The Computational Complexity of Computing Refunds

Anonymous Authors[1]

Abstract

We study a mechanism design setting where a seller offers an item to a buyer who is uncertain about her own valuation. Rather than using a simple take-it-or-leave-it price, the seller can extract significantly more revenue by offering refund options at different price levels. This setup, known as Sequential Screening, models a class of dynamic decision-making problems under uncertainty with applications ranging from airline ticket pricing to online education and healthcare diagnostics, where agents pay for flexible options under ambiguity. We investigate the power of deterministic mechanisms in this context and view them as a form of semi-adaptive preference elicitation, where the seller leverages knowledge of the buyer's value distribution to design refund-based menus that screen types indirectly. We show that the revenue gap between the optimal deterministic mechanism and simpler menus with bounded (possibly randomized) options can be arbitrarily large. We further establish that computing the revenue-optimal mechanism is NP-hard, and complement this result with a PTAS that computes approximately optimal refund schedules.

1. Introduction

Pricing under uncertainty is a central problem in economics and computer science, capturing the tension between a seller's desire to extract revenue and a buyer's evolving private information. Consider purchasing an airline ticket weeks in advance, uncertain about future travel plans; ordering clothing online without knowing fit or quality; or subscribing to a streaming service without knowing future consumption. In each case, the buyer commits before fully learning her valuation, and the seller may leverage this uncertainty through carefully designed pricing schemes.

A prominent framework capturing such dynamics is the *Sequential Screening* model introduced by (Courty & Li, 2000). In this model, a buyer initially knows only partial information about her valuation and later learns its realization. The seller, who observes only the buyer's type distribution, offers a menu of contracts—typically interpreted as upfront payments paired with refund options—to screen buyers as their information unfolds over time.

Despite its foundational role in dynamic pricing theory and its influence across economics, operations research, and dynamic mechanism design, a basic algorithmic question has remained open almost three decades:

Can the revenue-maximizing menu of refund contracts in the Sequential Screening model be computed efficiently?

This question was posed in the original work previously mentioned and was recently highlighted again in the computer science literature by (Papadimitriou et al., 2022).

In this paper, we resolve this problem by proving that computing an optimal Sequential Screening mechanism is NP-hard. We further complement this hardness result with a polynomial-time approximation scheme (PTAS) that computes near-optimal refund menus.

Before presenting our formal results, we provide a sequence of illustrative examples showing that optimal revenue extraction may rely predominantly on contingent payments (refund-based extraction), predominantly on non-refundable upfront payments, or on a delicate combination of both. These examples highlight the richness and intrinsic complexity of the Sequential Screening problem.

1.1. Our Contribution

We study the classical Sequential Screening model in which a seller offers a two-stage menu of refund contracts to a buyer who gradually learns her valuation. Although this framework has been extensively studied in economics, the computational complexity of computing optimal mechanisms had remained unresolved.

Our first main result establishes that computing the revenue-maximizing Sequential Screening menu is NP-hard, resolving a long-standing open problem. This places Sequential Screening among a growing class of dynamic pricing problems whose optimal solutions exhibit inherent computational intractability.

Our second main result is a polynomial-time approximation scheme (PTAS) for this problem. The algorithm combines structural properties of optimal refund menus with a carefully designed dynamic program based on geometric discretization and revenue-preserving rounding.

At a high level, our approach builds on the framework introduced by (Chawla et al., 2021) for static pricing over ordered value spaces, which combines combinatorial reductions with approximation algorithms. Extending these techniques to the Sequential Screening setting is substantially more challenging. Unlike the static model, buyer utilities here depend on nonlinear expressions involving integrals over valuation distributions, and menu entries simultaneously influence both the buyer's participation decision and her stage-two consumption rule. Moreover, the mechanism must screen buyers not only across different value magnitudes but also across different levels of uncertainty about future valuations. Our analysis develops new structural insights to handle these interactions in a two-stage dynamic environment.

Finally, we identify several special cases of practical relevance where optimal or near-optimal mechanisms can be computed efficiently. Together, our results characterize both the fundamental hardness and the algorithmic tractability frontier of pricing under buyer uncertainty.

1.2. Related Work

Our work builds on a rich literature spanning dynamic mechanism design in economics and algorithmic mechanism design in computer science.

1.2.1. Economic Foundations

The study of dynamic incentive problems with evolving private information began with the seminal work of Baron & Besanko (1984), which introduced core tools for analyzing intertemporal screening. A cornerstone of this literature is the Sequential Screening model of Courty & Li (2000), which formalized refund-based pricing under buyer uncertainty and has since influenced research across economics and operations research.

Subsequent work explored extensions and refinements of dynamic screening under various informational and participation constraints, including Eső & Szentes (2007) and Pavan et al. (2014). Related strands studied limited commitment and gradual information revelation, beginning with Laffont & Tirole (1987) and Laffont & Tirole (1988), and more recently Doval & Skreta (2018). Complementary perspectives appear in settings where the seller controls information disclosure, such as Babaioff et al. (2012).

1.2.2. Algorithmic Mechanism Design

From the computational perspective, a growing literature investigates the complexity of optimal mechanism design in multi-dimensional and dynamic environments. Foundational hardness results appear in Daskalakis et al. (2014) and are complemented by structural characterizations in Cai et al. (2012a;b; 2013). Approximation approaches for complex buyer preferences were developed in works such as Babaioff et al. (2014); Cai & Zhao (2017).

In dynamic settings, Mirrokni et al. (2016a;b) and Ashlagi et al. (2016) study multi-round mechanisms under ex-post participation guarantees. Most closely related to our work, Papadimitriou et al. (2022) analyzes the complexity of dynamic mechanism design and resolves several open questions while leaving the ex-ante Sequential Screening model open—a gap our results fill.

Our algorithmic techniques are inspired by the static ordered pricing framework of Chawla et al. (2021), which we extend to a significantly more intricate dynamic screening environment.

Prior to analyzing the computational complexity and approximation algorithm for Sequential Screening, we provide illustrative examples that underscore both the advantages and potential difficulties of extracting revenue through mechanisms that go beyond basic pricing schemes.

1.3. Examples highlighting the complexity of the problem

Two-Price Formulation and Equivalence to Refund Contracts. Throughout the paper, we work with a two-price representation of contracts: a menu option is a pair (p, ℓ) where the buyer pays p in stage one and, after realizing value v, may purchase the item in stage two by paying ℓ, proceeding iff $v \geq \ell$. This formulation is equivalent to the classical refund-based representation used in Courty & Li (2000). Indeed, a refund contract (a, k)—where the buyer pays a upfront and receives refund k upon returning the item—is equivalent to the two-price contract $(p, \ell) = (a - k, k)$. In both formulations, the buyer keeps the item iff $v \geq k$, and her utility is

$$(v - k)_+ - (a - k) = (v - \ell)_+ - p.$$

Similarly, the seller's expected revenue is

$$a - k \cdot \Pr[v < k] = p + \ell \cdot \Pr[v \geq \ell].$$

Thus utilities, incentive constraints, and revenue coincide exactly. We adopt the two-price formulation since it preserves the standard interpretation of prices as utility losses to the buyer and revenue gains to the seller, while simplifying notation in our algorithmic analysis.

Notation for Examples. A two-stage pricing contract is a pair (p, ℓ) where the buyer pays p in stage one and, after realizing value v, purchases the item in stage two iff $v \geq \ell$, paying ℓ.

The seller's expected revenue from type τ is

$$p + \ell \cdot \Pr_{v \sim G_\tau}[v \geq \ell].$$

A *single posted price* corresponds to a contract $(p, 0)$, extracting all revenue in stage one.

A *pure contingent pricing* corresponds to a contract $(0, \ell)$, extracting revenue only in stage two.

A *choice of two-stage pricings* refers to offering a menu $\{(p_i, \ell_i)\}$ allowing different types to self-select different thresholds.

Example 1 (Power of Sequential Screening). There are n buyer types indexed by $i = 1, \ldots, n$ with probabilities $\alpha_i = \frac{2^{-i}}{\sum_{k=1}^n a_k}$. Type i has valuation:

$$v_i = \begin{cases} 1 & \text{with probability } \pi_i = \frac{2^i}{2^n}, \\ 0 & \text{otherwise.} \end{cases}$$

Consider first any single posted price $(p, 0)$. Its expected revenue is simply p times the probability the buyer participates. Since $\mathbb{E}[v_i] = \pi_i = \frac{2^i}{2^n}$, individual rationality implies $p \leq \pi_i$ for each participating type, and hence:

$$\begin{aligned} \text{REV}_{\text{posted}}(p) &= p \cdot \sum_{i:\pi_i \geq p} \alpha_i \leq p \cdot \sum_{j \geq i} \frac{2^{-j}}{\sum_{k=1}^n a_k} \\ &\leq 2p \cdot \frac{2^{-i}}{\sum_{k=1}^n a_k} \leq \frac{2}{2^n \cdot \sum_{k=1}^n a_k}. \end{aligned}$$

Now consider the pure contingent pricing $(0, 1)$. Each type i pays exactly when $v_i = 1$, yielding revenue:

$$\begin{aligned} \text{REV}_{\text{contingent}} &= \sum_{i=1}^n \alpha_i \pi_i \\ &= \sum_{i=1}^n \frac{2^{-i}}{\sum_{k=1}^n a_k} \cdot \frac{2^i}{2^n} = \frac{n}{2^n \cdot \sum_{k=1}^n a_k}. \end{aligned}$$

Hence:

$$\frac{\text{REV}_{\text{posted}}}{\text{REV}_{\text{contingent}}} \leq \frac{2}{n} \xrightarrow[n \to \infty]{} 0.$$

Thus extracting revenue purely in stage two can be arbitrarily better than any posted price. This concludes Example 1. □

Example 2 (Non-triviality of Sequential Screening). There is a single buyer type with valuation:

$$v = \begin{cases} 1 & \text{with probability } 1/2, \\ 0 & \text{otherwise.} \end{cases}$$

Consider any two-stage pricing (p, ℓ). The buyer's expected utility is:

$$U(p, \ell) = \mathbb{E}[(v - \ell)_+] - p = \tfrac{1}{2}(1 - \ell) - p,$$

so individual rationality requires $p \leq \frac{1}{2}(1 - \ell)$.

The seller's revenue is:

$$\text{REV}(p, \ell) = p + \tfrac{\ell}{2} \leq \tfrac{1}{2}.$$

Thus any two-stage pricing yields revenue at most $1/2$.

This bound is achieved by the posted price $(p, 0) = (1/2, 0)$, extracting all revenue in stage one.

Hence, in this instance, optimal revenue is obtained without any contingent payment, showing that second-stage pricing is not always beneficial. This concludes Example 2. □

2. Preliminaries

We consider a dynamic pricing model we refer to as the SEQUENTIALSCREENINGMODEL, a computer science–inspired formulation of the classical Sequential Screening framework introduced by (Courty & Li, 2000). The model captures two-stage pricing mechanisms in settings where buyers gradually learn their valuation over time.

Model Setup. There are two stages. At the beginning of stage one, a buyer privately learns their type $\tau \in T$, where each type induces a valuation distribution G_τ with density f_τ. The seller knows all distributions but not the realized type.

The seller offers a menu $\mathcal{M} = \{(p_i, \ell_i)\}_{i=1}^n$, where p_i is a stage-one payment and ℓ_i is a stage-two price (threshold).

At the beginning of stage two, the buyer realizes a valuation $v \sim G_\tau$ and may purchase the item at price ℓ_i, proceeding if and only if $v \geq \ell_i$.

We assume unit demand: the buyer consumes at most one item.

Profit Objective with Cost. Let $c \geq 0$ denote a fixed production cost incurred whenever the item is delivered. If type τ selects option (p_i, ℓ_i), the seller's expected profit is

$$p_i + \ell_i \cdot \Pr[v \geq \ell_i] - c \cdot \Pr[v \geq \ell_i].$$

Equivalently, the profit is

$$p_i + (\ell_i - c) \cdot \Pr[v \geq \ell_i].$$

The seller seeks to maximize expected profit over all buyer types.

Buyer Utility. The expected utility of a type τ buyer selecting contract (p, ℓ) is

$$U_\tau(p,\ell) = \mathbb{E}_{v\sim G_\tau}[(v-\ell)_+] - p = \int_\ell^\infty (v-\ell) f_\tau(v)\, dv - p.$$

Constraints. For each type $\tau \in T$, let $(p(\tau), \ell(\tau))$ denote the contract selected by τ. The mechanism must satisfy ex-ante individual rationality and incentive compatibility:

- **Individual Rationality (IR):** for all $\tau \in T$,

$$\int_{\ell(\tau)}^\infty (v - \ell(\tau)) f_\tau(v)\, dv - p(\tau) \geq 0.$$

- **Incentive Compatibility (IC):** for all $\tau, \tau' \in T$,

$$\int_{\ell(\tau)}^\infty (v - \ell(\tau))\, f_\tau(v)\, dv - p(\tau) \geq \int_{\ell(\tau')}^\infty (v - \ell(\tau')) \cdot f_\tau(v)\, dv - p(\tau').$$

Cost-Free Normalization. For clarity of exposition, we henceforth focus on the case $c = 0$. In Lemma **??** we show that this restriction is without loss of generality: any instance with production cost can be transformed into an equivalent cost-free instance that preserves incentive constraints, optimal revenue, computational complexity, and approximation guarantees.

Before proceeding to the main technical part(s), one observation that may facilitate our overall understanding: The stage-two price/"refund" ℓ fully determines the stage-2 allocation rule: the buyer consumes iff $v \geq \ell$. Thus the sequential screening problem reduces to selecting a menu of cutoffs and upfront payments satisfying IC/IR.

3. Intractability of Optimal Refund Menus

In this section we provide our main result, a proof that it is NP-hard to compute revenue-maximizing first- and second-stage prices in the SEQUENTIALSCREENINGMODEL. Let SEQUENTIALSCREENINGPRICING denote the following optimization problem: Given a unit-demand buyer whose type τ is drawn from a known distribution over a finite set T (with $|T| = m$), where each type $\tau \in T$ induces a known stage-2 value distribution G_τ over $[0, \mathcal{R}]$, compute the maximum expected revenue achievable by any two-stage menu

$$\mathcal{M} = \{(p_i, \ell_i)\}_{i=1}^n,$$

where the buyer chooses a menu entry (p_i, ℓ_i) after learning τ, pays p_i in stage 1, then in stage 2 observes $v \sim G_\tau$ and purchases iff $v \geq \ell_i$, paying ℓ_i. Equivalently, the objective is to compute

$$\mathrm{OPT}(\mathcal{D}) \;=\; \sup_{\mathcal{M}} \sum_{\tau \in T} q_\tau \left(p(\tau) + \ell(\tau) \cdot \Pr_{v\sim G_\tau}[v \geq \ell(\tau)] \right),$$

subject to ex-ante individual rationality and incentive compatibility constraints, i.e., for all $\tau, \tau' \in T$,

$$U_\tau\big(p(\tau), \ell(\tau)\big) \geq 0 \qquad \text{and}$$
$$U_\tau\big(p(\tau), \ell(\tau)\big) \geq U_\tau\big(p(\tau'), \ell(\tau')\big),$$

where

$$U_\tau(p,\ell) \;:=\; \mathbb{E}_{v\sim G_\tau}\big[(v-\ell)_+\big] - p.$$

Beyond stating the theorem, we provide a high-level yet comprehensive description of the proof's construction.

Theorem 1. SEQUENTIALSCREENINGPRICING *is NP-hard, even when each (realized) buyer has at most three distinct value point masses.*

We prove the theorem via a reduction from MAX-CUT; complete details appear in Appendix B.
For any MAX-CUT instance with graph $G(V, E)$, let $n = |V| > 10$ be large enough. Consider an instance of SEQUENTIALSCREENINGPRICING with $n + 1$ menu options. Fix a parameter $\delta \geq 0.2$. We want an optimal menu with the following properties:

1. The optimal menu assigns to each option $j \in [n]$ one of the two pricing profiles $(p_j, \ell_j)_{j=1}^n = (15\delta \cdot 2^{-j}, 18\delta \cdot 2^j)_{j=1}^n$ or $(14\delta \cdot 2^{-j}, 22\delta \cdot 2^j)_{j=1}^n$. That is, under the first profile the k-th menu option charges a stage-one price $15\delta \cdot 2^{-k}$ and a stage-two price $18\delta \cdot 2^k$, with an analogous interpretation for the second profile;

2. $menu_{n+1} = (15\delta \cdot 2^{n+1}, 18\delta \cdot 2^{n+1})$ and there is a set of buyers, named Gamblers, purchasing menu option $n + 1$ with realization probability $q_G = 0.8$ independent of the graph structure that contribute to the total revenue $R_G(n)$;

3. $menu_i = (15\delta \cdot 2^{-i}, 18\delta \cdot 2^i)$ or $(14\delta \cdot 2^{-i}, 22\delta \cdot 2^i)$ and there is a set of buyer types, named Worker1 and Worker2, whose elements are purchasing menu options in $[n]$ with realization probability $q_{W_1} + q_{W_2} < \frac{q_G}{36\delta}$, independent of the graph structure that contribute to the total revenue $R_W(n)$;

4. For each $(i,j) \in E$ with $i < j$, there exists a set T_{ij} of buyer types, and real number $R_{ij} > 0$ that is irrelevant to the graph structure such that: if $menu_j - menu_i = (14\delta \cdot (2^{-j} - 2^{-i}), 22\delta \cdot (2^{-j} - 2^{-i}))$ or $menu_j - menu_i = (15\delta \cdot (2^{-j} - 2^{-i}), 18\delta \cdot (2^{-j} - 2^{-i}))$, then the revenue contribution from T_{ij} is $R_{ij}(n)$; else the revenue contribution from T_{ij} is $R_{ij}(n) + \frac{1}{e^n}$.

As already mentioned, the complete construction of such an instance can be found in Section B. We now establish that SEQUENTIALSCREENINGPRICING is NP-hard for instances with the above properties.
Let us consider an instance $\mathcal{I}$ with the structural properties described above. The contribution to the revenue from buyer types "Gambler" and "Worker" is given by $R_G(n) + R_W(n)$. Now, fix a cut $C = (V_1, V \setminus V_1)$ in the graph $G = (V, E)$. Suppose we assign the menu option for each $i \in V_1$ the price $(15\delta, 18\delta)$, and for each $j \in V \setminus V_1$ the price $(14\delta, 22\delta)$. Then the total revenue generated from "Combinatorial" types under this pricing is:

$$\sum_{(i,j)\in E} R_{ij}(n) + \frac{1}{e^n}|C|.$$

In particular, if $C_{\max}$ denotes a maximum cut of the graph G, then the maximum achievable revenue across all such menu assignments is:

$$\mathrm{Rev}(G) = R_G(n) + R_W(n) + \sum_{(i,j)\in E} R_{ij}(n) + \frac{1}{e^n} C_{\max}.$$

This establishes a direct correspondence between the size of the maximum cut in G and the optimal revenue obtainable in the associated instance $\mathcal{I}$ of SEQUENTIALSCREENING-PRICING. Consequently, computing the optimal pricing in $\mathcal{I}$ is as hard as computing the maximum cut in G, thereby proving NP-hardness.

4. Efficient Approximation of Optimal Refund Menus

Theorem 2 (PTAS for SEQUENTIALSCREENINGPRICING)**.** *Consider a unit-demand buyer whose type is drawn from a distribution supported on m types, and let $\mathcal{R}$ be the maximum possible value over all types. For any $\epsilon > 0$, there exists an algorithm that computes a two-stage menu with $n = O(\frac{\log \mathcal{R}}{\epsilon})$ entries in time $(m, n^{(1/\epsilon)}, \log \mathcal{R})$ whose revenue is at least a $(1-\epsilon)$-fraction of the optimal two-stage menu revenue.*

Proof sketch. We proceed in four steps; complete details appear in Appendix C.

Step 1 (Menu compression and geometric rounding). We first compress the value space into $n = O(\log \mathcal{R}/\epsilon)$ representative levels/masses and show that there exists a near-optimal menu in which all stage-1 prices are non-increasing and all stage-2 prices are non-decreasing powers of $(1+\epsilon^2)$, losing only a $(1 - O(\epsilon))$ factor in revenue.

Step 2 (Restriction to a small set of prices). We then show that it suffices to restrict both stage-1 and stage-2 prices to a computable set $\Pi^\star \subseteq \{0\} \cup \{(1+\epsilon^2)^t : t \in \mathbb{Z}\}$ of size $|\Pi^\star| = (1/\epsilon, \log \mathcal{R})$, again at a $(1 - O(\epsilon))$ multiplicative revenue loss.

Step 3 (Interval structure and unit-demand-to-additive reduction). The main technical obstacle is that, in the sequential screening model, the buyer is *unit-demand over menu options*: modifying the price of one option may change which single option is chosen, thereby globally altering incentives and revenue. This makes it difficult to optimize prices locally.

We overcome this by showing that there exists a near-optimal menu with a strong *interval structure* on prices, and by reducing the resulting unit-demand problem to an equivalent *additive* proxy problem.

Interval structure. After the discretization steps, consider a menu with n options indexed so that stage-one prices are non-increasing and stage-two prices are non-decreasing:

$$p_1 \geq p_2 \geq \cdots \geq p_n \quad \text{and} \quad \ell_1 \leq \ell_2 \leq \cdots \leq \ell_n.$$

We show that the menu can be modified (with only a $1-O(\epsilon)$ revenue loss) so that the indices $1, \ldots, n$ can be partitioned into contiguous stage-1 and stage-2 intervals $I_1^1, I_2^1, \ldots, I_K^1$ and $I_1^2, I_2^2, \ldots, I_L^2$ with the following *price-gap property*: within each interval, all corresponding prices are multiplicatively close, while prices across different intervals differ by large multiplicative factors.

This structure allows us to *compress* prices within each interval—replacing all prices in an interval by a small number of representative values—while preserving expected revenue up to a factor $1 - O(\epsilon)$. At the same time, the large gaps across intervals maintain the global monotonic ordering required for the reduction to an additive proxy problem.

From unit-demand to additive values. Fix such an interval-structured menu and consider the buyer's discretized unit-demand values $v_1 \leq v_2 \leq \cdots \leq v_n$, where choosing option i yields value v_i. We define an additive valuation $v^\oplus$ over n virtual items by setting

$$v_1^\oplus := v_1, \qquad v_i^\oplus := v_i - v_{i-1} \quad \text{for } i \geq 2.$$

Thus, the original unit-demand value v_i is exactly the total additive value of the prefix $\{1, 2, \ldots, i\}$ under $v^\oplus$.

Prefix–suffix pricing interpretation. Under the interval structure, we reinterpret the menu as follows. Paying the stage-two price ℓ_i corresponds to purchasing the prefix $\{1, \ldots, i\}$

of additive items, while paying the stage-one price p_i corresponds to purchasing the suffix $\{i, \ldots, n\}$. Thus, choosing option i in the original unit-demand menu is equivalent to selecting a prefix–suffix bundle in the additive instance whose total price is $p_i + \ell_i$ and whose total value is v_i.

Why this helps. In the additive proxy problem, revenue decomposes across intervals and prices can be optimized locally without affecting incentives elsewhere. Moreover, for interval-structured menus, we prove that the buyer selects the same effective option under the original unit-demand menu and under the additive proxy, and that the expected revenue is preserved up to a factor $1 - O(\epsilon)$. This reduction allows us to apply a dynamic programming algorithm in Step 4 to compute near-optimal prices for each interval independently.

Step 4 (Dynamic programming over intervals). Finally, for the additive proxy instance and the restricted set of prices Π', we compute the optimal interval-structured menu by dynamic programming. The DP optimizes interval-by-interval subject to monotonicity and price-gap constraints, and runs in $(m, n^{(1/\epsilon)}, |\Pi'|)$ time. Mapping the resulting solution back yields a feasible two-stage menu for the original instance with revenue at least $(1 - \epsilon)$ times optimal. □

5. Discussion and Future Work

Our work settles the long-standing open question of the computational hardness of the Sequential Screening model, but it also opens the door to several compelling research directions. A natural next step is to refine the complexity landscape: while we prove NP-hardness, it remains unknown whether the problem is strongly NP-hard. Establishing this would rule out the existence of a Fully Polynomial-Time Approximation Scheme and solidify the limits of efficient computation in this domain.

Another avenue stems from practical considerations around individual rationality. In many real-world markets, such as online ad auctions, large advance payments—often required to extract full surplus—are either disallowed or capped due to regulation or platform constraints. This motivates the design and analysis of ex post individually rational mechanisms in sequential settings, where the literature lacks mechanisms that achieve constant-factor approximations.

Finally, it would be natural to extend the sequential screening framework to more than two stages, capturing scenarios with evolving buyer information over time—such as subscriptions, installment purchases, or recurring service platforms. A simple generalization might involve offering a sequence of refundable options at increasing levels of commitment, mirroring multi-phase decision-making in real-world transactions.

References

Ashlagi, I., Daskalakis, C., and Haghpanah, N. Sequential mechanisms with ex-post participation guarantees. In *Proceedings of the 17th ACM Conference on Economics and Computation*. ACM, 2016.

Babaioff, M., Kleinberg, R., and Paes Leme, R. Optimal mechanisms for selling information. In *Proceedings of the 13th ACM Conference on Electronic Commerce*, pp. 92–109. ACM, 2012.

Babaioff, M., Immorlica, N., Lucier, B., and Weinberg, S. M. A simple and approximately optimal mechanism for an additive buyer. In *Proceedings of the 55th IEEE Symposium on Foundations of Computer Science*, pp. 21–30, 2014.

Baron, D. P. and Besanko, D. Regulation and information in a continuing relationship. *Information Economics and Policy*, 1(3):267–302, 1984.

Bergemann, D. and Said, M. Dynamic auctions. In *Wiley Encyclopedia of Operations Research and Management Science*. Wiley, 2011.

Bergemann, D. and Välimäki, J. Dynamic mechanism design: An introduction. *Journal of Economic Literature*, 57(2):235–274, 2019.

Briest, P., Chawla, S., Kleinberg, R., and Weinberg, S. M. Pricing lotteries. *Journal of Economic Theory*, 156:144–174, 2015.

Cai, Y. and Zhao, M. Simple mechanisms for subadditive buyers via duality. In *Proceedings of the 49th ACM Symposium on Theory of Computing*, pp. 170–183, 2017.

Cai, Y., Daskalakis, C., and Weinberg, S. M. An algorithmic characterization of multi-dimensional mechanisms. In *Proceedings of the 44th ACM Symposium on Theory of Computing*, pp. 459–478, 2012a.

Cai, Y., Daskalakis, C., and Weinberg, S. M. Optimal multi-dimensional mechanism design: Reducing revenue to welfare maximization. In *Proceedings of the 53rd IEEE Symposium on Foundations of Computer Science*, pp. 130–139, 2012b.

Cai, Y., Daskalakis, C., and Weinberg, S. M. Reducing revenue to welfare maximization. In *Proceedings of the 24th ACM-SIAM Symposium on Discrete Algorithms*, pp. 911–926, 2013.

Chawla, S., Rezvan, R., Teng, Y., and Tzamos, C. Pricing ordered items, 2021. arXiv:2106.04704.

Chen, X., Matikas, G., Paparas, D., and Yannakakis, M. On the complexity of simple and optimal deterministic mechanisms for an additive buyer. In *Proceedings of the ACM-SIAM Symposium on Discrete Algorithms*, pp. 2036–2049, 2018.

Courty, P. and Li, H. Sequential screening. *The Review of Economic Studies*, 67(4):697–717, 2000.

Daskalakis, C., Deckelbaum, A., and Tzamos, C. The complexity of optimal mechanism design. In *Proceedings of the ACM-SIAM Symposium on Discrete Algorithms*, pp. 1302–1318, 2014.

Daskalakis, C., Deckelbaum, A., and Tzamos, C. Strong duality for a multiple-good monopolist. *Econometrica*, 2016.

Doval, L. and Skreta, V. Mechanism design with limited commitment. *arXiv preprint arXiv:1811.03579*, 2018.

Eső, P. and Szentes, B. Optimal information disclosure in auctions and the handicap auction. *The Review of Economic Studies*, 74(3):705–731, 2007.

Gonczarowski, Y. A. and Weinberg, S. M. The sample complexity of up-to-ϵ multidimensional revenue maximization. *Journal of the ACM*, 68(3):1–28, 2021.

Hart, S. and Nisan, N. The menu-size complexity of auctions. Technical report, Center for the Study of Rationality, 2013.

Laffont, J.-J. and Tirole, J. Comparative statics of the optimal dynamic incentive contract. *European Economic Review*, 31(4):901–926, 1987.

Laffont, J.-J. and Tirole, J. The dynamics of incentive contracts. *Econometrica*, pp. 1153–1175, 1988.

Lee, E. Hardness of graph pricing through generalized maxdicut. In *Proceedings of the 47th ACM Symposium on Theory of Computing*, pp. 391–399, 2015.

Mirrokni, V., Paes Leme, R., Tang, P., and Zuo, S. Dynamic auctions with bank accounts. In *Proceedings of the International Joint Conference on Artificial Intelligence*, 2016a.

Mirrokni, V., Paes Leme, R., Tang, P., and Zuo, S. Optimal dynamic mechanisms with ex-post ir via bank accounts. *arXiv preprint arXiv:1605.08840*, 2016b.

Myerson, R. B. Optimal auction design. *Mathematics of Operations Research*, 6(1):58–73, 1981.

Papadimitriou, C., Pierrakos, G., Psomas, A., and Rubinstein, A. On the complexity of dynamic mechanism design. *Games and Economic Behavior*, 134:399–427, 2022.

Pavan, A., Segal, I., and Toikka, J. Dynamic mechanism design: A myersonian approach. *Econometrica*, 82(2): 601–653, 2014.

Sugaya, T. and Wolitzky, A. The revelation principle in multistage games. *The Review of Economic Studies*, 88 (3):1503–1540, 2021.

A. Proofs

B. Omitted proofs in Section 3

B.1. Outline

Proof. We prove the theorem via a reduction from MAX-CUT.
For any MAX-CUT instance with graph $G(V, E)$, let $n = |V| > 10$ be large enough. Fix a parameter $\delta \geq 0.2$. Consider an instance of SEQUENTIALSCREENINGPRICING with $n + 1$ menu options. We want an optimal menu with the following properties:

1. The optimal menu assigns to each option $i \in [n]$ either the pricing pair $(15\delta \cdot 2^{-i}, 18\delta \cdot 2^i)$ or $(14\delta \cdot 2^{-i}, 22\delta \cdot 2^i)$;
2. $menu_{n+1} = (15\delta \cdot 2^{n+1}, 18\delta \cdot 2^{n+1})$ and there is a set of buyers, named "Gambler", purchasing menu option $n + 1$ with realization probability $q_G = 0.8$ independent of the graph structure that contribute to the total revenue $R_G(n)$;
3. $menu_i = (15\delta \cdot 2^{-i}, 18\delta \cdot 2^i)$ or $(14\delta \cdot 2^{-i}, 22\delta \cdot 2^i)$ and there is a set of buyer types, named "Worker1" and "Worker2", purchasing menu options in $[n]$ with realization probability $q_{W_1} + q_{W_2} < \frac{q_G}{36\delta}$, independent of the graph structure that contribute to the total revenue $R_W(n)$;
4. For each $(i, j) \in E$ with $i < j$, there exists a set T_{ij} of buyer types, and real number $R_{ij} > 0$ that is irrelevant to the graph structure such that: if $menu_j - menu_i = (14\delta \cdot (2^{-j} - 2^{-i}), 22\delta \cdot (2^{-j} - 2^{-i}))$ or $menu_j - menu_i = (15\delta \cdot (2^{-j} - 2^{-i}), 18\delta \cdot (2^{-j} - 2^{-i}))$, then the revenue contribution from T_{ij} is $R_{ij}(n)$; else the revenue contribution from T_{ij} is $R_{ij}(n) + \frac{1}{e^n}$.

Before jumping into the details of the construction, we establish that SEQUENTIALSCREENINGPRICING is NP-hard for instances with the above properties.
Let us consider an instance $\mathcal{I}$ with the structural properties described above. The contribution to the revenue from buyer types labeled "Gambler" and "Worker" is given by $R_G(n) + R_W(n)$. Now, fix a cut $C = (V_1, V \setminus V_1)$ in the graph $G = (V, E)$. Suppose we assign the menu option for each $i \in V_1$ the price $(15\delta, 18\delta)$, and for each $j \in V \setminus V_1$ the price $(14\delta, 22\delta)$. Then the total revenue generated from "Combinatorial" types under this pricing is:

$$\sum_{(i,j)\in E} R_{ij}(n) + \frac{1}{e^n}|C|.$$

In particular, if $C_{\max}$ denotes a maximum cut of the graph G, then the maximum achievable revenue across all such menu assignments is:

$$\text{Rev}(G) = R_G(n) + R_W(n) + \sum_{(i,j)\in E} R_{ij}(n) + \frac{1}{e^n} C_{\max}.$$

This establishes a direct correspondence between the size of the maximum cut in G and the optimal revenue obtainable in the associated instance $\mathcal{I}$ of SEQUENTIALSCREENINGPRICING. Consequently, computing the optimal pricing in $\mathcal{I}$ is as hard as computing the maximum cut in G, thereby proving NP-hardness.

□

B.2. Creating 2^n equivalent pricings

In this subsection, we define two two-stage pricing schemes, $\boldsymbol{p}$ and $\boldsymbol{p'}$, each with n options, along with suitable valuation functions and corresponding frequencies. These are constructed so that assigning any menu option a price from either scheme yields the same revenue. This subconstruction will be a key component of the final construction.

Lemma 1 (Buyer that prefers a certain level of uncertainty)**.** *Suppose a buyer type τ with the following value distribution:*

$$\mathrm{v}_\tau(i) = \begin{cases} 33\delta \cdot 2^i & p_1 = \frac{1}{2^{2i}}, \\ 0 & p_2 = 1 - \frac{1}{2^{2i}} \end{cases}, i \in [n]$$

Then, the buyer will prefer menu option $(15\delta \cdot 2^{-i}, 18\delta \cdot 2^i)$ over all the other menu options under pricing $\boldsymbol{p} = (15\delta, 18\delta)$. Similarly, for pricing $\boldsymbol{p'} = (14\delta, 22\delta)$.

Proof. Let $(p_k, \ell_k) = (15\delta \cdot 2^{-k}, 18\delta \cdot 2^k)$ denote the $k-$th menu option under pricing $\boldsymbol{p}$.

Case 1: $k \leq i$

$$
\begin{aligned}
u_\tau((p_i, \ell_i)) \geq u_\tau((p_k, \ell_k)) \\
&\Leftrightarrow \sum_{v \geq \ell_i} v \cdot f_\tau(v) - p_i - \ell_i \cdot (1 - F_\tau(\ell_i)) \geq \sum_{v \geq \ell_k} v \cdot f_\tau(v) - p_k - \ell_k \cdot (1 - F_\tau(\ell_k)) \\
&\Leftrightarrow 15\delta \cdot 2^{-k} + 18\delta \cdot 2^{k-2i} - 15\delta \cdot 2^{-i} - 18\delta \cdot 2^{-i} \geq 0 \\
&\overset{k=i-1}{\Leftrightarrow} 15\delta \cdot 2^{-i} - 18\delta \cdot 2^{-i-1} \geq 0
\end{aligned}
$$

where the last line holds since ${}_k\{15\delta \cdot 2^{-k} + 18\delta \cdot 2^{k-2i} - 15\delta \cdot 2^{-i} - 18\delta \cdot 2^{-i}\} =_k \{15\delta \cdot 2^{-k}\} = i - 1$.

Case 2: $i < k$

In this case, the $k-$th menu option violates the individual rationality constraints, since $\ell_k \geq \ell_{i+1} = 36\delta \cdot 2^i > \mathrm{v}_\tau(i)$. □

Based on Lemma 1, we define the following buyer types that will be useful for our final construction:

Corollary 1 (Type Gambler). *Suppose a buyer type "Gambler" with the following value distribution:*

$$
\mathrm{v_G}(i) = \begin{cases} 33\delta \cdot 2^{2(n+1)} & p_1 = \frac{1}{2^{2(n+1)}}, \\ 0 & p_2 = 1 - \frac{1}{2^{2(n+1)}} \end{cases}
$$

Then, the buyer will prefer menu option $(15\delta \cdot 2^{-(n+1)}, 18\delta \cdot 2^{(n+1)})$ *over all menu options under pricing* $\mathbf{p} = (15\delta, 18\delta)$.

Corollary 2 (Type Worker 1). *Suppose a buyer type "Worker1" with the following value distribution:*

$$
\mathrm{v}_{\mathrm{W}_1}(i) = \begin{cases} 33\delta \cdot 2^i & p_1 = \frac{1}{2^{2i}}, \\ 0 & p_2 = 1 - \frac{1}{2^{2i}} \end{cases}, i \in [n]
$$

Then, the buyer will prefer menu option $(15\delta \cdot 2^{-i}, 18\delta \cdot 2^i)$ *over all the other menu options under pricing* $\mathbf{p} = (15\delta, 18\delta)$.

Corollary 3 (Type Worker 2). *Suppose a buyer type "Worker2" with the following value distribution:*

$$
\mathrm{v}_{\mathrm{W}_2}(i) = \begin{cases} 36\delta \cdot 2^i & p_1 = \frac{1}{2^{2i}}, \\ 0 & p_2 = 1 - \frac{1}{2^{2i}} \end{cases}, i \in [n]
$$

Then, the buyer will prefer menu option $(14\delta \cdot 2^{-i}, 22\delta \cdot 2^i)$ *over all the other menu options under pricing* $\mathbf{p'} = (14\delta, 22\delta)$.

Now, we proceed to state and prove the main result of the subsection, the engineering of 2^n equivalent pricings for n menu options in the presence of buyer types "Worker1" and "Worker2":

Lemma 2 (Engineering 2^n Equivalent Pricings). *For buyer types "Worker1" and "Worker2", which appear with probability* q_{W_1} *and* q_{W_2} *respectively, where:*

$$
\frac{q_{W_1}}{q_{W_2}} = \frac{1}{11}
$$

it holds that both menu options $(15\delta \cdot 2^{-i}, 18\delta \cdot 2^i)$ *and* $(14\delta \cdot 2^{-i}, 22\delta \cdot 2^i)$ *contribute the same revenue.*

Proof. It is easy to see that since $(15\delta \cdot 2^{-i}, 18\delta \cdot 2^i)$ and $(14\delta \cdot 2^{-i}, 22\delta \cdot 2^i)$ achieve to extract all the utilities from buyer types "Worker1" and "Worker2", any other pricing is suboptimal. Moreover, it holds that:

$$
\begin{aligned}
\mathrm{REV}_{(15\delta \cdot 2^{-i}, 18\delta \cdot 2^i)} &= 15\delta \cdot 2^{-i}(q_{W_1} + q_{W_2}) + 18\delta \cdot 2^{-i}(q_{W_1} + q_{W_2}) \\
&= \frac{12}{11} q_{w_2} 33\delta \cdot 2^{-i} \\
&= 14\delta \cdot 2^{-i}(q_{W_2}) + 22\delta \cdot 2^{-i}(q_{W_2}) \\
&= \mathrm{REV}_{(14\delta \cdot 2^{-i}, 22\delta \cdot 2^i)}
\end{aligned}
$$

□

Corollary 4 (Discretizing the solution space)**.** *Let buyer types G, W_1, W_2 with corresponding value distributions $v_G, v_{W1}(i), v_{W2}(i)$ and total realization probability $q_G = 0.8, q_{W1} = \frac{1}{550\delta}, q_{W2} = \frac{1}{50}\delta$ respectively. For the optimal menu: $menu_i = (15\delta \cdot 2^{-i}, 18\delta \cdot 2^i)$ or $(14\delta \cdot 2^{-i}, 22\delta \cdot 2^i), \forall i \in [n]$ and $menu_{n+1} = (15\delta \cdot 2^{-(n+1)}, 18\delta \cdot 2^{n+1})$ hold.*

B.3. Adding the combinatorial constraints

Lemma 3 (Combinatorial Type 1)**.** *Suppose a buyer with the following value distribution:*

$$v_{CT1}(i,j) = \begin{cases} 18\delta \cdot 2^i - 18\delta \cdot \frac{2^i}{2^{j-i}+1} + 15\delta \cdot \frac{2^j}{2^{j-i}+1} & p_1 = \frac{1}{2^{2i}} - \frac{1}{2^{2j}}, \\ 33\delta \cdot 2^j & p_2 = \frac{1}{2^{2j}}, \\ 0 & p_3 = 1 - \frac{1}{2^{2i}}, \end{cases}$$

Then the buyer is indifferent over $(15\delta \cdot 2^{-i}, 18\delta \cdot 2^i)$ or $(15\delta \cdot 2^{-j}, 18\delta \cdot 2^j)$. Moreover, they will prefer to refrain from purchasing under pricing $\boldsymbol{p'}$.

Proof. At first, let's "sandwich" the non-zero value point masses of the distribution between two consecutive stage-2 prices, a necessary step to continue further our analysis.

Lemma 4 (Sandwiching value masses of Combinatorial Type 1)**.** *Let $v_{CT1,1}$ and $v_{CT1,2}$ define the point masses in the first and second branch of v_{CT1} respectively. Then it holds that:*

$$\ell'_i = 22\delta \cdot 2^i \leq v_{CT1,1}(i,j) \leq 18\delta \cdot 2^{i+1} = \ell_{i+1} \quad (1)$$
$$\ell'_j = 22\delta \cdot 2^j \leq v_{CT1,2}(i,j) \leq 22\delta \cdot 2^{j+1} = \ell_{j+1} \quad (2)$$

Proof. We start our analysis with proving the left inequality of 1.

$$\begin{aligned} v_{CT1,1}(i,j) &= 18\delta \cdot 2^i - 18\delta \cdot \frac{2^i}{2^{j-i}+1} + 15\delta \cdot \frac{2^j}{2^{j-i}+1} \\ &\geq 18\delta \cdot 2^i - 18\delta \cdot \frac{2^i}{2^{j-i}+1} + 2 \cdot \frac{18\delta}{2} \cdot \frac{2^{j-1}}{2^{j-i}+1} + 6\delta \cdot \frac{2^j}{2^{j-i}+1} \\ &\Rightarrow v_{CT1,1}(i,j) \geq 22\delta \cdot 2^i \end{aligned}$$

Now for the right inequality of 1, we have that:

$$\begin{aligned} v_{CT1,1}(i,j) = 18\delta \cdot 2^i - 18\delta \cdot \frac{2^i}{2^{j-i}+1} + 15\delta \cdot \frac{2^j}{2^{j-i}+1} \leq 18\delta \cdot 2^{j+1} \\ \Leftrightarrow 18\delta \cdot 2^j + 18\delta \cdot 2^j - 18\delta \cdot 2^j + 15\delta \cdot 2^j \leq 18\delta \cdot 2^{j+1} \end{aligned}$$

For $v_{CT1,2}$, 2 obviously holds. □

Based on Lemma 4 we will proceed with proving the aforementioned preference relations.

At first, let's show the equivalence between (p_i, ℓ_i) and (p_j, ℓ_j):

$$
\begin{aligned}
u_{\mathrm{CT1}}((p_i,\ell_i)) &= \sum_{v\geq \ell_i} v\cdot f_{\mathrm{CT1}}(v) - p_i - \ell_i\cdot(1-F_{\mathrm{CT1}}(\ell_i))\\
&= \mathrm{v}_{\mathrm{CT1},1}\cdot p_1 + \mathrm{v}_{\mathrm{CT1},2}\cdot p_2 - p_0\cdot 2^{-i} - \ell_0\cdot 2^{i}\cdot(p_1+p_2)\\
&= \frac{(33\delta)\cdot 2^{j}}{2^{j-i}+1}\cdot(2^{-2i}-2^{-2j}) + 33\delta\cdot 2^{j}\cdot 2^{-2j} - 15\delta\cdot 2^{-i} - 18\delta\cdot 2^{i}\cdot 2^{-2i}\\
&= 0\\
&= (33\delta\cdot 2^{j})\cdot 2^{-2j} - (15\delta)\cdot 2^{-j} - (18\delta\cdot 2^{j}\cdot 2^{-2j})\\
&= \sum_{v\geq \ell_j} v\cdot f_{\mathrm{CT1}}(v) - p_j - \ell_j\cdot(1-F_{\mathrm{CT1}}(\ell_j))\\
&= u_{\mathrm{CT1}}((p_j,\ell_j))
\end{aligned}
$$

Next, we show the optimality - from the buyer's perspective - of choosing either (p_i,ℓ_i) or (p_j,ℓ_j) compared to any other menu option under $\boldsymbol{p}$:

Case 1: $k<i$

$$
\begin{aligned}
&u_{\mathrm{CT1}}((p_i,\ell_i)) \geq u_{\mathrm{CT1}}((p_k,\ell_k))\\
&\quad\Leftrightarrow \sum_{v\geq \ell_i} v\cdot f_{\mathrm{CT1}}(v) - p_i - \ell_i\cdot(1-F_{\mathrm{CT1}}(\ell_i)) \geq \sum_{v\geq \ell_k} v\cdot f_{\mathrm{CT1}}(v) - p_k - \ell_k\cdot(1-F_{\mathrm{CT1}}(\ell_k))\\
&\quad\Leftrightarrow 15\delta\cdot 2^{-k} + 18\delta\cdot 2^{k-2i} - 15\delta\cdot 2^{-i} - 18\delta\cdot 2^{-i} \geq 0\\
&\quad\overset{k=i-1}{\Leftrightarrow} 15\delta\cdot 2^{-i} - 18\delta\cdot 2^{-i-1} \geq 0
\end{aligned}
$$

where the last line holds since $_k\{15\delta\cdot 2^{-k} + 18\delta\cdot 2^{k-2i} - 15\delta\cdot 2^{-i} - 18\delta\cdot 2^{-i}\} =_k \{15\delta\cdot 2^{-k}\} = i-1$.

Case 2: $i<k<j$

$$
\begin{aligned}
&u_{\mathrm{CT1}}((p_j,\ell_j)) \geq u_{\mathrm{CT1}}((p_k,\ell_k))\\
&\quad\Leftrightarrow \sum_{v\geq \ell_j} v\cdot f_{\mathrm{CT1}}(v) - p_j - \ell_j\cdot(1-F_{\mathrm{CT1}}(\ell_j)) \geq \sum_{v\geq \ell_k} v\cdot f_{\mathrm{CT1}}(v) - p_k - \ell_k\cdot(1-F_{\mathrm{CT1}}(\ell_k))\\
&\quad\Leftrightarrow 15\delta\cdot 2^{-k} + 18\delta\cdot 2^{k-2j} - 15\delta\cdot 2^{-j} - 18\delta\cdot 2^{-j} \geq 0\\
&\quad\overset{k=j-1}{\Leftrightarrow} 15\delta\cdot 2^{-j} - 18\delta\cdot 2^{-j-1} \geq 0
\end{aligned}
$$

where the last line holds since $_k\{15\delta\cdot 2^{-k} + 18\delta\cdot 2^{k-2j} - 15\delta\cdot 2^{-j} - 18\delta\cdot 2^{-j}\} =_k \{15\delta\cdot 2^{-k}\} = j-1$.

Case 3: $j<k$
It is easy to see that $u_{\mathrm{CT1}}((p_k,\ell_k))<0$ for $k\geq j+1$, since $\ell_k \geq \mathrm{v}_{\mathrm{CT1},2}(\mathrm{i},\mathrm{j})$.

Let's proceed similarly for the 2 equivalent menu options compared to the menu options produced from price vector $\boldsymbol{p'}$:

Case 1: $k<i$

$$
\begin{aligned}
&u_{\mathrm{CT2}}((p_i,\ell_i)) \geq u_{\mathrm{CT2}}((p'_k,\ell'_k))\\
&\quad\Leftrightarrow \sum_{v\geq \ell_i} v\cdot f_{\mathrm{CT1}}(v) - p_i - \ell_i\cdot(1-F_{\mathrm{CT1}}(\ell_i)) \geq \sum_{v\geq \ell'_k} v\cdot f_{\mathrm{CT1}}(v) - p'_k - \ell'_k\cdot(1-F_{\mathrm{CT1}}(\ell'_k))\\
&\quad\Leftrightarrow 14\delta\cdot 2^{-k} + 22\delta\cdot 2^{k-2i} - 15\delta\cdot 2^{-i} - 18\delta\cdot 2^{-i} \geq 0\\
&\quad\overset{k=i-1}{\Leftrightarrow} 39\delta\cdot 2^{-i} - 33\delta\cdot 2^{-i} \geq 0
\end{aligned}
$$

where the last line holds since $_k\{14\delta \cdot 2^{-k} + 22\delta \cdot 2^{k-2i} - 15\delta \cdot 2^{-i} - 18\delta \cdot 2^{-i}\} =_k \{14\delta \cdot 2^{-k}\} = i - 1$.

Case 2: $i < k < j$

$$
\begin{aligned}
&u_{\mathrm{CT1}}((p_j, \ell_j)) \geq u_{\mathrm{CT1}}((p'_k, \ell'_k)) \\
&\qquad \Leftrightarrow \sum_{v \geq \ell_j} v \cdot f_{\mathrm{CT1}}(v) - p_j - \ell_j \cdot (1 - F_{\mathrm{CT1}}(\ell_j)) \geq \sum_{v \geq \ell'_k} v \cdot f_{\mathrm{CT1}}(v) - p'_k - \ell'_k \cdot (1 - F_{\mathrm{CT1}}(\ell'_k)) \\
&\qquad \Leftrightarrow 14\delta \cdot 2^{-k} + 22\delta \cdot 2^{k-2j} - 15\delta \cdot 2^{-j} - 18\delta \cdot 2^{-j} \geq 0 \\
&\qquad \overset{k=j-1}{\Leftrightarrow} 39\delta \cdot 2^{-j} - 33\delta \cdot 2^{-j} \geq 0
\end{aligned}
$$

where the last line holds since $_k\{14\delta \cdot 2^{-k} + 22\delta \cdot 2^{k-2j} - 15\delta \cdot 2^{-j} - 18\delta \cdot 2^{-j}\} =_k \{14\delta \cdot 2^{-k}\} = j - 1$.

Case 3: $j < k$
Obviously $u_{\mathrm{CT1}}((p_k, \ell_k)) < 0$ for $k \geq j + 1$, since $\ell_k \geq \mathrm{v_{CT1,2}(i,j)}$.

□

Lemma 5 (Combinatorial Type 2). *Suppose a buyer with the following value distribution:*

$$
\mathrm{v}_{\mathrm{CT2}}(i,j) = \begin{cases} 22\delta \cdot 2^i - 22\delta \cdot \frac{2^i}{2^{j-i}+1} + 14\delta \cdot \frac{2^j}{2^{j-i}+1} & p_1 = \frac{1}{2^{2i}} - \frac{1}{2^{2j}}, \\ 36\delta \cdot 2^j & p_2 = \frac{1}{2^{2j}}, \\ 0 & p_3 = 1 - \frac{1}{2^{2i}}, \end{cases}
$$

Then the buyer will prefer menu option i or j under either pricing $\boldsymbol{p}$, $\boldsymbol{p'}$. Moreover, she is indifferent over $(14\delta \cdot 2^{-i}, 22\delta \cdot 2^i)$ or $(14\delta \cdot 2^{-j}, 22\delta \cdot 2^j)$ and will prefer menu option i over j under pricing $\boldsymbol{p}$.

Proof. Similarly with our previous analysis, we need to prove the following lemma:

Lemma 6 (Sandwiching value masses of Combinatorial Type 2). *Let $\mathrm{v}_{\mathrm{CT2,1}}$ and $\mathrm{v}_{\mathrm{CT2,2}}$ define the point masses in the first and second branch of $\mathrm{v}_{\mathrm{CT2}}$ respectively. Then it holds that:*

$$
\ell'_i = 22\delta \cdot 2^i \leq \mathrm{v}_{\mathrm{CT2,1}}(i,j) \leq 18\delta \cdot 2^{i+1} = \ell_{i+1} \tag{3}
$$

$$
\ell'_j = 22\delta \cdot 2^j \leq \mathrm{v}_{\mathrm{CT2,2}}(i,j) \leq 22\delta \cdot 2^{j+1} = \ell_{j+1} \tag{4}
$$

Proof. We start our analysis with proving the left inequality of 3.

$$
\begin{aligned}
\mathrm{v}_{\mathrm{CT2,1}}(i,j) &= 22\delta \cdot 2^i - 22\delta \cdot \frac{2^i}{2^{j-i}+1} + 14\delta \cdot \frac{2^j}{2^{j-i}+1} \\
&\geq 22\delta \cdot 2^i - 22\delta \cdot \frac{2^i}{2^{j-i}+1} + 2 \cdot \frac{22\delta}{2} \cdot \frac{2^{j-1}}{2^{j-i}+1} \\
&\Rightarrow \mathrm{v}_{\mathrm{CT2,1}}(i,j) \geq 22\delta \cdot 2^i
\end{aligned}
$$

Now for the right inequality of 3, we have that:

$$
\begin{aligned}
\mathrm{v}_{\mathrm{CT2,1}}(i,j) = 22\delta \cdot 2^i - 22\delta \cdot \frac{2^i}{2^{j-i}+1} + 14\delta \cdot \frac{2^j}{2^{j-i}+1} &\leq 18\delta \cdot 2^{j+1} \\
\Leftrightarrow \frac{14\delta \cdot 2^j - 22\delta \cdot 2^i}{2^{j-i}+1} &\leq 18\delta \cdot 2^{j+1} - 22\delta \cdot 2^i \\
\Leftrightarrow 14\delta \cdot 2^j + 22\delta \cdot 2^j - 18\delta \cdot 2^{j+1} - 18\delta \cdot 2^{2j-i+1} &\leq 0
\end{aligned}
$$

For $\mathrm{v}_{\mathrm{CT2,2}}$, 4 obviously holds. □

Based on Lemma 6 we will proceed with proving the aforementioned preference relations.

At first, let's show the equivalence between (p'_i, ℓ'_i) and (p'_j, ℓ'_j):

$$
\begin{aligned}
u_{\text{CT2}}((p'_i, \ell'_i)) &= \sum_{v \geq \ell'_i} v \cdot f_{\text{CT2}}(v) - p'_i - \ell'_i \cdot (1 - F_{\text{CT2}}(\ell'_i)) \\
&= \mathrm{v}_{\text{CT2},1} \cdot p_1 + \mathrm{v}_{\text{CT2},2} \cdot p_2 - p'_0 \cdot 2^{-i} - \ell'_0 \cdot 2^{i} \cdot (p_1 + p_2) \\
&= \frac{(36\delta) \cdot 2^{j}}{2^{j-i} + 1} \cdot (2^{-2i} - 2^{-2j}) + 36\delta \cdot 2^{j} \cdot 2^{-2j} - 14\delta \cdot 2^{-i} - 22\delta \cdot 2^{i} \cdot 2^{-2i} \\
&= \frac{36\delta \cdot 2^{j-2i} - 36\delta \cdot 2^{-j}}{2^{j-i} + 1} + 36\delta \cdot 2^{-j} - 36\delta \cdot 2^{-i} \\
&= 0 \\
&= (36\delta \cdot 2^{j}) \cdot 2^{-2j} - (14\delta) \cdot 2^{-j} - (22\delta \cdot 2^{j}) \cdot 2^{-2j} \\
&= \sum_{v \geq \ell'_j} v \cdot f_{\text{CT2}}(v) - p'_j - \ell'_j \cdot (1 - F_{\text{CT2}}(\ell'_j)) \\
&= u_{\text{CT2}}((p'_j, \ell'_j))
\end{aligned}
$$

Next, we show the optimality - from the buyer's perspective - of choosing either (p'_i, ℓ'_i) or (p'_j, ℓ'_j) compared to any other menu option under $\boldsymbol{p'}$:

Case 1: $k < i$

$$
\begin{aligned}
&u_{\text{CT2}}((p'_i, \ell'_i)) \geq u_{\text{CT2}}((p'_k, \ell'_k)) \\
&\qquad \Leftrightarrow \sum_{v \geq \ell'_i} v \cdot f_{\text{CT2}}(v) - p'_i - \ell'_i \cdot (1 - F_{\text{CT2}}(\ell'_i)) \geq \sum_{v \geq \ell'_k} v \cdot f_{\text{CT2}}(v) - p'_k - \ell'_k \cdot (1 - F_{\text{CT2}}(\ell'_k)) \\
&\qquad \Leftrightarrow 14\delta \cdot 2^{-k} + 22\delta \cdot 2^{k-2i} - 14\delta \cdot 2^{-i} - 22\delta \cdot 2^{-i} \geq 0 \\
&\qquad \overset{k=i-1}{\Leftrightarrow} 14\delta \cdot 2^{-i} - 22\delta \cdot 2^{-i-1} \geq 0
\end{aligned}
$$

where the last line holds since ${}_k\{14\delta \cdot 2^{-k} + 22\delta \cdot 2^{k-2i} - 14\delta \cdot 2^{-i} - 22\delta \cdot 2^{-i}\} =_k \{14\delta \cdot 2^{-k}\} = i - 1$.

Case 2: $i < k < j$

$$
\begin{aligned}
&u_{\text{CT2}}((p'_j, \ell'_j)) \geq u_{\text{CT2}}((p'_k, \ell'_k)) \\
&\qquad \Leftrightarrow \sum_{v \geq \ell'_j} v \cdot f_{\text{CT2}}(v) - p'_j - \ell'_j \cdot (1 - F_{\text{CT2}}(\ell'_j)) \geq \sum_{v \geq \ell'_k} v \cdot f_{\text{CT2}}(v) - p'_k - \ell'_k \cdot (1 - F_{\text{CT2}}(\ell'_k)) \\
&\qquad \Leftrightarrow 14\delta \cdot 2^{-k} + 22\delta \cdot 2^{k-2j} - 14\delta \cdot 2^{-j} - 22\delta \cdot 2^{-j} \geq 0 \\
&\qquad \overset{k=j-1}{\Leftrightarrow} 14\delta \cdot 2^{-j} - 22\delta \cdot 2^{-j-1} \geq 0
\end{aligned}
$$

where the last line holds since ${}_k\{14\delta \cdot 2^{-k} + 22\delta \cdot 2^{k-2j} - 14\delta \cdot 2^{-j} - 22\delta \cdot 2^{-j}\} =_k \{14\delta \cdot 2^{-k}\} = j - 1$.

Case 3: $j < k$
It is easy to see that $u_{\text{CT2}}((p'_k, \ell'_k)) < 0$ for $k \geq j + 1$, since $\ell'_k \geq \mathrm{v}_{\text{CT2},2}(\mathrm{i}, \mathrm{j})$.

Let's proceed similarly for the 2 equivalent menu options compared to the menu options produced from price vector $\boldsymbol{p}$:

Case 1: $k < i$

$$
\begin{aligned}
&u_{\mathrm{CT2}}((p_i', \ell_i')) \geq u_{\mathrm{CT2}}((p_k, \ell_k)) \\
&\Leftrightarrow \sum_{v \geq \ell_i'} v \cdot f_{\mathrm{CT2}}(v) - p_i' - \ell_i' \cdot (1 - F_{\mathrm{CT2}}(\ell_i')) \geq \sum_{v \geq \ell_k} v \cdot f_{\mathrm{CT2}}(v) - p_k - \ell_k \cdot (1 - F_{\mathrm{CT2}}(\ell_k')) \\
&\Leftrightarrow 15\delta \cdot 2^{-k} + 18\delta \cdot 2^{k-2i} - 14\delta \cdot 2^{-i} - 22\delta \cdot 2^{-i} \geq 0 \\
&\overset{k=i-1}{\Leftrightarrow} 16\delta \cdot 2^{-i} - 13\delta \cdot 2^{-i} \geq 0
\end{aligned}
$$

where the last line holds since ${}_k\{15\delta \cdot 2^{-k} + 18\delta \cdot 2^{k-2i} - 14\delta \cdot 2^{-i} - 22\delta \cdot 2^{-i}\} =_k \{15\delta \cdot 2^{-k}\} = i - 1$.

Case 2: $i < k < j$

$$
\begin{aligned}
&u_{\mathrm{CT2}}((p_j', \ell_j')) \geq u_{\mathrm{CT2}}((p_k, \ell_k)) \\
&\Leftrightarrow \sum_{v \geq \ell_j'} v \cdot f_{\mathrm{CT2}}(v) - p_j' - \ell_j' \cdot (1 - F_{\mathrm{CT2}}(\ell_j')) \geq \sum_{v \geq \ell_k} v \cdot f_{\mathrm{CT2}}(v) - p_k - \ell_k \cdot (1 - F_{\mathrm{CT2}}(\ell_k)) \\
&\Leftrightarrow 15\delta \cdot 2^{-k} + 18\delta \cdot 2^{k-2j} - 14\delta \cdot 2^{-j} - 22\delta \cdot 2^{-j} \geq 0 \\
&\overset{k=j-1}{\Leftrightarrow} 16\delta \cdot 2^{-j} - 13\delta \cdot 2^{-j} \geq 0
\end{aligned}
$$

where the last line holds since ${}_k\{15\delta \cdot 2^{-k} + 18\delta \cdot 2^{k-2j} - 14\delta \cdot 2^{-j} - 22\delta \cdot 2^{-j}\} =_k \{15\delta \cdot 2^{-k}\} = j - 1$.

Case 3: $j < k$
Obviously $u_{\mathrm{CT2}}((p_k, \ell_k)) < 0$ for $k \geq j + 1$, since $\ell_k \geq \mathrm{v}_{\mathrm{CT2,2}}(\mathrm{i,j})$.

Finally, we show that $u_{\mathrm{CT2}}((p_i, \ell_i)) \geq u_{\mathrm{CT2}}((p_j, \ell_j))$

$$
\begin{aligned}
&u_{\mathrm{CT2}}((p_i, \ell_i)) \geq u_{\mathrm{CT2}}((p_j, \ell_j)) \\
&\Leftrightarrow \mathrm{v}_{\mathrm{CT2,1}} \cdot f_{\mathrm{CT2}}(v) - p_i - \ell_i \cdot (1 - F_{\mathrm{CT2}}(\ell_i)) \geq -p_j - \ell_j \cdot (1 - F_{\mathrm{CT2}}(\ell_j)) \\
&\Leftrightarrow 22\delta \cdot 2^i - 22\delta \cdot \frac{2^i}{2^{j-i} + 1} + 14\delta \cdot \frac{2^j}{2^{j-i} + 1} \geq 0 \\
&\overset{k=j-1}{\Leftrightarrow} 14\delta \cdot 2^{-j} - 22\delta \cdot 2^{-j-1} \geq 0
\end{aligned}
$$

□

Lemma 7 (Combinatorial Type 3)**.** *Suppose a buyer with the following value distribution:*

$$
\mathrm{v}_{\mathrm{CT3}}(i,j) = \begin{cases} 18\delta \cdot 2^i + \frac{18\delta \cdot 2^{i-2j} - 22\delta \cdot 2^{-j}}{2^{-2i} - 2^{-2j}} + \frac{15\delta \cdot 2^{-i} - 14\delta \cdot 2^{-j}}{2^{-2i} - 2^{-2j}} & p_1 = \frac{1}{2^{2i}} - \frac{1}{2^{2j}}, \\ 36\delta \cdot 2^j & p_2 = \frac{1}{2^{2j}}, \\ 0 & p_3 = 1 - \frac{1}{2^{2i}}, \end{cases}
$$

Then the buyer will prefer menu option i or j under either pricing $\boldsymbol{p}$, $\boldsymbol{p'}$. Moreover, she is indifferent over $(15\delta \cdot 2^{-i}, 18\delta \cdot 2^i)$ or $(14\delta \cdot 2^{-j}, 22\delta \cdot 2^j)$.

Proof. Similarly with our previous analysis, we need to prove the following lemma:

Lemma 8 (Sandwiching value masses of Combinatorial Type 3)**.** *Let $\mathrm{v}_{\mathrm{CT3,1}}$ and $\mathrm{v}_{\mathrm{CT3,2}}$ define the point masses in the first and second branch of $\mathrm{v}_{\mathrm{CT3}}$ respectively. Then it holds that:*

$$
\ell_i = 18\delta \cdot 2^i \leq \mathrm{v}_{\mathrm{CT3,1}}(i,j) \leq 18\delta \cdot 2^{i+1} = \ell_{i+1} \tag{5}
$$

$$
\ell_j' = 22\delta \cdot 2^j \leq \mathrm{v}_{\mathrm{CT3,2}}(i,j) \leq 22\delta \cdot 2^{j+1} = \ell_{j+1} \tag{6}
$$

Proof. We start our analysis with proving the left inequality of 5.

$$\begin{aligned}
\mathrm{v}_{\mathrm{CT3},1}(i,j) = 18\delta \cdot 2^{i} + \frac{18\delta \cdot 2^{i-2j} - 22\delta \cdot 2^{-j}}{2^{-2i} - 2^{-2j}} + \frac{15\delta \cdot 2^{-i} - 14\delta \cdot 2^{-j}}{2^{-2i} - 2^{-2j}} \geq 18\delta \cdot 2^{i} \\
\Leftrightarrow 18\delta \cdot 2^{i-2j} - 22\delta \cdot 2^{-j} + 15\delta \cdot 2^{-i} - 14\delta \cdot 2^{-j} \geq 0 \\
\overset{i=j-1}{\Leftrightarrow} 18\delta \cdot 2^{-j-1} - 36\delta \cdot 2^{-j} + 15\delta \cdot 2^{-j+1} \geq 0
\end{aligned}$$

where the last line holds since ${}_i\{18\delta \cdot 2^{i-2j} - 22\delta \cdot 2^{-j} + 15\delta \cdot 2^{-i} - 14\delta \cdot 2^{-j}\} =_i \{15\delta \cdot 2^{-i}\} = j - 1$.

Now for the right inequality of 5, we have that:

$$\begin{aligned}
\mathrm{v}_{\mathrm{CT3},1}(i,j) = 18\delta \cdot 2^{-i} - 22\delta \cdot 2^{-j} + 15\delta \cdot 2^{-i} - 14\delta \cdot 2^{-j} \leq 18\delta \cdot 2^{-i+1} - 18\delta \cdot 2^{i-2j+1} \\
\Leftrightarrow 0 \leq 3\delta \cdot 2^{-i} + 36\delta \cdot 2^{-j} - 18\delta \cdot 2^{i-2j+1}
\end{aligned}$$

For $\mathrm{v}_{\mathrm{CT3},2}$, 6 obviously holds. □

Based on Lemma 6 we will proceed with proving the aforementioned preference relations.

We show the equivalence between (p_i, ℓ_i) and (p'_j, ℓ'_j):

$$\begin{aligned}
u_{\mathrm{CT3}}((p_i,\ell_i)) &= \sum_{v \geq \ell_i} v \cdot f_{\mathrm{CT3}}(v) - p_i - \ell_i \cdot (1 - F_{\mathrm{CT3}}(\ell_i)) \\
&= \mathrm{v}_{\mathrm{CT3},1} \cdot p_1 + \mathrm{v}_{\mathrm{CT3},2} \cdot p_2 - p_0 \cdot 2^{-i} - \ell_0 \cdot 2^{i} \cdot (p_1 + p_2) \\
&= 33\delta \cdot 2^{-i} - 36\delta \cdot 2^{-j} + 36\delta \cdot 2^{-j} - 15\delta \cdot 2^{-i} - 18\delta \cdot 2^{i} \cdot 2^{-2i} \\
&= 0 \\
&= (36\delta \cdot 2^{j}) \cdot 2^{-2j} - (14\delta) \cdot 2^{-j} - (22\delta \cdot 2^{j} \cdot 2^{-2j}) \\
&= \sum_{v \geq \ell'_j} v \cdot f_{\mathrm{CT3}}(v) - p'_j - \ell'_j \cdot (1 - F_{\mathrm{CT3}}(\ell'_j)) \\
&= u_{\mathrm{CT3}}((p'_j, \ell'_j))
\end{aligned}$$

Now, for the optimality - from the buyer's perspective - of choosing (p_i, ℓ_i) or (p_j, ℓ_j) compared to any other menu option under $\boldsymbol{p}$ and also for the preference of menu option (p_j, ℓ_j) over (p_i, ℓ_i) (as equivalent of (p'_j, ℓ'_j)), the analysis is exactly the same as in Lemma 3. Moreover, for the optimality of choosing (p'_i, ℓ'_i) or (p'_j, ℓ'_j) compared to any other menu option under $\boldsymbol{p'}$ the details are deferred to Lemma 5. It remains to give an explicit proof that $u_{\mathrm{CT3}}((p'_j, \ell'_j)) \geq u_{\mathrm{CT3}}((p'_i, \ell'_i))$ since there is a slight "asymmetry" in this case. More specifically, it is easy to see that $\mathrm{v}_{\mathrm{CT3},1} \leq 22\delta \cdot 2^{i} = \ell'_i$. Based on that observation, we have:

$$\begin{aligned}
&u_{\mathrm{CT3}}((p'_j, \ell'_j)) \geq u_{\mathrm{CT3}}((p'_i, \ell'_i)) \\
&\quad \Leftrightarrow \mathrm{v}_{\mathrm{CT3},1} \cdot f_{\mathrm{CT3}}(v) - p'_j - \ell'_j \cdot (1 - F_{\mathrm{CT3}}(\ell'_j)) \geq -p_j - \ell_j \cdot (1 - F_{\mathrm{CT3}}(\ell_j)) \\
&\quad \Leftrightarrow 18\delta \cdot 2^{-i} - 22\delta \cdot 2^{-j} + 15\delta \cdot 2^{-i} - 14\delta \cdot 2^{-j} - 15\delta \cdot 2^{-i} - 18\delta \cdot 2^{-i} \geq -14\delta \cdot 2^{-i} - 22\delta \cdot 2^{i-2j} \\
&\quad \Leftrightarrow 14\delta \cdot 2^{-i} - 36\delta \cdot 2^{-j} + 22\delta \cdot 2^{i-2j} \geq 0 \\
&\quad \overset{i=j-1}{\Leftrightarrow} 14\delta \cdot 2^{-j+1} - 36\delta \cdot 2^{-j} + 22\delta \cdot 2^{-j-1} \geq 0
\end{aligned}$$

where the last line holds since ${}_i\{14\delta \cdot 2^{-i} - 36\delta \cdot 2^{j} + 22\delta \cdot 2^{i-2j}\} =_i \{14\delta \cdot 2^{-i}\} = j - 1$.

□

Lemma 9 (Combinatorial Type 4)**.** *Suppose a buyer with the following value distribution:*

$$\mathrm{v}_{\mathrm{CT4}}(i,j) = \begin{cases} 22\delta \cdot 2^i + \frac{22\delta\cdot 2^{i-2j} - 18\delta\cdot 2^{-j}}{2^{-2i}-2^{-2j}} + \frac{14\delta\cdot 2^{-i} - 15\delta\cdot 2^{-j}}{2^{-2i}-2^{-2j}} & p_1 = \frac{1}{2^{2i}} - \frac{1}{2^{2j}}, \\ 33\delta \cdot 2^j & p_2 = \frac{1}{2^{2j}}, \\ 0 & p_3 = 1 - \frac{1}{2^{2i}}, \end{cases}$$

Then the buyer will prefer menu option i *or* j *under either pricing* $\boldsymbol{p}, \boldsymbol{p'}$*. Moreover, she is indifferent over* $(14\delta\cdot 2^{-i}, \{22\delta\cdot 2^i\})$ *or* $(15\delta \cdot 2^{-j}, 18\delta \cdot 2^j)$.

Proof. Similarly with our previous analysis, we need to prove the following lemma:

Lemma 10 (Sandwiching value masses of Combinatorial Type 4)**.** *Let* $\mathrm{v}_{\mathrm{CT4},1}$ *and* $\mathrm{v}_{\mathrm{CT4},2}$ *define the point masses in the first and second branch of* $\mathrm{v}_{\mathrm{CT4}}$ *respectively. Then it holds that:*

$$\ell'_i = 22\delta \cdot 2^i \le \mathrm{v}_{\mathrm{CT4},1}(i,j) \le 18\delta \cdot 2^{i+1} = \ell_{i+1} \tag{7}$$

$$\ell'_j = 22\delta \cdot 2^j \le \mathrm{v}_{\mathrm{CT4},2}(i,j) \le 22\delta \cdot 2^{j+1} = \ell_{j+1} \tag{8}$$

Proof. We start our analysis with proving the left inequality of 7.

$$\begin{aligned} \mathrm{v}_{\mathrm{CT4},1}(i,j) = 22\delta \cdot 2^i + \frac{22\delta \cdot 2^{i-2j} - 18\delta \cdot 2^{-j}}{2^{-2i} - 2^{-2j}} + \frac{14\delta \cdot 2^{-i} - 15\delta \cdot 2^{-j}}{2^{-2i} - 2^{-2j}} \ge 22\delta \cdot 2^i \\ \Leftrightarrow 14\delta \cdot 2^{-i} - 33\delta \cdot 2^{-j} + 22\delta \cdot 2^{i-2j} \ge 0 \\ \overset{i=j-1}{\Leftrightarrow} 14\delta \cdot 2^{-j+1} - 33\delta \cdot 2^{-j} + 22\delta \cdot 2^{-j-1} \ge 0 \end{aligned}$$

where the last line holds since $_i\{14\delta \cdot 2^{-i} - 33\delta \cdot 2^{-j} + 22\delta \cdot 2^{i-2j}\} =_i \{14\delta \cdot 2^{-i}\} = j-1$.

Now for the right inequality of 7, we have that:

$$\begin{aligned} \mathrm{v}_{\mathrm{CT4},1}(i,j) = 22\delta \cdot 2^i + \frac{22\delta \cdot 2^{i-2j} - 18\delta \cdot 2^{-j}}{2^{-2i} - 2^{-2j}} + \frac{14\delta \cdot 2^{-i} - 15\delta \cdot 2^{-j}}{2^{-2i} - 2^{-2j}} \le 18\delta \cdot 2^{i+1} \\ \Leftrightarrow 22\delta \cdot 2^{i-2j} - 33\delta \cdot 2^{-j} + 14\delta \cdot 2^{-i} \le 14\delta \cdot 2^{-i} - 14\delta \cdot 2^{i-2j} \\ \Leftrightarrow 0 \le 33\delta \cdot 2^{-j} - 36\delta \cdot 2^{i-2j} \end{aligned}$$

For $\mathrm{v}_{\mathrm{CT4},2}$, 8 obviously holds. □

We show the equivalence between (p'_i, ℓ'_i) and (p_j, ℓ_j):

$$\begin{aligned} u_{\mathrm{CT4}}((p_i, \ell'_i)) &= \sum_{v \ge \ell'_i} v \cdot f_{\mathrm{CT4}}(v) - p'_i - \ell'_i \cdot (1 - F_{\mathrm{CT4}}(\ell'_i)) \\ &= \mathrm{v}_{\mathrm{CT4},1} \cdot p_1 + \mathrm{v}_{\mathrm{CT4},2} \cdot p_2 - p'_0 \cdot 2^{-i} - \ell'_0 \cdot 2^i \cdot (p_1 + p_2) \\ &= 36\delta \cdot 2^{-i} - 33\delta \cdot 2^{-j} + 33\delta \cdot 2^{-j} - 14\delta \cdot 2^{-i} - 22\delta \cdot 2^i \cdot 2^{-2i} \\ &= 0 \\ &= (36\delta \cdot 2^j) \cdot 2^{-2j} - (14\delta) \cdot 2^{-j} - (22\delta \cdot 2^j \cdot 2^{-2j}) \\ &= \sum_{v \ge \ell_j} v \cdot f_{\mathrm{CT4}}(v) - p_j - \ell_j \cdot (1 - F_{\mathrm{CT4}}(\ell_j)) \\ &= u_{\mathrm{CT4}}((p_j, \ell_j)) \end{aligned}$$

Based on Lemma 10 and the analysis from Lemmas 3 and 5, Lemma 9 follows.

□

B.4. Finalizing the construction

Property 1. To ensure that the optimal menu is composed of options of the form $(15\delta \cdot 2^{-i}, 18\delta \cdot 2^{i})$ or $(14\delta \cdot 2^{-i}, 22\delta \cdot 2^{i}), \forall i \in [n]$ and $(14\delta \cdot 2^{-(n+1)}, 22\delta \cdot 2^{n+1})$ we need to construct 3 buyer types with value distributions and corresponding probabilities as described in the hypothesis of Corollary 4.

Property 2. As explained in Property 1, we construct a buyer type "Gambler" with value $33\delta \cdot 2^{2(n+1)}$ with (value) probability $\frac{1}{2^{2(n+1)}}$ and 0 otherwise. As mentioned in Corollary 1 , this buyer will prefer $n+1$-th menu option under pricing $(15\delta \cdot 2^{-i}, 18\delta \cdot 2^{i})$. The buyer type appears with probability q_G, where this probability is very close to 1. We want to make sure that the optimal pricing will set exactly $(15\delta \cdot 2^{-(n+1)}, 18\delta \cdot 2^{n+1})$ as the $n+1$-th menu option. In order to achieve that, we want to construct the rest of the buyer types such that they will contribute no more than q_G in the total expected revenue. This can be done by letting the rest of the buyer types have maximum expected value $\leq 36\delta$ and appear with probability $< \frac{1}{36\delta} q_G$.

Property 3. For each menu option $i \in [n]$ we construct two buyer types "Worker1" and "Worker2" with corresponding value distributions $\mathrm{v}_{W_1}(i)$ and $\mathrm{v}_{W_2}(i)$, as described in Corollaries 2 and 3 which appear with probabilities $q_{w_1} = \frac{1}{550\delta}$ and $q_{W_2} = \frac{1}{50\delta}$ respectively. As mentioned in these Corollaries, these buyers will purchase item i under the only feasible pricings.

Property 4. For the combinatorial related part of the construction, consider edge $(i, j) \in E$, with $i < j$ and set T_{ij} of buyer types with value distributions $\mathrm{v}_{\mathrm{CT1}}(i, j), \mathrm{v}_{\mathrm{CT2}}(i, j), \mathrm{v}_{\mathrm{CT3}}(i, j), \mathrm{v}_{\mathrm{CT4}}(i, j)$ as defined in Lemmas 3,5,7,9 respectively. Intuitively, these four combinatorial types will choose menu options i or j or nothing, and they have similar values for menu options i and j. Now, we aim to determine the appearance probability of each buyer type in the value distribution, such that if $menu_j - menu_i = (14\delta \cdot (2^{-j} - 2^{-i}), \{22\delta \cdot (2^{-j} - 2^{-i})\})$ or $menu_j - menu_i = (15\delta \cdot (2^{-j} - 2^{-i}), \{18\delta \cdot (2^{-j} - 2^{-i})\})$, then the revenue contribution from T_{ij} is R_{ij}; else the revenue contribution from T_{ij} is $R_{ij}(n) + \frac{1}{e^n}$. For convenience, we shorten our notation:

$$a = 15\delta \cdot 2^{-i} + 18\delta \cdot 2^{-i}$$
$$b = 15\delta \cdot 2^{-j} + 18\delta \cdot 2^{-j}$$
$$c = 14\delta \cdot 2^{-i} + 22\delta \cdot 2^{-i}$$
$$d = 14\delta \cdot 2^{-j} + 22\delta \cdot 2^{-j}$$

Based on Lemmas 3,5,7,9, we have the following 4×4 outcome matrix, say M, whose elements express the revenue of a combinatorial buyer under a specific menu pricing:

Buyers' valuations for (i, j) options

Menu pricing for (i, j) options

	(a,b)	**(c,d)**	**(a,d)**	**(c,b)**
(a,b)	b	b	b	a
(c,d)	0	d	d	c
(a,d)	a	a	d	a
(c,b)	b	b	b	b

For any vector $\boldsymbol{z} \in \mathbb{R}^4_{\geq 0}$, $M\boldsymbol{z}$ correspond to the revenue vector produced from the 4 pricings $((p_i, \ell_i), (p_j, \ell_j)) = (a, b), (c, d), (a, d), (c, b)$, given that z_τ buyers of type τ appear. To satisfy Property 4, our task here is to find a vector $\boldsymbol{z}$ such that $M\boldsymbol{z} = (R_{ij}, R_{ij}, R_{ij} + \frac{1}{e^n}, R_{ij} + \frac{1}{e^n})^T$, for some $R_{ij} > 0$.
By solving $\boldsymbol{z}_1$ for $M\boldsymbol{z}_1 = (1, 1, 1, 1)^T$ and $\boldsymbol{z}_2$ for $M\boldsymbol{z}_2 = (0, 0, 1, 1)^T$, we get:

$$z_1 = (\frac{d-b}{bd}, \frac{d-b}{bd(a-d)} \cdot a, \frac{b(a-d)-(b-d)a}{bd(a-d)}, 0)^T.$$

$$z_2 = (\frac{ad-bc}{bd(a-b)}, \frac{bc(a-d)-(a-b)^2 d}{bd(a-b)(a-d)}, \frac{a-b}{b(a-d)}, \frac{1}{b-a})^T.$$

Taking $z = 2^{-j-n} \cdot \frac{18}{550\delta} z_1 + e^{-n} z_2$, it follows that:

$$Mz = (2^{-j-n} \cdot \frac{18}{550\delta}, 2^{-j-n} \cdot \frac{18}{550\delta}, 2^{-j-n} \cdot \frac{18}{550\delta} + e^{-n}, 2^{-j-n} \cdot \frac{18}{550\delta} + e^{-n})$$

Therefore, if with probability z_ℓ the buyer has type $v_{\mathrm{CT}\ell}$, the four buyer types contribute $R_{ij}(n) = 2^{-j} \cdot \frac{18}{550\delta}$ revenue if $menu_j - menu_i = (14\delta \cdot (2^{-j} - 2^{-i}), 22\delta \cdot (2^{-j} - 2^{-i}))$ or $menu_j - menu_i = (15\delta \cdot (2^{-j} - 2^{-i}), 18\delta \cdot (2^{-j} - 2^{-i}))$ and $R_{ij} = 2^{-j-n} \cdot \frac{18}{550\delta} + e^{-n}$ otherwise.
Now, for the total realization probability of the four buyer types in the distribution, it holds that: $||z||_1 = \frac{1}{b} \cdot (R_{ij} + f) \leq 2^{-j-n} \cdot \frac{36}{550\delta}$. Since there are less than $\frac{1}{2}n^2$ edges, the total realization probability of all the combinatorial types is less than $\frac{1}{2}n^2 \cdot 2^{-j-n} \cdot \frac{36}{550\delta}$ and also the total realization probability of all buyer types added in Properties 3 and 4 is less than $\frac{0.8}{36\delta}$, as required in Property 2. This completes the proof of correctness of the construction.

C. Ommited proofs in Section 4

C.1. Outline

We give a polynomial-time approximation scheme in four steps.

Step 1: Geometric rounding of prices. Given any feasible menu $(p_i, \ell_i)_{i=1}^n$, we round each first-stage price downward and each second-stage price upward to the nearest power of $(1+\epsilon^2)$. A global scaling factor is applied to restore incentive compatibility. We show that the resulting menu preserves feasibility and loses at most an $O(\epsilon)$ fraction of revenue.

Step 2: Finite price alphabet. We prove that prices below $\Theta(\epsilon^2)$ contribute negligibly to revenue and can be set to zero, while all remaining prices lie in a bounded geometric range $[\Theta(\epsilon^2), \mathcal{R}]$. This yields a finite candidate price set

$$\Pi^* = \{(1+\epsilon^2)^k : k_{\min} \leq k \leq k_{\max}\}$$

of size $\mathrm{poly}(1/\epsilon, \log \mathcal{R})$, such that restricting menus to Π^* preserves a $(1 - O(\epsilon))$ approximation.

Step 3: Additive reduction and interval pricing. For each buyer type τ, define the expected surplus of menu option i as

$$S_\tau(i) = \sum_{j: v_j \geq \ell_i} (v_j - \ell_i) f_j.$$

We transform the unit-demand choice into an additive valuation by defining marginal values $w_{\tau,i} = S_\tau(i) - S_\tau(i+1)$ and marginal prices $c_i = p_i - p_{i+1}$. Selecting menu option i is equivalent to purchasing suffix bundle $\{i, \ldots, n\}$.

We then partition menu indices into intervals so that prices within each interval differ by at most a $(1 + O(\epsilon))$ factor, while prices across intervals differ by at least a polynomial factor. This interval structure allows local price perturbations without affecting buyer incentives and ensures that only $O(1/\epsilon^2)$ effective price blocks need be considered.

Step 4: Dynamic programming. For the additive instance with interval structure and prices drawn from Π^*, we define a dynamic program whose states correspond to the current interval boundary and price level. Transitions enumerate the next breakpoint and assign a candidate marginal price consistent with gap constraints. The revenue contribution of each interval is computed independently using additive values.

The dynamic program runs in $\mathrm{poly}(m, n^{\mathrm{poly}(1/\epsilon)}, \log \mathcal{R})$ time and returns an optimal structured pricing. Mapping this pricing back through Steps 3–1 yields a feasible two-stage menu achieving at least $(1-\epsilon)$ of the optimal revenue.

C.2. Step 1: (Pre-)Preprocessing

Before delving into the main analysis, we first preprocess the value distribution by transforming it into n discretized value points with non-zero probability mass, ordered from lowest to highest. This step is essential both for applying our discretization scheme and for enabling the dynamic program that computes the final pricing menu.

Lemma 11 (Menu Compression Lemma)**.** *Let F be a distribution over values in $[0, \mathcal{R}]$. For any $\epsilon > 0$, there exists a set of $n = O(\frac{\log \mathcal{R}}{\epsilon})$ prices $v_1, \ldots, v_n$, such that any mechanism $\mathcal{M}$ can by approximated by a mechanism $\mathcal{M}'$ that only offers these n prices, such that:*

$$\mathrm{REV}(\mathcal{M}') \geq (1 - O(\epsilon))\mathrm{REV}(\mathcal{M})$$

Proof. Let $\mathcal{M}$ denote the revenue maximizing mechanism for distribution F.
Define price levels:

$$P_k = \{0\} \cup \{(1+\epsilon)^k | 0 \leq k \leq \log_{1+\epsilon} \mathcal{R}\}$$

Then $|P| = O(\frac{\log \mathcal{R}}{\epsilon})$. Imagine partitioning the value space into geometrically growing buckets of the form $[(1+\epsilon)^i, (1+\epsilon)^{i+1}]$. Each bucket contains values that are close to each other. By merging options within a bucket into a single representative we define a mechanism $\mathcal{M}'$. In this way, we discretize the pricing scheme while introducing a factor $1 \pm \epsilon$ distortion in utility and revenue. Therefore, for each bucket we lose at most a factor of $1 - \epsilon$ in revenue and thus, it holds $\mathrm{REV}(\mathcal{M}') \geq (1 - O(\epsilon))\mathrm{REV}(\mathcal{M})$. □

After approximating the original value distribution by n distinct, ordered value points with small revenue loss, we are now ready to present the main lemma of this subsection. This lemma allows us to significantly reduce the space of feasible first- and second-stage prices, again with only a small multiplicative loss in revenue. Intuitively, it shows that without much loss, each price can be rounded to a nearby power-of-$(1+\epsilon^2)$, enabling us to discretize the solution space and apply a dynamic programming approach in the next steps.

Lemma 12. *For any $\epsilon > 0$, let p, q^p and ℓ, q^ℓ be two stage-1 and stage-2 price functions respectively, satisfying $q^p(\lambda) \leq p(\lambda) \leq (1+\epsilon)q^p(\lambda)$ and $q^\ell(\lambda) \leq \ell(\lambda) \leq (1+\epsilon)q^\ell(\lambda)$ for all random allocations $\lambda \in \Delta(2^{[n]})$. Then for scaling factor $\alpha = (1+\epsilon)^{-\frac{1}{\sqrt{\epsilon}}}$ and any valuation function v:*

$$\mathrm{REV}(\mathcal{M}') \geq (1 - O(\epsilon))\mathrm{REV}(\mathcal{M})$$

Proof. Suppose that under pricing $(p, \{\ell\})$ buyer purchases allocation λ, while under $(p', \{\ell'\}) = (aq^p, \{aq^\ell\})$ buyer purchases λ'.

The buyer has higher utility for λ than λ' under $(p, \{\ell\})$, hence:

$$\int_{\ell(\lambda)}^{\infty} v f(v)\, dv - p(\lambda) - \ell(\lambda) \cdot \int_{\ell(\lambda)}^{\infty} v f(v)\, dv \geq \int_{\ell(\lambda')}^{\infty} v f(v)\, dv - p(\lambda') - \ell(\lambda') \cdot \int_{\ell(\lambda')}^{\infty} v f(v)\, dv$$

On the other hand, the buyer has higher utility for λ' than λ under $(aq^p, \{aq^\ell\})$, thus:

$$\int_{\ell'(\lambda')}^{\infty} v f(v)\, dv - p'(\lambda') - \ell'(\lambda') \cdot \int_{\ell'(\lambda')}^{\infty} f(v)\, dv \geq \int_{\ell'(\lambda)}^{\infty} v f(v)\, dv - p'(\lambda) - \ell'(\lambda) \cdot \int_{\ell'(\lambda)}^{\infty} f(v)\, dv$$

Adding the two inequalities above, it follows:

$$\begin{aligned}(p(\lambda') + \ell(\lambda') \cdot \int_{\ell(\lambda')}^{\infty} f(v)\, dv) \geq (p(\lambda) + \ell(\lambda) \cdot \int_{\ell(\lambda)}^{\infty} f(v)\, dv) + (p'(\lambda') + \ell'(\lambda') \cdot \int_{\ell'(\lambda')}^{\infty} f(v)\, dv) \\ - (p'(\lambda) + \ell'(\lambda) \cdot \int_{\ell'(\lambda)}^{\infty} f(v)\, dv) + (\int_{\ell'(\lambda)}^{\ell(\lambda)} v f(v)\, dv - \int_{\ell'(\lambda')}^{\ell(\lambda')} v f(v)\, dv)\end{aligned} \tag{9}$$

Moreover, we have:

$$-\int_{\ell'(\lambda')}^{\ell(\lambda')} vf(v)\,dv \geq -\ell(\lambda')\int_{\ell'(\lambda')}^{\ell(\lambda')} f(v)\,dv \tag{10}$$

and

$$\int_{\ell'(\lambda)}^{\ell(\lambda)} vf(v)\,dv \geq \ell'(\lambda)\int_{\ell'(\lambda)}^{\ell(\lambda)} f(v)\,dv \tag{11}$$

Also, before finalizing our analysis, observe that:

$$\alpha = (1+\epsilon)^{-\frac{1}{\sqrt{\epsilon}}} < 1-\sqrt{\epsilon}+2\epsilon \tag{12}$$

and

$$\alpha = (1+\epsilon)^{-\frac{1}{\sqrt{\epsilon}}} > 1-\sqrt{\epsilon} \tag{13}$$

Now, we are ready to establish our lower bound for the revenue of the new mechanism:

$$\begin{aligned}
9 \xrightarrow{10,11} (p(\lambda') + \ell(\lambda') \cdot \int_{\ell(\lambda')}^{\infty} f(v)\,dv) &\geq (p(\lambda) + \ell(\lambda) \cdot \int_{\ell(\lambda)}^{\infty} f(v)\,dv) + (p'(\lambda') + \ell'(\lambda') \cdot \int_{\ell'(\lambda')}^{\infty} f(v)\,dv) \\
&- (p'(\lambda) + \ell'(\lambda) \cdot \int_{\ell'(\lambda)}^{\infty} f(v)\,dv) + \ell'(\lambda) \cdot \int_{\ell'(\lambda)}^{\ell(\lambda)} f(v)\,dv - \ell(\lambda') \cdot \int_{\ell'(\lambda')}^{\ell(\lambda')} f(v)\,dv \\
&\implies (p(\lambda') + \ell(\lambda') \cdot \int_{\ell'(\lambda')}^{\infty} f(v)\,dv) \geq (p(\lambda) + \ell(\lambda) \cdot \int_{\ell(\lambda)}^{\infty} f(v)\,dv) \\
&+ (p'(\lambda') + \ell'(\lambda') \cdot \int_{\ell'(\lambda')}^{\infty} f(v)\,dv) - (p'(\lambda) + \ell'(\lambda) \cdot \int_{\ell(\lambda)}^{\infty} f(v)\,dv) \\
&\xrightarrow{\text{hyp.}} (\alpha^{-1}(1+\epsilon) - 1) \cdot (p(\lambda') + \ell(\lambda') \cdot \int_{\ell'(\lambda')}^{\infty} f(v)\,dv) \geq (1-\alpha) \cdot (p(\lambda) + \ell(\lambda) \cdot \int_{\ell(\lambda)}^{\infty} f(v)\,dv) \\
&\implies (p(\lambda') + \ell(\lambda') \cdot \int_{\ell'(\lambda')}^{\infty} f(v)\,dv) \geq \frac{1-\alpha}{\alpha^{-1}(1+\epsilon)-1} \cdot (p(\lambda) + \ell(\lambda) \cdot \int_{\ell(\lambda)}^{\infty} f(v)\,dv) \\
&\xrightarrow{12,13} (p(\lambda') + \ell(\lambda') \cdot \int_{\ell'(\lambda')}^{\infty} f(v)\,dv) \geq \frac{(1-\sqrt{\epsilon}) \cdot (\sqrt{\epsilon}) - 2\epsilon)}{\epsilon + \sqrt{\epsilon})} \cdot (p(\lambda) + \ell(\lambda) \cdot \int_{\ell(\lambda)}^{\infty} f(v)\,dv) \\
&\implies (p(\lambda') + \ell(\lambda') \cdot \int_{\ell'(\lambda')}^{\infty} f(v)\,dv) \geq (1-4\sqrt{\epsilon}) \cdot (p(\lambda) + \ell(\lambda) \cdot \int_{\ell(\lambda)}^{\infty} f(v)\,dv)
\end{aligned}$$

Hence $\mathrm{REV}_{(\alpha\cdot q^p, \{\alpha\cdot q^\ell\})}(v) \geq (1-4\sqrt{\epsilon})\mathrm{REV}_{(p,\{\ell\})}(v)$. □

Let $(p, \{\ell\})$ be the optimal menu pricing. Let $(q^p, \{q^\ell\})$ be the menu pricing that rounds each 1-st and 2-nd stage menu price to the closest integral power of $(1+\epsilon^2)$. Then $q^p(\lambda) \leq p(\lambda) \leq (1+\epsilon^2)q^p(\lambda)$ and $q^\ell(\lambda) \leq \ell(\lambda) \leq (1+\epsilon^2)q^\ell(\lambda), \forall\lambda \in \Delta(2^{[n]})$. Now, we are ready to apply Lemma 12 in order to prove the Theorem of Step 1. Set $\epsilon \to \epsilon^2$, hence $\alpha = (1+\epsilon^2)^{-\frac{1}{\epsilon}}$, $q^p(\lambda) \leq p(\lambda) \leq (1+\epsilon^2)q^p(\lambda)$ and $q^\ell(\lambda) \leq \ell(\lambda) \leq (1+\epsilon^2)q^\ell(\lambda), \forall\lambda \in \Delta(2^{[n]})$. Then $(q^{p,(1)}, \{q^{\ell,(1)}\}) = ((1+\epsilon^2)^{-\frac{1}{\epsilon}}q^p, \{(1+\epsilon^2)^{-\frac{1}{\epsilon}}q^\ell\})$ is a menu pricing with powers-of-$(1+\epsilon^2)$ 1-st and 2-nd stage prices for each menu option, achieving $(1-4\epsilon) = (1-O(\epsilon))$ function of the revenue of $(p, \{\ell\})$.

C.3. Step 2: Discretizing the solution space

It suffices to show that there exists a menu pricing with all stage-1 and stage-2 prices being either 0 or bounded in range $[\Omega(\epsilon^2), \mathcal{R}]$ that achieves an $1 - O(\epsilon)$ function of the revenue of $(q^{p,(1)}, \{q^{\ell,(1)}\})$. Let's define $(q^{p,(2)}, \{q^{\ell,(2)}\})$ as follows:

$$q^{p,(2)}(\lambda) = \begin{cases} 0, & \text{if } q^{p,(1)}(\lambda) \le \epsilon^2 \\ (1-\epsilon)q^{p,(1)}(\lambda), & \text{otherwise} \end{cases}, \quad q^{\ell,(2)}(\lambda) = \begin{cases} 0, & \text{if } q^{\ell,(1)}(\lambda) \le \epsilon^2 \\ (1-\epsilon)q^{\ell,(1)}(\lambda), & \text{otherwise} \end{cases}$$

The proof employs a similar perturbation technique as in Lemma **??**. Before the main analysis, notice that:

$$\begin{aligned} (1-\epsilon)(q^{p,(1)} - \epsilon^2) \le & \begin{cases} 0, & \text{if } q^{p,(1)}(\lambda) \le \epsilon^2 \\ (1-\epsilon)q^{p,(1)}(\lambda), & \text{otherwise} \end{cases} \\ & \xrightarrow{\text{def.}} (1-\epsilon)(q^{p,(1)} - \epsilon^2) \le q^{p,(2)} \end{aligned} \tag{14}$$

Similarly, it holds:

$$(1-\epsilon)(q^{\ell,(1)} - \epsilon^2) \le q^{\ell,(2)} \tag{15}$$

Also, from the definition, it holds that:

$$q^{p,(2)} \le (1-\epsilon) \cdot q^{p,(1)} \tag{16}$$

and

$$q^{\ell,(2)} \le (1-\epsilon) \cdot q^{\ell,(1)} \tag{17}$$

Like in Step 1, suppose that under pricing $(q^{p,(1)}, \{q^{\ell,(1)}\})$ buyer purchases allocation λ, while under $(q^{p,(2)}, \{q^{\ell,(2)}\})$ she purchases λ'.

The buyer has higher utility for λ than λ' under $(q^{p,(1)}, \{q^{\ell,(1)}\})$, hence:

$$\begin{aligned} \int_{q^{\ell,(1)}(\lambda)}^{\infty} v f(v)\, dv - q^{p,(1)}(\lambda) - q^{\ell,(1)}(\lambda) \cdot \int_{q^{\ell,(1)}(\lambda)}^{\infty} f(v)\, dv \ge \\ \int_{q^{\ell,(1)}(\lambda')}^{\infty} v f(v)\, dv - q^{p,(1)}(\lambda') - q^{\ell,(1)}(\lambda') \cdot \int_{q^{\ell,(1)}(\lambda')}^{\infty} f(v)\, dv \end{aligned}$$

On the other hand, the buyer has higher utility for λ' than λ under $(q^{p,(2)}, \{q^{\ell,(2)}\})$, thus:

$$\begin{aligned} \int_{q^{\ell,(2)}(\lambda')}^{\infty} v f(v)\, dv - q^{p,(2)}(\lambda') - q^{\ell,(2)}(\lambda') \cdot \int_{q^{\ell,(2)}(\lambda')}^{\infty} f(v)\, dv \ge \\ \int_{q^{\ell,(2)}(\lambda)}^{\infty} v f(v)\, dv - q^{p,(2)}(\lambda') - q^{\ell,(2)}(\lambda) \cdot \int_{q^{\ell,(2)}(\lambda)}^{\infty} f(v)\, dv \end{aligned}$$

Adding the two inequalities above, it follows:

$$\begin{aligned} (p(\lambda') + \ell(\lambda') \cdot \int_{\ell(\lambda')}^{\infty} f(v)\, dv) \ge (p(\lambda) + \ell(\lambda) \cdot \int_{\ell(\lambda)}^{\infty} f(v)\, dv) + (p'(\lambda') + \ell'(\lambda') \cdot \int_{\ell'(\lambda')}^{\infty} f(v)\, dv) \\ - (p'(\lambda) + \ell'(\lambda) \cdot \int_{\ell'(\lambda)}^{\infty} f(v)\, dv) + (\int_{\ell'(\lambda)}^{\ell(\lambda)} v f(v)\, dv - \int_{\ell'(\lambda')}^{\ell(\lambda')} v f(v)\, dv) \end{aligned} \tag{18}$$

Moreover, we have:

$$\int_{q^{\ell,(2)}(\lambda)}^{q^{\ell,(1)}(\lambda)} v f(v)\, dv \ge -q^{\ell,(2)}(\lambda) \cdot \int_{q^{\ell,(1)}(\lambda)}^{q^{\ell,(2)}(\lambda)} f(v)\, dv \tag{19}$$

and

$$\int_{q^{\ell,(2)}(\lambda)}^{q^{\ell,(1)}(\lambda)} vf(v)\,dv \geq -q^{\ell,(2)}(\lambda)\cdot \int_{q^{\ell,(1)}(\lambda)}^{q^{\ell,(2)}(\lambda)} f(v)\,dv \tag{20}$$

With the aim of the relations above, we can derive the lower bound for our theorem:

$$\begin{aligned}
18 \xrightarrow{19,20} & (q^{p,(1)}(\lambda') + q^{\ell,(1)}(\lambda')\cdot \int_{q^{\ell,(2)}(\lambda')}^{\infty} f(v)\,dv) \geq (q^{p,(1)}(\lambda) + q^{\ell,(1)}(\lambda)\cdot \int_{q^{\ell,(1)}(\lambda)}^{\infty} f(v)\,dv) \\
& + (q^{p,(2)}(\lambda') + q^{\ell,(2)}(\lambda')\cdot \int_{q^{\ell,(2)}(\lambda')}^{\infty} f(v)\,dv) - (q^{p,(2)}(\lambda) + q^{\ell,(2)}(\lambda)\cdot \int_{q^{\ell,(1)}(\lambda)}^{\infty} f(v)\,dv) \\
& \xrightarrow{14,15,16,17} (1-\epsilon)^{-1}\cdot(q^{p,(2)}(\lambda') + q^{\ell,(2)}(\lambda')\cdot \int_{q^{\ell,(2)}(\lambda')}^{\infty} f(v)\,dv) + \epsilon^2(1+\int_{q^{\ell,(2)}(\lambda')}^{\infty} f(v)\,dv) \geq \\
& (q^{p,(1)}(\lambda) + q^{\ell,(1)}(\lambda)\cdot \int_{q^{\ell,(1)}(\lambda)}^{\infty} f(v)\,dv) + (q^{p,(2)}(\lambda') + q^{\ell,(2)}(\lambda')\cdot \int_{q^{\ell,(2)}(\lambda')}^{\infty} f(v)\,dv) \\
& - (1-\epsilon)\cdot(q^{p,(1)}(\lambda) + q^{\ell,(1)}(\lambda)\cdot \int_{q^{\ell,(1)}(\lambda)}^{\infty} f(v)\,dv) \\
& \implies (q^{p,(2)}(\lambda') + q^{\ell,(2)}(\lambda')\cdot \int_{q^{\ell,(2)}(\lambda')}^{\infty} f(v)\,dv) + \epsilon^2(1-\epsilon) + \epsilon^2(1-\epsilon)\cdot \int_{q^{\ell,(2)}(\lambda')}^{\infty} f(v)\,dv) \geq \\
& \epsilon(1-\epsilon)(q^{p,(1)}(\lambda) + q^{\ell,(1)}(\lambda)\cdot \int_{q^{\ell,(1)}(\lambda)}^{\infty} f(v)\,dv) + (1-\epsilon)(q^{p,(2)}(\lambda') + q^{\ell,(2)}(\lambda')\cdot \int_{q^{\ell,(2)}(\lambda')}^{\infty} f(v)\,dv) \\
& \implies \epsilon(q^{p,(2)}(\lambda') + q^{\ell,(2)}(\lambda')\cdot \int_{q^{\ell,(2)}(\lambda')}^{\infty} f(v)\,dv) + \epsilon^2(1-\epsilon) + \epsilon^2(1-\epsilon)\cdot \int_{q^{\ell,(2)}(\lambda')}^{\infty} f(v)\,dv \geq \\
& \epsilon(1-\epsilon)\cdot(q^{p,(1)}(\lambda) + q^{\ell,(1)}(\lambda)\cdot \int_{q^{\ell,(1)}(\lambda)}^{\infty} f(v)\,dv) \\
& \implies q^{p,(2)}(\lambda') + q^{\ell,(2)}(\lambda') \int_{q^{\ell,(2)}(\lambda')}^{\infty} f(v)\,dv \geq (1-\epsilon)(q^{p,(1)}(\lambda) + q^{\ell,(1)}(\lambda) \int_{q^{\ell,(1)}(\lambda)}^{\infty} f(v)\,dv) \\
& \qquad - 2\epsilon + \epsilon^2
\end{aligned}$$

Hence $\mathrm{REV}_{(q^{p,(2)},\{q^{\ell,(2)}\})}(v) \geq (1-O(\epsilon))\mathrm{REV}_{(q^{p,(1)},\{q^{\ell,(1)}\})}(v)$.
All stage-1 and stage-2 prices of $(q^{p,(2)},\{q^{\ell,(2)}\}) \in \Pi^{*n}\times\Pi^{*n}$, where Π^{*n} is some set with $|\Pi^{*n}| = O(\frac{1}{\epsilon}\log\frac{\mathcal{R}}{\epsilon})$ since all menu prices in $(q^{p,(2)},\{q^{\ell,(2)}\})$ are in range $[\Omega(\epsilon^2),\mathcal{R}]$ and are powers of $(1+\epsilon^2)$ in $(q^{p,(1)},\{q^{\ell,(1)}\})$ multiplied in $(1-\epsilon)$.

C.4. Step 3: From unit-demand to additive

After Steps 1–2, we restrict attention to menus of length $n = O(\log\mathcal{R}/\epsilon)$ with first-stage prices $p_1 \geq p_2 \geq \cdots \geq p_n$ and second-stage prices (thresholds) $\ell_1 \leq \ell_2 \leq \cdots \leq \ell_n$, each drawn from the finite set Π^*.

For a buyer of type τ with discretized value support $v_1 \leq v_2 \leq \cdots \leq v_n$ and corresponding probabilities $f_1,\ldots,f_n$, define the expected surplus obtained from menu option i as:

$$S_\tau(i) = \sum_{j:v_j\geq\ell_i} (v_j-\ell_i)f_j.$$

Since ℓ_i is nondecreasing, the sequence $S_\tau(1) \geq S_\tau(2) \geq \cdots \geq S_\tau(n) \geq 0$ is nonincreasing.

A unit-demand buyer selecting option i obtains utility:

$$U_\tau(i) = S_\tau(i) - p_i.$$

We now define marginal surplus values:

$$w_{\tau,i} = S_\tau(i) - S_\tau(i+1), \quad i = 1, \ldots, n,$$

with $S_\tau(n+1) = 0$, so that:

$$S_\tau(i) = \sum_{j=i}^{n} w_{\tau,j}.$$

Similarly, define marginal prices:

$$c_i = p_i - p_{i+1}, \quad i = 1, \ldots, n,$$

with $p_{n+1} = 0$, so that:

$$p_i = \sum_{j=i}^{n} c_j.$$

We interpret this as an additive valuation over items $\{1, \ldots, n\}$, where item i has value $w_{\tau,i}$ and price c_i. Purchasing the suffix bundle $\{i, i+1, \ldots, n\}$ yields total value $S_\tau(i)$ and total price p_i.

Hence, the utility of purchasing the suffix bundle is exactly:

$$S_\tau(i) - p_i,$$

which coincides with the utility of selecting menu option i in the original unit-demand problem.

Therefore, optimizing over unit-demand menu choices is equivalent to optimizing over suffix bundles under additive valuations, with utilities and revenue preserved exactly.

C.5. Step 4: Dynamic Programming

Following Step 3, the problem reduces to computing optimal suffix pricing for an additive buyer with item values $w_{\tau,1}, \ldots, w_{\tau,n}$ and prices chosen from the finite set Π'.

Moreover, by the interval pricing structure enforced in Step 3, prices change only at a bounded number of breakpoints and remain nearly constant within each interval, while differing by large multiplicative factors across intervals.

Let the indices

$$1 = i_0 < i_1 < i_2 < \cdots < i_K = n+1$$

define the interval boundaries, where each interval $[i_{k-1}, i_k - 1]$ shares a common marginal price $c^{(k)}$.

The total price of suffix $\{i, \ldots, n\}$ is therefore:

$$p_i = \sum_{k:\, i < i_k} c^{(k)}.$$

Since the buyer is additive, revenue contributed by each interval is independent.

We define a dynamic program where the state:

$$DP[t, z]$$

denotes the maximum expected revenue obtainable from items $\{t, \ldots, n\}$ when the next marginal price is fixed to $z \in \Pi'$.

For each state, we enumerate the next breakpoint $j > t$ and next price level $z' \in \Pi'$ satisfying the interval gap constraints, and add the revenue contribution of interval $[t, j-1]$:

$$DP[t, z] = \max_{j > t,\, z'} \Big\{ \mathrm{Rev}(t, j-1, z') + DP[j, z'] \Big\}.$$

Here $\mathrm{Rev}(t, j-1, z')$ denotes the expected revenue obtained from assigning marginal price z' to all items in interval $[t, j-1]$, which equals

$$z' \cdot \sum_{\tau} q_\tau \sum_{i=t}^{j-1} \mathbf{1}[w_{\tau,i} \geq z'] \,.$$

The base case is $DP[n+1, \cdot] = 0$.

Since the number of intervals is $O(1/\epsilon^2)$, the price alphabet Π' has size $\mathrm{poly}(1/\epsilon, \log \mathcal{R})$, and each transition enumerates polynomially many possibilities, the dynamic program runs in

$$\mathrm{poly}\big(m, n^{\mathrm{poly}(1/\epsilon)}, \log \mathcal{R}\big)$$

time.

The resulting suffix pricing is translated back to a menu pricing (p_i, ℓ_i) using the inverse transformations of Steps 3–1, incurring only a $(1 - O(\epsilon))$ revenue loss.