sim\mathbf{p}(\cdot|\boldsymbol{x})}\sum_{i=1}^{N}\mathbf{p}_t[i]^2\right).$$

Proof. By homophily,

$$\begin{aligned}
&\mathbb{E}\left[\sum_{t=1}^{n} \mathop{\mathbb{P}}_{y_t\sim\hat{\mathbf{p}}(\cdot|\boldsymbol{x},\boldsymbol{y}_{1:t-1})}[y_t \in G]\right] \\
=&\sum_{t=1}^{n} \mathop{\mathbb{E}}_{G,\boldsymbol{y}_{1:t-1}\sim\hat{\mathbf{p}}(\cdot|\boldsymbol{x})}\left[\mathop{\mathbb{P}}_{y_t\sim\hat{\mathbf{p}}(\cdot|\boldsymbol{x},\boldsymbol{y}_{1:t-1})}[y_t \in G]\right] \\
\geq&\sum_{t=1}^{n} \mathop{\mathbb{E}}_{G,\boldsymbol{y}_{1:t-1}\sim\mathbf{p}(\cdot|\boldsymbol{x})}\left[\mathop{\mathbb{P}}_{y_t\sim\hat{\mathbf{p}}(\cdot|\boldsymbol{x},\boldsymbol{y}_{1:t-1})}[y_t \in G]\right] \\
=&\sum_{t=1}^{n} \mathop{\mathbb{E}}_{\boldsymbol{y}_{1:t-1}\sim\mathbf{p}(\cdot|\boldsymbol{x})}\mathop{\mathbb{E}}_{G}\left[\frac{e^{\delta}\,\mathbb{P}_{y_t\sim\mathbf{p}_t(\cdot|\boldsymbol{y}_{1:t-1})}[y_t \in G]}{1+(e^{\delta}-1)\,\mathbb{P}_{y_t\sim\mathbf{p}_t(\cdot|\boldsymbol{y}_{1:t-1})}[y_t \in G]}\middle|\boldsymbol{y}_{1:t-1}\right]
\end{aligned} \tag{10}$$

By Lemma E.13, we can decompose (10). Also observe that $\mathbb{E}_G\left[\mathbf{p}_t(G)\middle|\boldsymbol{y}_{1:t-1}\right] = \gamma$ where $\mathbf{p}_t(G) := \mathbb{P}_{y_t\sim\mathbf{p}_t(\cdot|\boldsymbol{y}_{1:t-1})}[y_t \in G]$ is short hand for clarity. To see the second observation, notice that y_t is independent to G, thus we can apply Statement 1 of Theorem B.7).

Apply the two observations to (10), we have

$$\begin{aligned}
(10) \geq& \sum_{t=1}^{n}\mathop{\mathbb{E}}_{\boldsymbol{y}_{1:t-1}\sim\mathbf{p}(\cdot|\boldsymbol{x})}\mathop{\mathbb{E}}_{G}\left[\frac{e^{\delta}\gamma}{1+(e^{\delta}-1)\gamma} + \frac{e^{\delta}(\mathbf{p}_t(G)-\gamma)}{(1+(e^{\delta}-1)\gamma)^2} - e^{\delta}(\mathbf{p}_t(G)-\gamma)^2\middle|\boldsymbol{y}_{1:t-1}\right] \\
=& \frac{e^{\delta}n\gamma}{1+(e^{\delta}-1)\gamma} + \sum_{t=1}^{n}\mathop{\mathbb{E}}_{\boldsymbol{y}_{1:t-1}\sim\mathbf{p}(\cdot|\boldsymbol{x})}\left[\frac{e^{\delta}(\mathbb{E}_G[\mathbf{p}_t(G)|\boldsymbol{y}_{1:t-1}]-\gamma)}{(1+(e^{\delta}-1)\gamma)^2} - e^{\delta}\mathop{\mathbb{E}}_{G}[(\mathbf{p}_t(G)-\gamma)^2|\boldsymbol{y}_{1:t-1}]\right] \\
=& \frac{e^{\delta}n\gamma}{1+(e^{\delta}-1)\gamma} - \sum_{t=1}^{n}e^{\delta}\mathop{\mathbb{E}}_{\boldsymbol{y}_{1:t-1}\sim\mathbf{p}(\cdot|\boldsymbol{x})}\mathrm{Var}_G[\mathbf{p}_t(G)|\boldsymbol{y}_{1:t-1}].
\end{aligned}$$

By the variance formula for sampling without replacement (N choose $N\gamma$),

$$\mathrm{Var}_G[\mathbf{p}_t(G)|\boldsymbol{y}_{1:t-1}] = \gamma N \frac{1}{N}\sum_{i=1}^{N}(\mathbf{p}_t[i]^2 - N^{-2})(1 - \frac{\gamma N - 1}{N-1}) \le \gamma(1-\gamma)\sum_{i=1}^{N}\mathbf{p}_t[i]^2.$$

Thus it follows that

$$\begin{aligned}(10) &\ge \frac{e^{\delta} n\gamma}{1+(e^{\delta}-1)\gamma} - \sum_{t=1}^{n} e^{\delta} \mathop{\mathbb{E}}_{\boldsymbol{y}_{1:t-1}\sim\mathbf{p}(\cdot|\boldsymbol{x})} \gamma(1-\gamma)\sum_{i=1}^{N}\mathbf{p}_t[i]^2 \\ &= n\gamma\left(\frac{e^{\delta}}{1+(e^{\delta}-1)\gamma} - \frac{(1-\gamma)e^{\delta}}{n}\sum_{t=1}^{n}\mathop{\mathbb{E}}_{\boldsymbol{y}_{1:t-1}\sim\mathbf{p}(\cdot|\boldsymbol{x})}\sum_{i=1}^{N}\mathbf{p}_t[i]^2\right).\end{aligned}$$

□

F. Analysis of Kirchenbauer et al. (2023)

F.1. Soft watermarking scheme of Kirchenbauer et al. (2023)

This section illustrates the soft watermarking scheme proposed by Kirchenbauer et al. (2023). This straightforward algorithm only requires access to the language model's logits at each time step. Let $\boldsymbol{y} = [y_1, \dots, y_n]$ represent the output sentence of language model $\mathcal{M}$ given the prompt $\boldsymbol{x}$. The watermarking scheme generates $\boldsymbol{y}_{1:n}$ by hashing y_{t-1} to a partition of the token space (Green List and Red List) and amplifies the probability of tokens on the Green List. Specifically, $[y_1, \dots, y_n]$ is derived from the following Markov chain:

1. $y_1 \sim \mathrm{Softmax}\big(\mathrm{logits}_{\mathcal{M}}(y_1 = \cdot|x)\big)$
2. For $t = 2 : n$,
$$y_t \sim \mathrm{Softmax}\big(\mathrm{logits}_{\mathcal{M}}(y_t = \cdot|[\boldsymbol{x}, y_1 \dots, y_{t-1}]) + \delta\mathbf{1}(\cdot \in \mathrm{Green}(y_{t-1}))\big)$$

Typically, $\gamma|\mathcal{V}|$ tokens are selected to form a Green List, where γ symbolizes the fraction of tokens to be watermarked (by default, $\gamma = 0.5$). The logit value for each green token is augmented by a constant δ (default value $= 2$), which denotes the watermark strength. This elevation enhances the likelihood of sampling green, watermarked tokens, particularly for high-entropy distributions.

Validation of whether a text was generated by a watermarked language model is achievable given knowledge of the hash function and tokenizer. The adversary constructs $\boldsymbol{u} = [u_1, \dots, u_m]$ from $\boldsymbol{x}, \boldsymbol{y}_{1:n}$ and any auxiliary input. The detection algorithm calculates the quantity of green tokens $|\boldsymbol{u}|_G = \sum_{t=2}^{m} \mathbf{1}(u_t \in \mathrm{Green}(u_{t-1}))$. One can assume the null hypothesis, denoted as H_0*: The text sequence is produced independently of the green list rule*. Following this, a z-statistic score is computed as $z = (|\boldsymbol{u}|_G - \gamma m) / \sqrt{m\gamma(1-\gamma)}$. If the z-score exceeds a predetermined threshold, the algorithm declares, "This was generated from $\hat{\mathcal{M}}$!".

F.2. Security property of Kirchenbauer et al. (2023)

We also demonstrate the robustness property of the soft watermarking algorithm in Kirchenbauer et al. (2023) in the following Theorem F.1

Theorem F.1 (Robustness to editing in the watermarking scheme of Kirchenbauer et al. (2023))**.** *Let $\boldsymbol{y} = [y_1, \dots, y_n]$ represent the watermarked sequence. Suppose the adversary $\mathcal{A}$ follows the definition 2.2 and outputs a modified text $\boldsymbol{u} = [u_1, \dots, u_m]$. Following Equation 2, we calculate the z-score of the soft watermarking (Kirchenbauer et al., 2023) $z_{\boldsymbol{y}}$ and $z_{\boldsymbol{u}}$. Then we have*

$$z_{\boldsymbol{u}} \ge z_{\boldsymbol{y}} - \max\{\frac{(2+\gamma/2)\eta}{\sqrt{n}}, \frac{(2-\gamma/2)\eta}{\sqrt{n-\eta}}\}.$$

Proof. The proof is similar to that of Theorem B.1 except that the maximum perturbation to $|\mathbf{y}|_G$ is now 2η rather than η. We now justify that the maximum perturbation has really doubled below, but ignore the part that is the same as in the proof of Theorem B.1.

Let $\mathsf{BiGrams}(\boldsymbol{u}) = \{\{u_1, u_2\}, \{u_2, u_3\}, ..., \{u_{n-1}, u_n\}\}$ and similarly $\mathsf{BiGrams}(\boldsymbol{y})$ enumerates the set of all two grams in sequence $\boldsymbol{y}_{1:m}$.

We claim that each edit can modify at most two elements in the above set. To see this, consider "insertion", "deletion", and "edit" separately.

- If we "insert" one token $\tilde{u}$ at t, then $\{u_{t-1}, u_t\}$ and $\{u_t, u_{t+1}\}$ become $\{u_{t-1}, \tilde{u}\}$, $\{\tilde{u}, u_t\}$ and $\{u_t, u_{t+1}\}$. Only one element of $\mathsf{BiGrams}(\boldsymbol{u})$ is modified — $\{u_{t-1}, u_t\}$.
- For "deletion" at t, $\{u_{t-1}, u_t\}$ and $\{u_t, u_{t+1}\}$ become $\{u_{t-1}, u_{t+1}\}$. So two elements from $\mathsf{BiGrams}(\boldsymbol{u})$ are gone.
- For "edit" at t, $\{u_{t-1}, u_t\}$ and $\{u_t, u_{t+1}\}$ become $\{u_{t-1}, \tilde{u}\}$ and $\{\tilde{u}, u_{t+1}\}$. Thus again only two elements from $\mathsf{BiGrams}(\boldsymbol{u})$ are gone.

It follows that when $\boldsymbol{y}$ is obtained after up to η edits

$$|\mathsf{BiGrams}(\boldsymbol{u}) \cap \mathsf{BiGrams}(\boldsymbol{y})| \geq |\mathsf{BiGrams}(\boldsymbol{u})| - 2\eta$$

Observe that $\sum_{t=2}^{n} \mathbf{1}(u_t \in \text{Green}(u_{t-1}))$ counts the number of qualifying elements in $\mathsf{BiGrams}(\boldsymbol{u})$, which completes the proof. □

For this reason, our watermark is twice as robust as that of Kirchenbauer et al. (2023). This provides the theoretical guarantee to our empirical results presented in the experiments!

Remark F.2. We can view our watermark as a trivial Markovian watermarking scheme with $k = 0$, and what Kirchenbauer et al. (2023) proposed to be $k = 1$. For the more general k-Markovian watermarking scheme that depends on a prefix of length k, the robustness deteriorates by a factor of k, as the maximum perturbation will become $\frac{((k+1)+\gamma/2)\eta}{\sqrt{n}}$. To say it differently, choosing $k = 0$ gives the maximum robustness and maximum simplicity at the same time, and the benefit leads to significant gains in our experiments, especially against paraphrasing attacks.

The other approach leverages the randomness in the language model rollout. In particular, we fix G and construct a martingale sequence $X_1, X_2, ..., X_n$ where $X_0 = 0$ and

$$X_t = X_{t-1} + \mathbf{1}(y_t \in G) - \mathop{\mathbb{P}}_{y_t \sim \hat{p}(\cdot|x, \boldsymbol{y}_{1:t-1})}(y_t \in G).$$

Check that $\mathbb{E}[X_t | \boldsymbol{y}_{1:t-1}] = X_{t-1}$. By the fact that $|X_t - X_{t-1}| \leq 1$ we can apply Azuma-Hoeffding's inequality and get

$$\mathbb{P}\left[|X_n - \mathbb{E}[X_n]| \geq u\right] \leq 2e^{-\frac{u^2}{2n}}.$$

Check that by an inductive argument $\mathbb{E}[X_n] = 0$. So we get that with probability at least $1 - \delta$

$$|X_n| = \left| \sum_{t=1}^{n} \mathbf{1}(y_t \in G) - \sum_{t=1}^{n} \mathop{\mathbb{P}}_{y_t \sim \hat{p}(\cdot|x, \boldsymbol{y}_{1:t-1})}(y_t \in G) \right| \leq \sqrt{2n \log(2/\delta)}.$$

The issue is that both $\sum_{t=1}^{n} \mathbf{1}(y_t \in G)$ and $\sum_{t=1}^{n} \mathbb{P}_{y_t \sim \hat{p}(\cdot|x, \boldsymbol{y}_{1:t-1})}(y_t \in G)$ are random. While Azuma-Hoeffding allows us to bound their differences whp, we still have to show that $\sum_{t=1}^{n} \mathbb{P}_{y_t \sim \hat{p}(\cdot|x, \boldsymbol{y}_{1:t-1})}(y_t \in G)$ is a constant factor larger than $n\gamma$ (the factor depends on δ and should be like $1 + \delta$). It doesn't happen for any fixed G and maybe randomness over G will need to be taken into account and that makes it a bit complicated. One observation we can make even in the fixed G case is that if we know x, $y_{1:n}$ and the language model, then we can compute $\sum_{t=1}^{n} \mathbb{P}_{y_t \sim \hat{p}(\cdot|x, \boldsymbol{y}_{1:t-1})}(y_t \in G)$ exactly and whether the deviation is substantially larger than random can be observed.

G. Technical Lemmas

Lemma G.1 (Bernstein-style inequality for random permutation (Albert, 2019, Proposition 2.2)). *Let $\{a_{i,j}\}_{1 \leq i,j \leq n}$ be a collection of non-negative numbers and Π_n be a random uniform permutation. Let $Z_n = \sum_{i=1}^{n} a_{i,\Pi_n(i)}$. Then, for any $t > 0$*

$$\mathbb{P}\left[|Z_n - \mathbb{E}[Z_n]| \geq 2\sqrt{\frac{t}{n}\sum_{i,j=1}^{n} a_{i,j}^2} + \max_{1 \leq i,j \leq n}\{a_{i,j}\}t\right] \leq 8e^{1/16}e^{-\frac{t}{16}}.$$

Lemma G.2 (Variance for sampling without replacement)**.** *Let $x_1, ..., x_N \in \mathbb{R}$. For any sample size $1 \leq n \leq N$, and π be a random permutation of $\{1, 2, ..., N\}$. The variance of $X = \frac{1}{n}\sum_{i=1}^{n} x_{\pi(i)}$ satisfies*

$$\mathrm{Var}(X) = \frac{1}{nN}\sum_{i=1}^{N}(x_i - \bar{x})^2(1 - \frac{n-1}{N-1}).$$

Definition G.3 (Martingale)**.** A sequence of random variables $(X_n)_{n\in\mathbb{N}}$ is called a *martingale* if it satisfies the following conditions:

1. $\mathbb{E}[|X_n|] < \infty$ for all $n \in \mathbb{N}$.
2. $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$ for all $n \in \mathbb{N}$.

where $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq ... \subseteq \mathcal{F}_n \subseteq \mathcal{F}_{n+1} \subseteq ...$ is a filtration. Specifically, $\mathcal{F}_n$ can be the sigma-algebra generated by another sequence of random variable $Y_1, ..., Y_n$, i.e., $\mathcal{F}_n = \sigma(Y_{1:n})$ and X_n can be a function of $Y_{1:n}$.

Lemma G.4 (Azuma-Hoeffding Inequality)**.** *Let $(X_n)_{n\in\mathbb{N}}$ be a martingale such that $|X_{n+1} - X_n| \leq c_n$ for some constants c_n and all $n \in \mathbb{N}$. Then for all $t > 0$ and $n \in \mathbb{N}$, we have*

$$\mathbb{P}\left(|X_n - X_0| \geq t\right) \leq 2\exp\left(-\frac{t^2}{2\sum_{i=1}^{n} c_i^2}\right).$$