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Numerical analysis of a fractional-order chaotic system based on conformable fractional-order derivative

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Abstract. In this paper, the numerical solutions of conformable fractional-order linear and nonlinear equations are obtained by employing the constructed conformable Adomian decomposition method (CADM). We found that CADM is an effective method for numerical solution of conformable fractional-order differential equations. Taking the conformable fractional-order simplified Lorenz system as an example, the numerical solution and chaotic behaviors of the conformable fractional-order simplified Lorenz system are investigated. It is found that rich dynamics exist in the conformable fractional-order simplified Lorenz system, and the minimum order for chaos is even less than 2. The results are validated by means of bifurcation diagram, Lyapunov characteristic exponents and phase portraits.

1 Introduction

Fractional calculus is a valuable tool in the modeling of many phenomena, and it has become a topic of great interest in science and engineering [1–15]. Currently, the most commonly used definitions about fractional calculus are Riemann-Liouville definition and Caputo definition [4]. Although both of them are used most in practice, there exist some problems. For instance, they do not satisfy some properties that the integer-order derivative satisfies, such as the chain rule and the properties for the product and quotient. To overcome these difficulties, recently Khalil *et al.* [5] proposed a new fractional derivative, that is prominently compatible with the classical derivative, and is called the conformable fractional-order derivative. Mathematical properties of conformable fractional-order derivative were investigated in [6], and it also indicated in the paper that conformable definition is the simplest one to recognize the fractional derivative and the integral derivative. Meanwhile, this new fractional definition also has some physical applications such as the conformable fractional-order heat equation [7], conformable fractional-order Newonian mechanics [8] and conformable fractional-order Burgers's equation [9]. In fact, the research about conformable fractional derivative is still at an early stage, and many further studies can be carried out in this new fractional derivative.

In recent years, dynamical analysis, synchronization and practical applications of fractional-order chaotic systems have attracted much attention [10–15]. For instance, Chen *et al.* [10] analyzed the dynamics of the fractional-order Chen system, and the minimum order they found to have chaos is 2.1. Maheri and Arifin [11] investigated synchronization between two fractional-order chaotic systems with unknown parameters using a robust adaptive nonlinear controller. He *et al.* [13] proposed a method for color image encryption based on fractional-order hyperchaotic systems. However, the definition of the fractional-order derivative in these articles is the Riemann-Liouville definition or the Caputo definition. As far as we know, there are no reports about conformable fractional-order chaotic system. It is interesting and necessary to apply the conformable fractional-order derivative to chaotic systems and to analyze chaotic behaviors in these fractional-order chaotic systems.

Particularly, we need to find a numerical solution method for conformable fractional-order chaotic systems. At present, numerical solutions of the fractional-order chaotic systems with Riemann-Liouville definition or Caputo definition are obtained by frequency algorithm [16], Adms-Bashforth-Moulton algorithm (ABM) [17] or Adomian decomposition method (ADM) [18]. Moreover, to solve conformable fractional-order differential equations, some efforts were made [7–9, 19, 20]. However, these methods are used according to the particular occasion. Among the above numerical

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algorithms, ADM is a proper choice [21–23]. In this paper, conformable ADM (CADM) is developed to obtain the numerical solution of conformable fractional-order equations, including the solution of the conformable fractional-order chaotic system.

In this study, the numerical solution and dynamics of a conformable fractional-order chaotic system are investigated. The rest of this paper is organized as follows. In sect. 2, the conformable Adomian decomposition method (CADM) is proposed, and it is used to solve conformable fractional-order linear and nonlinear equations. In sect. 3, numerical solution of a conformable fractional-order chaotic system is obtained by employing CADM, and dynamics of the fractional-order simplified Lorenz system is investigated. Finally, the results are summarized in sect. 4.

2 Solution of conformable fractional differential equations

2.1 Definitions and fundamentals

Recently, Khalil *et al.* [5] introduced a new definition of fractional derivative, which is known as conformable fractional derivative.

Definition 1 ([5]). The conformable fractional derivative is given by

$$T_t^q f(t) = \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon t^{1-q}) - f(t)}{\varepsilon}\,, \tag{1}$$

where $t > 0$, $q \in (0,1]$. Obviously, when $q = 1$, it is an integer-order derivative.

Some properties of the conformable fractional derivative are listed as follows.

Theorem 1 ([5]). *Let $q \in (0,1]$, and suppose $f(\cdot)$ and $g(\cdot)$ are q-differentiable at point $t > 0$. Then*

i) $T_t^q(af(t) \pm bg(t)) = aT_t^q f(t) \pm bT_t^q g(t)$, *where* $a, b \in R$;

ii) $T_t^q(f(t)g(t)) = g(t)T_t^q f(t) + f(t)T_t^q g(t)$;

iii) $T_t^q(\frac{f(t)}{g(t)}) = \frac{g(t)T_t^q f(t) - f(t)T_t^q g(t)}{g(t)^2}$;

iv) $T_t^q T_t^\alpha f(t) = T_t^{q+\alpha} f(t)$;

v) $T_t^q f(g(t)) = t^{1-q} g'(t) f'(g(t))$.

Remark 1. Riemann-Liouville definition and Caputo definition *do not* satisfy the properties as shown in theorem 1.

According to definition 1 and theorem 1, the conformable fractional derivatives of some functions are given as shown in theorem 2.

Theorem 2 ([5]). *Let $q \in (0,1]$, and suppose $f(\cdot)$ is q-differentiable at point $t > 0$, then*

i) $T_t^q f(t) = t^{1-q}\frac{\mathrm{d}f(t)}{\mathrm{d}t}$;

ii) $T_t^q(t^\mu) = t^{1-q}\mu t^{\mu-1} = \mu t^{\mu-q}$, $f(t) = t^\mu$, $\mu \in R$;

iii) $T_t^q(\exp(\frac{1}{q}t^q)) = \exp(\frac{1}{q}t^q)$;

iv) $T_t^q(\sin\frac{1}{q}t^q) = \cos\frac{1}{q}t^q$;

v) $T_t^q(\cos(\frac{1}{q}t^q)) = -\sin(\frac{1}{q}t^q)$;

vi) $T_t^q(\frac{1}{q}t^q) = 1$.

Definition 2 ([6]). The conformable fractional integral of function $f(\cdot)$ starting from $t_0 \geq 0$ is defined as

$$I_{t_0}^q f(t) = \int_{t_0}^t \frac{f(x)}{(x-t_0)^{1-q}}\mathrm{d}x, \tag{2}$$

where $t > t_0$, $f(\cdot)$ is q-differentiable at $(t_0, t]$ and $q \in (0,1]$.

Here, two useful Lemmas about conformable fractional calculus are given and they will be used to solve the conformable fractional differential equations.

Lemma 1. *Let $t_0 > 0$, $q \in (0,1]$, and $f(\cdot)$ is q-differentiable at $[t_0, \infty)$, so we have*

$$I_{t_0}^q T_{t_0}^q f(t) = f(t) - f(t_0), \tag{3}$$

Proof. When $t > t_0$, theorem 2i) now is given by [6]

$$T_{t_0}^q f(t) = (t - t_0)^{1-q} \frac{\mathrm{d}f(t)}{\mathrm{d}t}, \tag{4}$$

According to definition 2, we have

$$\begin{aligned} I_{t_0}^q T_{t_0}^q f(t) &= \int_{t_0}^t (x - t_0)^{q-1} T_x^q f(x)\mathrm{d}x \\ &= \int_{t_0}^t (x - t_0)^{q-1} (x - t_0)^{1-q} f'(x)\mathrm{d}x \\ &= f(t) - f(t_0). \end{aligned} \tag{5}$$

Lemma 2. *If C is a constant, $q_1, q_2, \ldots, q_n \in (0,1]$, we have*

$$I_{t_0}^{q_n} \ldots I_{t_0}^{q_2} I_{t_0}^{q_1} C = C \frac{(t-a)^{q_1+\ldots+q_n}}{q_1(q_1+q_2)\ldots(q_1+\ldots+q_n)}. \tag{6}$$

Proof.

$$I_{t_0}^{q_1} C = \int_{t_0}^t \frac{C}{(\tau - t_0)^{1-q_1}} \mathrm{d}\tau = C \frac{(t-t_0)^{q_1}}{q_1}, \tag{7}$$

$$I_{t_0}^{q_2} I_{t_0}^{q_1} C = \int_{t_0}^t \frac{C}{q_1} \frac{(\tau - t_0)^{q_1}}{(\tau - t_0)^{1-q_2}} \mathrm{d}\tau = \frac{C(\tau - t_0)^{q_1+q_2}}{q_1(q_1+q_2)}. \tag{8}$$

The rest can be done in the same manner, thus we have

$$I_a^{q_n} \ldots I_a^{q_2} I_a^{q_1} C = C \frac{(t-a)^{q_1+\ldots+q_n}}{q_1(q_1+q_2)\ldots(q_1+\ldots+q_n)}. \tag{9}$$

Particularly, when $q_1 = q_2 = \ldots = q_n = q$, $q \in (0,1]$, the following equation is obtained:

$$I_a^q \ldots I_a^q I_a^q C = C \frac{(t-a)^{nq}}{n! q^n}. \tag{10}$$

2.2 Conformable Adomian decomposition method

The Adomian decomposition method (ADM) [18] is developed to obtain numerical solution of conformable fractional-order equations. Let us assume that the conformable fractional-order equation is presented by

$$T_t^q \mathbf{x}(t) = f(\mathbf{x}(t)) + \mathbf{g}, \tag{11}$$

where T_t^q is the $q \in (0,1]$ order conformable fractional-order derivative, $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ are state variables, and g is the constant. Firstly, divide the equation into three parts, which now is given by

$$T_t^q \mathbf{x}(t) = L\mathbf{x}(t) + N\mathbf{x}(t) + \mathbf{g}. \tag{12}$$

Here, $L\mathbf{x}(t)$ is the linear term and $N\mathbf{x}(t)$ is the nonlinear term. By applying the conformable fractional integral operator on both sides of eq. (11), according to lemma 1, we get

$$\mathbf{x}(t) = I_{t_0}^q L\mathbf{x}(t) + I_{t_0}^q N\mathbf{x}(t) + I_{t_0}^q \mathbf{g} + \mathbf{x}(t_0). \tag{13}$$

The nonlinear terms are decomposed based on the following method [22,23]:

$$\begin{cases} A_j^i = \dfrac{1}{i!} \left[\dfrac{\mathrm{d}^i}{\mathrm{d}\lambda^i} N(v_j^i(\lambda)) \right]_{\lambda=0}, \\ v_j^i(\lambda) = \displaystyle\sum_{k=0}^i (\lambda)^k x_j^k, \end{cases}. \tag{14}$$

where $i = 0, 1, 2, \ldots, \infty$, and $j = 1, 2, \ldots, n$. Thus the nonlinear terms can be presented as

$$N\mathbf{x} = \sum_{i=0}^{\infty} A^i \left(\mathbf{x}^0, \mathbf{x}^1, \ldots, \mathbf{x}^i\right). \tag{15}$$

According to ADM, the solution of eq. (12) is given by

$$\mathbf{x} = \sum_{i=0}^{\infty} \mathbf{x}^i = I_{t_0}^q L \sum_{i=0}^{\infty} \mathbf{x}^i + I_{t_0}^q \sum_{i=0}^{\infty} \mathbf{A}^i + I_{t_0}^q \mathbf{g} + \mathbf{x}(t_0). \tag{16}$$

By employing the following method, $\mathbf{x}^i$ can be calculated:

$$\begin{cases} \mathbf{x}^0 = I_{t_0}^q \mathbf{g} + \mathbf{x}(t_0) \\ \mathbf{x}^1 = I_{t_0}^q L\mathbf{x}^0 + I_{t_0}^q \mathbf{A}^0(\mathbf{x}^0) \\ \mathbf{x}^2 = I_{t_0}^q L\mathbf{x}^1 + I_{t_0}^q \mathbf{A}^1(\mathbf{x}^0, \mathbf{x}^1) \\ \qquad \cdots \\ \mathbf{x}^i = I_{t_0}^q L\mathbf{x}^{i-1} + I_{t_0}^q \mathbf{A}^{i-1}(\mathbf{x}^0, \mathbf{x}^1, \ldots, \mathbf{x}^{i-1}) \\ \qquad \cdots \end{cases}. \tag{17}$$

2.3 Solution of conformable fractional-order equations

In this section, we deduce the solutions of two conformable fractional-order differential equations by applying the CADM. Moreover, we compare these solutions with their corresponding exact solutions.

Case 1. The conformable fractional-order exponential function. The equation given by $T_{t_0}^q f(t) = f(t)$, with $f(t_0) = f(0) = 1$ and $q \in (0, 1]$ is calculated by employing CADM. The exact solution of this conformable fractional-order equation is

$$f(t) = \exp\left(\frac{1}{q} t^q\right). \tag{18}$$

According to CADM, the solution is given by

$$f(t) = \sum_{i=0}^{\infty} f^i. \tag{19}$$

Firstly, the initial condition is

$$f^0 = f(0) = 1. \tag{20}$$

According to eq. (16) and lemma 2, the following items are calculated:

$$f^1 = I_{t_0}^q f^0 = \frac{t^q}{q}, \tag{21}$$

$$f^2 = I_{t_0}^q f^1 = \frac{t^{2q}}{2q^2}, \tag{22}$$

$$f^n = I_{t_0}^q f^{n-1} = \frac{t^{nq}}{n! q^n}, \tag{23}$$

$$f(t) = \sum_{i=0}^{\infty} f^0 \frac{t^{iq}}{i! q^i} = \sum_{i=0}^{\infty} \frac{t^{iq}}{i! q^i} = \exp\left(\frac{1}{q} t^q\right). \tag{24}$$

When we calculate infinite items, the solution obtained is the same exact solution. To illustrate the precision of the CADM solutions, the error between CADM solutions with 6 items and the exact solution is calculated as show in fig. 1. The order q is chosen as 0.7, 0.8, 0.9 and 1.0, respectively. It shows that the accuracy of the CADM solutions increases with the increase of order q, but it decreases with the increase of time. It means that CADM can get more accurate results when q is closer to one. Moreover, for smaller order q, the simulation time should be shorter when we want more satisfying results.

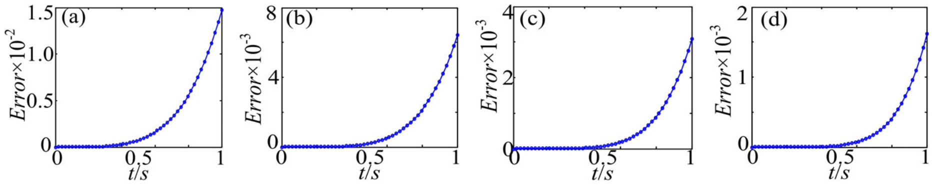

Fig. 1. Error between CADM solution (6 items) and exact solution with different q: (a) $q = 0.7$; (b) $q = 0.8$; (c) $q = 0.9$; (d) $q = 1.0$.

Case 2. The conformable fractional-order Riccati equation. This equation is defined as

$$T_a^q f(t) = 1 - (f(t))^2, \qquad f(0) = 0. \tag{25}$$

The exact solution is

$$f(t) = \frac{\exp(\frac{1}{q}t^q) - 1}{\exp(\frac{1}{q}t^q) + 1}. \tag{26}$$

In conformable fractional-order Riccati equation, there is a nonlinear term $-f(t) \times f(t)$. According to eq. (13), we define

$$f(t) = f^0 + \lambda f^1 + \lambda^2 f^2 + \lambda^3 f^3 + \dots . \tag{27}$$

Then, the nonlinear item can be calculated as

$$\begin{aligned} F(t) &= -f(t) \times f(t) \\ &= -f^0 f^0 - 2f^1 f^0 \lambda - \left(2f^2 f^0 - f^1 f^1\right)\lambda^2 - \left(2f^3 f^0 - 2f^2 f^1\right)\lambda^3 - \dots . \end{aligned} \tag{28}$$

Thus

$$A^0 = -\frac{1}{0!}\left[\frac{\mathrm{d}^0}{\mathrm{d}\lambda^0}F(t)\right]_{\lambda=0} = -f^0 f^0, \tag{29}$$

$$A^1 = -\frac{1}{1!}\left[\frac{\mathrm{d}}{\mathrm{d}\lambda}F(t)\right]_{\lambda=0} = -2f^1 f^0, \tag{30}$$

$$A^2 = -\frac{1}{2!}\left[\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}F(t)\right]_{\lambda=0} = -2f^2 f^0 - f^1 f^1. \tag{31}$$

By applying the above method, A^i can be obtained. By applying the CADM, the following items are computed:

$$f^0 = I_{t_0}^q g = g\frac{t^q}{q} = \frac{t^q}{q}, \tag{32}$$

$$f^1 = -I_{t_0}^q (f^0)^2 = -I_{t_0}^q \frac{t^{2q}}{q^2} = -\frac{t^{3q}}{3q^3}, \tag{33}$$

$$f^2 = -I_{t_0}^q (2f^1 f^0) = -I_{t_0}^q \left(-\frac{2t^{4q}}{3q^3}\right) = \frac{2t^{5q}}{15q^5}, \tag{34}$$

$$f^3 = -I_{t_0}^q (2f^2 f^0 + f^1 f^1) = -I_{t_0}^q \left(\frac{4t^{6q}}{15q^6} + \frac{t^{6q}}{9q^6}\right) = -\frac{17t^{7q}}{315q^7}. \tag{35}$$

Based on the CADM, the solution is given by

$$f(t) = \sum_{i=0}^{\infty} f^i = \frac{t^q}{q} - \frac{t^{3q}}{3q^3} + \frac{2t^{5q}}{15q^5} - \frac{17t^{7q}}{315q^7} + \frac{62t^{9q}}{2835q^9} - \dots . \tag{36}$$

Obviously, the CADM solution obtained above is the power series expansion of the exact solution as shown in eq. (26). Figure 2 shows the relationship between the exact solution and the CADM solution. As shown in fig. 2, the solution is more accurate when the order q is larger. In addition, fig. 2 shows that the CADM solution has high precision when time is small.

In fact, more items can improve the accuracy of the CADM solution. Here, we take case 1 as an example, the accuracy with different number of terms is calculated and the results are shown in fig. 3, where $q = 0.7$, the number of terms is seven, eight, nine and ten. It is shown that the solution with more terms is more accurate.

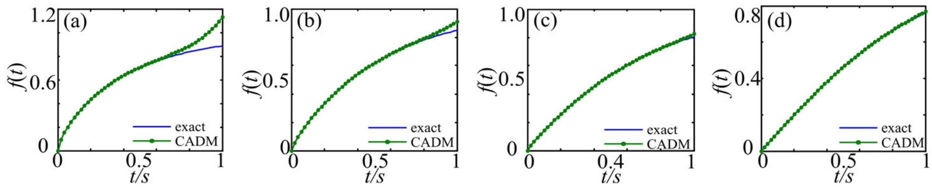

Fig. 2. The relationship between CADM solution and exact solution (solid lines), (a) $q = 0.7$; (b) $q = 0.8$; (c) $q = 0.9$; (d) $q = 1.0$.

3 Conformable fractional-order simplified Lorenz system

3.1 Solution and its accuracy

In ref. [23] and [24], the fractional-order simplified Lorenz system with Caputo definition is investigated by applying ADM and ABM, respectively. It shows that the fractional-order simplified Lorenz system has rich dynamics under both cases. In this paper, we introduce the conformable definition to the simplified Lorenz system, and the conformable fractional-order simplified Lorenz system is defined as

$$\begin{cases} T_t^q x_1 = 10(x_2 - x_1), \\ T_t^q x_2 = (24 - 4c)x_1 + cx_2 - x_1x_3, \\ T_t^q x_3 = x_1x_2 - 8x_3/3, \end{cases} \tag{37}$$

where q is the order and c is the system parameter. The linear terms and nonlinear terms of the system are

$$\begin{bmatrix} L_{x1} \\ L_{x2} \\ L_{x3} \end{bmatrix} = \begin{bmatrix} 10(x_2 - x_1) \\ (24 - 4c)x_1 + cx_2 \\ -8x_3/3 \end{bmatrix}, \qquad \begin{bmatrix} N_{x1} \\ N_{x2} \\ N_{x3} \end{bmatrix} = \begin{bmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \end{bmatrix}. \tag{38}$$

According to eq. (14), the nonlinear terms can be represented as

$$\begin{cases} A_2^0 = -x_1^0x_3^0, \\ A_2^1 = -x_1^1x_3^0 - x_1^0x_3^1, \\ A_2^2 = -x_1^2x_3^0 - x_1^1x_3^1 - x_1^0x_3^2, \\ A_2^3 = -x_1^3x_3^0 - x_1^2x_3^1 - x_1^2x_3^2 - x_1^0x_3^3, \\ A_2^4 = -x_1^4x_3^0 - x_1^3x_3^1 - x_1^1x_3^3 - x_1^2x_3^2 - x_1^0x_3^4, \\ A_2^5 = -x_1^5x_3^0 - x_1^4x_3^1 - x_1^1x_3^4 - x_1^3x_3^2 - x_1^2x_3^3 - x_1^0x_3^5, \end{cases} \tag{39}$$

$$\begin{cases} A_3^0 = x_1^0x_2^0, \\ A_3^1 = x_1^1x_2^0 + x_1^0x_2^1, \\ A_3^2 = x_1^2x_2^0 + x_1^1x_2^1 + x_1^0x_2^2, \\ A_3^3 = x_1^3x_2^0 + x_1^2x_2^1 + x_1^2x_2^2 + x_1^0x_2^3, \\ A_3^4 = x_1^4x_2^0 + x_1^2x_2^2 + +x_1^3x_2^1 + x_1^2x_2^3 + x_1^0x_2^4, \\ A_3^5 = x_1^5x_2^0 + x_1^4x_2^1 + x_1^1x_2^4 + x_1^3x_2^2 + x_1^2x_2^3 + x_1^0x_2^5. \end{cases} \tag{40}$$

Let us assume the initial condition is $\mathbf{x}_0 = (x_1(t_0), x_2(t_0), x_3(t_0))$, then the first term is

$$\begin{cases} x_1^0 = x_1(t_0), \\ x_2^0 = x_2(t_0), \\ x_3^0 = x_3(t_0). \end{cases} \tag{41}$$

Let $c_1^0 = x_1^0$, $c_2^0 = x_2^0$ and $c_3^0 = x_3^0$, thus we have $\mathbf{c}^0 = (c_1^0, c_2^0, c_3^0)$. According to eq. (17), we have $\mathbf{x}^1 = I_{t_0}^q L\mathbf{x}^0 + I_{t_0}^q \mathbf{A}^0$. Base on the method as shown in eq. (17), the second term of each state variable is shown as follows:

$$\begin{cases} x_1^1 = 10(c_2^0 - c_1^0)\dfrac{(t-t_0)^q}{q}, \\ x_2^1 = \left[(24-4c)c_1^0 + cc_2^0 - c_1^0 c_3^0\right]\dfrac{(t-t_0)^q}{q}, \\ x_3^1 = \left(-\dfrac{8}{3}c_3^0 + c_1^0 c_2^0\right)\dfrac{(t-t_0)^q}{q}. \end{cases} \tag{42}$$

If we let

$$\begin{cases} c_1^1 = 10(c_2^0 - c_1^0), \\ c_2^1 = (24-4c)c_1^0 + cc_2^0 - c_1^0 c_3^0, \\ c_3^1 = -(8/3)c_3^0 + c_1^0 c_2^0, \end{cases} \tag{43}$$

the second term can be defined as

$$\mathbf{x}^1 = \mathbf{c}^1 \frac{(t-t_0)^q}{q}. \tag{44}$$

Similarly, the third, fourth, fifth, sixth and seventh set of coefficients are, respectively,

$$\begin{cases} c_1^2 = 10(c_2^1 - c_1^1), \\ c_2^2 = (24-4c)c_1^1 + cc_2^1 - c_1^1 c_3^0 - c_1^0 c_3^1, \\ c_3^2 = -\frac{8}{3}c_3^1 + c_1^1 c_2^0 + c_1^0 c_2^1, \end{cases} \tag{45}$$

$$\begin{cases} c_1^3 = 10(c_2^2 - c_1^2), \\ c_2^3 = (24-4c)c_1^2 + cc_2^2 - c_1^2 c_3^0 - 2c_1^1 c_3^1 - c_1^0 c_3^2, \\ c_3^3 = -\dfrac{8}{3}c_3^2 + c_1^2 c_2^0 + 2c_1^1 c_2^1 + c_1^0 c_2^2, \end{cases} \tag{46}$$

$$\begin{cases} c_1^4 = 10(c_2^3 - c_1^3), \\ c_2^4 = (24-4c)c_1^3 + cc_2^3 - c_1^3 c_3^0 - 3(c_1^2 c_3^1 + c_1^1 c_3^2) - c_1^0 c_3^3, \\ c_3^4 = -\dfrac{8}{3}c_3^3 + c_1^3 c_2^0 + 3(c_1^1 c_2^2 + c_1^2 c_2^1) + c_1^0 c_2^3, \end{cases} \tag{47}$$

$$\begin{cases} c_1^5 = 10(c_2^4 - c_1^4), \\ c_2^5 = (24-4c)c_1^4 + cc_2^4 - c_1^4 c_3^0 - 4(c_1^3 c_3^1 + c_1^1 c_3^3) - 6c_1^2 c_3^2 - c_1^0 c_3^4, \\ c_3^5 = -\dfrac{8}{3}c_3^4 + c_1^4 c_2^0 + 4(c_1^1 c_2^3 + c_1^3 c_2^1) + 6c_1^2 c_2^2 + c_1^0 c_2^4, \end{cases} \tag{48}$$

$$\begin{cases} c_1^6 = 10(c_2^5 - c_1^5), \\ c_2^6 = (24-4c)c_1^5 + cc_2^5 - c_1^5 c_3^0 - 5(c_1^4 c_3^1 + c_1^1 c_3^4) - c_1^0 c_3^5 - 10(c_1^2 c_3^3 + c_1^3 c_3^2), \\ c_3^6 = -\dfrac{8}{3}c_3^5 + c_1^5 c_2^0 + 5(c_1^1 c_2^4 + c_1^4 c_2^1) + 10(c_1^2 c_2^3 + c_1^3 c_2^2) + c_1^0 c_2^5. \end{cases} \tag{49}$$

So the CADM solution of the conformable fractional-order simplified Lorenz system with seven terms is

$$\tilde{x}_j(t) = \sum_{i=0}^{6} c_j^5 \frac{(t-t_0)^{iq}}{i!q^i}. \tag{50}$$

As is shown in sect. 2, the analytical solution may be inaccurate as time goes on. Here, we let $q = 1$, $\mathbf{x}_0 = [1, 1, 1]$, $c = 2$. As there are no exact solution for chaotic system, the accuracy of CADM is compared to the numerical solution by employing the fourth-order Runge-Kutta method (RK_4) with $h = 0.01$. As the CADM solution is an

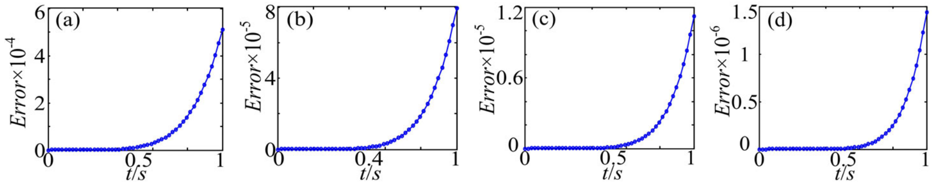

Fig. 3. Error between exact solution and CADM solution with different number of terms: (a) 7 terms; (b) 8 terms; (c) 9 terms; (d) 10 terms.

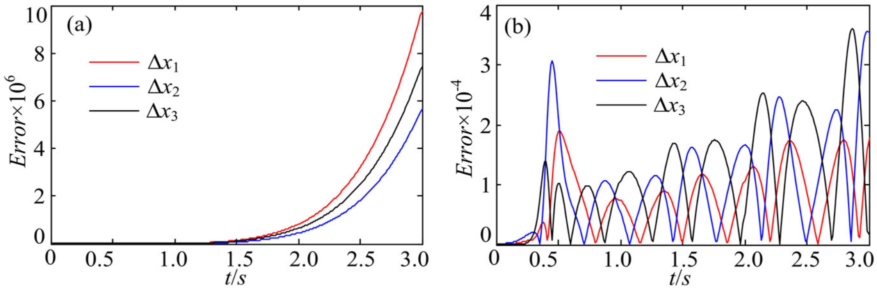

Fig. 4. Differences between CADM solution and RK_4 solution. (a) Analytical solution of CADM; (b) discrete solution of CADM.

analytical solution, we just calculate the solution of some points in time. Specifically, according to eq. (50), we calculate $x_{j\text{CADM}}(t = 0), x_{j\text{CADM}}(t = 0.01), \ldots, x_{j\text{CADM}}(t = 3.0)$, where $j = 1, 2, 3$. Thus two solutions can be one-to-one correspondent. The differences between CADM solution and RK_4 solution are given in fig. 4(a), where

$$\Delta x_j = |x_{j\text{Rk}_4} - x_{j\text{CADM}}|\,. \tag{51}$$

The simulation carried out in this paper is between $t = 0\,\text{s}$ to $t = 3.0\,\text{s}$. As can be seen from fig. 4(a), the results of CADM are far away from those of RK_4. So, we need to divide the computation interval $[t_0, t]$ into N parts with step size of h, and each subinterval can be presented as $[t_m, t_{m+1}]$, where $h = (t_{m+1} - t_m)/N$. Namely, we now have subintervals $[t_0, t_1], [t_1, t_2], [t_2, t_3], \ldots, [t_m, t_{m+1}], \ldots, [t_{N-1}, t_N]$, where $t_N = t$. Taking subinterval $[t_m, t_{m+1}]$ as an example, according to $\mathbf{x}(t_m)$ and eq. (50), $\mathbf{x}(t_{m+1})$ can be calculated. For conformable fractional-order simplified Lorenz system, the initial values are $\mathbf{x}(t_0) = [x_1(t_0), x_2(t_0), x_3(t_0)]$. Thus we can get $\mathbf{x}(t_1)$ according to $\mathbf{x}(t_0)$ based on eq. (50) in subintervals $[t_0, t_1]$, and so on. By adopting this method, we obtain $\mathbf{x}(t_{m+1}) = F_{\text{CADM}}(\mathbf{x}(t_m))$. Thus the solution can also be expressed as [25]

$$\mathbf{x}\,(m+1) = F_{\text{CADM}}\,(\mathbf{x}(m))\,, \tag{52}$$

which is a typical discrete map. In our study, we fix $h = 0.01$. It is shown in fig. 4(b) that the discrete solution agrees well with that of RK_4. Thus in this paper, the conformable fractional-order simplified Lorenz is solved by CADM with discrete form.

3.2 Dynamical analysis

Lyapunov characteristic exponents (LCEs) of the conformable fractional-order simplified Lorenz system are calculated based on eq. (52) and QR decomposition method [26]. Firstly, the Jacobian matrix of eq. (52) is computed. Let us suppose that the solution is given by

$$\mathbf{x}\,(m+1) = F_{\text{CADM}}\,(\mathbf{x}(m)) = \begin{cases} F_{1\text{CADM}}\,(\mathbf{x}(m))\,, \\ F_{2\text{CADM}}\,(\mathbf{x}(m))\,, \\ F_{3\text{CADM}}\,(\mathbf{x}(m))\,. \end{cases} \tag{53}$$

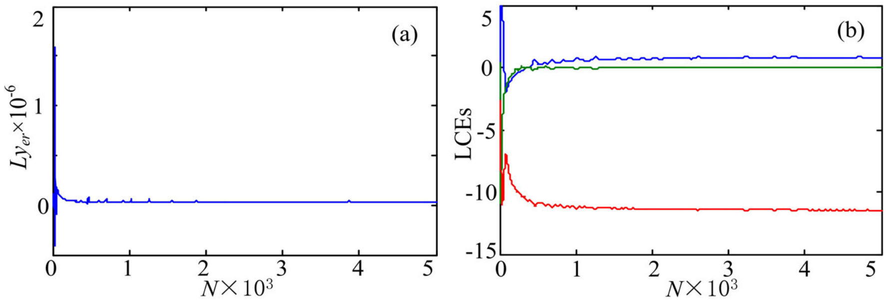

Fig. 5. LCEs results with $q = 1$ and $c = 2$. (a) Error curve with different iteration number; (b) LCEs curve *versus* different iteration numbers.

The Jacobian matrix of discrete solution as given by eq. (53) is defined as

$$\mathbf{J} = \begin{bmatrix} \frac{\partial F_{1\text{CADM}}}{\partial x_1} & \frac{\partial F_{1\text{CADM}}}{\partial x_2} & \frac{\partial F_{1\text{CADM}}}{\partial x_3} \\ \frac{\partial F_{2\text{CADM}}}{\partial x_1} & \frac{\partial F_{2\text{CADM}}}{\partial x_2} & \frac{\partial F_{2\text{CADM}}}{\partial x_3} \\ \frac{\partial F_{3\text{CADM}}}{\partial x_1} & \frac{\partial F_{3\text{CADM}}}{\partial x_2} & \frac{\partial F_{3\text{CADM}}}{\partial x_3} \end{bmatrix}, \tag{54}$$

which can be obtained by using mathematical software. In this paper, function Jacobian $(\cdot)$ of Matlab is applied. Then, QR decomposition method is applied for LCEs calculation. The method is shown as follows:

$$\begin{aligned} qr\left(\mathbf{J}_M\mathbf{J}_{M-1}\dots\mathbf{J}_1\right) &= qr\left(\mathbf{J}_M\mathbf{J}_{M-1}\dots\mathbf{J}_2(\mathbf{J}_1\mathbf{Q}_0)\right) \\ &= qr\left(\mathbf{J}_M\mathbf{J}_{M-1}\dots\mathbf{J}_3(\mathbf{J}_2\mathbf{Q}_1)\right)\mathbf{R}_1 \\ &= \dots \\ &= qr\left(\mathbf{J}_M\mathbf{J}_{M-1}\dots\mathbf{J}_i(\mathbf{J}_{i-1}\mathbf{Q}_{i-2})\right)\mathbf{R}_{i-1}\dots\mathbf{R}_2\mathbf{R}_1 \\ &= \dots \\ &= \mathbf{Q}_M\mathbf{R}_M\dots\mathbf{R}_2\mathbf{R}_1. \end{aligned} \tag{55}$$

Here, $qr(\cdot)$ is the QR decomposition function. Thus LCEs is calculated according to

$$Ly_n = \frac{1}{Nh}\sum_{i=1}^{N}\ln\left|\mathbf{R}_i(n,n)\right|, \tag{56}$$

where $n = 1, 2, 3$. If $q = 1$, the summation of LCEs of simplified Lorenz system is $\triangle = -10+c-8/3$. Let the difference between $\triangle$ and summation of obtained LCEs be

$$Ly_{er} = \left|\sum_{i=1}^{3} Ly_i + 10 + 8/3 - c\right|. \tag{57}$$

Let $c = 2$, $M = 5000$, $\mathbf{x}_0 = [1, 2, 3]$, the result is illustrated in fig. 5. In this case, LCEs are $(Ly_1, Ly_2, Ly_3) = (0.8135, 0, -11.4802)$. It means that the system is chaotic. Meanwhile, it shows that LCEs calculated in this paper have high accuracy and fast convergence speed.

Next, bifurcation diagram, LCEs are employed to investigate dynamical behaviors of the conformable fractional-order simplified Lorenz system. Two cases are analyzed.

Case 1. Dynamics with $c = 2$ and q varying.

Fix $c = 2$, let the derivative order q vary from 0.55 to 1 with step size of 0.0009 and the initial values of state variables $\mathbf{x}_0 = [1, 2, 3]$. It shows in fig. 6 that the system is chaotic over the interval $q \in [0.5743, 1]$. Meanwhile, the maximum Lyapunov characteristic exponent decreases with the increase of derivative order q. Phase portraits for $q = 0.5743$ are presented in fig. 7. It is shown, in fig. 7(a), that the phase diagram is not smooth, which means the solution is not accurate when $q = 0.5743$ with $h = 0.01$. To get smoother result, smaller h should be used. In fig. 7(b), $h = 0.001$ is used for calculation, and it is shown that a much more satisfying result is obtained. So in this case, the minimum order for chaos is $q = 0.5743$.

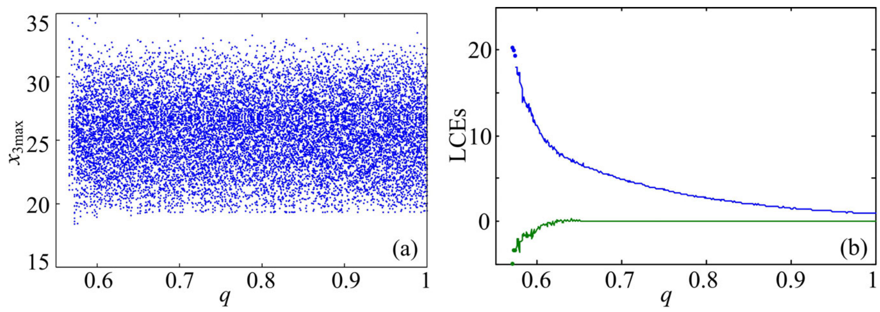

Fig. 6. Dynamics analysis results with $c = 2$ and q varying. (a) Bifurcation diagram; (b) LCEs.

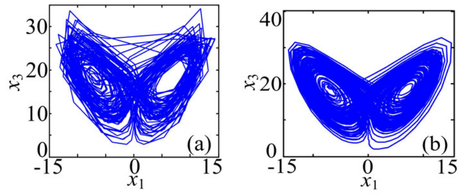

Fig. 7. Phase diagrams for $q = 0.5743$ and $c = 2$. (a) $h = 0.01$ (not smooth); (b) $h = 0.001$ (smooth).

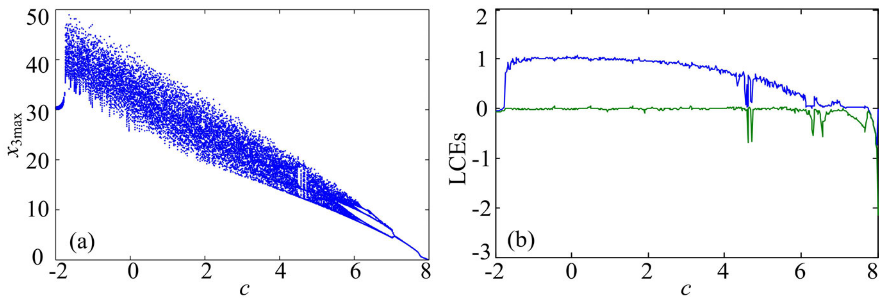

Fig. 8. Dynamics analysis results with $q = 0.98$ and c varying. (a) Bifurcation diagram; (b) LCEs.

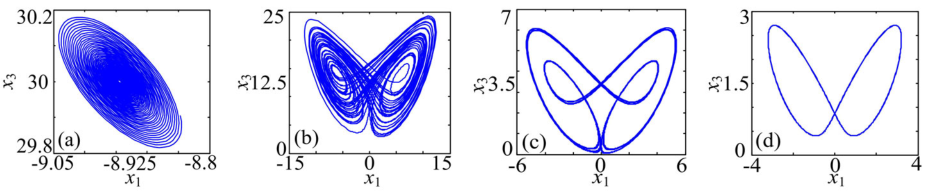

Fig. 9. Phase diagrams for $q = 0.98$ and different c: (a) $c = -2$; (b) $c = 3.5$; (c) $c = 6$; (d) $c = 7.5$.

Case 2. Dynamics with $q = 0.98$ *and* c *varying.*

Fix $q = 0.98$, and vary the system parameter c from -2 to 8 with step size of 0.02, and the initial values are the same as above. The bifurcation diagram is shown in fig. 8(a), and its corresponding LCEs result is illustrated in fig. 8(b). The chaotic zone covers most of the range $c \in [-1.7395, 7.018]$, excepting a periodic window near $c = 4.6$. With the decrease of c, the system enters chaos by periodic double bifurcation, then the system is drawn out from chaotic movement abruptly and the convergent state is observed at $c = -1.74$. Phase diagrams in different stage are plotted as shown in fig. 9, where $c = -2$, 3.5, 6 and 7.5. Convergent state, chaotic attractor, periodic circles are observed, respectively. Thus the system has rich dynamics.

4 Conclusion

In this paper, the numerical solution of the conformable fractional-order chaotic system is investigated for the first time. The conformable Adomian decomposition method (CADM) is constructed to find the numerical solution of conformable fractional-order differential equations. We apply CADM to the conformable fractional-order exponential function and conformable fractional-order Riccati equation. In these two cases, the series solution obtained can be written with respect to the exact solution. This indicates that CADM is an effective method for the numerical solution of conformable fractional-order differential equations. The dynamics of conformable fractional-order simplified Lorenz system is investigated. Different states, including convergent state, chaotic attractor and periodic circles, are observed, which means that the conformable fractional-order simplified Lorenz system has rich dynamics. The minimum order for chaos is 1.7223 for $c = 2$ and $h = 0.01$. This lays the foundation for the extensive applications research of conformable fractional-order chaotic systems in the future.

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