Nearly tight weighted 2-designs in complex projective spaces of every dimension

John Jasper
Department of Mathematics & Statistics
Air Force Institute of Technology
Wright-Patterson AFB, Ohio, USA

Dustin G. Mixon
Department of Mathematics
The Ohio State University
Columbus, Ohio, USA

***Abstract*—We use dense Sidon sets to construct small weighted projective 2-designs. This represents quantitative progress on Zauner's conjecture.**

I. Introduction

In his PhD thesis [19], Zauner conjectured that for every $d \in \mathbb{N}$, there exists an arrangement of d^2 distinct points in $\mathbb{CP}^{d-1}$ with the property that every pair of points has the same Fubini–Study distance. Such arrangements are known as *symmetric informationally complete positive operator–valued measures* (or SICs) in quantum information theory, where they find use in quantum state tomography.

Over the last decade, there have been three main approaches to make progress on Zauner's conjecture. First, computational investigations have produced numerical approximations of putative SICs (as well as some exact SICs) in finitely many dimensions [18]. Further analysis in [3] then established that the coordinates of each of these exact SICs reside in an abelian extention of a particular real quadratic extension of $\mathbb{Q}$. This discovery prompted a second approach to Zauner's conjecture, which leverages conjectures in algebraic number theory (such as those due to Stark) to find additional exact SICs and even a conditional proof of Zauner's conjecture [14], [1], [5], [2].

As a third approach, [15] reformulated Zauner's conjecture in terms of the entanglement breaking rank $n(d)$ of a certain quantum channel over $\mathbb{C}^{d\times d}$.

Proposition 1. *For each $d \in \mathbb{N}$, $n(d) \geq d^2$, with equality precisely when there exists a SIC in $\mathbb{CP}^{d-1}$.*

Importantly, this allows for quantitative progress on Zauner's conjecture: For each $d \in \mathbb{N}$, find an upper bound on $n(d)$. Then the closer this upper bound is to d^2, the "closer" we are to a proof of Zauner's conjecture in dimension d. Soon after the release of [15], a follow-up paper [13] showed that $n(d)$ equals the size of the smallest *weighted* 2*-design* for $\mathbb{CP}^{d-1}$, thereby identifying Proposition 1 above with Theorem 4 in [17].

Definition 2. The unit vectors $x_1, \ldots, x_n \in \mathbb{C}^d$ are said to form a **weighted 2-design for** $\mathbb{CP}^{d-1}$ if there exist weights $w_1, \ldots, w_n \geq 0$ such that the weighted combination

$$\sum_{k=1}^{n} w_k (x_k^{\otimes 2})(x_k^{\otimes 2})^*$$

equals the orthogonal projection onto the subspace of symmetric tensors in $(\mathbb{C}^d)^{\otimes 2}$.

The following summarizes all of the best known upper bounds on $n(d)$:

Proposition 3.

(a) *$n(d) = d^2$ whenever a SIC in $\mathbb{CP}^{d-1}$ is known to exist.*
(b) *$n(d) \leq kd^2 + 2d$ whenever $kd + 1$ is a prime power with $k \in \mathbb{N}$.*
(c) *$n(d) \leq d^2 + 1$ whenever $d - 1$ is a prime power.*
(d) *$n(d) \leq d^2 + d - 1$ whenever d is a prime power.*
(e) *$n(d) \leq \binom{d+1}{2}^2$.*

Proof. First, (a) follows from the fact that SICs are weighted projective 2-designs of minimal size. Next, (b) follows from combining Theorem 4.1 with Proposition 4.2 in [16]. Also, (c) and (d) follow from Corollaries 4.4 and 4.6 in [6]. Finally, (e) follows from Corollary 7 in [13]. □

The weighted projective 2-designs that imply Proposition 3(c) and (d) are instances of the same *Bodmann–Haas construction* [6]. In this paper, we identify a much larger class of weighted projective 2-designs that arise from this construction, which in turn implies our main result:

Theorem 4. $n(d) \leq d^2 + O(d^{1.525})$.

This is the first known general upper bound on $n(d)$ that is $o(d^4)$, let alone sharp up to lower-order terms. In the next section, we present the Bodmann–Haas construction and show that applying it to *Sidon sets* results in weighted projective 2-designs. Next, Section 3 reviews the densest known Sidon sets and uses them to construct weighted projective 2-designs that are *nearly tight* (meaning their size is close to the lower bound d^2). We conclude in Section 4 with a discussion.

(After posting the original version of this paper on the arXiv, we were alerted that our results can also be derived from results in the recent paper [12].)

II. The Bodmann–Haas construction

In what follows, we describe a construction technique due to Bodmann and Haas [6] that was originally obtained by generalizing a particular construction of mutually unbiased bases due to Godsil and Roy [9].

Definition 5. Fix a finite abelian group G. The **Bodmann–Haas construction** is a map that receives a subset $S \subseteq G$ and returns a sequence of $|G| + |S|$ unit vectors in $\mathbb{C}^S$. In particular, each character $\alpha \in \hat{G}$ determines the vector $x_\alpha \in \mathbb{C}^S$ defined by

$$x_\alpha(s) = \frac{1}{\sqrt{|S|}}\alpha(s) \qquad (s \in S),$$

while each $r \in S$ determines $e_r \in \mathbb{C}^S$ defined in terms of the Kronecker delta by

$$e_r(s) = \delta_{r,s} \qquad (s \in S).$$

Overall, given $S \subseteq G$, the Bodmann–Haas construction returns $\{x_\alpha\}_{\alpha\in\hat{G}} \cup \{e_r\}_{r\in S}$.

Bodmann and Haas [6] used this to construct projective codes that achieve equality in the *orthoplex bound*, and then seemingly as an afterthought, they verified that these codes are weighted projective 2-designs. In this section, we find many more instances in which the Bodmann–Haas construction returns a weighted projective 2-design.

Definition 6. A **Sidon set** is a subset S of a finite abelian group G with the property that the map $\{a, b\} \mapsto a + b$ is injective over $a, b \in S$ (allowing $a = b$).

Theorem 7. *Fix a finite abelian group G. Then for any Sidon set $S \subseteq G$, the Bodmann–Haas construction applied to S returns a weighted 2-design for $\mathbb{CP}^{|S|-1}$.*

Proof. Consider the transpose map T over $(\mathbb{C}^S)^{\otimes 2}$ defined by taking $T(x \otimes y) = y \otimes x$ and extending linearly. The orthogonal projection onto the subspace of symmetric tensors is then given by $P = \frac{1}{2}(I+T)$, where I denotes the identity map. Identifying P with its matrix representation relative to the basis $\{e_s \otimes e_{s'}\}_{s,s'\in S}$, then the matrix entries are given by

$$\begin{aligned}P_{(s,s'),(t,t')} &= (e_s \otimes e_{s'})^* \frac{1}{2}(I+T)(e_t \otimes e_{t'}) \\ &= \frac{1}{2}(e_s^* \otimes e_{s'}^*)(e_t \otimes e_{t'} + e_{t'} \otimes e_t) \\ &= \frac{1}{2}(\delta_{s,t}\delta_{s',t'} + \delta_{s,t'}\delta_{s',t}).\end{aligned}$$

This leads us to consider the matrices A and B defined by

$$A_{(s,s'),(t,t')} = \begin{cases} 1 & s \neq s',\, t \neq t',\, \{s,s'\} = \{t,t'\}, \\ 0 & \text{else}, \end{cases}$$

$$B_{(s,s'),(t,t')} = \begin{cases} 1 & s = s',\, t = t',\, \{s,s'\} = \{t,t'\}, \\ 0 & \text{else}, \end{cases}$$

since then $P = \frac{1}{2}A + B$.

Given a Sidon set $S \subseteq G$, the Bodmann–Haas construction returns the sequence $\{x_\alpha\}_{\alpha\in\hat{G}} \cup \{e_r\}_{r\in S}$. Consider the matrices

$$X := \sum_{\alpha\in\hat{G}} (x_\alpha^{\otimes 2})(x_\alpha^{\otimes 2})^*, \quad E := \sum_{r\in S} (e_r^{\otimes 2})(e_r^{\otimes 2})^*.$$

We will express P as a nonnegative combination of X and E, thereby demonstrating that $\{x_\alpha\}_{\alpha\in\hat{G}} \cup \{e_r\}_{r\in S}$ is a weighted projective 2-design. We proceed by computing matrix entries:

$$\begin{aligned}X_{(s,s'),(t,t')} &= (e_s \otimes e_{s'})^* X (e_t \otimes e_{t'}) \\ &= \sum_{\alpha\in\hat{G}} (e_s \otimes e_{s'})^* (x_\alpha^{\otimes 2})(x_\alpha^{\otimes 2})^* (e_t \otimes e_{t'}) \\ &= \sum_{\alpha\in\hat{G}} (e_s^* x_\alpha)(e_{s'}^* x_\alpha)(x_\alpha^* e_t)(x_\alpha^* e_{t'}) \\ &= \frac{1}{|S|^2} \sum_{\alpha\in\hat{G}} \alpha(s)\alpha(s')\overline{\alpha(t)\alpha(t')} \\ &= \frac{1}{|S|^2} \sum_{\alpha\in\hat{G}} \alpha(s + s' - t - t') \\ &= \begin{cases} \frac{|G|}{|S|^2} & s + s' = t + t', \\ 0 & \text{else}. \end{cases}\end{aligned}$$

In particular, since S is Sidon by assumption, we have $X = \frac{|G|}{|S|^2}(A + B)$. Next,

$$\begin{aligned}E_{(s,s'),(t,t')} &= (e_s \otimes e_{s'})^* E (e_t \otimes e_{t'}) \\ &= \sum_{r\in S} (e_s \otimes e_{s'})^* (e_r^{\otimes 2})(e_r^{\otimes 2})^* (e_t \otimes e_{t'}) \\ &= \sum_{r\in S} (e_s^* e_r)(e_{s'}^* e_r)(e_r^* e_t)(e_r^* e_{t'}) \\ &= \sum_{r\in S} \delta_{s,r}\delta_{s',r}\delta_{r,t}\delta_{r,t'} \\ &= \begin{cases} 1 & s = s' = t = t', \\ 0 & \text{else}, \end{cases}\end{aligned}$$

and so $E = B$. Overall, we have

$$P = \frac{1}{2}A + B = \frac{|S|^2}{2|G|}X + \frac{1}{2}E,$$

and so $\{x_\alpha\}_{\alpha\in\hat{G}} \cup \{e_r\}_{r\in S}$ is a weighted projective 2-design with $w_\alpha = \frac{|S|^2}{2|G|}$ for $\alpha \in \hat{G}$ and $w_r = \frac{1}{2}$ for $r \in S$. $\square$

III. Nearly tight weighted projective 2-designs

Recall that we seek a sharp upper bound on $n(d)$. Letting $m(d)$ denote the size of the smallest group that contains a Sidon set of size d, then the following bound is an immediate consequence of Theorem 7:

Corollary 8. $n(d) \le m(d) + d$.

As we will soon see, Corollary 8 is the sharpest known upper bound on $n(d)$ for all but finitely many $d \in \mathbb{N}$. To obtain a tight upper bound on $m(d)$, we seek *dense* Sidon sets, i.e., large Sidon sets relative to their parent groups. Note that the equivalence

$$a + b = c + d \qquad \Longleftrightarrow \qquad a - d = c - b$$

implies that $S \subseteq G$ is a Sidon set precisely when all pairwise differences between distinct members of S are different. The pigeonhole principle then gives the necessary condition

$$|S|\big(|S| - 1\big) \le |G| - 1, \qquad (1)$$

i.e., $|S| \le (1+o(1))\sqrt{|G|}$. Meanwhile, every known infinite family of Sidon sets that satisfies $|S| \ge (1-o(1))\sqrt{|G|}$ stems from the following constructions; see [7] and references therein:

Proposition 9. *In what follows,*[1] *q is a prime power and S is a Sidon set in G.*

(a) ***Erdős–Turán.*** (q^2, q). $G = (\mathbb{F}_q, +)^2$, $S = \{(x, x^2) : x \in \mathbb{F}_q\}$, $\mathrm{char}(q) > 2$.
(b) ***Singer.*** $(q^2+q+1, q+1)$. $G = \mathbb{F}_{q^3}^\times / \mathbb{F}_q^\times$, $S = \{[x] : x \in \mathbb{F}_{q^3}^\times, \mathrm{tr}\, x = 0\}$.
(c) ***Bose.*** (q^2-1, q). $G = \mathbb{F}_{q^2}^\times$, $S = \{x \in \mathbb{F}_{q^2}^\times : \mathrm{tr}\, x = 0\}$.
(d) ***Spence.*** $(q(q-1), q-1)$. $G = \mathbb{F}_q^\times \times (\mathbb{F}_q, +)$, $S = \{(x, x) : x \in \mathbb{F}_q^\times\}$.
(e) ***Hughes.*** $((q-1)^2, q-2)$. $G = (\mathbb{F}_q^\times)^2$, $S = \{(x, y) : x, y \ne 0, x + y = 1\}$.

Some of these Sidon sets have already met the Bodmann–Haas construction. First, Godsil and Roy [9] used a relative difference set isomorphic to the Erdös–Turán Sidon set to construct mutually unbiased bases. Later, Bodmann and Haas [6] used the Singer Sidon set as well as a relative difference set isomorphic to the Bose Sidon set to construct projective codes and designs. We are not aware of the Spence or Hughes Sidon sets appearing previously in the literature on codes and designs.

Since subsets of Sidon sets are also Sidon sets, we may bound $m(d)$ by the size of the smallest group described in Proposition 9 whose Sidon set has size at least d. This allows us to prove our main result:

[1]For convenience, we report the parameters $(|G|, |S|)$ of each construction in terms of q.

Proof of Theorem 4. It suffices to demonstrate $m(d) \le d^2 + O(d^{1.525})$, since then the result follows from Corollary 8. Considering Proposition 9(a), it holds that $m(d)$ is at most $p(d)^2$, where $p(d)$ denotes the smallest prime $\ge d$. Finally, the main result in [4] gives $p(d) \le d + O(d^{0.525})$, and so we are done. □

We conclude this section by discussing Table I, which compares our bound based on Corollary 8 and Proposition 9 to the previous bounds in Proposition 3. Implementing Proposition 3(a) is cumbersome since the known SICs are not maintained in a public database. We first collected the dimensions listed in [8], which represents a complete survey of known dimensions as of September 2017. Then we included the dimensions 23 (due to [14]), 52 (due to [5]), and 67 (due to [1]). As far as we know, this gives all dimensions ≤ 100 for which an exact SIC has been published. We do not include any of the unpublished exact SICs that were announced in [14], [10], [11].

In Table I, the d^2 column gives the best known lower bound on $n(d)$, while the next two columns give competing upper bounds on $n(d)$. We highlight the better of the two upper bounds in yellow, and when this matches the lower bound, we also highlight the d^2 column. At times, the Sidon set we use is obtained by removing any k points from the Hughes Sidon set, which we denote by $\mathrm{H}(q) - k$.

We make a few observations from Table I. For every $d \le 100$, exactly one of three things happens: either a SIC exists, or the upper bounds tie, or our upper bound is strictly better by using (a subset of) the Hughes Sidon set. In this regime, none of the bounds from Proposition 3 make use of part (e). Similarly, $m(d)$ is never achieved by the Erdős-Turán Sidon set since there is always Bose Sidon set of the same size but in a smaller group. When the upper bounds tie, it's frequently because our use of the Singer and Bose Sidon sets align with Proposition 3(c) and (d), respectively, as these correspond to the original application of the Bodmann–Haas construction. However, the upper bounds also tie between Proposition 3(b) with $k = 1$ and our use of Spence Sidon sets. In hindsight, these ties are made possible because the parameters match, but we don't know of a deeper relationship between these weighted projective 2-designs. One of the main takeaways from Table I is that the improvements we provide are due to (subsets of) the Hughes Sidon sets.

IV. Discussion

In this paper, we used Sidon sets to construct weighted projective 2-designs, which in turn represents quantitative progress towards Zauner's conjecture. In this section, we highlight some fundamental limits of our approach.

Table I
BEST KNOWN BOUNDS ON $n(d)$

d	d^2	Prior bound on $n(d)$	Best known bound on $m(d)+d$	Sidon set
1	1	1	2	{0}
2	4	4	5	B(2)
3	9	9	10	Si(2)
4	16	16	17	Si(3)
5	25	25	26	Si(4)
6	36	36	37	Si(5)
7	49	49	55	B(7)
8	64	64	65	Si(7)
9	81	81	82	Si(8)
10	100	100	101	Si(9)
11	121	121	131	B(11)
12	144	144	145	Si(11)
13	169	169	181	B(13)
14	196	196	197	Si(13)
15	225	225	255	Sp(16)
16	256	256	271	B(16)
17	289	289	290	Si(16)
18	324	324	325	Si(17)
19	361	361	379	B(19)
20	400	400	401	Si(19)
21	441	441	505	H(23)
22	484	528	528	Sp(23)
23	529	529	551	B(23)
24	576	576	577	Si(23)
25	625	649	649	B(25)
26	676	677	677	Si(25)
27	729	755	755	B(27)
28	784	784	785	Si(27)
29	841	869	869	B(29)
30	900	900	901	Si(29)
31	961	961	991	B(31)
32	1024	1025	1025	Si(31)
33	1089	1090	1090	Si(32)
34	1156	3536	1330	H(37) − 1
35	1225	1225	1331	H(37)
36	1296	1368	1368	Sp(37)
37	1369	1369	1405	B(37)
38	1444	1445	1445	Si(37)
39	1521	1521	1639	H(41)
40	1600	1680	1680	Sp(41)
41	1681	1721	1721	B(41)
42	1764	1765	1765	Si(41)
43	1849	1849	1891	B(43)
44	1936	1937	1937	Si(43)
45	2025	8190	2161	H(47)
46	2116	2208	2208	Sp(47)
47	2209	2255	2255	B(47)
48	2304	2304	2305	Si(47)
49	2401	2449	2449	B(49)
50	2500	2501	2501	Si(49)

d	d^2	Prior bound on $n(d)$	Best known bound on $m(d)+d$	Sidon set
51	2601	5304	2755	H(53)
52	2704	2704	2808	Sp(53)
53	2809	2861	2861	B(53)
54	2916	2917	2917	Si(53)
55	3025	18260	3419	H(59) − 2
56	3136	6384	3420	H(59) − 1
57	3249	13110	3421	H(59)
58	3364	3480	3480	Sp(59)
59	3481	3539	3539	B(59)
60	3600	3601	3601	Si(59)
61	3721	3781	3781	B(61)
62	3844	3845	3845	Si(61)
63	3969	4095	4095	Sp(64)
64	4096	4159	4159	B(64)
65	4225	4226	4226	Si(64)
66	4356	4488	4488	Sp(67)
67	4489	4489	4555	B(67)
68	4624	4625	4625	Si(67)
69	4761	9660	4969	H(71)
70	4900	5040	5040	Sp(71)
71	5041	5111	5111	B(71)
72	5184	5185	5185	Si(71)
73	5329	5401	5401	B(73)
74	5476	5477	5477	Si(73)
75	5625	11400	6159	H(79) − 2
76	5776	17480	6160	H(79) − 1
77	5929	35728	6161	H(79)
78	6084	6240	6240	Sp(79)
79	6241	6319	6319	B(79)
80	6400	6401	6401	Si(79)
81	6561	6641	6641	B(81)
82	6724	6725	6725	Si(81)
83	6889	6971	6971	B(83)
84	7056	7057	7057	Si(83)
85	7225	21845	7829	H(89) − 2
86	7396	14964	7830	H(89) − 1
87	7569	30450	7831	H(89)
88	7744	7920	7920	Sp(89)
89	7921	8009	8009	B(89)
90	8100	8101	8101	Si(89)
91	8281	49868	9307	H(97) − 4
92	8464	25576	9308	H(97) − 3
93	8649	34782	9309	H(97) − 2
94	8836	26696	9310	H(97) − 1
95	9025	18240	9311	H(97)
96	9216	9408	9408	Sp(97)
97	9409	9505	9505	B(97)
98	9604	9605	9605	Si(97)
99	9801	19800	10099	H(101)
100	10000	10200	10200	Sp(101)

Note that the necessary condition (1) gives that any group containing a Sidon set of size d must have cardinality at least $d^2 - d + 1$. The weighted projective 2-design resulting from the Bodmann–Haas construction then has size at least d^2+1. As such, our approach is not powerful enough to establish Zauner's conjecture that $n(d) = d^2$. We suspect that the easiest way to improve the bounds on $n(d)$ in Table I is to find more exact SICs, of which several have been announced in [14], [10], [11].

Short of a proof of Zauner's conjecture, it would be interesting to improve Theorem 4. To this end, the Bodmann–Haas construction is somewhat limiting, since we believe $m(d)$ is at least nearly achieved by subsets of the Sidon sets in Proposition 9. As such, any improvement must come from better bounds on gaps between primes. Heuristics suggest that the best possible bound is given by Cramér's conjecture, and so an estimate of the form $n(d) \leq d^2 + o(d \log^2 d)$ would likely require a different approach.

ACKNOWLEDGMENTS

The authors thank Emily King for a helpful discussion of relevant literature. JJ was supported by NSF DMS 2220320. DGM was supported by NSF DMS 2220304. The views expressed are those of the authors and do not reflect the official guidance or position of the United States Government, the Department of Defense, the United States Air Force, or the United States Space Force.

REFERENCES

[1] M. Appleby, I. Bengtsson, M. Grassl, M. Harrison, G. McConnell, SIC-POVMs from Stark units: Prime dimensions $n^2 + 3$, J. Math. Phys. 63 (2022) 112205.

[2] M. Appleby, S. T. Flammia, G. S. Kopp, A Constructive Approach to Zauner's Conjecture via the Stark Conjectures, arXiv:2501.03970 (2025).

[3] M. Appleby, S. Flammia, G. McConnell, J. Yard, SICs and algebraic number theory, Found. Phys. (2017) 1–18.

[4] R. C. Baker, G. Harman, J. Pintz, The difference between consecutive primes, II, Proc. London Math. Soc. 83 (2001) 532–562.

[5] I. Bengtsson, M. Grassl, G. McConnell, SIC-POVMs from Stark units: Dimensions $n^2 + 3 = 4p$, p prime, arXiv:2403.02872 (2024).

[6] B. G. Bodmann, J. Haas, Achieving the orthoplex bound and constructing weighted complex projective 2-designs with Singer sets, Linear Algebra Appl. 511 (2016) 54–71.

[7] S. Eberhard, F. Manners, The Apparent Structure of Dense Sidon Sets, Electronic J. Combin. 30 (2023).

[8] S. Flammia, Exact SIC fiducial vectors, physics.usyd.edu.au/~sflammia/SIC/.

[9] C. Godsil, A. Roy, Equiangular lines, mutually unbiased bases, and spin models, European J. Combin. 30 (2009) 246–262.

[10] M. Grassl, Computing numerical and exact SIC-POVMs, Chaos and Quantum Chaos Seminar, Jagiellonian University, March 29, 2021, youtube.com/watch?v=CGNxSRcqWts.

[11] M. Grassl, Computing SIC-POVMs using permutation symmetries and Stark units, Codes and Expansions Seminar, October 26, 2021, youtube.com/watch?v=2vzS55SjaZI.

[12] J. T. Iosue, T. C. Mooney, A. Ehrenberg, A. V. Gorshkov, Projective toric designs, quantum state designs, and mutually unbiased bases, Quantum 8 (2024) 1546.

[13] J. W. Iverson, E. J. King, D. G. Mixon, A note on tight projective 2-designs, J. Combin. Designs 29 (2021) 809–832.

[14] G. S. Kopp, SIC-POVMs and the Stark conjectures, Int. Math. Res. Not. IMRN 18 (2021) 13812–13838.

[15] S. K. Pandey, V. I. Paulsen, J. Prakash, M. Rahaman, Entanglement breaking rank and the existence of SIC POVMs, J. Math. Phys. 61 (2020) 042203.

[16] A. Roy, A. J. Scott, Weighted complex projective 2-designs from bases: Optimal state determination by orthogonal measurements, J. Math. Phys. 48 (2007) 072110.

[17] A. J. Scott, Tight informationally complete quantum measurements, J. Phys. A 39 (2006) 13507.

[18] A. J. Scott, M. Grassl, Symmetric informationally complete positive-operator-valued measures: A new computer study, J. Math. Phys. 51 (2010) 042203.

[19] G. Zauner, Quantum designs, PhD thesis, U. Vienna, 1999.