RLHF from Heterogeneous Feedback via Personalization and Preference Aggregation

Chanwoo Park
MIT
cpark97@mit.edu

Mingyang Liu
MIT
liumy19@mit.edu

Dingwen Kong
MIT
dingwenk@mit.edu

Kaiqing Zhang
University of Maryland, College Park
kaiqing@umd.edu

Asuman Ozdaglar
MIT
asuman@mit.edu

Abstract

Reinforcement learning from human feedback (RLHF) has been an effective technique for aligning AI systems with human values, with remarkable successes in fine-tuning large-language models recently. Most existing RLHF paradigms make the underlying assumption that human preferences are relatively *homogeneous*, and can be encoded by a single reward model. In this paper, we focus on addressing the issues due to the inherent *heterogeneity* in human preferences, as well as their potential *strategic* behavior in providing feedback. Specifically, we propose two frameworks to address heterogeneous human feedback in principled ways: personalization-based one and preference-aggregation-based one. For the former, we propose two approaches based on representation learning and clustering, respectively, for learning *multiple* reward models that trade-off the bias (due to preference heterogeneity) and variance (due to the use of fewer data for learning each model by personalization). We then establish sample complexity guarantees for both approaches. For the latter, we aim to adhere to the single-model framework, as already deployed in the current RLHF paradigm, by carefully *aggregating* diverse and truthful preferences from humans. We propose two approaches based on reward and preference aggregation, respectively: the former utilizes social choice theory to aggregate individual reward models, with sample complexity guarantees; the latter directly aggregates the human feedback in the form of probabilistic opinions. Under the probabilistic-opinion-feedback model, we also develop an approach to handle strategic human labelers who may bias and manipulate the aggregated preferences with untruthful feedback. Based on the ideas in mechanism design, our approach ensures truthful preference reporting, with the induced aggregation rule maximizing social welfare functions.

1 Introduction

As AI models are becoming more powerful, there is greater emphasis on aligning their performance and priorities with the preferences of human users. In this context, reinforcement learning from human feedback (RLHF) has emerged as a promising approach, because it combines pre-trained large language models with direct human feedback (Ziegler et al., 2019; Ouyang et al., 2022; Bai et al., 2022). RLHF utilizes human feedback in the form of *preferences* over multiple responses in order to fine-tune the output of a pre-trained model, for example, by encouraging certain responses or types of output. The finetuning can be done by either learning a user reward model over user preference data, or by using the preference data directly (through direct preference optimization (Rafailov et al., 2024)). In either case, accurately approximating user preferences is an important task, which becomes way more challenging when the target group of users is heterogeneous (Figure 1) (Pollak and Wales, 1992; Boxall and Adamowicz, 2002).

Workshop on Aligning Reinforcement Learning Experimentalists and Theorists (ARLET 2024).

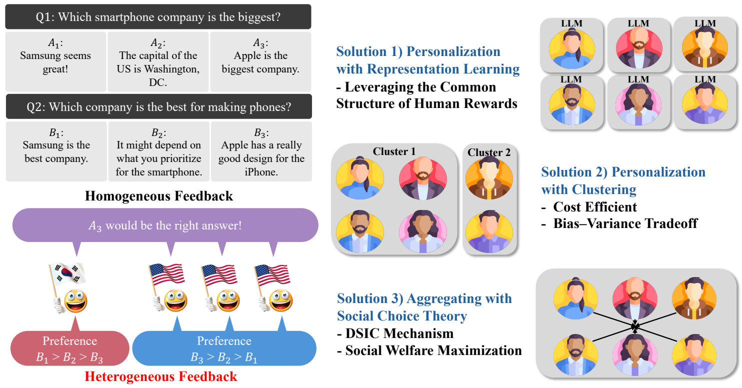

Figure 1: We demonstrate a setting where humans might have heterogeneous feedback. We provide a personalization-based framework and a human preference aggregation-based framework.

This paper contributes to this literature by providing a holistic study of learning (different) reward models from heterogeneous user preference data. There are two major challenges in this context. The first (**C1**) is a **pure learning one:** preference data from each individual might not be sufficiently rich to construct an accurate model of heterogeneous users. The second (**C2**) is after learning different reward models for heterogeneous users, **how to aggregate** them carefully to learn a single model. Moreover, with humans (who are oftentimes viewed as rational decision-makers) involved in the loop, they might *strategically misreport* their preferences to manipulate this aggregated model. For example, in online rating systems, users may provide extreme feedback to disproportionately influence the overall ratings toward their viewpoint. Our approach develops ways of tackling these challenges.

To address (**C1**), we adopt two approaches based on representation learning, which assume that individual reward functions share a structure through a common representation. We model each reward function as the inner product of a common representation and a parameter vector. Given the lack of sufficient individual feedback, having a shared structure by representation helps articulate each user's reward model. The first approach **constructs a personalized reward model** for each user. In this approach, we find a common representation and learn each individual's parameter vector by pooling every individual feedback. The second approach **segments user preferences into clusters** and learns a reward model for each cluster. This approach is useful when individual reward functions might not be available due to insufficient data. By assuming "diversity of user's parameter vectors", which means that individual parameter vectors span the entire space of parameters (a common assumption in multi-task learning), we show that this approach enables better sample complexity results. Leveraging data from all users helps learn the common representation, as the diversity assumption guarantees sufficient information about every dimension of the representation.

To address (**C2**), we first estimate the parameters for each individual's reward model using the individual's preference comparison data. Then, we aggregate reward models using a family of reward aggregation rules, which follows six pivotal axioms from social choice theory. We then provide sample complexities of the policy induced from the single aggregated reward model. We additionally provide a model with a different feedback type - probabilistic opinion. Concretely, instead of choosing a single answer from a pool of candidate answers, we allow the human labeler to choose a probability distribution over the answers, which indicates how much the labeler likes those answers. This type of feedback can arguably express the labeler's preference more accurately. Moreover, probabilistic opinion feedback does not require the relationship between the human reward model and preference. We consider various aggregation rules to aggregate their probabilistic opinion vectors into one. We showed that our suggested probabilistic opinion aggregation rule is equivalent to reward aggregation rules following six pivotal axioms, under the Plackett-Luce model (Plackett, 1975; Luce, 2005).

To deal with the *strategic misreport* problem, we adopt a mechanism design approach whereby users correctly reporting their preferences is incentivized. We model each human labeler's utility as a quasi-linear function, considering both the distance between her probabilistic opinion vector and the aggregated opinion vector, and the associated costs. Under this model, we show that our proposed aggregation rule maximizes *social welfare*. Lastly, we design an incentive-compatible mechanism to guarantee truthful reporting by inducing proper cost in the human feedback collection process.

1.1 Related Work

We defer a detailed related work and comparison with recent work to Appendix E.

Notation. The matrix $\mathbf{O}$ denotes an all-zero matrix, while I stands for an identity matrix, of proper dimensions. We use $A \succ \mathbf{O}$ to denote that matrix A is a positive definite matrix. The function σ represents the Sigmoid function, defined by $\sigma(x) = 1/(1 + \exp(-x))$. The notation $[K]$ denotes the set $\{1, 2, \ldots, K\}$. $\Delta(\mathcal{A})$ refers to a probability vector in $\mathbb{R}^{|\mathcal{A}|}$. The term $\sigma_k^2(A)$ denotes the k-th largest singular value of matrix A. $\|x\|_2$ refers to the ℓ_2-norm. We also define $\|x\|_\Sigma = \sqrt{x^\intercal \Sigma x}$ for a positive definite matrix Σ. For a matrix M, the norm $\|M\|_F$ denotes the Frobenius norm of M.

2 Preliminaries

Most existing RLHF processes (for language model fine-tuning) consist of two main stages: (1) learning a model of human rewards (oftentimes from preference data), and (2) fine-tuning with the reference policy through Reinforcement Learning algorithms, e.g., Proximal Policy Optimization (PPO) (Schulman et al., 2017). It may also be possible to avoid the explicit learning of reward functions while fine-tuning the policy directly from preference data (Rafailov et al., 2024).

Markov Decision Processes. We define the state s as an element of the set of possible prompts or questions, denoted by $\mathcal{S}$, and the set of actions a, contained in $\mathcal{A}$, as the potential answers or responses to these questions. Consider an RLHF setting with N human labelers (or users), each of whom has their own reward function. This setting can be characterized by a Markov Decision Process (MDP) with N reward functions, represented by the tuple $M = (\mathcal{S}, \mathcal{A}, H, (P_h)_{h\in[H]}, \boldsymbol{r} = (r_i)_{i\in[N]})$, where H denotes the length of the horizon, $P_h : \mathcal{S} \times \mathcal{A} \mapsto \Delta(\mathcal{S})$ is the state transition probability at step $h \in [H]$, $\mathcal{T} := (\mathcal{S} \times \mathcal{A})^H$ denotes the set of all possible trajectories, and $r_i : \mathcal{T} \to \mathbb{R}$ is the reward function for individual i and trajectory $\tau \in \mathcal{T}$, representing the utility of human user i from a sequence of responses to a given prompt. We assume $-R_{\max} \le r_i(\tau) \le R_{\max}$ for every $\tau \in \mathcal{T}$ and $i \in [N]$, for some $R_{\max} > 0$. This reward model also covers the case that $r_i(\tau) = \sum_{h\in[H]} r_{h,i}(s_h, a_h)$, where $r_{h,i} : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$ denotes the state-action reward function for each step h and individual i, and $\tau = (s_1, a_1, s_2, a_2, \ldots, s_H, a_H)$. The MDP concludes at an absorbing termination state with zero reward after H steps. A policy $\pi_h : (\mathcal{S} \times \mathcal{A})^{h-1} \times \mathcal{S} \to \Delta(\mathcal{A})$ is defined as a function mapping trajectories to distributions over actions for each step $h \in [H]$ within the horizon H. We define the history-dependent policy class as Π. The collection of these policies across all steps is denoted by $\pi = (\pi_h)_{h=1}^{H-1}$. The expected cumulative reward of a policy π is given by $J(\pi; r_i) := \mathbb{E}_{\tau,\pi}[r_i(\tau)]$ where the expectation in the formula is taken over the distribution of the trajectories under the policy π. Trajectory occupancy measures, denoted by $d_\pi : \mathcal{T} \to [0, 1]$, are defined as $d_\pi(\tau) := \mathbb{P}_\pi(\tau)$, which denotes the probability of generating trajectory τ following policy π.

Relationship between Preference and Reward Function. For the MDP with $M = (\mathcal{S}, \mathcal{A}, H, (P_h)_{h\in[H]}, \boldsymbol{r} = (r_i)_{i\in[N]})$, if we compare two trajectories τ_0 and τ_1, we define some random variable o such that $o = 0$ if $\tau_0 \succ \tau_1$, and $o = 1$ if $\tau_0 \prec \tau_1$. Here, $\tau_0 \succ \tau_1$ indicates that τ_0 is preferred than τ_1. We assume that $P_{r_i}(o = 0 \mid \tau_0, \tau_1) = \Phi(r_i(\tau_0) - r_i(\tau_1))$ for all $i \in [N]$, where $\Phi : \mathbb{R} \to [0, 1]$ is a monotonically increasing function, which satisfy $\Phi(x) + \Phi(-x) = 1$ and $\log \Phi(x)$ is a strongly convex function. For example, $\Phi(x) = \sigma(x)$ indicates the BTL model (Definition 4.1 below), a frequently used model for the relationship between preference and reward. Also, we define $P_{\boldsymbol{r}}(\cdot \mid \tau_0, \tau_1) := (P_{r_1}(\cdot \mid \tau_0, \tau_1)^\intercal, \ldots, P_{r_N}(\cdot \mid \tau_0, \tau_1)^\intercal)^\intercal$. We call $P_{\boldsymbol{r}}$ and P_{r_i} a preference probability vector induced by the reward vector $\boldsymbol{r}$ and the reward r_i.

3 Provable Personalized RLHF via Representation Learning

3.1 Learning Personalized Reward Model

In this subsection, we provide the first approach in the personalization-based framework, based on representation learning.

Reward Function Class. We will assume that we have access to a pre-trained feature function $\phi : \mathcal{T} \to \mathbb{R}^d$, which encodes a trajectory of states and actions (i.e., questions and answers) to a d-dimensional feature vector. This covers the case where feature $\phi_h : \mathcal{S} \times \mathcal{A} \to \mathbb{R}^d$ is defined at each state-action pair, i.e., $\phi(\tau) := \sum_{h\in[H]} \phi_h(s_h, a_h)$ for trajectory $\tau = (s_1, a_1, \ldots, s_H, a_H)$. For example, it is common to use the penultimate layer of an existing pre-trained LLM or other pre-trained backbones to encode a long sentence to a feature vector (Donahue et al., 2014; Gulshan et al., 2016; Tang et al., 2016).

Our first goal is to learn multiple reward models for each human user using preference datasets. First, we define the reward function class as

$$\mathcal{G}_{\boldsymbol{r}} = \left\{ (\langle \psi_\omega(\phi(\cdot)), \theta_i \rangle)_{i\in[N]} \,\middle|\, \psi_\omega \in \Psi, \theta_i \in \mathbb{R}^k \text{ and } \|\theta_i\|_2 \le B \text{ for all } i \in [N] \right\},$$

for some $B > 0$, where Ψ is the set of representation functions parameterized by $\omega \in \Omega$, i.e., $\Psi = \{\psi_\omega \mid \omega \in \Omega\}$, where $\psi_\omega : \mathbb{R}^d \to \mathbb{R}^k$. We assume that $d \gg k$. We denote $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_N)$, and to emphasize the relationship between reward and $(\omega, \boldsymbol{\theta})$, we will write $r_{\omega,\theta_i}(\cdot) := \langle \psi_\omega(\phi(\cdot)), \theta_i \rangle$ for each individual $i \in [N]$ and $\boldsymbol{r}_{\omega,\boldsymbol{\theta}}(\cdot) := (r_{\omega,\theta_1}(\cdot), \cdots, r_{\omega,\theta_N}(\cdot))^\intercal \in \mathbb{R}^N$. From this section, we will write $\boldsymbol{r}^\star = (r_1^\star, \ldots, r_N^\star)$ as the underlying human reward functions.

Assumption 1 (Realizability)**.** *We assume that the underlying true reward can be represented as $r_i^\star(\cdot) = \langle \psi^\star(\phi(\cdot)), \theta_i^\star \rangle$ for some representation function $\psi^\star \in \Psi$ (in other words, there exists some $\omega^\star \in \Omega$ such that $\psi_{\omega^\star} = \psi^\star$) and $\|\theta_i^\star\|_2 \leq B$ for each individual $i \in [N]$.*

To emphasize $(\omega, \boldsymbol{\theta})$, we define shorthand notation $P_{\omega,\boldsymbol{\theta}} := P_{r_{\omega,\boldsymbol{\theta}}}$ as the preference probability induced by $r_{\omega,\boldsymbol{\theta}}$. We also write $P_{\omega,\theta} := P_{\langle \psi_\omega(\phi(\cdot)), \theta \rangle}$, which is the probability induced by $\langle \psi_\omega(\phi(\cdot)), \theta \rangle$.

3.1.1 Algorithms

We introduce our algorithm for learning personalized policy. Compared to traditional RLHF algorithms (Ziegler et al., 2019; Ouyang et al., 2022; Zhu et al., 2023), we consider personalized reward function by representation learning.

Algorithm 1 outputs a joint estimation of $\psi^\star$ and $\boldsymbol{\theta}^\star$ with maximum likelihood estimation (MLE), together with personalized policies. The input of the algorithm is $\widehat{\mathcal{D}} = \cup_{i \in [N]} \widehat{\mathcal{D}}_i$ where $\widehat{\mathcal{D}}_i = \{(o_i^{(j)}, \tau_{i,0}^{(j)}, \tau_{i,1}^{(j)})_{j \in [N_p]}\}$. Here, $\tau_{i,t}^{(j)}$ is sampled from the distribution μ_t for $t = 0, 1$, and $o_i^{(j)} \sim P_{r_i^\star}(\cdot | \tau_0^{(j)}, \tau_1^{(j)})$. First, we estimate the reward function of human users. After estimating the reward functions, we construct a confidence set for the reward function as follows: Confidence set (Equation (F.1)) with $\zeta' = C_8 \left(k \frac{\xi^2 \kappa^2 \log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/(NN_p))/\delta)}{\eta^2 N N_p} + \frac{\xi^2 (k + \log(N/\delta))}{\eta^2 N_p} + \lambda B^2 \right)$, where $C_8, \lambda > 0$ are constants, $\xi := \max_{x \in [-2R_{\max}, 2R_{\max}]} \left| \frac{\Phi'(x)}{\Phi(x)} \right|$, $\kappa := (\min_{x \in [-2R_{\max}, 2R_{\max}]} \Phi'(x))^{-1}$, and $\eta := \min_{x \in [-2R_{\max}, 2R_{\max}]} \left(\frac{\Phi'(x)^2 - \Phi''(x)\Phi(x)}{\Phi(x)^2} \right)$. In the case that $\Phi(x) = \sigma(x)$ (i.e. Φ is a Sigmoid), $\xi \leq 1$ and $\kappa = \eta = \frac{1}{2 + \exp(-2R_{\max}) + \exp(2R_{\max})}$. This confidence set will be related to Theorem 3.1. Lastly, we find the best policy based on the pessimistic expected value function. $\mu_{i,\text{ref}}$ in Algorithm 1 is a known reference trajectory distribution for individual $i \in [N]$, and it can be set as μ_1. We defer Algorithm 5 which addresses a scenario where a new human user, who was not a labeler before, aims to learn their own reward models.

3.1.2 Results and Analyses

For ease of analysis, we consider the case where the sizes of preference datasets for each individual $i \in \{0\} \cup [N]$ are identical, i.e., $\widehat{\mathcal{D}}_i = \{(o_i^{(j)}, \tau_{i,0}^{(j)}, \tau_{i,1}^{(j)})_{j \in [N_p]}\}$, satisfies $|\widehat{\mathcal{D}}_i| = N_p$ for all $i \in \{0\} \cup [N]$. The result in this section can also be extended to the case with $|\widehat{\mathcal{D}}_i| = N_{p,i}$ for each individual i. We defer all the proofs of this section to Appendix G.

Definition 3.1 (Concentrability Coefficient)**.** *The concentrability coefficient, w.r.t a reward vector class $\mathcal{G}_{\boldsymbol{r}}$, human user i, a target policy π_{tar} (which policy to compete with, which potentially can be the optimal policy $\pi_i^\star$ corresponding to $r_i^\star$), and a reference policy μ_{ref}, is defined as follows:* $C_{\boldsymbol{r}}(\mathcal{G}_{\boldsymbol{r}}, \pi_{tar}, \mu_{ref}, i) := \max \left\{ 0, \sup_{\boldsymbol{r} \in \mathcal{G}_{\boldsymbol{r}}} \frac{\mathbb{E}_{\tau_0 \sim \pi_{tar}, \tau_1 \sim \mu_{ref}} [r_i^\star(\tau_0) - r_i^\star(\tau_1) - r_i(\tau_0) + r_i(\tau_1)]}{\sqrt{\mathbb{E}_{\tau_0 \sim \mu_0, \tau_1 \sim \mu_1} \left[|r_i^\star(\tau_0) - r_i^\star(\tau_1) - r_i(\tau_0) + r_i(\tau_1)|^2 \right]}} \right\}$. *We also define the concentrability coefficient ($C_r(\mathcal{G}_r, \pi_{tar}, \mu_{ref})$) of the reward scalar class in Appendix D.*

(Zhan et al., 2023) provides an interpretation of concentrability coefficient. For example, if $\mu_{\text{ref}} = \mu_1$, the value of $C_{\boldsymbol{r}}(\mathcal{G}_{\boldsymbol{r}}, \pi_{\text{tar}}, \mu_1, i) \leq \sqrt{\max_{\tau \in \mathcal{T}} \frac{d_{\pi_{\text{tar}}}(\tau)}{\mu_0(\tau)}}$, so this reflects the concept of "single-policy concentrability" (Rashidinejad et al., 2021; Zanette et al., 2021; Ozdaglar et al., 2023), which is commonly assumed to be bounded in the offline RL literature.

We consider the case that $(\theta_i)_{i \in [N]}$ are diverse (Assumption 2), which is critical for improving the sample complexity of Algorithm 1 by outputting $(\widehat{\pi}_i')_{i \in [N]}$. We will additionally assume the uniqueness of the representation up to the orthonormal linear transformation (Assumption 3), and uniform concentration of covariance (Assumption 4). These assumptions are commonly used in multi-task learning (Du et al., 2021; Tripuraneni et al., 2021; Lu et al., 2021)

Assumption 2 (Diversity)**.** *The matrix $\Theta^\star = [\theta_1^\star, \cdots, \theta_N^\star] \in \mathbb{R}^{k \times N}$ satisfies $\sigma_k^2(\Theta^\star) \geq \Omega(N/k)$.*

Assumption 2 means that θ_i is evenly distributed in $\mathbb{R}^d$ space for $i \in [N]$, which indicates "diverse" human reward function.

Assumption 3 (Uniqueness of Representation (up to Orthonormal-Transformation))**.** *For any representation functions $\psi, \psi' \in \Psi$ and $\epsilon > 0$, if there exists $\{v_i\}_{i=1}^T, \{v_i'\}_{i=1}^T$, and a trajectory distribution μ that satisfy $\frac{1}{T}\sum_{i\in[T]} \mathbb{E}_{\tau\sim\mu}\|\psi(\phi(\tau))^\top v_i - \psi'(\phi(\tau))^\top v_i'\|^2 \leq \epsilon$, where $W = [v_1, v_2, \cdots, v_T] \in \mathbb{R}^{k\times T}$ satisfies $\sigma_k^2(W) \geq \Omega(T/k)$, and $\|v_i\|_2 \leq B$ for all $i \in [T]$. Then, there exists a constant orthonormal matrix P such that*

$$\|\psi(\phi(\tau)) - P\psi'(\phi(\tau))\|^2 \leq ck\epsilon/B$$

for all trajectory τ where $c > 0$ is a constant.

This assumption posits that if two representation functions, ψ and ψ', yield sufficiently small differences in expected squared norms of their inner products with corresponding vectors over trajectory distributions, then they are related by a constant orthonormal transformation. If $\psi_\omega(\phi(s,a)) := \omega\phi(s,a)$ where ω is $k \times d$ orthonormal matrix, we can prove that Assumption 3 holds with non-degenerate $\phi(s,a)$ distribution (Appendix G.4.2).

Definition 3.2. *Given distributions μ_0, μ_1 and two representation functions $\psi, \psi' \in \Psi$, define the covariance between ψ and ψ' with respect to μ_0, μ_1 to be*

$$\Sigma_{\psi,\psi'}(\mu_0,\mu_1) := \mathbb{E}_{\tau_0\sim\mu_0,\tau_1\sim\mu_1}[(\psi(\phi(\tau_0)) - \psi(\phi(\tau_1)))(\psi'(\phi(\tau_0)) - \psi'(\phi(\tau_1)))^\intercal] \in \mathbb{R}^{k\times k}.$$

Define the symmetric covariance as

$$\Lambda_{\psi,\psi'}(\mu_0,\mu_1) = \begin{bmatrix} \Sigma_{\psi,\psi}(\mu_0,\mu_1) & \Sigma_{\psi,\psi'}(\mu_0,\mu_1) \\ \Sigma_{\psi',\psi}(\mu_0,\mu_1) & \Sigma_{\psi,\psi'}(\mu_0,\mu_1) \end{bmatrix}.$$

We make the following assumption on the concentration property of the representation covariances.

Assumption 4. (Uniform Concentrability). *For any $\delta \in (0,1]$, there exists a number $N_{\textit{unif}}(\Psi, \mu_0, \mu_1, \delta)$ such that for any $n \geq N_{\textit{unif}}(\Psi, \mu_0, \mu_1, \delta)$, the empirical estimation $\widehat{\Lambda}_{\psi,\psi'}(\mu_0,\mu_1)$ of $\Lambda_{\psi,\psi'}(\mu_0,\mu_1)$ based on n independent trajectory sample pairs from distributions (μ_0,μ_1), with probability at least $1-\delta$, will satisfy the following inequality for all $\psi, \psi' \in \Psi$:*

$$1.1\Lambda_{\psi,\psi'}(\mu_0,\mu_1) \succeq \widehat{\Lambda}_{\psi,\psi'}(\mu_0,\mu_1) \succeq 0.9\Lambda_{\psi,\psi'}(\mu_0,\mu_1).$$

Assumption 4 means that the empirical estimate $\widehat{\Lambda}_{\psi,\psi'}(\mu_0,\mu_1)$ closely approximates the true $\Lambda_{\psi,\psi'}(\mu_0,\mu_1)$ with high probability. Similarly, if $\psi_\omega(\phi(\tau)) := \omega\phi(\tau)$, $N_{\text{point}}(\Psi,\mu_0,\mu_1,\delta) = \widetilde{\mathcal{O}}(d)$ (Du et al., 2021, Claim A.1). If distributions μ_0, μ_1 are clear from the context, we omit the notation μ_0, μ_1 for $\Sigma_{\psi,\psi'}(\mu_0,\mu_1)$ and $\Lambda_{\psi,\psi'}(\mu_0,\mu_1)$. Moreover, we also write $\Sigma_{\psi,\psi}$ as Σ_ψ for notational convenience.

We present the gap of the expected value function between the target policy $\pi_{i,\text{tar}}$ and the estimated policy $\widehat{\pi}_i$ for each individual $i \in [N]$. Here, $\pi_{i,\text{tar}}$, which may be the optimal policy $\pi_i^\star$ over $r_i^\star$, serves as the policy that $\widehat{\pi}_i$ will compare with.

Theorem 3.1. (Expected Value Function Gap). *Suppose Assumptions 1, 2, 3, and 4 hold. For any $\delta \in (0,1]$, all $i \in [N]$ and $\lambda > 0$, with probability at least $1-\delta$, the output $\widehat{\pi}_i'$ of Algorithm 1 satisfies*

$$\begin{aligned} &J(\pi_{i,\textit{tar}}; r_i^\star) - J(\widehat{\pi}_i'; r_i^\star) \\ &\leq \sqrt{cC_{\boldsymbol{r}}(\mathcal{G}_{\boldsymbol{r}}, \pi_{i,\textit{tar}}, \mu_{i,\textit{ref}}, i)^2 \left(k\frac{\xi^2\kappa^2 \log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/(NN_p))/\delta)}{\eta^2 NN_p} + \frac{\xi^2(k+\log(N/\delta))}{\eta^2 N_p} + \lambda B^2\right)} \end{aligned} \tag{3.1}$$

where $c > 0$ is a constant.

Lastly, we can also use the learned representation for a new human user in Appendix G.1.1.

Remark 1 (Sample Complexity)**.** *For Theorem 3.1, if we naively learn the personalization model without representation learning, $\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/(NN_p))$ will be very large. For example, if we use linear representation $\phi_\omega(x) = \omega x$ and ω is a $d \times k$ orthonormal matrix, then* $\log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/NN_p)/\delta) \leq \mathcal{O}\left((dk+Nk)\log\left(R_{\max}NN_p/\delta\right)\right)$ *while naive personalization with*

$$\mathcal{G}_{\boldsymbol{r}}' = \left\{(\langle\phi(\cdot),\theta_i\rangle)_{i\in[N]}\rangle \,\middle|\, \theta_i \in \mathbb{R}^d \text{ and } \|\theta_i\|_2 \leq B \text{ for all } i \in [N]\right\}$$

provides $\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}'}(1/(NN_p)) \leq \mathcal{O}\left(Nd\log\left(R_{\max}NN_p/\delta\right)\right)$. Since $d \gg k$, the bound of Equation (3.1)'s right-hand side has a significant improvement when we use representation learning. If the representation function class is an MLP class, we can use a known bracket number by Bartlett et al. (2017).

We also point out that the existing technique from representation learning literature does not cover the case with general representation function learning with a log-likelihood loss function with $\mathcal{O}(1/N_p)$ rate, to the best of our knowledge. The technical results are thus of independent interest.

Lastly, we examine the tightness of our analysis by the theoretical lower bound of the sub-optimality gap of personalization.

Theorem 3.2 (Informal, Lower Bound for the Sub-Optimality Gap of Personalization)**.** *For any $k > 6$ and large N_p, there exists a representation function $\phi(\cdot)$ and $C > 0$ so that*

$$\min_{i\in[N]} \inf_{\widehat{\pi}} \sup_{Q\in\mathrm{CB}} \left(\max_{\pi^*\in\Pi} J(\pi^*; r_{\omega,\theta_i}) - J(\widehat{\pi}; r_{\omega,\theta_i}) \right) \geq C \cdot \sqrt{\frac{k}{N_p}},$$

where CB is a family of MDP with N reward functions.

Our approach for personalized reward lower bound builds upon (Zhu et al., 2023, Theorem 3.10). By Theorem 3.2, for general representation function class, we establish that Algorithm 1 is near-optimal for the sub-optimality of the induced personalization policy, as $\log(\mathcal{N}_{\mathcal{G}_r}(1/NN_p))$ can be small so that $\sqrt{k\frac{\log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/(NN_p))/\delta)}{NN_p}}$ can be dominated by $\sqrt{\frac{k}{N_p}}$. Note that if Ψ is a linear representation class, Theorem 3.1 still has a $\sqrt{k}$ gap compared to the lower bound (Theorem 3.2). This gap is also observed in (Tripuraneni et al., 2020). We will leave the sharpening of this $\sqrt{k}$ factor for future.

3.2 Personalized RLHF via Human User Clustering

We now provide the second approach in the personalization-based framework, through human user clustering. In particular, fine-tuning an LLM for each individual may be impractical. We thus propose an alternative approach that segments human users into clusters and fine-tunes an LLM for *each cluster*. This strategy entails deploying K clustered models, which can be smaller than the number of human users N. A critical aspect of this methodology is the way to generate clusters. This clustering-based personalization has also been studied in the federated (supervised) learning literature (Mansour et al., 2020; Ghosh et al., 2020; Sattler et al., 2020). We introduce our algorithm next, based on the algorithmic idea in Mansour et al. (2020).

3.2.1 Algorithms

We partition all the N human users into K clusters and find the best parameters for each cluster:

$$\max_{(r_{(k)})_{k\in[K]}} \sum_{i\in[N]} \frac{1}{N} \max_{k\in[K]} \sum_{j\in[N_p]} \log P_{r_{(k)}} \left(o_i^{(j)} \mid \tau_{i,0}^{(j)}, \tau_{i,1}^{(j)} \right). \tag{3.2}$$

Algorithm 2 outputs K clustered policies and a map from human users to clusters. The input of the algorithm is $\widehat{\mathcal{D}} = \cup_{i\in[N]}\widehat{\mathcal{D}}_i$ where $\widehat{\mathcal{D}}_i = \{(o_i^{(j)}, \tau_{i,0}^{(j)}, \tau_{i,1}^{(j)})_{j\in[N_p]}\}$, which is the same as Algorithm 1. After estimating the representation parameter $\widehat{\omega}$, the algorithm will estimate the reward function parameters $(\widehat{\theta}_{(k)})_{k\in[K]}$ with Equation (F.3). Lastly, we find the best policy based on the expected value function. We also provide a practical algorithm that uses DPO (Rafailov et al., 2024) (and also refer to Appendix D) and EM (Moon, 1996) algorithms to solve Equation (F.3) in Algorithm 3.

3.2.2 Results and Analyses

To analyze the clustering-based personalization approach, we adapt the notion of *label discrepancy* in Mohri and Muñoz Medina (2012) to our RLHF setting, for preference data and a given reward function class. We defer all the proofs of this section to Appendix H.

Definition 3.3 (Label Discrepancy)**.** *Label discrepancy for preference distribution $\boldsymbol{D}_i$ and $\boldsymbol{D}_j$, which are distributions of (o, τ_0, τ_1), with reward function class $\mathcal{G}_r$ is defined as follows:*

$$\mathtt{disc}(\boldsymbol{D}_i, \boldsymbol{D}_j, \mathcal{G}_r) = \max_{\boldsymbol{r}\in\mathcal{G}_r} \left| \mathbb{E}_{\boldsymbol{D}_i} \log P_r(o \mid \tau_1, \tau_0) - \mathbb{E}_{\boldsymbol{D}_j} \log P_r(o \mid \tau_1, \tau_0) \right|.$$

The discrepancy is defined as the supremum value of the difference between the log-likelihood of the preference data when taking expectations over two human dataset distributions. This quantity will be used in the analysis to characterize the gap between the log-likelihood of the estimated parameters and the underlying parameters. A similar concept is frequently used in domain adaptation (Mansour et al., 2009) and federated learning (Mansour et al., 2020).

Theorem 3.3. (Total Expected Value Function Gap). *Suppose Assumptions 1, 2, 3, and 4 hold. Also, assume that $C_r(\mathcal{G}_r, \pi, \mu_{i,ref}, i) \leq C'_{max}$ for all policy π and $i \in [N]$. For any $\delta \in (0, 1]$, all $i \in [N]$ and $\lambda > 0$, with probability at least $1 - \delta$, the output $((\widehat{\pi}_{(k)})_{k\in[K]}, \widehat{f})$ of Algorithm 2 satisfies*

$$\sum_{i\in[N]} \left(J(\pi_{i,tar}; r_i^\star) - J(\widehat{\pi}_{\widehat{f}(i)}; r_i^\star) \right) \leq cN\kappa \Bigg(\underbrace{\frac{\log(2K/\delta) + kK\log(N_p/k)}{N_p} + \frac{k\xi^2\kappa^2 \log(\mathcal{N}_{\mathcal{G}_r}(1/(NN_p))/\delta)}{NN_p}}_{(i)}$$

$$+ \underbrace{\left(\sum_{i\in[N]} \frac{1}{N} \mathtt{disc}(\mathcal{D}_i, \mathcal{C}_{\widehat{f}(i)}, \mathcal{G}_{\psi^\star}) \right)^2}_{(ii)} + \left(\frac{\log(\mathcal{N}_{\mathcal{G}_{\psi^\star}}(1/NN_p)/\delta)}{NN_p} \right)^2 \Bigg)^{1/4},$$

where $c > 0$ is a constant.

Note that Theorem 3.3 addresses the bias-variance tradeoff: as the number of clusters (K) increases, term (i) (variance) increases, while term (ii) (bias) decreases.

4 Reward and Preference Aggregation

This section adheres to the RLHF setting with a single LLM, while handling the heterogeneous human feedback by reward/preference aggregation. For reward aggregation, we first estimate individual reward functions and then aggregate these functions to form a unified reward model. In comparison, for preference aggregation, we introduce a novel framework termed "probabilistic opinion pooling". Specifically, instead of relying on binary comparison data, human users provide feedback as *probability vectors*. This approach eliminates the need to aggregate heterogeneous preferences via reward functions, allowing for the direct aggregation of probabilistic opinions provided by users.

4.1 Reward Aggregation

We introduce the following reward aggregation rules (Equations (4.1) and (4.2)), which are favorable as they satisfy several pivotal axioms in social choice theory. These axioms – monotonicity, symmetry, continuity, independence of unconcerned agents, translation independence, and the Pigou-Dalton transfer principle – are crucial for ensuring fairness and consistency in the decision-making process (List, 2013; Skiadas, 2009, 2016). We present the definition of these axioms in Appendix I.1 for completeness. The aggregation rules are presented as follows:

$$\mathrm{Agg}_\alpha(\boldsymbol{r}) = \begin{cases} \frac{1}{\alpha}\log\left(\frac{1}{N}\sum_{i\in[N]}\exp(\alpha r_i)\right) & \alpha \neq 0 \\ \frac{1}{N}\sum_{i\in[N]} r_i & \alpha = 0 \end{cases} \quad (4.1)$$

$$\mathrm{Agg}'_\alpha(\boldsymbol{r}) = \begin{cases} \frac{1}{N\alpha}\sum_{i\in[N]}(\exp(\alpha r_i) - 1) & \alpha \neq 0 \\ \frac{1}{N}\sum_{i\in[N]} r_i & \alpha = 0 \end{cases} \quad (4.2)$$

where $\boldsymbol{r} = (r_1, \ldots, r_N)^\intercal$ is a reward vector with trajectory input. Note that Equation (4.1) and Equation (4.2) are equivalent in the sense of the associated optimal policy, as $\log(x)$ is monotonically increasing. We can verify that $\lim_{\alpha\to-\infty}\mathrm{Agg}_\alpha(\boldsymbol{r}) = \min_{i\in[N]} r_i$ and $\lim_{\alpha\to\infty}\mathrm{Agg}_\alpha(\boldsymbol{r}) = \max_{i\in[N]} r_i$. This implies that when α is small (or large), the reward aggregation rule emphasizes $\min_{i\in[N]}(\text{or } \max_{i\in[N]})r_i$, respectively. When $\alpha = 0$, Equation (4.1) represents utilitarianism, and when $\alpha \to -\infty$, Equation (4.1) represents a Leximin-based aggregation rule (List, 2013).

4.1.1 Algorithm and Analysis

Algorithm 4 outputs a joint estimation of $\psi^\star$ and $\boldsymbol{\theta}^\star$ with maximum likelihood estimation as Algorithm 1. The procedure is overall the same as Algorithm 1, except the last step for estimating the best policy for the pessimistic expected value function associated with the aggregated reward function.

Theorem 4.1. (Expected Value Function Gap). *Suppose Assumptions 1, 2, 3, and 4 hold. For any $\delta \in (0, 1]$, all $i \in [N]$ and $\lambda > 0$, with probability at least $1 - \delta$, the output $\widehat{\pi}$ of Algorithm 4 satisfies*

$$\begin{aligned} &J(\pi_{tar}; Agg_\alpha(\boldsymbol{r}^\star)) - J(\widehat{\pi}; Agg_\alpha(\boldsymbol{r}^\star)) \\ &\leq \sqrt{c_\alpha C_{\boldsymbol{r}}(\mathcal{G}_{\boldsymbol{r}}, \pi_{tar}, \mu_{ref})^2\left(\frac{k\kappa^2\log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/(NN_p))/(\delta/N))}{NN_p} + \frac{\xi^2(k + \log(N/\delta))}{\eta^2 N_p} + \lambda B^2\right)} \end{aligned}$$

where $c_\alpha > 0$ is a constant depending on α, and other constants are defined in Section 3.1.1.

We defer the proof of Theorem 4.1 to Appendix I.3. Lastly, we prove the tightness and near-optimality of our analysis for Algorithm 4 by providing a theoretical lower bound of the sub-optimality gap of aggregation, which is deferred to Appendix I.2.

4.2 Preference Aggregation with Probabilistic Opinion Data

Consider a set of questions $\{s^{(j)}\}_{j\in[N_p]}$, and for each question $s^{(j)}$, there are K potential answers denoted by $\mathcal{A}^{(j)} := \{a_k^{(j)}\}_{k\in[K]}$. Traditional RLHF methods involve human labelers $i \in [N]$ selecting a preferred answer from $\mathcal{A}^{(j)}$. This approach limits the human feedback to a singular choice, which, though being simple, restricts the expressiveness of human preferences.

To address this, we introduce a new setting whereby human labelers provide feedback as a probability vector $q_i^{(j)} \in \Delta(\mathcal{A}^{(j)})$, which is also called *probabilistic opinion* in social choice theory (Stone, 1961; Lehrer and Wagner, 2012). Here, $\Delta(\mathcal{A}^{(j)})$ represents the set of all possible distributions over the answers in $\mathcal{A}^{(j)}$. This allows labelers to quantify their preferences across multiple answers rather than selecting only one, and can be implemented in practice without increasing too much of overload for feedback collection.

Our setup does not assume a predefined relationship between each reward function for every human labeler and their preferences. Instead, we aggregate the diverse probabilistic preferences of multiple labelers into a consensus probability distribution over the answers. We define an aggregation function (or a *probabilistic opinion pooling function*), Agg-p$_\alpha(\boldsymbol{P})$, which takes a tuple of human preference distributions $\boldsymbol{P} = (P_1, \dots, P_N) \in (\Delta(\mathcal{A}))^N$ and maps it to a single probability distribution in $\Delta(\mathcal{A})$ where $\mathcal{A}$ is the potential answer set. For each $a \in \mathcal{A}$,

$$\text{Agg-p}_\alpha(\mathbf{P})(a) := \begin{cases} \frac{\left(\sum_{i\in[N]}(P_i(a))^\alpha\right)^{1/\alpha}}{\sum_{a'\in\mathcal{A}}\left(\sum_{i\in[N]}(P_i(a'))^\alpha\right)^{1/\alpha}} & \alpha \neq 0 \\ \frac{\left(\prod_{i\in[N]} P_i(a)\right)^{1/N}}{\sum_{a'\in\mathcal{A}}\left(\prod_{i\in[N]} P_i(a')\right)^{1/N}} & \alpha = 0 \end{cases}. \tag{4.3}$$

The case where $\alpha = 0$ is referred to as the geometric pooling function (McConway, 1978)[♮]. Interestingly, Equation (4.3), which describes the aggregation of probabilistic preferences, has a connection to Equation (4.2), concerning reward aggregation, under the assumption of the Plackett-Luce model for the relationship between reward functions and preference models (Definition 4.1). We then formalize the connection between the probabilistic opinion pooling in Equation (4.3) and the reward aggregation rule in Equation (4.1). We defer the proof of Theorem 4.2 to Appendix I.6.

Definition 4.1. *The Plackett-Luce (PL) model (Plackett, 1975; Luce, 2005) quantifies the likelihood that a trajectory τ_k is preferred over all other pairs in the set $\{\tau_k\}_{k\in[K]}$ by assigning it a probability defined as*

$$P_r\left(\tau_k \succ \tau_{k'} \forall k' \neq k \,\middle|\, (\tau_k)_{k\in[K]}\right) = \frac{\exp(r(\tau_k))}{\sum_{k'\in[K]}\exp(r(\tau_{k'}))}$$

where r is the reward function for a human labeler. In the case where $k = 2$, this formulation simplifies to the Bradley-Terry-Luce (BTL) Model (Bradley and Terry, 1952).

Theorem 4.2. (Relationship between Reward Aggregation and Preference Aggregation). *Suppose human preferences are modeled by the PL model, and all human labelers share a common lower bound on their reward functions. Let $(R_i(a))_{a\in\mathcal{A}}$ represent the reward function associated with action $a \in \mathcal{A}$ and $P_i \in \Delta(\mathcal{A})$ denote the corresponding probabilistic opinion for individual $i \in [N]$. Then, the preference aggregation Agg-p$_\alpha(\mathbf{P})$, is equivalent to the preference derived under the PL model with the aggregated rewards $(\text{Agg}_\alpha(\mathbf{R}(a)))_{a\in\mathcal{A}}$ for any $\alpha \in [-\infty, \infty]$.*

While we generally do not presuppose any specific relationship between probabilistic opinions and reward functions, Theorem 4.2 shows that under the classical choice model of Plackett-Luce, these two aggregation rules can coincide.

We defer the algorithm for probabilistic opinion pooling in Appendix I.7. We can use the aggregated probabilistic opinions to fine-tune the policy using DPO-based algorithm.

5 Mechanism Design for Preference Aggregation

Suppose that human labeler i ($i \in [N]$) provides preference data by probabilistic opinion $P_i \in \Delta(\mathcal{A})$. We now consider the natural scenario where the labelers may be *strategic* – given they are human beings with (certain degree of) rationality. In particular, knowing the form of preference aggregation (and the fact that they may affect the process), human labelers may provide *untruthful* feedback of their preference, in order to benefit more in terms of their *actual* utility/preference. In particular, the untruthful preference may *bias* the aggregated preference (that LLM will be fine-tuned over) towards their own preference, and thus manipulates the LLM output.

An Example with Untruthful Feedback. Consider a set of N labelers evaluating two answers, where each labeler expresses a probabilistic opinion on the answers (a_1, a_2). Specifically, suppose labeler N believes that a_1 is slightly preferable to a_2, represented by the probability vector $P_N = (0.6, 0.4)^\intercal$. Conversely, all other labelers $i \in [N-1]$ have probabilistic opinion favoring the second answer, represented by $P_i = (0.2, 0.8)^\intercal$. We assume that the aggregation of these opinions employs the Agg-p$_{-\infty}$ rule, defined as Agg-p$_{-\infty}(\boldsymbol{P})(a_t) = \frac{\min_{i\in[N]} P_i(a_t)}{\min_{i\in[N]} P_i(a_1)+\min_{i\in[N]} P_i(a_2)}$ for $t = 1, 2$, where $\boldsymbol{P}$ represents the matrix of probabilistic opinions across all labelers and answers. Under truthful reporting, the aggregated result would be calculated as Agg-p$_{-\infty}(\boldsymbol{P}) = (1/3, 2/3)^\intercal$. However, labeler N can strategically provide an untruthful probabilistic opinion to distort the aggregated result toward his original view: If labeler N reports a distorted opinion of $P'_N = (13/15, 2/15)^\intercal$ instead of $(0.6, 0.4)^\intercal$, the new aggregated opinion becomes Agg-p$_{-\infty}(\boldsymbol{P}') = (0.6, 0.4)^\intercal$, where $\boldsymbol{P}' = (P_1, \dots, P_{N-1}, P'_N)$, which aligns exactly with labeler N's probabilistic opinion, while further deviating from other labelers' actual preference. This example underscores the potential of

[♮]We refer Appendix I.5 for the discussion of the case with $\alpha = 0$.

strategic behavior in the aggregation of probabilistic opinions, and thus highlights the importance of incentivizing truthful preference reporting.

To address the untruthful feedback issue, we resort to the ideas in mechanism design (Nisan and Ronen, 1999; Börgers, 2015; Roughgarden, 2010). Specifically, we will develop mechanisms that can impose some *cost* on human labelers, so that they do not have the incentive to report untruthful preferences. For a given question s, we assume that labeler i's probabilistic opinion vector is p_i, and the aggregated vector being $p = \text{Agg-p}(\boldsymbol{P})$ where $\boldsymbol{P} = (p_1, \cdots, p_N)$. We additionally impose cost $c_i > 0$ to labeler i based on the reports and aggregation results. We assume the following quasi-linear utility of labeler $i : u_i(\boldsymbol{P}) := d(p_i, \text{Agg-p}(\boldsymbol{P})) - c_i(\boldsymbol{P})$ where $d(\cdot, \cdot)$ measures the distance between two probability distribution. Under this utility model, we design an incentive-compatible mechanism to elicit truthful reports.

Theorem 5.1 (Informal)**.** *For any distance function $d(\cdot, \cdot)$ in a given class, there exists an aggregation rule in (4.3) maximizes social welfare Welfare$(\boldsymbol{P}) := \sum_{i=1}^N d(p_i, \text{Agg-p}(\boldsymbol{P}))$. Moreover, inspired by the Vickery-Clarke-Groves mechanism (Vickrey, 1961; Clarke, 1971; Groves, 1973), we can design proper cost function $c_i \geq 0$, which makes truthful reporting incentive-compatible.*

Here incentive-compatibility means for labeler i, whatever the other labelers' reports are, truthful reporting will always maximize her own utility function. Intuitively, our mechanism punishes labeler i through cost c_i for the externality she posed on other labelers, i.e., if she makes other labelers worse off based on the aggregation outcome. We defer detailed explanation of this section in Appendix I.9.

Imposing Cost for Human Feedback Collection. Though not being enforced in most existing RLHF frameworks, we believe it is reasonable and possible to incorporate it in the feedback collection, especially in scenarios where a single reward model (and thus a single LLM) is mandated. For example, the future large models may be regulated by some administrative agency, e.g., the government. These agencies' objective is for social good, despite the heterogeneity in human preferences, and also possess the power to enforce cost to human labelers, e.g., via taxing. It may also be possible for big technology companies who train LLMs, e.g., OpenAI, to incentivize truthful feedback through personalized and strategic (negative) payment (which corresponds to the cost here) to human labelers.

Remark 2. *An analog of Theorem 5.1 can also be applied to reward aggregation. Additionally, under the PL model, the mechanism design for reward aggregation and preference aggregation coincide.*

6 Experiments

We now conduct an empirical evaluation of our methods' performance on a text summarization task, using the Reddit TL;DR summarization dataset and the Reddit TL;DR human feedback dataset (`comparison` and `axes evals`) (Stiennon et al., 2020). We used GPT-J 6B (Wang and Komatsuzaki, 2021) and LLaMA3 8B models (Meta, 2024) in our experiments. We defer details in Appendix J.

6.1 Experiment 1: Reward Model Performance with `comparison` Dataset

In Experiment 1, we compared our Algorithm 1 and Algorithm 2 with naive RLHF methods. We constructed a reward model using a supervised fine-tuned language model and added a linear layer to represent individual reward functions, as in our model in Section 3.1. We provide a detailed discussion of the reward model structure for general representation and linear representation in Appendix J. We used two clusters for the personalized reward model with user clustering. We evaluated the reward models based on their accuracy in correctly assigning higher rewards to chosen summaries over rejected summaries in the validation set. Our results, shown in Figure 2, indicate that clustering methods can efficiently learn the personalized reward model. Furthermore, personalization with general representation learning is necessary, as indicated by the performance gap compared to personalization with linear representation learning. Notably, for LLaMA3 8B, the performance differences between the Naive method and both PG and CG are statistically significant by t-test ($p < 0.006$).

6.2 Experiment 2: Output Examples of Reward Aggregation with `axes` Dataset

In Experiment 2, we aggregated three axes rewards using Equation (4.1) with $\alpha = -\infty, -1, 0, 1, \infty$. We included representative outputs from these aggregated results in Appendix J.1.

References

ABRAMSON, J., AHUJA, A., CARNEVALE, F., GEORGIEV, P., GOLDIN, A., HUNG, A., LANDON, J., LHOTKA, J., LILLICRAP, T., MULDAL, A. ET AL. (2022). Improving multimodal interactive agents with reinforcement learning from human feedback. *arXiv preprint arXiv:2211.11602*.

AGARWAL, A., KAKADE, S., KRISHNAMURTHY, A. and SUN, W. (2020). Flambe: Structural complexity and representation learning of low rank mdps. *Conference on Neural Information Processing Systems (NeurIPS)*.

ANANDALINGAM, G. and FRIESZ, T. L. (1992). Hierarchical optimization: An introduction. *Annals of Operations Research*, **34** 1–11.

ARROW, K. J. (1951). Alternative approaches to the theory of choice in risk-taking situations. *Econometrica: Journal of the Econometric Society* 404–437.

BAI, Y., JONES, A., NDOUSSE, K., ASKELL, A., CHEN, A., DASSARMA, N., DRAIN, D., FORT, S., GANGULI, D., HENIGHAN, T. ET AL. (2022). Training a helpful and harmless assistant with reinforcement learning from human feedback. *arXiv preprint arXiv:2204.05862*.

BARTLETT, P. L., FOSTER, D. J. and TELGARSKY, M. J. (2017). Spectrally-normalized margin bounds for neural networks. *Conference on Neural Information Processing Systems (NeurIPS)*.

BAXTER, J. (2000). A model of inductive bias learning. *Journal of artificial intelligence research*, **12** 149–198.

BÖRGERS, T. (2015). *An introduction to the theory of mechanism design*. Oxford University Press, USA.

BOSE, A., DU, S. S. and FAZEL, M. (2024). Offline multi-task transfer rl with representational penalization. *arXiv preprint arXiv:2402.12570*.

BOXALL, P. C. and ADAMOWICZ, W. L. (2002). Understanding heterogeneous preferences in random utility models: a latent class approach. *Environmental and resource economics*, **23** 421–446.

BRADLEY, R. A. and TERRY, M. E. (1952). Rank analysis of incomplete block designs: I. the method of paired comparisons. *Biometrika*, **39** 324–345.

CHAKRABORTY, S., QIU, J., YUAN, H., KOPPEL, A., HUANG, F., MANOCHA, D., BEDI, A. S. and WANG, M. (2024). Maxmin-rlhf: Towards equitable alignment of large language models with diverse human preferences. *arXiv preprint arXiv:2402.08925*.

CLARKE, E. H. (1971). Multipart pricing of public goods. *Public choice* 17–33.

DIETRICH, F. and LIST, C. (2016). Probabilistic opinion pooling.

DONAHUE, J., JIA, Y., VINYALS, O., HOFFMAN, J., ZHANG, N., TZENG, E. and DARRELL, T. (2014). Decaf: A deep convolutional activation feature for generic visual recognition. In *International Conference on Machine Learning (ICML)*. PMLR.

DU, S. S., HU, W., KAKADE, S. M., LEE, J. D. and LEI, Q. (2021). Few-shot learning via learning the representation, provably. *ICLR*.

FINN, C., LEVINE, S. and ABBEEL, P. (2016). Guided cost learning: Deep inverse optimal control via policy optimization. In *International Conference on Machine Learning (ICML)*. PMLR.

GENEST, C. (1984). A characterization theorem for externally bayesian groups. *The Annals of Statistics* 1100–1105.

GHOSH, A., CHUNG, J., YIN, D. and RAMCHANDRAN, K. (2020). An efficient framework for clustered federated learning. *Conference on Neural Information Processing Systems (NeurIPS)*.

GROVES, T. (1973). Incentives in teams. *Econometrica: Journal of the Econometric Society* 617–631.

GULSHAN, V., PENG, L., CORAM, M., STUMPE, M. C., WU, D., NARAYANASWAMY, A., VENUGOPALAN, S., WIDNER, K., MADAMS, T., CUADROS, J. ET AL. (2016). Development and validation of a deep learning algorithm for detection of diabetic retinopathy in retinal fundus photographs. *JAMA*, **316** 2402–2410.

HAVRILLA, A., ZHURAVINSKYI, M., PHUNG, D., TIWARI, A., TOW, J., BIDERMAN, S., ANTHONY, Q. and CASTRICATO, L. (2023). trlx: A framework for large scale reinforcement learning from human feedback. *Conference on Empirical Methods in Natural Language Processing (EMNLP)*.

HSU, D., KAKADE, S. and ZHANG, T. (2012). A tail inequality for quadratic forms of subgaussian random vectors.

HWANG, M., WEIHS, L., PARK, C., LEE, K., KEMBHAVI, A. and EHSANI, K. (2024). Promptable behaviors: Personalizing multi-objective rewards from human preferences. *Conference on Computer Vision and Pattern Recognition (CVPR)*.

ISHFAQ, H., NGUYEN-TANG, T., FENG, S., ARORA, R., WANG, M., YIN, M. and PRECUP, D. (2024). Offline multitask representation learning for reinforcement learning. *arXiv preprint arXiv:2403.11574*.

JANG, J., KIM, S., LIN, B. Y., WANG, Y., HESSEL, J., ZETTLEMOYER, L., HAJISHIRZI, H., CHOI, Y. and AMMANABROLU, P. (2023). Personalized soups: Personalized large language model alignment via post-hoc parameter merging. *arXiv preprint arXiv:2310.11564*.

LEHRER, K. and WAGNER, C. (2012). *Rational consensus in science and society: A philosophical and mathematical study*, vol. 24. Springer Science & Business Media.

LI, X., LIPTON, Z. C. and LEQI, L. (2024). Personalized language modeling from personalized human feedback. *arXiv preprint arXiv:2402.05133*.

LIST, C. (2013). Social choice theory.

LIU, Q., CHUNG, A., SZEPESVÁRI, C. and JIN, C. (2022). When is partially observable reinforcement learning not scary? In *Conference on Learning Theory*. PMLR.

LIU, Q., NETRAPALLI, P., SZEPESVARI, C. and JIN, C. (2023). Optimistic mle: A generic model-based algorithm for partially observable sequential decision making. In *Proceedings of the 55th Annual ACM Symposium on Theory of Computing*.

LOSHCHILOV, I. and HUTTER, F. (2018). Fixing weight decay regularization in adam.

LU, R., HUANG, G. and DU, S. S. (2021). On the power of multitask representation learning in linear mdp. *arXiv preprint arXiv:2106.08053*.

LUCE, R. D. (2005). *Individual choice behavior: A theoretical analysis*. Courier Corporation.

MADANSKY, A. (1964). *Externally bayesian groups*. Rand Corporation.

MANSOUR, Y., MOHRI, M., RO, J. and SURESH, A. T. (2020). Three approaches for personalization with applications to federated learning. *arXiv preprint arXiv:2002.10619*.

MANSOUR, Y., MOHRI, M. and ROSTAMIZADEH, A. (2009). Domain adaptation: Learning bounds and algorithms. *arXiv preprint arXiv:0902.3430*.

MCCONWAY, K. J. (1978). *The combination of experts' opinions in probability assessment: some theoretical considerations*. Ph.D. thesis, University College London (University of London).

META (2024). Meta llama 3.
https://ai.meta.com/blog/meta-llama-3/.

MOHRI, M. and MUÑOZ MEDINA, A. (2012). New analysis and algorithm for learning with drifting distributions. In *Algorithmic Learning Theory*.

MOON, T. K. (1996). The expectation-maximization algorithm. *IEEE Signal processing magazine*, **13** 47–60.

MOULIN, H. (2004). *Fair division and collective welfare*. MIT press.

NISAN, N. and RONEN, A. (1999). Algorithmic mechanism design. In *Proceedings of the thirty-first annual ACM symposium on Theory of computing*.

OUYANG, L., WU, J., JIANG, X., ALMEIDA, D., WAINWRIGHT, C., MISHKIN, P., ZHANG, C., AGARWAL, S., SLAMA, K., RAY, A. ET AL. (2022). Training language models to follow instructions with human feedback. *Conference on Neural Information Processing Systems (NeurIPS)*.

OZDAGLAR, A. E., PATTATHIL, S., ZHANG, J. and ZHANG, K. (2023). Revisiting the linear-programming framework for offline rl with general function approximation. In *International Conference on Machine Learning (ICML)*. PMLR.

PLACKETT, R. L. (1975). The analysis of permutations. *Journal of the Royal Statistical Society Series C: Applied Statistics*, **24** 193–202.

POLLAK, R. A. and WALES, T. J. (1992). *Demand system specification and estimation*. Oxford University Press, USA.

RAFAILOV, R., SHARMA, A., MITCHELL, E., MANNING, C. D., ERMON, S. and FINN, C. (2024). Direct preference optimization: Your language model is secretly a reward model. *Conference on Neural Information Processing Systems (NeurIPS)*.

RASHIDINEJAD, P., ZHU, B., MA, C., JIAO, J. and RUSSELL, S. (2021). Bridging offline reinforcement learning and imitation learning: A tale of pessimism. *Conference on Neural Information Processing Systems (NeurIPS)*.

REN, T., ZHANG, T., SZEPESVÁRI, C. and DAI, B. (2022). A free lunch from the noise: Provable and practical exploration for representation learning. In *Uncertainty in Artificial Intelligence*. PMLR.

ROUGHGARDEN, T. (2010). Algorithmic game theory. *Communications of the ACM*, **53** 78–86.

SATTLER, F., MÜLLER, K.-R. and SAMEK, W. (2020). Clustered federated learning: Model-agnostic distributed multitask optimization under privacy constraints. *IEEE transactions on neural networks and learning systems*, **32** 3710–3722.

SCHULMAN, J., WOLSKI, F., DHARIWAL, P., RADFORD, A. and KLIMOV, O. (2017). Proximal policy optimization algorithms. *arXiv preprint arXiv:1707.06347*.

SEN, A. (2018). *Collective Choice and Social Welfare*. Harvard University Press.

SKIADAS, C. (2009). *Asset pricing theory*. Princeton University Press.

SKIADAS, C. (2016). Scale or translation invariant additive preferences. *Unpublished manuscript*.

STIENNON, N., OUYANG, L., WU, J., ZIEGLER, D., LOWE, R., VOSS, C., RADFORD, A., AMODEI, D. and CHRISTIANO, P. F. (2020). Learning to summarize with human feedback. *Conference on Neural Information Processing Systems (NeurIPS)*.

STONE, M. (1961). The opinion pool. *The Annals of Mathematical Statistics* 1339–1342.

TANG, D., QIN, B., FENG, X. and LIU, T. (2016). Effective lstms for target-dependent sentiment classification. In *Proceedings of COLING 2016, the 26th International Conference on Computational Linguistics: Technical Papers*.

TRIPURANENI, N., JIN, C. and JORDAN, M. (2021). Provable meta-learning of linear representations. In *International Conference on Machine Learning (ICML)*. PMLR.

TRIPURANENI, N., JORDAN, M. and JIN, C. (2020). On the theory of transfer learning: The importance of task diversity. *Conference on Neural Information Processing Systems (NeurIPS)*.

UEHARA, M., ZHANG, X. and SUN, W. (2021). Representation learning for online and offline rl in low-rank mdps. In *International Conference on Learning Representations*.

VICKREY, W. (1961). Counterspeculation, auctions, and competitive sealed tenders. *The Journal of finance*, **16** 8–37.

VÖLSKE, M., POTTHAST, M., SYED, S. and STEIN, B. (2017). Tl; dr: Mining reddit to learn automatic summarization. In *Proceedings of the Workshop on New Frontiers in Summarization*.

WANG, B. and KOMATSUZAKI, A. (2021). Gpt-j-6b: A 6 billion parameter autoregressive language model.

WANG, Y., LIU, Q. and JIN, C. (2024). Is rlhf more difficult than standard rl? a theoretical perspective. *Conference on Neural Information Processing Systems (NeurIPS)*.

WARNELL, G., WAYTOWICH, N., LAWHERN, V. and STONE, P. (2018). Deep tamer: Interactive agent shaping in high-dimensional state spaces. *Association for the Advancement of Artificial Intelligence (AAAI)*.

WU, Z., HU, Y., SHI, W., DZIRI, N., SUHR, A., AMMANABROLU, P., SMITH, N. A., OSTENDORF, M. and HAJISHIRZI, H. (2024). Fine-grained human feedback gives better rewards for language model training. *Conference on Neural Information Processing Systems (NeurIPS)*.

XIONG, W., DONG, H., YE, C., ZHONG, H., JIANG, N. and ZHANG, T. (2024). Gibbs sampling from human feedback: A provable kl-constrained framework for rlhf. *International Conference on Machine Learning (ICML)*.

YU, Y., WANG, T. and SAMWORTH, R. J. (2015). A useful variant of the davis–kahan theorem for statisticians. *Biometrika*, **102** 315–323.

ZANETTE, A., WAINWRIGHT, M. J. and BRUNSKILL, E. (2021). Provable benefits of actor-critic methods for offline reinforcement learning. *Conference on Neural Information Processing Systems (NeurIPS)*, **34** 13626–13640.

ZHAN, W., UEHARA, M., KALLUS, N., LEE, J. D. and SUN, W. (2023). Provable offline preference-based reinforcement learning. *International Conference on Learning Representations (ICLR)*.

ZHAN, W., UEHARA, M., SUN, W. and LEE, J. D. (2022). Pac reinforcement learning for predictive state representations. *arXiv preprint arXiv:2207.05738*.

ZHONG, H., DENG, Z., SU, W. J., WU, Z. S. and ZHANG, L. (2024). Provable multi-party reinforcement learning with diverse human feedback. *arXiv preprint arXiv:2403.05006*.

ZHU, B., JORDAN, M. and JIAO, J. (2023). Principled reinforcement learning with human feedback from pairwise or k-wise comparisons. *International Conference on Machine Learning (ICML)*.

ZIEGLER, D. M., STIENNON, N., WU, J., BROWN, T. B., RADFORD, A., AMODEI, D., CHRISTIANO, P. and IRVING, G. (2019). Fine-tuning language models from human preferences. *arXiv preprint arXiv:1909.08593*.

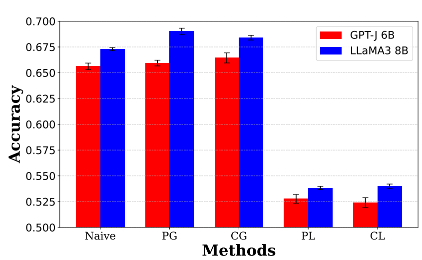

Figure 2: Accuracy of different methods with 3 times experiments. P: Personalized, C: Clustered, G(L): General (Linear) representation. Naive RLHF: original training method.

Supplementary Materials for

"RLHF from Heterogeneous Feedback via Personalization and Preference Aggregation"

Contents

A Societal Impact

Our work is mainly theoretical, and aimed at better understanding RLHF with heterogeneous feedback, with principles, algorithms, and analyses. As such, we do not anticipate any direct positive or negative societal impact from this research.

B Limitations

Our works provided overall theoretical analysis and experimental validation. However, due to the computational issue, we experimented on the 6B and 8B models, and also we did not calculate the penalty for the pessimism in our Algorithms.

C Table of Notation

Notation	Definition
N	Number of Individuals
$\mathcal{S}$	State Space
$\mathcal{A}$	Action Set
H	Horizon Length
P_h	Transition Probability at Horizon h
$\boldsymbol{r}$	Reward
$\mathcal{T}$	Trajectory Set
τ	Trajectory
$J(\pi; r_i)$	$\mathbb{E}_{\tau,\pi}[r_i(\tau)]$
$d_\pi(\tau)$	Occupancy Measure: $\mathbb{P}_\pi(\tau)$
$\Phi\colon \mathbb{R} \to [0,1]$	Strongly Convex Function Mapping Reward to Preference
$\sigma(x)$	Sigmoid Function: $\frac{e^x}{1+e^x}$
$P_{r_i}(o = 0 \mid \tau_0, \tau_1)$	$\Phi(r_i(\tau_0) - r_i(\tau_1))$
$\psi_\omega : \mathbb{R}^d \to \mathbb{R}^k$	Representation Function
Ψ	$\{\psi_\omega \mid \omega \in \Omega\}$
$\mathcal{G}_{\boldsymbol{r}}$	Set of Reward Functions: $\{(\langle \psi_\omega(\phi(\cdot)), \theta_i \rangle)_{i\in[N]} \mid \psi_\omega \in \Psi, \theta_i \in \mathbb{R}^k$ and $\|\theta_i\|_2 \le B$ for all $i \in [N]\}$
$\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(\epsilon)$	Bracket Number of $\mathcal{G}_{\boldsymbol{r}}$ Associated with ϵ
$r_{\omega,\theta_j}(\cdot)$	$\langle \psi_\omega(\phi(\cdot)), \theta_j \rangle$
$\boldsymbol{r}_{\omega,\boldsymbol{\theta}}(\cdot)$	$(r_{\omega,\theta_1}(\cdot), \cdots, r_{\omega,\theta_N}(\cdot)) \in \mathbb{R}^N$
$r_i^\star(\cdot)$	Ground-truth Reward: $\langle \psi^\star(\phi(\cdot)), \theta_i^\star \rangle$
$\psi^\star (= \psi_{\omega^\star})$	Ground-truth Representation Function
$R_{\max}$	$-R_{\max} \le r_i^\star(\tau) \le R_{\max}$
$\widehat{\mathcal{D}}$	$\cup_{i\in[N]} \widehat{\mathcal{D}}_i$
$\widehat{\mathcal{D}}_i$	$\{(o_i^{(j)}, \tau_{i,0}^{(j)}, \tau_{i,1}^{(j)})_{j\in[N_p]}\}$
N_p	$N_p = \lvert\widehat{\mathcal{D}}_1\rvert = \lvert\widehat{\mathcal{D}}_2\rvert = ... = \lvert\widehat{\mathcal{D}}_N\rvert$
$C_r(\mathcal{G}_{\boldsymbol{r}}, \pi_{\text{tar}}, \mu_{\text{ref}}, i)$	Defined in Definition 3.1
$C'_{\boldsymbol{r}}(\mathcal{G}_{\boldsymbol{r}}, \pi^\star, \mu_1, i)$	Defined in Equation (G.10)
$\text{Agg}_\alpha(\boldsymbol{r})$	Defined in Equation (4.2)
$\text{Agg-p}_\alpha(\boldsymbol{p})(a)$	Defined in Equation (4.3)

D Deferred Definition

Bracketing Number. We modify and adopt the definition of the bracketing number of preferences introduced by (Zhan et al., 2023), with some adjustments. Consider $\mathcal{G}_{\boldsymbol{r}}$ as the class of functions representing sets of reward vectors, where each reward vector is denoted by $(r_i)_{i\in[N]}$. Assume g_1 and g_2 maps $(\tau_0, \tau_1) \in \mathcal{T} \times \mathcal{T}$ to $2N$-dimensional vectors. A pair (g_1, g_2) constitutes an ϵ-bracket if for every pair of trajectories (τ_0, τ_1) and for each $i \in [N]$, it holds that $g_1(\cdot \mid \tau_0, \tau_1) \le g_2(\cdot \mid \tau_0, \tau_1)$

and $\|g_1(\cdot \mid \tau_0,\tau_1) - g_2(\cdot \mid \tau_0,\tau_1)\|_1 \le \epsilon$. The ϵ-bracketing number of $\mathcal{G}_{\boldsymbol{r}}$, denoted by $\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(\epsilon)$, is defined as the minimum number of ϵ-brackets $(g_{b,1}, g_{b,2})_{b\in[\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}(\epsilon)}]}$ required such that for any reward vector $\boldsymbol{r} \in \mathcal{G}_{\boldsymbol{r}}$, there exists at least one bracket $b \in [\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}(\epsilon)}]$ such that for all pairs of trajectories (τ_0,τ_1), $g_{b,1}(\cdot \mid \tau_0,\tau_1) \le P_{\boldsymbol{r}}(\cdot \mid \tau_0,\tau_1) \le g_{b,2}(\cdot \mid \tau_0,\tau_1)$ holds.

Concentrability Coefficient for a Reward Scalar Class This definition is exactly the same with the concentrability coefficient of preference as outlined by (Zhan et al., 2023).

Definition D.1 (Zhan et al. (2023))**.** *The concentrability coefficient, with a reward vector class $\mathcal{G}_r$, a target policy π_{tar} (which policy to compete with (potentially optimal policy $\pi^\star$)), and a reference policy μ_{ref}, is defined as follows:*

$$C_r\left(\mathcal{G}_r, \pi_{tar}, \mu_{ref}\right) := \max\left\{0, \sup_{r\in\mathcal{G}_r} \frac{\mathbb{E}_{\tau_0\sim\pi_{tar},\tau_1\sim\mu_{ref}}\left[r^\star(\tau_0) - r^\star(\tau_1) - r(\tau_0) + r(\tau_1)\right]}{\sqrt{\mathbb{E}_{\tau_0\sim\mu_0,\tau_1\sim\mu_1}\left[\left|r^\star(\tau_0) - r^\star(\tau_1) - r(\tau_0) + r(\tau_1)\right|^2\right]}}\right\}.$$

Direct Preference Optimization (DPO) (Rafailov et al., 2024). Consider the case with Markovian reward and policy, i.e., the reward $r : \mathcal{S}\times\mathcal{A}\to\mathbb{R}$ is a function of state s and action a, and the policy $\pi : \mathcal{S}\to\Delta(\mathcal{A})$ is also depending only on the state s. Also, assume that we compare actions for each state rather than the whole trajectories. In the fine-tuning phase using RL, when KL-regularization with the reference policy π^{old} is employed, the optimal policy is given by:

$$\pi(a \mid s) = \frac{1}{Z(s)}\pi^{\text{old}}(a \mid s)\exp\left(\frac{r(s,a)}{\beta}\right),$$

where $Z(s)$ serves as a normalization factor that is independent of the answer a, and β represents the coefficient for KL regularization. Integrating the BTL model into this framework yields:

$$\pi^{\text{RLHF}} = \arg\min_{\pi} -\mathbb{E}_{(s,a_0)\succ(s,a_1)}\left[\log\sigma\left(\beta\log\frac{\pi(a_0 \mid s)}{\pi^{\text{old}}(a_0 \mid s)} - \beta\log\frac{\pi(a_1 \mid s)}{\pi^{\text{old}}(a_1 \mid s)}\right)\right],$$

where σ denotes the Sigmoid function (Rafailov et al., 2024). This formulation bypasses the step of explicitly estimating the reward function.

E Related Work

Reinforcement Learning from Human Feedback. Empirical evidence has demonstrated the efficacy of incorporating human preferences into reinforcement learning (RL) for enhancing robotics (Abramson et al., 2022; Hwang et al., 2024) and for refining large-scale language models (Ziegler et al., 2019; Ouyang et al., 2022; Bai et al., 2022). These human inputs take various forms, such as rankings (Ziegler et al., 2019; Ouyang et al., 2022; Bai et al., 2022), demonstrations (Finn et al., 2016), and scalar ratings (Warnell et al., 2018). A few approaches have been explored empirically to personalize RLHF. For example, assigning fine-grained rewards to small text segments to enhance the training process (Wu et al., 2024), or training each human labeler's reward model with Multi-Objective Reinforcement Learning perspective (Jang et al., 2023; Hwang et al., 2024) have been proposed. Moreover, (Li et al., 2024) suggested the training of each human labeler's reward model directly using personalized feedback with human embedding obtained by the human model, and also an approach for the clustering with finding cluster embedding.

On the theory front, the studies of RLHF have received increasing research interest. The most related prior works are (Zhu et al., 2023; Zhan et al., 2023; Wang et al., 2024), where (Zhu et al., 2023) investigated the Bradley-Terry-Luce (BTL) model (Bradley and Terry, 1952) within the context of a linear reward framework; while (Zhan et al., 2023) generalized the results to encompass more general classes of reward functions. Both works concern the setting with offline preference data. (Xiong et al., 2024) provided a theoretical analysis for KL-regularized RLHF. In the online setting, (Wang et al., 2024) established a correlation between online preference learning and online RL through a *preference-to-reward* interface. Yet, to the best of our knowledge, there is no prior work that has analyzed RLHF with heterogeneous feedback with theoretical guarantees (except the recent independent works discussed in detail below).

Representation Learning. Early work of (Baxter, 2000) established a generalization bound that hinges on the concept of a task generative model within the representation learning framework. More recently, (Tripuraneni et al., 2021; Du et al., 2021) demonstrated that, in the setup with linear representations and squared loss functions, task diversity can significantly enhance the efficiency of learning representations. Moreover, (Tripuraneni et al., 2020) provided a representation learning with general representation and general loss functions. Representation learning has been extended to the reinforcement learning setting as well. For low-rank Markov Decision Processes, where both the reward function and the probability kernel are represented through the inner products of state and action representations with certain parameters, (Agarwal et al., 2020; Ren et al., 2022; Uehara et al., 2021) explored the theoretical foundations for learning these representations. Also, (Ishfaq et al., 2024; Bose et al., 2024) analyzed the sample complexity of multi-task offline RL.

Reward and Preference Aggregation. Preference aggregation is the process by which multiple humans' preference orderings of various social alternatives are combined into a single, collective preference or choice (List, 2013). Arrow's Impossibility Theorem demonstrates that no aggregation rule for preference orderings can simultaneously meet specific criteria essential for ensuring a fair and rational aggregation of each human user's preferences into a collective decision (Arrow, 1951). Therefore, people considered replacing preference orderings with assigning real numbers to social alternatives (Sen, 2018; Moulin, 2004), which is sometimes called a reward (welfare) function [†] for each human user. (Skiadas, 2016; Moulin, 2004) provided reward (welfare) aggregation rules which satisfy several desirable properties. Furthermore, an alternative method to circumvent Arrow's impossibility theorem involved aggregating preferences via probabilistic opinion (Stone, 1961; Lehrer and Wagner, 2012). In this approach, opinions are represented as probability assignments to specific events or propositions of interest.

Comparison with Recent Works. While preparing the present work, we noticed two recent independent works that are closely related. Firstly, (Chakraborty et al., 2024) considered the aggregation of reward models with heterogeneous preference data, focusing on aligning with the Egalitarian principle in social choice theory. In contrast, we provide a framework with various aggregation rules and also prove that the aggregation rules we considered are also welfare-maximizing. More importantly, we design mechanisms for human feedback providers so that they can truthfully report their preferences even when they may be strategic. Moreover, we also develop another framework to handle heterogeneous preferences: the personalization-based one. Finally, we establish near-optimal sample complexity analyses for the frameworks we developed.

More recently, (Zhong et al., 2024) provided a theoretical analysis of reward aggregation in RLHF, focusing primarily on linear representations. Our work, in comparison, considers general representation functions and general relationships between reward function and preference. Unlike (Zhong et al., 2024), where they focused on reward aggregation, we focus on personalization for every human labeler and also employ clustering techniques for personalization. (Zhong et al., 2024) and our paper also both investigated the case that reward and preference are not related. Our paper suggested a probabilistic opinion pooling with a mechanism design to effectively elicit truthful human preferences, presuming human labelers may be strategic. In contrast, (Zhong et al., 2024) analyzed an algorithm for a von Neumann winner policy, where a von Neumann winner policy is a policy that has at least a 50% chance of being preferred compared to any other policy. Moreover, (Zhong et al., 2024) also explored the Pareto efficiency of the resulting policy.

Fundamentals of Auction Theory. Consider the sealed-bid auction mechanism (Vickrey, 1961), where each participant $i \in [N]$ privately submits a bid $b_i(x)$ for every possible outcome $x \in X$, whose true value is $p_i(x) \in \mathbb{R}$. An auction is termed a Dominant Strategic Incentive-Compatible (DSIC) auction (Roughgarden, 2010) if revealing each participant's true valuation is a weakly dominant strategy, i.e., an individual's optimal strategy is to bid their true valuation of the item, $b_i(x) = p_i(x)$ for all $x \in X$, irrespective of the bids $b_{-i}(x)$ submitted by others for all $x \in X$. This mechanism is also called a *truthful* mechanism (Roughgarden, 2010). An auction has a social-welfare-maximizing allocation rule (Roughgarden, 2010) if the outcome x is $\arg\max_{x \in X} \sum_{i \in [N]} p_i(x)$.

[†]In our paper, we regard the reward function as a welfare function in social choice theory.

F Deferred Pseudocode of Algorithms

Algorithm 1 Personalized RLHF via Representation Learning

Input: Dataset $\widehat{\mathcal{D}} = \cup_{i\in[N]}\widehat{\mathcal{D}}_i$ where $\widehat{\mathcal{D}}_i = \{(o_i^{(j)}, \tau_{i,0}^{(j)}, \tau_{i,1}^{(j)})_{j\in[N_p]}\}$ is the preference dataset for the ith individual.

Estimate $\omega^\star$ and $\boldsymbol{\theta}^\star$ by

$$(\widehat{\omega}, \widehat{\boldsymbol{\theta}}) \leftarrow \underset{\omega\in\Omega, \|\theta_i\|_2\leq B \text{ for all } i\in[N]}{\arg\max} \sum_{i\in[N]}\sum_{j\in[N_p]} \log P_{\omega,\theta_i}(o_i^{(j)} \mid \tau_{i,0}^{(j)}, \tau_{i,1}^{(j)})$$

Construct a confidence set of the reward function by

$$\mathcal{R}'(\widehat{\mathcal{D}}) \leftarrow \cap_{i\in[N]}\left\{\boldsymbol{r}_{\omega,\boldsymbol{\theta}} \,\middle|\, \frac{1}{N_p}\sum_{j\in[N_p]} \left|(r_{\widehat{\omega},\widehat{\theta}_i}(\tau_{i,0}^{(j)}) - r_{\widehat{\omega},\widehat{\theta}_i}(\tau_{i,1}^{(j)})) - (r_{\omega,\theta_i}(\tau_{i,0}^{(j)}) - r_{\omega,\theta_i}(\tau_{i,1}^{(j)}))\right|^2 \leq \zeta'\right\} \tag{F.1}$$

Compute policy with respect to $\mathcal{R}(\widehat{\mathcal{D}})$ (or $\mathcal{R}'(\widehat{\mathcal{D}})$) for all $i \in [N]$ by

$$\widehat{\pi}_i' \leftarrow \underset{\pi\in\Pi}{\arg\max} \min_{\boldsymbol{r}\in\mathcal{R}'(\widehat{\mathcal{D}})} \big(J(\pi; r_i) - \mathbb{E}_{\tau\sim\mu_{i,\text{ref}}}[r_i(\tau)]\big) \tag{F.2}$$

Output: $(\widehat{\omega}, \widehat{\boldsymbol{\theta}}, (\widehat{\pi}_i')_{i\in[N]})$.

Algorithm 2 Personalized RLHF via Clustering

Input: Dataset $\widehat{\mathcal{D}} = \cup_{i\in[N]}\widehat{\mathcal{D}}_i$ where $\widehat{\mathcal{D}}_i = \{(o_i^{(j)}, \tau_{i,0}^{(j)}, \tau_{i,1}^{(j)})_{j\in[N_p]}\}$ is the preference dataset for the ith individual, and $\widehat{\omega}$ form Algorithm 1.

Learn $\theta_{(i)}$ and the clustering map $f : [N] \to [K]$ by

$$(\widehat{\theta}_{(k)})_{k\in[K]} \leftarrow \underset{\|\theta_{(k)}\|_2\leq B \text{ for all } k\in[K]}{\arg\max} \sum_{i\in[N]} \max_{k\in[K]} \sum_{j\in[N_p]} \log P_{\widehat{\omega},\theta_{(k)}}(o_i^{(j)} \mid \tau_{i,0}^{(j)}, \tau_{i,1}^{(j)}) \tag{F.3}$$

$$\widehat{f}(i) \leftarrow \underset{k\in[K]}{\arg\max} \sum_{j\in[N_p]} \log P_{\widehat{\omega},\widehat{\theta}_{(k)}}(o_i^{(j)} \mid \tau_{i,0}^{(j)}, \tau_{i,1}^{(j)}) \text{ for all } i \in [N]$$

For each $k \in [K]$,

$$\widehat{\pi}_{(k)} \leftarrow \underset{\pi\in\Pi}{\arg\max} \Big(J(\pi; r_{\widehat{\omega},\widehat{\theta}_{(k)}}) - \mathbb{E}_{\tau\sim\mu_1}[r_{\widehat{\omega},\widehat{\theta}_{(k)}}(\tau)]\Big).$$

Output: $((\widehat{\pi}_{(k)})_{k\in[K]}, (\widehat{\theta}_{(k)})_{k\in[K]}, \widehat{\omega}, \widehat{f})$.

Algorithm 3 ClusterDPO: Learning K clustered policies by DPO

Input: Dataset $\widehat{\mathcal{D}} = \cup_{i\in[N]}\widehat{\mathcal{D}}_i$ where $\widehat{\mathcal{D}}_i = \{a_{i,0}^{(j)} \succ a_{i,1}^{(j)}, s_i^{(j)})_{j\in[N_p]}\}$ is the preference dataset for the ith individual, β is a parameter for DPO

Randomly select K human users $p_1,\dots,p_K$ and initialize $\pi_{(k)}^0$ for all $k\in[K]$ as

$$\pi_{(k)}^0 \leftarrow \arg\max_{\pi\in\Pi} \sum_{j\in[N_p]} \log\sigma\left(\beta\log\frac{\pi(a_{p_k,0}^{(j)}\mid s_{p_k}^{(j)})}{\pi^{\mathrm{old}}(a_{p_k,0}^{(j)}\mid s_{p_k}^{(j)})} - \beta\log\frac{\pi(a_{p_k,1}^{(j)}\mid s_{p_k}^{(j)})}{\pi^{\mathrm{old}}(a_{p_k,1}^{(j)}\mid s_{p_k}^{(j)})}\right)$$

Randomly initialize $f^0(i)$ for $i\notin\{p_1,\dots,p_K\}$

for $t\in[T]$ **do**

  Randomly select K human users $p_1,\dots,p_K$.

  for $i\in[N]$ **do**

    if $i\notin\{p_1,\dots,p_K\}$ **then**

      Define $f^t(i)\leftarrow f^{t-1}(i)$

    end if

  end for

  Assign $f^t(p_k)$ for all $k\in[K]$ as

$$f^t(p_k) \leftarrow \arg\max_{s\in[K]} \sum_{j\in[N_p]} \log\sigma\left(\beta\log\frac{\pi_{(s)}^{t-1}(a_{p_k,0}^{(j)}\mid s_{p_k}^{(j)})}{\pi^{\mathrm{old}}(a_{p_k,0}^{(j)}\mid s_{p_k}^{(j)})} - \beta\log\frac{\pi_{(s)}^{t-1}(a_{p_k,1}^{(j)}\mid s_{p_k}^{(j)})}{\pi^{\mathrm{old}}(a_{p_k,1}^{(j)}\mid s_{p_k}^{(j)})}\right) \tag{F.4}$$

  Run a few steps of optimization to update $\pi_{(s)}^{t-1}$ for all $s\in[K]$ (for example, gradient ascent or Adam) to maximize

$$\sum_{f(p_k)=s}\sum_{j\in[N_p]} \log\sigma\left(\beta\log\frac{\pi(a_{p_k,0}^{(j)}\mid s_{p_k}^{(j)})}{\pi^{\mathrm{old}}(a_{p_k,0}^{(j)}\mid s_{p_k}^{(j)})} - \beta\log\frac{\pi(a_{p_k,1}^{(j)}\mid s_{p_k}^{(j)})}{\pi^{\mathrm{old}}(a_{p_k,1}^{(j)}\mid s_{p_k}^{(j)})}\right)$$

  and obtain $\pi_{(s)}^t$ for all $s\in[K]$.

end for

Assign $f^{T+1}(i)$ for all $i\in[N]$ as

$$f^{T+1}(i) \leftarrow \arg\max_{s\in[K]} \sum_{j\in[N_p]} \log\sigma\left(\beta\log\frac{\pi_{(s)}^{T}(a_{i,0}^{(j)}\mid s_{i}^{(j)})}{\pi^{\mathrm{old}}(a_{i,0}^{(j)}\mid s_{i}^{(j)})} - \beta\log\frac{\pi_{(s)}^{T}(a_{i,1}^{(j)}\mid s_{i}^{(j)})}{\pi^{\mathrm{old}}(a_{i,1}^{(j)}\mid s_{i}^{(j)})}\right)$$

Output: $(\pi_{(k)}^T)_{k\in[K]}$ and f^{T+1}

Algorithm 4 RLHF with Reward Aggregation

Input: Dataset $\widehat{\mathcal{D}} = \cup_{i\in[N]}\widehat{\mathcal{D}}_i$ where $\widehat{\mathcal{D}}_i = \{(o_i^{(j)}, \tau_{i,0}^{(j)}, \tau_{i,1}^{(j)})_{j\in[N_p]}\}$ is the preference dataset for the ith human, $\lambda>0$, and $\widehat{\omega}$ from Algorithm 1. We also use Equation (F.1) for constructing a confidence set of reward function $\mathcal{R}'(\widehat{\mathcal{D}})$.

Compute policy with respect to $\mathcal{R}'(\widehat{\mathcal{D}})$ for all $i\in[N]$ by

$$\widehat{\pi} \leftarrow \arg\max_{\pi\in\Pi}\min_{\boldsymbol{r}\in\mathcal{R}'(\widehat{\mathcal{D}})} \left(J(\pi;\mathrm{Agg}_\alpha(r_1,\dots,r_N)) - \mathbb{E}_{\tau\sim\mu_{\mathrm{ref}}}[\mathrm{Agg}_\alpha(r_1,\dots,r_N)(\tau)]\right). \tag{F.5}$$

Output: $(\widehat{\omega}, \widehat{\boldsymbol{\theta}}, \widehat{\pi})$.

G Deferred Proofs in Section 3.1

G.1 Deferred Explanation of Algorithm 5 for a New Human User

Algorithm 5 addresses a scenario where a new human user, who was not a labeler before, aims to learn their own reward models using representations previously learned by other human users, focusing solely on learning $\theta_0^\star$. They leverage the learned representation $\psi_{\widehat{\omega}}$ from Algorithm 1. The input of the algorithm is $\widehat{\mathcal{D}}_0 = \{(o_0^{(j)}, \tau_{0,0}^{(j)}, \tau_{0,1}^{(j)})_{j\in[N_p]}\}$. Algorithm 5 provides an estimation of $\theta_0^\star$ with MLE using the frozen representation $\psi_{\widehat{\omega}}$. Similarly, after estimating the reward function, we construct confidence set for the MLE estimation with $\zeta = C_8\left(k\frac{\xi^2\kappa^2\log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/(NN_p))/\delta)}{\eta^2 NN_p} + \frac{\xi^2(k+\log(1/\delta))}{\eta^2 N_p} + \lambda B^2\right)$ for a constant $C_8 > 0$. Lastly, we find the best policy based on the pessimistic expected value function. $\mu_{0,\text{ref}}$ in Algorithm 5 is a known reference trajectory distribution.

Algorithm 5 Transferable RLHF for a New Human User via Representation Learning

---

Input: Dataset $\widehat{\mathcal{D}}_0 = \{(o_0^{(j)}, \tau_{0,0}^{(j)}, \tau_{0,1}^{(j)})_{j\in[N_p]}\}$ and $\widehat{\omega}$ from Algorithm 1.
Estimate $\theta_0^\star$ by

$$\widehat{\theta}_0 \leftarrow \underset{\|\theta_0\|_2\le B}{\arg\max} \sum_{j\in[N_p]} \log P_{\widehat{\omega},\theta_0}(o_0^{(j)} \mid \tau_{0,0}^{(j)}, \tau_{0,1}^{(j)})$$

Construct a confidence set of the reward function by

$$\mathcal{R}(\widehat{\mathcal{D}}) \leftarrow \left\{ r_{\omega,\theta_0} \mid \frac{1}{N_p}\sum_{j\in[N_p]} \left|(r_{\widehat{\omega},\widehat{\theta}_0}(\tau_{0,0}^{(j)}) - r_{\widehat{\omega},\widehat{\theta}_0}(\tau_{i,1}^{(j)})) - (r_{\omega,\theta_0}(\tau_{0,0}^{(j)}) - r_{\omega,\theta_0}(\tau_{0,1}^{(j)}))\right|^2 \le \zeta \right\}$$

Compute policy with respect to $\mathcal{R}(\widehat{\mathcal{D}})$ by

$$\widehat{\pi}_0 \leftarrow \underset{\pi\in\Pi}{\arg\max} \min_{r_0\in\mathcal{R}(\widehat{\mathcal{D}}_0)} \left(J(\pi; r_0) - \mathbb{E}_{\tau\sim\mu_{0,\text{ref}}}[r_0(\tau)]\right)$$

Output: $(\widehat{\pi}_i)_{i\in[N]}$.

G.1.1 Expected Value Function Gap for a New Human User

We have expected value function gap for a new human user as follows:

Theorem G.1. (Expected Value Function Gap for a New Human User). *Suppose Assumptions 1, 2, 3, and 4 hold. For any $\delta \in (0,1]$ and $\lambda > 0$, with probability at least $1-\delta$, the output $\widehat{\pi}_0$ of Algorithm 5 satisfies*

$$\begin{aligned}
&J(\pi_{0,tar}; r_0^\star) - J(\widehat{\pi}_0; r_0^\star) \\
&\le \sqrt{cC_{\boldsymbol{r}}(\mathcal{G}_{\boldsymbol{r}}, \pi_{i,tar}, \mu_{i,ref}, i)^2\left(k\frac{\xi^2\kappa^2\log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/(NN_p))/\delta)}{\eta^2 NN_p} + \frac{\xi^2(k+\log(1/\delta))}{\eta^2 N_p} + \lambda B^2\right)}
\end{aligned}$$

where $c > 0$ is a constant.

We defer this proof to Appendix G.6.

G.2 Expected Value Function Gap without Diversity Assumption

Firstly, we provide an algorithm for each reward function learning without Assumptions 2, 3, and 4.

- Confidence set (Equation (G.1)) for the MLE estimation as (Liu et al., 2022), which is also used in (Liu et al., 2023; Zhan et al., 2023; Wang et al., 2024; Zhan et al., 2022), with $\zeta = C_1\log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/(NN_p))/\delta)$ for a constant $C_1 > 0$, which will be related to Theorem G.2. the definition of bracketing number ($\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}$) is deferred to Appendix D.

We will provide the expected value function gap of the output of Algorithm 6 and the reference policy.

Algorithm 6 Personalized RLHF via Representation Learning - without Diversity Assumption

Input: Dataset $\widehat{\mathcal{D}} = \cup_{i\in[N]} \widehat{\mathcal{D}}_i$ where $\widehat{\mathcal{D}}_i = \{(o_i^{(j)}, \tau_{i,0}^{(j)}, \tau_{i,1}^{(j)})_{j\in[N_p]}\}$ is the preference dataset for the ith individual.
Estimate $\omega^\star$ and $\boldsymbol{\theta}^\star$ by

$$(\widehat{\omega}, \widehat{\boldsymbol{\theta}}) \leftarrow \underset{\omega\in\Omega, \|\theta_i\|_2\le B \text{ for all } i\in[N]}{\arg\max} \sum_{i\in[N]}\sum_{j\in[N_p]} \log P_{\omega,\theta_i}(o_i^{(j)} \mid \tau_{i,0}^{(j)}, \tau_{i,1}^{(j)})$$

Construct a confidence set of the reward function by

$$\mathcal{R}(\widehat{\mathcal{D}}) \leftarrow \left\{ \boldsymbol{r}_{\omega,\boldsymbol{\theta}} \,\middle|\, \sum_{i\in[N]}\sum_{j\in[N_p]} \log P_{\omega,\theta_i}(o_i^{(j)} \mid \tau_{i,0}^{(j)}, \tau_{i,1}^{(j)}) \ge \sum_{i\in[N]}\sum_{j\in[N_p]} \log P_{\widehat{\omega},\widehat{\theta}_i}(o_i^{(j)} \mid \tau_{i,0}^{(j)}, \tau_{i,1}^{(j)}) - \zeta \right\} \tag{G.1}$$

Compute policy with respect to $\mathcal{R}(\widehat{\mathcal{D}})$ (or $\mathcal{R}'(\widehat{\mathcal{D}})$) for all $i \in [N]$ by

$$\widehat{\pi}_i \leftarrow \underset{\pi\in\Pi}{\arg\max} \min_{\boldsymbol{r}\in\mathcal{R}(\widehat{\mathcal{D}})} \left(J(\pi; r_i) - \mathbb{E}_{\tau\sim\mu_{i,\text{ref}}}[r_i(\tau)] \right) \tag{G.2}$$

Output: $(\widehat{\omega}, \widehat{\boldsymbol{\theta}}, (\widehat{\pi}_i)_{i\in[N]})$.

Theorem G.2. (Total Expected Value Function Gap). *Suppose Assumption 1 holds. For any $\delta \in (0,1]$, with probability at least $1-\delta$, the output $(\widehat{\pi}_i)_{i\in[N]}$ of Algorithm 1 satisfies*

$$\sum_{i\in[N]} \left(J(\pi_{i,tar}; r_i^\star) - J(\widehat{\pi}_i; r_i^\star)\right) \le \sqrt{\frac{c\kappa^2 N C_{\max}^2 \log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/NN_p)/\delta)}{N_p}},$$

where $C_{\max} := \max_{i\in[N]} C_{\boldsymbol{r}}(\mathcal{G}_{\boldsymbol{r}}, \pi_{i,tar}, \mu_{i,ref}, i)$ and $c > 0$ is a constant.

Corollary G.1. (Expected Value Function Gap). *Suppose Assumption 1 holds. For any $\delta \in (0,1]$ and all $i \in [N]$, with probability at least $1-\delta$, the output $\widehat{\pi}_i$ of Algorithm 1 satisfies*

$$J(\pi_{i,tar}; r_i^\star) - J(\widehat{\pi}_i; r_i^\star) \le \sqrt{\frac{c\kappa^2 C_{\boldsymbol{r}}(\mathcal{G}_{\boldsymbol{r}}, \pi_{i,tar}, \mu_{i,ref}, i)^2 \log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/NN_p)/\delta)}{N_p}},$$

where $c > 0$ is a constant.

Note that the results above do not need any assumption on $(\theta_i^\star)_{i\in[N]}$. Still, as $N_p \to \infty$, $\widehat{\pi}_i$ has comparable or better performance than the comparator policy $\pi_{i,\text{tar}}$, which approaches the optimal policy if $\pi_{i,\text{tar}} = \pi_i^\star$. **We will leverage the proof of Theorem G.2 to prove Theorem 3.1.** To be specific, we will improve the bound for Corollary G.1, as the gap of the expected value function does not decay with N, which is the number of human users. We defer the proofs of Theorem G.2 and Corollary G.1 to Appendix G.3.

G.3 Proof of Theorem G.2 and Corollary G.1

Theorem G.2. (Total Expected Value Function Gap). *Suppose Assumption 1 holds. For any $\delta \in (0,1]$, with probability at least $1-\delta$, the output $(\widehat{\pi}_i)_{i\in[N]}$ of Algorithm 1 satisfies*

$$\sum_{i\in[N]} \left(J(\pi_{i,tar}; r_i^\star) - J(\widehat{\pi}_i; r_i^\star)\right) \le \sqrt{\frac{c\kappa^2 N C_{\max}^2 \log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/NN_p)/\delta)}{N_p}},$$

where $C_{\max} := \max_{i\in[N]} C_{\boldsymbol{r}}(\mathcal{G}_{\boldsymbol{r}}, \pi_{i,tar}, \mu_{i,ref}, i)$ and $c > 0$ is a constant.

Corollary G.1. (Expected Value Function Gap). *Suppose Assumption 1 holds. For any $\delta \in (0,1]$ and all $i \in [N]$, with probability at least $1-\delta$, the output $\widehat{\pi}_i$ of Algorithm 1 satisfies*

$$J(\pi_{i,tar}; r_i^\star) - J(\widehat{\pi}_i; r_i^\star) \le \sqrt{\frac{c\kappa^2 C_{\boldsymbol{r}}(\mathcal{G}_{\boldsymbol{r}}, \pi_{i,tar}, \mu_{i,ref}, i)^2 \log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/NN_p)/\delta)}{N_p}},$$

where $c > 0$ is a constant.

Before having a proof of Theorem G.2 and Corollary G.1, we provide two general properties of MLE estimates, which is a slightly modified version of (Zhan et al., 2023) and (Liu et al., 2022).

Lemma 1 ((Zhan et al. (2023), Lemma 1, reward vector version)). *For any $\delta \in (0,1]$, if $\boldsymbol{r} \in \mathcal{G}_{\boldsymbol{r}}$, with dataset $\widehat{\mathcal{D}} = \cup_{i\in[N]} \widehat{\mathcal{D}}_i$ where $\widehat{\mathcal{D}}_i = \{(o_i^{(j)}, \tau_{i,0}^{(j)}, \tau_{i,1}^{(j)})_{j\in[N_p]}\}$, $\tau_{i,0}^{(j)} \sim \mu_0$, $\tau_{i,1}^{(j)} \sim \mu_1$, and $o_i^{(j)} \sim P_{r_i^\star}(\cdot|\tau_0^{(j)}, \tau_1^{(j)})$, there exist $C_1 > 0$ such that*

$$\sum_{i\in[N]}\sum_{j\in[N_p]} \log\left(\frac{P_{r_i}(o_i^{(j)} \mid \tau_{i,0}^{(j)}, \tau_{i,1}^{(j)})}{P_{r_i^\star}(o_i^{(j)} \mid \tau_{i,0}^{(j)}, \tau_{i,1}^{(j)})}\right) \le C_1 \log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/(NN_p))/\delta)$$

holds.

Lemma 2 ((Liu et al. (2022), Proposition 14, scalar version)). *For any $\delta \in (0,1]$, with probability at least $1-\delta$, if $r \in \mathcal{G}_r'$, with dataset $\widehat{\mathcal{D}} = \{(o^{(j)}, \tau_0^{(j)}, \tau_1^{(j)})_{j\in[M]}\}$ where $\tau_0^{(j)} \sim \mu_0$, $\tau_1^{(j)} \sim \mu_1$, and $o^{(j)} \sim P_{r^\star}(\cdot|\tau_0^{(j)}, \tau_1^{(j)})$,*

$$\mathbb{E}_{\mu_0,\mu_1}\left[\|P_r(\cdot \mid \tau_0^{(j)}, \tau_1^{(j)}) - P_{r^\star}(\cdot \mid \tau_0^{(j)}, \tau_1^{(j)})\|_1^2\right] \le \frac{C_2}{M}\left(\sum_{j\in[M]} \log\left(\frac{P_{r^\star}(o^{(j)} \mid \tau_0^{(j)}, \tau_1^{(j)})}{P_r(o^{(j)} \mid \tau_0^{(j)}, \tau_1^{(j)})}\right) + \log(\mathcal{N}_{\mathcal{G}_r'}(1/M)/\delta)\right)$$

holds where $C_2 > 0$ is a constant.

Lemma 3 ((Liu et al. (2022), Proposition 14, vector version)). *For any $\delta \in (0,1]$, with probability at least $1-\delta$, if $\boldsymbol{r} \in \mathcal{G}_{\boldsymbol{r}}'$, with dataset $\widehat{\mathcal{D}} = \cup_{i\in[N]} \widehat{\mathcal{D}}_i$ where $\widehat{\mathcal{D}}_i = \{(o_i^{(j)}, \tau_{i,0}^{(j)}, \tau_{i,1}^{(j)})_{j\in[N_p]}\}$, $\tau_{i,0}^{(j)} \sim \mu_0$, $\tau_{i,1}^{(j)} \sim \mu_1$, and $o_i^{(j)} \sim P_{r_i^\star}(\cdot|\tau_0^{(j)}, \tau_1^{(j)})$,*

$$\begin{aligned}
&\frac{1}{N}\sum_{i\in[N]} \mathbb{E}_{\mu_0,\mu_1}\left[\|P_{r_i}(\cdot \mid \tau_0^{(j)}, \tau_1^{(j)}) - P_{r_i^\star}(\cdot \mid \tau_0^{(j)}, \tau_1^{(j)})\|_1^2\right] \\
&\qquad \le \frac{C_2}{NN_p}\left(\sum_{i\in[N]}\sum_{j\in[N_p]} \log\left(\frac{P_{r_i^\star}(o^{(j)} \mid \tau_0^{(j)}, \tau_1^{(j)})}{P_{r_i}(o^{(j)} \mid \tau_0^{(j)}, \tau_1^{(j)})}\right) + \log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}'}(1/(NN_p))/\delta)\right)
\end{aligned}$$

holds where $C_2 > 0$ is a constant.

Note that $\boldsymbol{r}^\star$ does not need to be in $\mathcal{G}_{\boldsymbol{r}}'$ for the above lemmas. Lemma 1 states that the log-likelihood $\log P_{\boldsymbol{r}}$ for a preference dataset generated by the reward model $\boldsymbol{r}^\star$ cannot exceed the log-likelihood $\log P_{\boldsymbol{r}^\star}$ for a preference dataset generated by the reward model $\boldsymbol{r}^\star$, with a gap related to the bracket number of $\mathcal{G}_{\boldsymbol{r}}$. Lemma 3 states that the ℓ_1 distance between likelihood function $P_{\boldsymbol{r}^\star}$ and $P_{\boldsymbol{r}}$ for all $\boldsymbol{r} \in \mathcal{G}_{\boldsymbol{r}}'$ can be bounded with the difference between log-likelihood $\log P_{\boldsymbol{r}^\star}$ and $\log P_{\boldsymbol{r}}$ for a preference dataset generated by the reward model $\boldsymbol{r}^\star$ with a gap related to the bracket number of $\mathcal{G}_{\boldsymbol{r}}'$.

Proof of Theorem G.2 and Corollary G.1. We define the event $\mathcal{E}_1, \mathcal{E}_2$ as satisfying (Lemma 1, Lemma 3) with $\delta \leftarrow \delta/2$, respectively, so we have $\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2) > 1-\delta$. We will only consider the under event $\mathcal{E}_1 \cap \mathcal{E}_2$. Then, we can guarantee that

$$\begin{aligned}
&\sum_{i\in[N]}\sum_{j\in[N_p]} \log P_{\widehat{\omega},\widehat{\theta}_i}(o_i^{(j)} \mid \tau_{i,0}^{(j)}, \tau_{i,1}^{(j)}) \\
&\qquad \le \sum_{i\in[N]}\sum_{j\in[N_p]} \log P_{\omega^\star,\theta_i^\star}(o_i^{(j)} \mid \tau_{i,0}^{(j)}, \tau_{i,1}^{(j)}) + C_1 \log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/(NN_p))/\delta),
\end{aligned}$$

which indicates that $\boldsymbol{r}^\star (= \boldsymbol{r}_{\omega^\star,\boldsymbol{\theta}^\star}) \in \mathcal{R}(\widehat{\mathcal{D}})$. Moreover, by the definition of Equation (G.1), if $\boldsymbol{r}_{\omega,\boldsymbol{\theta}}, \boldsymbol{r}_{\omega',\boldsymbol{\theta}'} \in \mathcal{R}(\widehat{\mathcal{D}})$,

$$\begin{aligned}
&\Big|\sum_{i\in[N]}\sum_{j\in[N_p]} \log P_{\omega,\theta_i}(o_i^{(j)} \mid \tau_{i,0}^{(j)}, \tau_{i,1}^{(j)}) - \sum_{i\in[N]}\sum_{j\in[N_p]} \log P_{\omega',\theta_i'}(o_i^{(j)} \mid \tau_{i,0}^{(j)}, \tau_{i,1}^{(j)})\Big| \\
&\qquad \le C_1 \log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/(NN_p))/\delta)
\end{aligned}$$

holds, since $\sum_{i\in[N]}\sum_{j\in[N_p]}\log P_{\omega,\theta_i}(o_i^{(j)} \mid \tau_{i,0}^{(j)},\tau_{i,1}^{(j)})$ is bounded by $\sum_{i\in[N]}\sum_{j\in[N_p]}\log P_{\widehat{\omega},\widehat{\theta}_i}(o_i^{(j)} \mid \tau_{i,0}^{(j)},\tau_{i,1}^{(j)})$ by definition of $\widehat{\omega},\widehat{\boldsymbol{\theta}}$ if $r_{\omega,\boldsymbol{\theta}} \in \mathcal{G}_{\boldsymbol{r}}$. Therefore, by Lemma 3, we have

$$
\begin{aligned}
&\frac{1}{N}\sum_{i\in[N]}\mathbb{E}_{\mu_0,\mu_1}\left[\|P_{\omega,\theta_i}(\cdot \mid \tau_{i,0}^{(j)},\tau_{i,1}^{(j)}) - P_{\omega^\star,\theta_i^\star}(\cdot \mid \tau_{i,0}^{(j)},\tau_{i,1}^{(j)})\|_1^2\right] \\
&\le \frac{C_2}{NN_p}\left(\sum_{i\in[N]}\sum_{j\in[N_p]}\log\left(\frac{P_{\omega^\star,\theta_i^\star}(o_i^{(j)} \mid \tau_{i,0}^{(j)},\tau_{i,1}^{(j)})}{P_{\omega,\theta_i}(o_i^{(j)} \mid \tau_{i,0}^{(j)},\tau_{i,1}^{(j)})}\right) + \log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/(NN_p))/\delta)\right) \\
&\le \frac{C_2}{NN_p}\left(C_1\log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/(NN_p))/\delta) + \log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/(NN_p))/\delta)\right) \\
&= \frac{C_3}{NN_p}\log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/(NN_p))/\delta)
\end{aligned}
$$

for any $\boldsymbol{r}_{\omega,\boldsymbol{\theta}} \in \mathcal{R}(\widehat{\mathcal{D}})$, where $C_3 = C_2(C_1+1)$. Then, by the mean value theorem, for any $\boldsymbol{r}_{\omega,\boldsymbol{\theta}} \in \mathcal{R}(\widehat{\mathcal{D}})$, we have

$$
\begin{aligned}
&\frac{1}{N}\sum_{i\in[N]}\mathbb{E}_{\mu_0,\mu_1}\left[\left|(r_{\omega,\theta_i}(\tau_{i,0}) - r_{\omega,\theta_i}(\tau_{i,1})) - (r_i^\star(\tau_{i,0}) - r_i^\star(\tau_{i,1}))\right|^2\right] \\
&\le \frac{\kappa^2}{N}\sum_{i\in[N]}\mathbb{E}_{\mu_0,\mu_1}\left[\|P_{\omega,\boldsymbol{\theta}}(\cdot \mid \tau_{i,0}^{(j)},\tau_{i,1}^{(j)},i) - P_{\omega^\star,\boldsymbol{\theta}^\star}(\cdot \mid \tau_{i,0}^{(j)},\tau_{i,1}^{(j)},i)\|_1^2\right] \\
&\le \frac{C_3\kappa^2}{NN_p}\log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/(NN_p))/\delta).
\end{aligned}
\tag{G.3}
$$

Now, we define for all policy π,

$$
r_\pi^{i,\mathrm{inf}} := \operatorname*{arg\,min}_{\boldsymbol{r}\in\mathcal{R}(\mathcal{D})}\left(J(\pi,r_i) - \mathbb{E}_{\tau\sim\mu_{i,\mathrm{ref}}}[r_i(\tau)]\right).
$$

Then, we can bound the difference of the expected cumulative reward of a policy $\pi_{i,\mathrm{tar}}$ and $\widehat{\pi}_i$ by

$$
\begin{aligned}
&J(\pi_{i,\mathrm{tar}};r_i^\star) - J(\widehat{\pi}_i;r_i^\star) \\
&= (J(\pi_{i,\mathrm{tar}};r_i^\star) - \mathbb{E}_{\tau\sim\mu_{i,\mathrm{ref}}}[r_i^\star(\tau)]) - (J(\widehat{\pi}_i;r_i^\star) - \mathbb{E}_{\tau\sim\mu_{i,\mathrm{ref}}}[r_i^\star(\tau)]) \\
&\underset{(i)}{\le} (J(\pi_{i,\mathrm{tar}};r_i^\star) - \mathbb{E}_{\tau\sim\mu_{i,\mathrm{ref}}}[r_i^\star(\tau)]) \\
&\qquad - (J(\pi_{i,\mathrm{tar}};r_{\pi_{i,\mathrm{tar}}}^{i,\mathrm{inf}}) - \mathbb{E}_{\tau\sim\mu_{i,\mathrm{ref}}}[r_{\pi_{i,\mathrm{tar}}}^{i,\mathrm{inf}}(\tau)]) + (J(\widehat{\pi}_j;r_{\widehat{\pi}_i}^{i,\mathrm{inf}}) - \mathbb{E}_{\tau\sim\mu_{i,\mathrm{ref}}}(r_{\widehat{\pi}_i}^{i,\mathrm{inf}}(\tau))) \\
&\qquad - (J(\widehat{\pi}_i;r_i^\star) - \mathbb{E}_{\tau\sim\mu_{i,\mathrm{ref}}}[r_i^\star(\tau)]) \\
&\underset{(ii)}{\le} (J(\pi_{i,\mathrm{tar}};r_i^\star) - \mathbb{E}_{\tau\sim\mu_{i,\mathrm{ref}}}[r_i^\star(\tau)]) - (J(\pi_{i,\mathrm{tar}};r_{\pi_{i,\mathrm{tar}}}^{i,\mathrm{inf}}) - \mathbb{E}_{\tau\sim\mu_{i,\mathrm{ref}}}[r_{\pi_{i,\mathrm{tar}}}^{i,\mathrm{inf}}(\tau)]) \\
&= \mathbb{E}_{\tau_{i,0}\sim\pi_{i,\mathrm{tar}},\tau_{i,1}\sim\mu_{i,\mathrm{ref}}}[(r_i^\star(\tau_{i,1}) - r_i^\star(\tau_{i,0})) - (r_{\pi_{i,\mathrm{tar}}}^{i,\mathrm{inf}}(\tau_{i,1}) - r_{\pi_{i,\mathrm{tar}}}^{i,\mathrm{inf}}(\tau_{i,0}))] \\
&\le C_{\boldsymbol{r}}(\mathcal{G}_{\boldsymbol{r}},\pi_{i,\mathrm{tar}},\mu_{i,\mathrm{ref}},i)\sqrt{\mathbb{E}_{\mu_0,\mu_1}\left[\left|(r_i^\star(\tau_{i,1}) - r_i^\star(\tau_{i,0})) - (r_{\pi_{i,\mathrm{tar}}}^{i,\mathrm{inf}}(\tau_{i,1}) - r_{\pi_{i,\mathrm{tar}}}^{i,\mathrm{inf}}(\tau_{i,0}))\right|^2\right]}
\end{aligned}
\tag{G.4}
$$

Here, (i) holds since $\widehat{\pi}_j$ is a distributional robust policy for $\mathcal{R}(\widehat{\mathcal{D}})$ (Equation (G.1)) and (ii) holds due to the definition of $r^{i,\inf}_{\widehat{\pi}_i}$. Therefore, if we sum Equation (G.4) over $i \in [N]$, we have

$$
\begin{aligned}
&\sum_{i\in[N]} \left(J(\pi_{i,\text{tar}}; r_i^\star) - J(\widehat{\pi}_i; r_i^\star)\right) \\
&\leq C_{\max} \sum_{i\in[N]} \sqrt{\mathbb{E}_{\mu_0,\mu_1}\left[\left|(r_i^\star(\tau_{i,1}) - r_i^\star(\tau_{i,0})) - (r^{i,\inf}_{\pi_{i,\text{tar}}}(\tau_{i,1}) - r^{i,\inf}_{\pi_{i,\text{tar}}}(\tau_{i,0}))\right|^2\right]} \\
&\leq C_{\max} \sqrt{N \sum_{i\in[N]} \mathbb{E}_{\mu_0,\mu_1}\left[\left|(r_i^\star(\tau_{i,1}) - r_i^\star(\tau_{i,0})) - (r^{i,\inf}_{\pi_{i,\text{tar}}}(\tau_{i,1}) - r^{i,\inf}_{\pi_{i,\text{tar}}}(\tau_{i,0}))\right|^2\right]} \\
&\leq C_{\max} \sqrt{\frac{C_3 N \kappa^2 \log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/NN_p)/\delta)}{N_p}},
\end{aligned}
$$

which proves Theorem G.2. Moreover, we have

$$
\begin{aligned}
&J(\pi_{i,\text{tar}}; r_i^\star) - J(\widehat{\pi}_i; r_i^\star) \\
&\leq C_{\boldsymbol{r}}(\mathcal{G}_{\boldsymbol{r}}, \pi_{i,\text{tar}}, \mu_{i,\text{ref}}, i) \sqrt{\mathbb{E}_{\mu_0,\mu_1}\left[\left|(r_i^\star(\tau_{i,1}) - r_i^\star(\tau_{i,0})) - (r^{i,\inf}_{\pi_{i,\text{tar}}}(\tau_{i,1}) - r^{i,\inf}_{\pi_{i,\text{tar}}}(\tau_{i,0}))\right|^2\right]} \\
&\leq C_{\boldsymbol{r}}(\mathcal{G}_{\boldsymbol{r}}, \pi_{i,\text{tar}}, \mu_{i,\text{ref}}, i) \sqrt{\sum_{i\in[N]} \mathbb{E}_{\mu_0,\mu_1}\left[\left|(r_i^\star(\tau_{i,1}) - r_i^\star(\tau_{i,0})) - (r^{i,\inf}_{\pi_{i,\text{tar}}}(\tau_{i,1}) - r^{i,\inf}_{\pi_{i,\text{tar}}}(\tau_{i,0}))\right|^2\right]} \\
&\leq C_{\boldsymbol{r}}(\mathcal{G}_{\boldsymbol{r}}, \pi_{i,\text{tar}}, \mu_{i,\text{ref}}, i) \sqrt{\frac{C_3 \kappa^2 \log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/NN_p)/\delta)}{N_p}}
\end{aligned}
$$

which proves Corollary G.1. $\square$

G.4 Discussion on Assumption 3

G.4.1 Comparing with (Lu et al., 2021, Assumption 6.4)

Assumption 5 ((Lu et al. (2021), Assumption 6.4))**.** *For any representation functions $\psi, \psi' \in \Psi$ and $\epsilon > 0$, if there exists $v, v' \in \mathbb{R}^d$ that satisfy*

$$\mathbb{E}\|\psi(x)^\top v - \psi'(x)^\top v'\|^2 \leq \epsilon$$

Then there exists a constant invertible matrix P such that

$$\|\psi(x) - P\psi'(x)\|^2 \leq o(\epsilon/\|v\|^2) = o(\epsilon/\|v'\|^2).$$

for all x.

Assumption 3 bears similarity to Assumption 5; however, the latter is notably more stringent. For instance, consider the case where $v = v' = e_1$ without loss of generality. If it holds that $\mathbb{E}\|\psi_1(x) - \psi_1'(x)\|^2 \leq \epsilon$, then it implies $\psi \sim P\psi'$. In this context, ψ_1 and ψ_1' represent the first coordinates of ψ and ψ', respectively. The assumption that similarity in the first coordinate necessitates equivalence of the entire representations ($\psi \sim P\psi'$) is a strong assumption.

G.4.2 Case Study (Linear Representation): $\psi_\omega(x) = \omega x$ and ω is an Orthonormal Matrix

Proposition 1. *Assume that $\psi_\omega(\phi(\tau)) = \omega\phi(\tau)$ where ω is a $k \times d$ orthornormal matrix. For any representation functions $\psi_\omega, \psi_{\omega'} \in \Psi$ and $\epsilon > 0$, if there exists $\{v_i\}_{i=1}^T, \{v_i'\}_{i=1}^T$, and a trajectory distribution μ that satisfy*

$$\frac{1}{T}\sum_{i\in[T]} \mathbb{E}_{\tau\sim\mu}\|\psi_\omega(\phi(\tau))^\top v_i - \psi_{\omega'}(\phi(\tau))^\top v_i'\|^2 \leq \epsilon \tag{G.5}$$

dand $V = [v_1, v_2, \cdots, v_T] \in \mathbb{R}^{k\times T}$ *satisfies* $\sigma_k^2(W) \geq \Omega(T/k)$*, and* $\|v_i\|_2 \leq B$ *for all* $i \in [T]$*. If* $\Sigma := \mathbb{E}_\mu[\phi(\tau)\phi(\tau)^\intercal] \succ \mathbf{O}$*, then there exists a constant invertible matrix* P *such that*

$$\|\psi_\omega(\phi(\tau)) - P\psi_{\omega'}(\phi(\tau))\|^2 \leq ck\epsilon/B$$

where $c > 0$ *is a constant.*

Proof. By Equation (G.5), we have

$$(\omega^\intercal V - (\omega')^\intercal V')^\intercal \Sigma(\omega^\intercal V - (\omega')^\intercal V') \leq T\epsilon,$$

where $V' = [v'_1, \ldots, v'_T] \in \mathbb{R}^{k\times T}$. Since $\Sigma \succ \mathbf{O}$, we have

$$\|\omega^\intercal V - (\omega')^\intercal V'\|^2 \leq T\epsilon.$$

By (Yu et al., 2015, Theorem 4), there exist an orthonormal matrix P such that

$$\|\omega - P(\omega')^\intercal\|^2 \leq ck\epsilon$$

where $c > 0$ is a constant, which concludes Proposition 1. □

G.5 Proof of Corollary G.2

With Assumption 2 and Assumption 3, $\psi^\star$ and ψ_ω are close up to an orthonormal matrix transformation, as asserted below:

Corollary G.2. (Closeness between $\psi^\star$ and ψ_ω). *Suppose Assumptions 1, 2, and 3 hold. For any* $\delta \in (0, 1]$*, with probability at least* $1 - \delta$*, if* $\boldsymbol{r}_{\omega,\boldsymbol{\theta}} \in \mathcal{R}'(\mathcal{D})$ *as specified in Algorithm 1, then there exists an orthonormal matrix* P_ω *such that*

$$\left[\|\psi^\star(\phi(\tau_0)) - \psi^\star(\phi(\tau_1)) - P_\omega(\psi_\omega(\phi(\tau_0)) - \psi_\omega(\phi(\tau_1)))\|^2\right] \leq k\frac{c_{rep}\kappa^2\log(\mathcal{N}_{\mathcal{G}_r}(1/(NN_p))/\delta)}{NN_pB^2}$$

for all τ_0, τ_1*, where* $c_{rep} > 0$ *is a constant.*

Proof. By Equation (G.3), if we use Assumption 3 with $\Theta^\star/B$, we can find an orthonormal matrix P_ω such that

$$\left[\|\psi^\star(\phi(\tau_0)) - \psi^\star(\phi(\tau_1)) - P_\omega(\psi_\omega(\phi(\tau_0)) - \psi_\omega(\phi(\tau_1)))\|^2\right] \leq k\frac{c_{\text{rep}}\kappa^2\log(\mathcal{N}_{\mathcal{G}_r}(1/(NN_p))/\delta)}{NN_pB^2}$$

for all τ_0, τ_1, where $c_{\text{rep}} > 0$ is a constant. □

G.6 Proof of Theorem 3.1

Lemma 4. *Suppose Assumptions 1, 2 and 3 hold. For any* $\delta \in (0, 1]$ *and* $\lambda > 0$*, with probability at least* $1 - \delta$*,* $\boldsymbol{r}^\star \in \mathcal{R}'(\widehat{\mathcal{D}})$*, i.e., the underlying reward functions are an element of Equation (F.1).*

Proof. Assume that Corollary G.2 holds with probability $1 - \delta/2$ for $\widehat{\omega}$, i.e.,

$$\left[\|\psi^\star(\phi(\tau_0)) - \psi^\star(\phi(\tau_1)) - P_{\widehat{\omega}}(\psi_{\widehat{\omega}}(\phi(\tau_0)) - \psi_{\widehat{\omega}}(\phi(\tau_1)))\|^2\right] \leq k\frac{c_{\text{rep}}\kappa^2\log(\mathcal{N}_{\mathcal{G}_r}(1/(NN_p))/\delta)}{NN_pB^2}. \tag{G.6}$$

We only consider the event that Equation (G.6) holds. We will use this $P_{\widehat{\omega}}$ for the proof of Theorem 3.1. We will approach similarly with the proof of (Zhu et al., 2023). Consider the following optimization problem:

$$\underset{\|\theta\|_i \leq B}{\text{maximize}} f(\theta_i) := \frac{1}{N_p}\sum_{j\in[N_p]}\log P_{\widehat{\omega},\theta_i}(o_i^{(j)} \mid \tau_{i,0}^{(j)}, \tau_{i,1}^{(j)}).$$

Then, we have $\widehat{\theta}_i = \underset{\|\theta\|_i \le B}{\arg\max} f(\theta_i)$ and

$$\nabla f(\theta_i) = \frac{1}{N_p}\sum_{j\in[N_p]}\Bigg(\frac{\Phi'(\langle\psi_{\widehat{\omega}}(\phi(\tau_{i,0}^{(j)})) - \psi_{\widehat{\omega}}(\phi(\tau_{i,1}^{(j)})), \theta_i\rangle)}{\Phi(\psi_{\widehat{\omega}}(\phi(\tau_{i,0}^{(j)})) - \psi_{\widehat{\omega}}(\phi(\tau_{i,1}^{(j)})), \theta_i\rangle)}\mathbf{1}(o_i^{(j)}=0)$$
$$- \frac{\Phi'(\langle\psi_{\widehat{\omega}}(\phi(\tau_{i,1}^{(j)})) - \psi_{\widehat{\omega}}(\phi(\tau_{i,0}^{(j)})), \theta_i\rangle)}{\Phi(\psi_{\widehat{\omega}}(\phi(\tau_{i,1}^{(j)})) - \psi_{\widehat{\omega}}(\phi(\tau_{i,0}^{(j)})), \theta_i\rangle)}\mathbf{1}(o_i^{(j)}=1)\Bigg)\left(\psi_{\widehat{\omega}}(\phi(\tau_{i,0}^{(j)})) - \psi_{\widehat{\omega}}(\phi(\tau_{i,1}^{(j)}))\right)$$

$$\nabla^2 f(\theta_i) = \frac{1}{N_p}\sum_{j\in[N_p]}\frac{\Phi''(x_i^{(j)})\Phi(x_i^{(j)}) - \Phi'(x_i^{(j)})^2}{\Phi(x_i^{(j)})^2}\left(\psi_{\widehat{\omega}}(\phi(\tau_{i,0}^{(j)})) - \psi_{\widehat{\omega}}(\phi(\tau_{i,1}^{(j)}))\right)\left(\psi_{\widehat{\omega}}(\phi(\tau_{i,0}^{(j)})) - \psi_{\widehat{\omega}}(\phi(\tau_{i,1}^{(j)}))\right)^\intercal$$

where $x_i^{(j)} = \langle\psi_{\widehat{\omega}}(\phi(\tau_{i,0}^{(j)})) - \psi_{\widehat{\omega}}(\phi(\tau_{i,1}^{(j)})), \theta_i\rangle$. Here, we also define $\psi_{\widehat{\omega}}(\widehat{\mathcal{D}}_i) \in \mathbb{R}^{N_p\times k}$ such as every $j\in[N_p]$th row is $\left(\psi_\omega(\phi(\tau_{i,0}^{(j)})) - \psi_\omega(\phi(\tau_{i,1}^{(j)}))\right)$.

Then, we have

$$\nabla^2 f(\theta_i) \preceq -\eta\widehat{\Sigma}_{\psi_{\widehat{\omega}}} := -\frac{\eta}{N_p}\sum_{j\in[N_p]}\left(\psi_{\widehat{\omega}}(\phi(\tau_{i,0}^{(j)})) - \psi_{\widehat{\omega}}(\phi(\tau_{i,1}^{(j)}))\right)\left(\psi_{\widehat{\omega}}(\phi(\tau_{i,0}^{(j)})) - \psi_{\widehat{\omega}}(\phi(\tau_{i,1}^{(j)}))\right)^\intercal$$

where $\eta := \min_{x\in[-2R_{\max},2R_{\max}]}\left(\frac{\Phi'(x)^2 - \Phi''(x)\Phi(x)}{\Phi(x)^2}\right)$. For example, if $\Phi(x) = \sigma(x)$, then $\eta = \frac{1}{2+\exp(-2R_{\max})+\exp(2R_{\max})}$.

Then, by the Taylor expansion of f, we have

$$f(\widehat{\theta}_i) - f(P_{\widehat{\omega}}^\intercal\theta_i^\star) - \langle\nabla f(P_{\widehat{\omega}}^\intercal\theta_i^\star), \widehat{\theta}_i - P_{\widehat{\omega}}^\intercal\theta_i^\star\rangle \le -\frac{\eta}{2}\|\widehat{\theta}_i - P_{\widehat{\omega}}^\intercal\theta_i^\star\|^2_{\widehat{\Sigma}_{\psi_{\widehat{\omega}}}}.$$

Since $\widehat{\theta}_i = \underset{\|\theta\|_i \le B}{\arg\max} f(\theta_i)$, for any $\lambda > 0$, we have

$$\|\nabla f(P_{\widehat{\omega}}^\intercal\theta_i^\star)\|_{(\widehat{\Sigma}_{\psi_{\widehat{\omega}}}+\lambda I)^{-1}}\|\widehat{\theta}_i - P_{\widehat{\omega}}^\intercal\theta_i^\star\|_{\widehat{\Sigma}_{\psi_{\widehat{\omega}}}+\lambda I} \ge \langle\nabla f(P_{\widehat{\omega}}^\intercal\theta_i^\star), \widehat{\theta}_i - P_{\widehat{\omega}}^\intercal\theta_i^\star\rangle \ge \frac{\eta}{2}\|\widehat{\theta}_i - P_{\widehat{\omega}}^\intercal\theta_i^\star\|^2_{\widehat{\Sigma}_{\psi_{\widehat{\omega}}}}. \tag{G.7}$$

We define a random vector $V \in \mathbb{R}^{N_p}$ as follows:

$$V_j = \begin{cases} \frac{\Phi'(\langle\psi^\star(\phi(\tau_{i,0}^{(j)})) - \psi^\star(\phi(\tau_{i,1}^{(j)})), \theta_i^\star\rangle)}{\Phi(\psi^\star(\phi(\tau_{i,0}^{(j)})) - \psi^\star(\phi(\tau_{i,1}^{(j)})), \theta_i^\star\rangle)} & \text{w.p.} \quad \Phi(\psi^\star(\phi(\tau_{i,0}^{(j)})) - \psi^\star(\phi(\tau_{i,1}^{(j)})), \theta_i^\star\rangle) \\ -\frac{\Phi'(\langle\psi^\star(\phi(\tau_{i,1}^{(j)})) - \psi^\star(\phi(\tau_{i,0}^{(j)})), \theta_i^\star\rangle)}{\Phi(\psi^\star(\phi(\tau_{i,1}^{(j)})) - \psi^\star(\phi(\tau_{i,0}^{(j)})), \theta_i^\star\rangle)} & \text{w.p.} \quad \Phi(\psi^\star(\phi(\tau_{i,1}^{(j)})) - \psi^\star(\phi(\tau_{i,0}^{(j)})), \theta_i^\star\rangle) \end{cases}$$

for all $j\in[N_p]$. Define $\xi = \max_{x\in[-2R_{\max},2R_{\max}]}\left|\frac{\Phi'(x)}{\Phi(x)}\right|$. If $\Phi(x) = \sigma(x)$, $\xi \le 1$. Then, we can verify that $\mathbb{E}[V] = 0$ and $|V_j| \le \xi$ for all $j \in [N_p]$.

Also, define $V' \in \mathbb{R}^{N_p}$ as follows:

$$V_j' = \begin{cases} \frac{\Phi'(\langle\psi_{\widehat{\omega}}(\phi(\tau_{i,0}^{(j)})) - \psi_{\widehat{\omega}}(\phi(\tau_{i,1}^{(j)})), P_{\widehat{\omega}}^\intercal\theta_i^\star\rangle)}{\Phi(\psi_{\widehat{\omega}}(\phi(\tau_{i,0}^{(j)})) - \psi_{\widehat{\omega}}(\phi(\tau_{i,1}^{(j)})), P_{\widehat{\omega}}^\intercal\theta_i^\star\rangle)} & \text{w.p.} \quad \Phi(\psi^\star(\phi(\tau_{i,0}^{(j)})) - \psi^\star(\phi(\tau_{i,1}^{(j)})), \theta_i^\star\rangle) \\ -\frac{\Phi'(\langle\psi_{\widehat{\omega}}(\phi(\tau_{i,1}^{(j)})) - \psi_{\widehat{\omega}}(\phi(\tau_{i,0}^{(j)})), P_{\widehat{\omega}}^\intercal\theta_i^\star\rangle)}{\Phi(\psi_{\widehat{\omega}}(\phi(\tau_{i,1}^{(j)})) - \psi_{\widehat{\omega}}(\phi(\tau_{i,0}^{(j)})), P_{\widehat{\omega}}^\intercal\theta_i^\star\rangle)} & \text{w.p.} \quad \Phi(\psi^\star(\phi(\tau_{i,1}^{(j)})) - \psi^\star(\phi(\tau_{i,0}^{(j)})), \theta_i^\star\rangle) \end{cases}$$

for all $j\in[N_p]$. $\nabla f(P_{\widehat{\omega}}^\intercal\theta_i^\star)$ can be written as

$$\nabla f(P_{\widehat{\omega}}^\intercal\theta_i^\star) = \frac{1}{N_p}\psi_{\widehat{\omega}}(\widehat{\mathcal{D}}_i)^\intercal V_i' = \frac{1}{N_p}\psi_{\widehat{\omega}}(\widehat{\mathcal{D}}_i)^\intercal V_i + \frac{1}{N_p}\psi_{\widehat{\omega}}(\widehat{\mathcal{D}}_i)^\intercal(V_i' - V_i).$$

Therefore, we can bound $\|\nabla f(P_{\widehat{\omega}}^\intercal\theta_i^\star)\|_{(\widehat{\Sigma}_{\psi_{\widehat{\omega}}}+\lambda I)^{-1}}$ by

$$\|\nabla f(P_{\widehat{\omega}}^\intercal\theta_i^\star)\|_{(\widehat{\Sigma}_{\psi_{\widehat{\omega}}}+\lambda I)^{-1}} \le \underbrace{\|\frac{1}{N_p}\psi_{\widehat{\omega}}(\widehat{\mathcal{D}}_i)^\intercal V_i\|_{(\widehat{\Sigma}_{\psi_{\widehat{\omega}}}+\lambda I)^{-1}}}_{(i)} + \underbrace{\|\frac{1}{N_p}\psi_{\widehat{\omega}}(\widehat{\mathcal{D}}_i)^\intercal(V_i' - V_i)\|_{(\widehat{\Sigma}_{\psi_{\widehat{\omega}}}+\lambda I)^{-1}}}_{(ii)}.$$

Step 1: Bounding (i).
Define $M = \frac{1}{N_p^2}\psi_{\widehat{\omega}}(\widehat{\mathcal{D}}_i)(\widehat{\Sigma}_{\psi_{\widehat{\omega}}} + \lambda I)^{-1}\psi_{\widehat{\omega}}(\widehat{\mathcal{D}}_i)^\intercal$, then we have

$$\|\frac{1}{N_p}\psi_{\widehat{\omega}}(\widehat{\mathcal{D}}_i)^\intercal V_i\|_{(\widehat{\Sigma}_{\psi_{\widehat{\omega}}}+\lambda I)^{-1}} = V^\intercal M V.$$

We can check

$$\mathrm{Tr}(M) \le \frac{k}{N_p}, \qquad \mathrm{Tr}\left(M^2\right) \le \frac{k}{N_p^2}, \qquad \|M\|_F = \sigma_1(M) \le \frac{1}{N_p}$$

in the same way with (Zhu et al., 2023, Page 19). Therefore, as V's components are bounded, independent, and $\mathbb{E}V = \mathbf{0}$, we can use Bernstein's inequality in quadratic form (for example, (Hsu et al., 2012, Theorem 2.1) and (Zhu et al., 2023, Page 19)), so we have

$$\|\frac{1}{N_p}\psi_{\widehat{\omega}}(\widehat{\mathcal{D}}_i)^\intercal V_i\|_{(\widehat{\Sigma}_{\psi_{\widehat{\omega}}}+\lambda I)^{-1}} \le \xi C_4\sqrt{\frac{k+\log(N/\delta)}{N_p}} \tag{G.8}$$

for a constant $C_4 > 0$ with probability at least $1-\delta/(2N)$.

Step 2: Bounding (ii).
We have $\left|\frac{\Phi'(x)}{\Phi(x)} - \frac{\Phi'(y)}{\Phi(y)}\right| \le \xi|x-y|$ by the mean value theorem if $x, y \in [-2R_{\max}, 2R_{\max}]$, so

$$\begin{aligned}|V_i - V_i'| &\le \max_{\tau_0,\tau_1}\xi|\langle(\psi^\star(\phi(\tau_0)) - \psi^\star(\phi(\tau_1))) - (P_{\widehat{\omega}}\psi_{\widehat{\omega}}(\phi(\tau_0)) - P_{\widehat{\omega}}\psi_{\widehat{\omega}}(\phi(\tau_1))), \theta_i^\star\rangle| \\ &\le \xi\sqrt{k\frac{c_{\mathrm{rep}}\kappa^2\log(\mathcal{N}_{\mathcal{G}_r}(1/(NN_p))/\delta)}{NN_p}}.\end{aligned}$$

Therefore, we have

$$\|\frac{1}{N_p}\psi_{\widehat{\omega}}(\widehat{\mathcal{D}}_i)^\intercal(V_i' - V_i)\|_{(\widehat{\Sigma}_{\psi_{\widehat{\omega}}}+\lambda I)^{-1}} \le \frac{\xi C_5}{\sqrt{N_p}}\sqrt{k\frac{\kappa^2\log(\mathcal{N}_{\mathcal{G}_r}(1/(NN_p))/\delta)}{NN_p}} \tag{G.9}$$

where $C_5 > 0$ is a constant.

Step 3: Combining (i) and (ii).

Combining Equation (G.8) and Equation (G.9), we have

$$\begin{aligned}\|\nabla f(P_{\widehat{\omega}}^\intercal\theta_i^\star)\|_{(\widehat{\Sigma}_{\psi_{\widehat{\omega}}}+\lambda I)^{-1}} &\le \frac{\xi C_5}{\sqrt{N_p}}\sqrt{k\frac{\kappa^2\log(\mathcal{N}_{\mathcal{G}_r}(1/(NN_p))/\delta)}{NN_p}} + \xi C_4\sqrt{\frac{k+\log(N/\delta)}{N_p}} \\ &\le C_6\sqrt{k\frac{\xi^2\kappa^2\log(\mathcal{N}_{\mathcal{G}_r}(1/(NN_p))/\delta)}{NN_p} + \frac{\xi^2(k+\log(N/\delta))}{N_p}}\end{aligned}$$

for a constant $C_6 > 0$ with probability at least $1-\delta/N$ and Equation (G.7) provides

$$\|\widehat{\theta}_i - P_{\widehat{\omega}}^\intercal\theta_i^\star\|_{\widehat{\Sigma}_{\psi_{\widehat{\omega}}}} \le C_7\sqrt{k\frac{\xi^2\kappa^2\log(\mathcal{N}_{\mathcal{G}_r}(1/(NN_p))/\delta)}{\eta^2 NN_p} + \frac{\xi^2(k+\log(N/\delta))}{\eta^2 N_p} + \lambda B^2},$$

which is equivalent to

$$\begin{aligned}&\frac{1}{N_p}\sum_{j\in[N_p]}\left|\langle(\psi_{\widehat{\omega}}(\phi(\tau_{i,0}^{(j)})) - \psi_{\widehat{\omega}}(\phi(\tau_{i,1}^{(j)}))), \widehat{\theta}_i - P_{\widehat{\omega}}^\intercal\theta_i^\star\rangle\right|^2 \\ &\qquad\le C_7^2\left(k\frac{\xi^2\kappa^2\log(\mathcal{N}_{\mathcal{G}_r}(1/(NN_p))/\delta)}{\eta^2 NN_p} + \frac{\xi^2(k+\log(N/\delta))}{\eta^2 N_p} + \lambda B^2\right),\end{aligned}$$

with probability at least $1-\delta/N$.

Now, we will bound $\frac{1}{N_p}\sum_{j\in[N_p]}\left|(r_{\widehat{\omega},\widehat{\theta}_i}(\tau_{i,0}^{(j)}) - r_{\widehat{\omega},\widehat{\theta}_i}(\tau_{i,1}^{(j)})) - (r_i^\star(\tau_{i,0}^{(j)}) - r_i^\star(\tau_{i,1}^{(j)}))\right|^2$:

$$
\begin{aligned}
&\frac{1}{N_p}\sum_{j\in[N_p]}\left|(r_{\widehat{\omega},\widehat{\theta}_i}(\tau_{i,0}^{(j)}) - r_{\widehat{\omega},\widehat{\theta}_i}(\tau_{i,1}^{(j)})) - (r_i^\star(\tau_{i,0}^{(j)}) - r_i^\star(\tau_{i,1}^{(j)}))\right|^2 \\
&= \frac{1}{N_p}\sum_{j\in[N_p]}\left|\langle \psi_{\widehat{\omega}}(\phi(\tau_{i,0}^{(j)})) - \psi_{\widehat{\omega}}(\phi(\tau_{i,1}^{(j)})), \widehat{\theta}_i\rangle - \langle \psi^\star(\phi(\tau_{i,0}^{(j)})) - \psi^\star(\phi(\tau_{i,1}^{(j)})), \theta_i^\star\rangle\right|^2 \\
&\le \frac{2}{N_p}\sum_{j\in[N_p]}\left|\langle(\psi_{\widehat{\omega}}(\phi(\tau_{i,0}^{(j)})) - \psi_{\widehat{\omega}}(\phi(\tau_{i,1}^{(j)}))), \widehat{\theta}_i - P_{\widehat{\omega}}^\intercal\theta_i^\star\rangle\right|^2 \\
&\quad + \frac{2}{N_p}\sum_{j\in[N_p]}\left|\langle \psi_{\widehat{\omega}}(\phi(\tau_{i,0}^{(j)})) - \psi_{\widehat{\omega}}(\phi(\tau_{i,1}^{(j)})) - P_{\widehat{\omega}}(\psi^\star(\phi(\tau_{i,0}^{(j)})) - \psi^\star(\phi(\tau_{i,1}^{(j)}))), \theta_i^\star\rangle\right|^2 \\
&\le 2C_7\left(k\frac{\xi^2\kappa^2\log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/(NN_p))/\delta)}{\eta^2 NN_p} + \frac{\xi^2(k+\log(N/\delta))}{\eta^2 N_p} + \lambda B^2\right) \\
&\quad + \frac{2}{N_p}N_p k\frac{c_{\text{rep}}\kappa^2\log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/(NN_p))/\delta)}{NN_p} \\
&\le C_8\left(k\frac{\xi^2\kappa^2\log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/(NN_p))/\delta)}{\eta^2 NN_p} + \frac{\xi^2(k+\log(N/\delta))}{\eta^2 N_p} + \lambda B^2\right)
\end{aligned}
$$

for a constant $C_8 > 0$. Combining this result for all $i \in [N]$, Lemma 4 holds. ☐

Lemma 5. *Suppose Assumptions 1, 2, 3, and 4 hold. For any $\delta \in (0,1]$, with probability at least $1-\delta$, for any $\boldsymbol{r}_{\omega,\boldsymbol{\theta}} \in \mathcal{R}'(\widehat{\mathcal{D}})$,*

$$
\begin{aligned}
&\mathbb{E}_{\mu_0,\mu_1}\left[\left|(r_{\omega,\theta_i}(\tau_{i,0}) - r_{\omega,\theta_i}(\tau_{i,1})) - (r_i^\star(\tau_{i,0}) - r_i^\star(\tau_{i,1}))\right|^2\right] \\
&\qquad \le C_9\left(k\frac{\xi^2\kappa^2\log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/(NN_p))/\delta)}{\eta^2 NN_p} + \frac{\xi^2(k+\log(N/\delta))}{\eta^2 N_p} + \lambda B^2\right)
\end{aligned}
$$

where $C_9 > 0$ is a constant.

Proof. For any τ_0, τ_1, by Assumption 4, with large $N_p \ge N_{\text{unif}}(\Psi, \mu_0, \mu_1, \delta)$, we have the analog of Equation (G.3):

$$
\begin{aligned}
&\mathbb{E}_{\mu_0,\mu_1}\left[\left|(r_{\omega,\theta_i}(\tau_{i,0}) - r_{\omega,\theta_i}(\tau_{i,1})) - (r_i^\star(\tau_{i,0}) - r_i^\star(\tau_{i,1}))\right|^2\right] \\
&= \begin{bmatrix}\theta_i \\ -\theta_i^\star\end{bmatrix}^\intercal \Lambda_{\phi_\omega,\phi_{\psi^\star}}(\mu_0,\mu_1)\begin{bmatrix}\theta_i \\ -\theta_i^\star\end{bmatrix} \le 1.1\begin{bmatrix}\theta_i \\ -\theta_i^\star\end{bmatrix}^\intercal \widehat{\Lambda}_{\phi_\omega,\phi_{\psi^\star}}(\mu_0,\mu_1)\begin{bmatrix}\theta_i \\ -\theta_i^\star\end{bmatrix} \\
&\le 1.1C_8\left(k\frac{\xi^2\kappa^2\log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/(NN_p))/\delta)}{\eta^2 NN_p} + \frac{\xi^2(k+\log(N/\delta))}{\eta^2 N_p} + \lambda B^2\right)
\end{aligned}
$$

which concludes the proof. ☐

Theorem 3.1. (Expected Value Function Gap). *Suppose Assumptions 1, 2, 3, and 4 hold. For any $\delta \in (0,1]$, all $i \in [N]$ and $\lambda > 0$, with probability at least $1-\delta$, the output $\widehat{\pi}_i'$ of Algorithm 1 satisfies*

$$
\begin{aligned}
&J(\pi_{i,tar}; r_i^\star) - J(\widehat{\pi}_i'; r_i^\star) \\
&\le \sqrt{cC_{\boldsymbol{r}}(\mathcal{G}_{\boldsymbol{r}}, \pi_{i,tar}, \mu_{i,ref}, i)^2\left(k\frac{\xi^2\kappa^2\log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/(NN_p))/\delta)}{\eta^2 NN_p} + \frac{\xi^2(k+\log(N/\delta))}{\eta^2 N_p} + \lambda B^2\right)}
\end{aligned} \tag{3.1}
$$

where $c > 0$ is a constant.

Proof. We have

$$
\begin{aligned}
&J(\pi_{i,\text{tar}}; r_i^\star) - J(\widehat{\pi}_i'; r_i^\star) \\
&\le C_{\boldsymbol{r}}(\mathcal{G}_{\boldsymbol{r}}, \pi_{i,\text{tar}}, \mu_{i,\text{ref}}, i)\sqrt{\mathbb{E}_{\mu_0,\mu_1}\left[\left|(r_i^\star(\tau_{i,1}) - r_i^\star(\tau_{i,0})) - (r_{\pi_{i,\text{tar}}}^{i,\text{inf}}(\tau_{i,1}) - r_{\pi_{i,\text{tar}}}^{i,\text{inf}}(\tau_{i,0}))\right|^2\right]} \\
&\le \sqrt{cC_{\boldsymbol{r}}(\mathcal{G}_{\boldsymbol{r}}, \pi_{i,\text{tar}}, \mu_{i,\text{ref}}, i)^2\left(\frac{k\kappa^2\log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/(NN_p))/\delta)}{NN_p} + \frac{\xi^2(k+\log(N/\delta))}{\eta^2 N_p} + \lambda B^2\right)}
\end{aligned}
$$

where $c > 0$ is a constant, which is similar to the proof of Theorem G.2. □

In the exactly same way, we can prove Theorem G.1, so we omit the proof of Theorem G.1

G.7 Proof of Theorem 3.2

We present the formal version of Theorem 3.2.

Theorem G.3. (Lower Bound for the Sub-Optimality Gap of Personalization). *For any $k > 6$, $N_p \geq Ck\Lambda^2$ and $\Lambda \geq 2$, there exists a representation function $\phi(\cdot)$ so that*

$$\min_{i\in[N]} \inf_{\widehat{\boldsymbol{\pi}}} \sup_{Q\in\mathrm{CB}(\Lambda)} \left(\max_{\pi^*\in\Pi} J(\pi^*; r_{\omega,\theta_i}) - J(\widehat{\pi}; r_{\omega,\theta_i}) \right) \geq C\Lambda \cdot \sqrt{\frac{k}{N_p}},$$

where

$$\mathrm{CB}(\Lambda) := \left\{ Q := \left(\{\mu_0, \mu_1\}, \{\tau_{i,0}^{(j)}, \tau_{i,1}^{(j)}\}_{i\in[N], j\in[N_p]}, \omega, \boldsymbol{\theta} \right) \mid C'_{\boldsymbol{r}}(\mathcal{G}_{\boldsymbol{r}}, \pi^\star, \mu_1, i) \leq \Lambda \text{ for all } i \in [N] \right\}$$

is the family of MDP with N reward functions and $H = 1$ instances, where

$$C'_{\boldsymbol{r}}(\mathcal{G}_{\boldsymbol{r}}, \pi^\star, \mu_1, i) := \max\left\{ 0, \sup_{\boldsymbol{r}\in\mathcal{G}_{\boldsymbol{r}}} \frac{\mathbb{E}_{\tau_0\sim\pi^\star, \tau_1\sim\mu_1}\left[r_i^\star(\tau_0) - r_i^\star(\tau_1) - r_i(\tau_0) + r_i(\tau_1)\right]}{\sqrt{\frac{1}{N_p}\sum_{j=1}^{N_p}\left[\left| r_i^\star\left(\tau_{i,0}^{(j)}\right) - r_i^\star\left(\tau_{i,1}^{(j)}\right) - r_i\left(\tau_{i,0}^{(j)}\right) + r_i\left(\tau_{i,1}^{(j)}\right)\right|^2\right]}} \right\}. \tag{G.10}$$

All results in this paper still hold for the new concentrability coefficient $C'_{\boldsymbol{r}}$.

Proof of Theorem 3.2. We follow the construction in Theorem 3.10 of (Zhu et al., 2023).

We will only consider $H = 1$ case. Assume k can be divided by 3 without loss of generality. Let $\mathcal{S} := \{0, 1, ..., k/3 - 1\}$ and $\mathcal{A} := \{a_1, a_2, a_3\}$. Let $\psi_\omega(\phi(s, a_1)) = e_{3s}$, $\psi_\omega(\phi(s, a_2)) = e_{3s+1}$, and $\psi_\omega(\phi(s, a_3)) = 0$. Also, let $v_{-1} := \{1/d, 1/d + \Delta, -2/d - \Delta\}$ and $v_1 := \{1/d + 2\Delta, 1/d + \Delta, -2/d - 3\Delta\}$. We construct $2^{|\mathcal{S}|}$ instances in CB. Let $w \in \{\pm 1\}^{|\mathcal{S}|}$ and $\theta_w := [v_{w_1}, v_{w_2}, ..., v_{w_{|\mathcal{S}|}}]$. Let $\mu_0(s, a_1) = \frac{1-2\Lambda^2}{|\mathcal{S}|}$, $\mu_0(s, a_2) = \frac{2\Lambda^2}{|\mathcal{S}|}$, and $\mu_1(s, a_3) = 1$ for any $s \in \mathcal{S}$.

According to (Zhu et al., 2023), $\left\|\Sigma_{\mathcal{D}}^{-1/2}\mathbb{E}_{s\sim\rho}\left[\psi_\omega(\phi(s, \pi^\star(s)))\right]\right\|_2 \leq \Lambda$, where ρ is the uniform distribution over $\mathcal{S}$. At the same time, for any θ_w we have $\|\theta_w\|_2 \in \Theta_B$ when taking $B = 1$, $d > 6$ and $\Delta < 1/(6d)$.

Next, we will show that $C'_{\boldsymbol{r}}(\mathcal{G}_{\boldsymbol{r}}, \pi^\star, \mu_1, i) \leq \Lambda$. By definition, we have

$$\left\|\Sigma_{\mathcal{D}}^{-1/2}\mathbb{E}_{s\sim\rho}\left[\psi_\omega(\phi(s, \pi^\star(s)))\right]\right\|_2 = \left\|\Sigma_{\mathcal{D}}^{-1/2}\mathbb{E}_{s\sim\rho, a\sim\pi^\star(\cdot\,|\,s), (s',a')\sim\mu_1}\left[\psi_\omega(\phi(s, \pi^\star(s))) - \psi_\omega(\phi(s', a')\right]\right\|_2,$$

since $a' \equiv a_3$ by definition of μ_1 and $\psi_\omega(\phi(\cdot, a_3)) \equiv 0$. Then, by Section D.1. of (Zhan et al., 2023), we have $C'_{\boldsymbol{r}}(\mathcal{G}_{\boldsymbol{r}}, \pi^\star, \mu_1, i) \leq \left\|\Sigma_{\mathcal{D}}^{-1/2}\mathbb{E}_{s\sim\rho}\left[\psi_\omega(\phi(s, \pi^\star(s)))\right]\right\|_2 \leq \Lambda$. Therefore, combined with Theorem 3.10 of (Zhu et al., 2023), we finished the proof. □

H Proof of Section 3.1.2

Corollary G.2 holds with probability $1 - \delta/3$, so we have

$$\max_{\tau_0,\tau_1} \|(\psi^\star(\phi(\tau_0)) - \psi^\star(\phi(\tau_1))) - P_{\widehat{\omega}}(\psi_{\widehat{\omega}}(\phi(\tau_0)) - \psi_{\widehat{\omega}}(\phi(\tau_1)))\|^2 \leq k\frac{C_3\kappa^2 \log(\mathcal{N}_{\mathcal{G}_r}(1/(NN_p))/\delta)}{NN_pB^2}$$

Lemma 6 (Mansour et al. (2020)). *For any $\delta \in (0,1]$, with probability at least $1-\delta$, the output $((\widehat{\pi}_{(k)})_{k\in[K]}, (\widehat{\theta}_{(k)})_{k\in[K]}, \widehat{\omega}, \widehat{f})$ of Algorithm 2 satisfies*

$$\max_{\|\theta_i'\|\le B \text{ for all } i\in[N]} \sum_{i\in[N]}\sum_{j\in[N_{p,i}]} \log\left(\frac{P_{\widehat{\omega},\theta_i'}(o_i^{(j)} \mid \tau_{i,0}^{(j)}, \tau_{i,1}^{(j)})}{P_{\widehat{\omega},\widehat{\theta}_{\widehat{f}(i)}}(o_i^{(j)} \mid \tau_{i,0}^{(j)}, \tau_{i,1}^{(j)})}\right)$$
$$\le C_{\textit{cluster}} N N_p \left(\sqrt{\frac{\log(2K/\delta)}{N_p}} + \sqrt{\frac{kK\log(N_p/k)}{N_p}} + \sum_{i\in[N]} \frac{1}{N} \boldsymbol{disc}(\mathcal{D}_i, \mathcal{C}_{\widehat{f}(i)}, \mathcal{G}_{\psi_{\widehat{\omega}}})\right),$$

where $\mathcal{C}_k := \cup_{\widehat{f}(i)=k}\mathcal{D}_i$, $C_{\textit{cluster}} > 0$ is a constant, and $\mathcal{G}_{\psi_\omega} := \{r_{\omega,\theta} \mid \|\theta\| \le B\}$ for all $\omega \in \Omega$.

Claim 1. *For any $\delta \in (0,1]$, with probability at least $1-\delta$, for arbitrary $\mathcal{D}_i$ and $\mathcal{D}_j$, the gap between label discrepency with reward function class $\mathcal{G}_{\psi_{\widehat{\omega}}}$ and $\mathcal{G}_{\psi^\star}$ is bounded as follows:*

$$\left|\boldsymbol{disc}(\boldsymbol{D}_i, \boldsymbol{D}_j, \mathcal{G}_{\psi_{\widehat{\omega}}}) - \boldsymbol{disc}(\boldsymbol{D}_i, \boldsymbol{D}_j, \mathcal{G}_{\psi^\star})\right| \le 2C_{10}\sqrt{\frac{k\xi^2\kappa^2\log(\mathcal{N}_{\mathcal{G}_r}(1/(NN_p))/\delta)}{NN_p}}$$

for $i,j \in [N]$ where $C_{10} > 0$ is a constant. We recall the definition of $\mathcal{G}_{\psi_w} = \{\langle \psi_w, \theta\rangle \mid \|\theta\|_2 \le B\}$.

Proof.

$$\begin{aligned}
&\left|\mathbb{E}_{\boldsymbol{D}_i} \log P_{\widehat{\omega}, P_{\widehat{\omega}}^{\intercal}\theta}(o \mid \tau_1, \tau_0) - \mathbb{E}_{\boldsymbol{D}_i} \log P_{\omega^\star,\theta}(o \mid \tau_1, \tau_0)\right| \\
&\le \mathbb{E}_{\boldsymbol{D}_i}\left|\log P_{\widehat{\omega}, P_{\widehat{\omega}}^{\intercal}\theta}(o \mid \tau_1, \tau_0) - \log P_{\omega^\star,\theta}(o \mid \tau_1, \tau_0)\right| \\
&\le \xi \mathbb{E}_{\boldsymbol{D}_i}\left|\langle P_{\widehat{\omega}}(\psi_{\widehat{\omega}}(\phi(\tau_1)) - \psi_{\widehat{\omega}}(\phi(\tau_0))) - (\psi^\star(\phi(\tau_1)) - \psi^\star(\phi(\tau_0))), \theta\rangle\right| \\
&\le C_{10}\sqrt{\frac{k\xi^2\kappa^2\log(\mathcal{N}_{\mathcal{G}_r}(1/(NN_p))/\delta)}{NN_p}}
\end{aligned}$$

where $\xi := \max_{x\in[-R_{\max}, R_{\max}]}\left|\frac{\Phi'(x)}{\Phi(x)}\right|$, which is also defined in Appendix G. ☐

Theorem 3.3. (Total Expected Value Function Gap). *Suppose Assumptions 1, 2, 3, and 4 hold. Also, assume that $C_r(\mathcal{G}_r, \pi, \mu_{i,\textit{ref}}, i) \le C'_{\textit{max}}$ for all policy π and $i \in [N]$. For any $\delta \in (0,1]$, all $i \in [N]$ and $\lambda > 0$, with probability at least $1-\delta$, the output $((\widehat{\pi}_{(k)})_{k\in[K]}, \widehat{f})$ of Algorithm 2 satisfies*

$$\sum_{i\in[N]}\left(J(\pi_{i,\textit{tar}}; r_i^\star) - J(\widehat{\pi}_{\widehat{f}(i)}; r_i^\star)\right) \le cN\kappa\Bigg(\underbrace{\frac{\log(2K/\delta) + kK\log(N_p/k)}{N_p} + \frac{k\xi^2\kappa^2\log(\mathcal{N}_{\mathcal{G}_r}(1/(NN_p))/\delta)}{NN_p}}_{(i)}$$
$$+ \underbrace{\left(\sum_{i\in[N]}\frac{1}{N}\boldsymbol{disc}(\mathcal{D}_i, \mathcal{C}_{\widehat{f}(i)}, \mathcal{G}_{\psi^\star}))\right)^2}_{(ii)} + \left(\frac{\log(\mathcal{N}_{\mathcal{G}_{\psi^\star}}(1/NN_p)/\delta)}{NN_p}\right)^2\Bigg)^{1/4},$$

where $c > 0$ is a constant.

We note that due to the $\sqrt{kK/N_p}$ order on the right-hand side of Lemma 6, we have a slower rate in Theorem 3.3 than Theorem 3.1. This gap is mainly due to the fact that the analysis of Lemma 6 should cover uniformly for arbitrary $\widehat{f}$, and also due to a difference between $\max$ and expectation of $\max$, which is bounded using McDiarmid's inequality.

Proof. By Claim 1 with Lemma 6, we have

$$\sum_{i\in[N]}\sum_{j\in[N_{p,i}]} \log\left(\frac{P_{\omega^\star,\theta_i^\star}(o_i^{(j)} \mid \tau_{i,0}^{(j)}, \tau_{i,1}^{(j)})}{P_{\widehat{\omega},\widehat{\theta}_{\widehat{f}(i)}}(o_i^{(j)} \mid \tau_{i,0}^{(j)}, \tau_{i,1}^{(j)})}\right)$$

$$
\begin{aligned}
&\leq \max_{\|\theta_i'\|\leq B \text{ for all } i\in[N]} \sum_{i\in[N]}\sum_{j\in[N_{p,i}]} \log\left(\frac{P_{\omega^\star,\theta_i'}(o_i^{(j)} \mid \tau_{i,0}^{(j)},\tau_{i,1}^{(j)})}{P_{\widehat{\omega},\widehat{\theta}_{\widehat{f}(i)}}(o_i^{(j)} \mid \tau_{i,0}^{(j)},\tau_{i,1}^{(j)})}\right)\\
&\underset{(i)}{\leq} \max_{\|\theta_i'\|\leq B \text{ for all } i\in[N]} \sum_{i\in[N]}\sum_{j\in[N_{p,i}]} \log\left(\frac{P_{\widehat{\omega},\theta_i'}(o_i^{(j)} \mid \tau_{i,0}^{(j)},\tau_{i,1}^{(j)})}{P_{\widehat{\omega},\widehat{\theta}_{\widehat{f}(i)}}(o_i^{(j)} \mid \tau_{i,0}^{(j)},\tau_{i,1}^{(j)})}\right)\\
&\qquad + NN_pC_{10}\sqrt{\frac{k\xi^2\kappa^2\log(\mathcal{N}_{\mathcal{G}_r}(1/(NN_p))/\delta)}{NN_p}}\\
&\leq C_{\text{cluster}}NN_p\Bigg(\sqrt{\frac{\log(2K/\delta)}{N_p}}+\sqrt{\frac{kK\log(N_p/k)}{N_p}}+\sqrt{\frac{k\xi^2\kappa^2\log(\mathcal{N}_{\mathcal{G}_r}(1/(NN_p))/\delta)}{NN_p}}\\
&\qquad\qquad + \sum_{i\in[N]}\frac{1}{N}\texttt{disc}(\mathcal{D}_i,\mathcal{C}_{\widehat{f}(i)},\mathcal{G}_{\psi_{\widehat{\omega}}})\Bigg)\\
&\leq C_{11}NN_p\Bigg(\sqrt{\frac{\log(2K/\delta)}{N_p}}+\sqrt{\frac{kK\log(N_p/k)}{N_p}}+\sqrt{\frac{k\xi^2\kappa^2\log(\mathcal{N}_{\mathcal{G}_r}(1/(NN_p))/\delta)}{NN_p}}\\
&\qquad\qquad + \sum_{i\in[N]}\frac{1}{N}\texttt{disc}(\mathcal{D}_i,\mathcal{C}_{\widehat{f}(i)},\mathcal{G}_{\psi^\star}))\Bigg),
\end{aligned}
$$

where $\widehat{\omega}$ is a learned parameter from the representation learning, and $C_{11} > 0$ is a constant. Here, (i) came from the same reason with Claim 1. Therefore, by Lemma 2, we have

$$
\begin{aligned}
&\mathbb{E}_{\mu_0,\mu_1}\left[\|P_{\widehat{\omega},\widehat{\theta}_{f(i)}}(\cdot \mid \tau_{i,0}^{(j)},\tau_{i,1}^{(j)}) - P_{w^\star,\theta^\star}(\cdot \mid \tau_{i,0}^{(j)},\tau_{i,1}^{(j)})\|_1^2\right]\\
&\leq C_{11}\Bigg(\sqrt{\frac{\log(2K/\delta)}{N_p}}+\sqrt{\frac{kK\log(N_p/k)}{N_p}}+\sqrt{\frac{k\xi^2\kappa^2\log(\mathcal{N}_{\mathcal{G}_r}(1/(NN_p))/\delta)}{NN_p}}\\
&\qquad\qquad + \sum_{i\in[N]}\frac{1}{N}\texttt{disc}(\mathcal{D}_i,\mathcal{C}_{\widehat{f}(i)},\mathcal{G}_{\psi^\star})) + \frac{\log(\mathcal{N}_{\mathcal{G}_{\psi_{\widehat{\omega}}}}(1/NN_p)/\delta)}{NN_p}\Bigg).
\end{aligned}
$$

Here, we used $\mathcal{N}_{\mathcal{G}_{\psi_{\widehat{\omega}}}}(1/NN_p) = \mathcal{N}_{\mathcal{G}_{\psi^\star}}(1/NN_p)$. Now, we get the similar bound with Equation (G.3):

$$
\begin{aligned}
&\frac{1}{N}\sum_{i\in[N]}\mathbb{E}_{\mathcal{D}_i}\left[\left|(r_{\widehat{\omega},\widehat{\theta}_{\widehat{f}(i)}}(\tau_{i,0}) - r_{\widehat{\omega},\widehat{\theta}_{\widehat{f}(i)}}(\tau_{i,1})) - (r_i^\star(\tau_{i,0}) - r_i^\star(\tau_{i,1}))\right|^2\right]\\
&\leq C_{11}\kappa^2\Bigg(\sqrt{\frac{\log(2K/\delta)}{N_p}}+\sqrt{\frac{kK\log(N_p/k)}{N_p}}+\sqrt{\frac{k\xi^2\kappa^2\log(\mathcal{N}_{\mathcal{G}_r}(1/(NN_p))/\delta)}{NN_p}}\\
&\qquad\qquad + \sum_{i\in[N]}\frac{1}{N}\texttt{disc}(\mathcal{D}_i,\mathcal{C}_{\widehat{f}(i)},\mathcal{G}_{\psi^\star})) + \frac{\log(\mathcal{N}_{\mathcal{G}_{\psi^\star}}(1/NN_p)/\delta)}{NN_p}\Bigg).
\end{aligned}
$$

Lastly, we use the following:

$$
\begin{aligned}
&J(\pi_{i,\text{tar}}; r_i^\star) - J(\widehat{\pi}_i; r_i^\star)\\
&= (J(\pi_{i,\text{tar}}; r_i^\star) - \mathbb{E}_{\tau\sim\mu_{i,\text{ref}}}(r_i^\star(\tau))) - (J(\widehat{\pi}_i; r_i^\star) - \mathbb{E}_{\tau\sim\mu_{i,\text{ref}}}(r_i^\star(\tau)))\\
&= (J(\pi_{i,\text{tar}}; r_i^\star) - \mathbb{E}_{\tau\sim\mu_{i,\text{ref}}}(r_i^\star(\tau))) - (J(\pi_{i,\text{tar}}; \widehat{r}_i) - \mathbb{E}_{\tau\sim\mu_{i,\text{ref}}}(\widehat{r}_i(\tau)))\\
&\quad + (J(\pi_{i,\text{tar}}; \widehat{r}_i) - \mathbb{E}_{\tau\sim\mu_{i,\text{ref}}}(\widehat{r}_i(\tau))) - (J(\widehat{\pi}_j; \widehat{r}_i) - \mathbb{E}_{\tau\sim\mu_{i,\text{ref}}}(\widehat{r}_i(\tau)))\\
&\quad + (J(\widehat{\pi}_i; \widehat{r}_i) - \mathbb{E}_{\tau\sim\mu_{i,\text{ref}}}(\widehat{r}_i(\tau))) - (J(\widehat{\pi}_i; r_i^\star) - \mathbb{E}_{\tau\sim\mu_{i,\text{ref}}}(r_i^\star(\tau)))\\
&\leq 2C'_{\max}\sqrt{\mathbb{E}_{\mu_0,\mu_1}\left[\left|(r_i^\star(\tau_{i,0}) - r_i^\star(\tau_{i,1})) - (\widehat{r}_i(\tau_{i,0}) - \widehat{r}_i(\tau_{i,1}))\right|^2\right]}
\end{aligned}
$$

where the last inequality came from the fact that $\widehat{\pi}_i$ is the best policy with respect to $\widehat{r}_{f(i)}$. Therefore, summing the above relationship with $i \in [N]$ provides

$$
\begin{aligned}
&\sum_{i\in[N]} \left(J(\pi_{i,\text{tar}}; r_i^\star) - J(\widehat{\pi}_i; r_i^\star)\right) \\
&\le C_{12} N\kappa \Bigg(\frac{\log(2K/\delta)}{N_p} + \frac{kK\log(N_p/k)}{N_p} + \frac{k\xi^2\kappa^2 \log(\mathcal{N}_{\mathcal{G}_r}(1/(NN_p))/\delta)}{NN_p} \\
&\qquad + \left(\sum_{i\in[N]} \frac{1}{N}\texttt{disc}(\mathcal{D}_i, \mathcal{C}_{\widehat{f}(i)}, \mathcal{G}_{\psi^\star})\right)^2 + \left(\frac{\log(\mathcal{N}_{\mathcal{G}_{\psi^\star}}(1/NN_p)/\delta)}{NN_p}\right)^2 \Bigg)^{1/4}.
\end{aligned}
$$

□

Remark 3. *In contrast to the results in Section 3.1, we additionally assume $C_r(\mathcal{G}_r, \pi, \mu_1, i) \le C'_{max}$ in Theorem 3.3. To adopt a pessimistic approach, constructing a confidence set for clustered reward functions across all clusters is necessary. However, the ambiguity of which human user belongs to which cluster complicates this analysis, as pessimism would need to be applied to every potential cluster. Consequently, defining a confidence set for every possible clustering scenario is required, significantly complicating the analysis of the algorithm.*

H.1 Why do We Provide Algorithm 3?

Given the inherent complexity of this hierarchical optimization problem, which presents more challenges than standard optimization tasks (Anandalingam and Friesz, 1992), we propose a novel algorithm that circumvents the need for explicit reward function estimation in Algorithm 3. Our approach begins by randomly assigning each human user to a cluster. Subsequently, we reassign random human users to the cluster where the policy most effectively maximizes their empirical DPO loss (Equation (F.4)). Finally, we refine our solution by optimizing the DPO loss function for the selected human users within each cluster, thereby enhancing the overall policy effectiveness.

I Proof of Section 4

I.1 Six Pivotal Axioms for Reward Aggregation

For the completeness of the paper, we introduce six pivotal axioms for reward aggregation (Moulin, 2004).

- **Monotonicity**: For two reward vectors, $\boldsymbol{r} = (r_1, \ldots, r_N)^\top$ and $\boldsymbol{r}' = (r'_1, \ldots, r'_N)^\top$ such that $r_i = r'_i$ for $i \neq j$ and $r_j > r'_j$ for some $j \in [N]$, then $\boldsymbol{r} \succ \boldsymbol{r}'$. This is related to Pareto optimality, indicating that if one vector is strictly better than another in at least one dimension and no worse in any other, it is considered superior.
- **Symmetry**: The reward aggregation function should treat all individuals equally. The outcome should not depend on the identities of the individuals but only on their rewards.
- **Independence of Unconcerned Agents**: If for an individual $j \in [N]$, $r_j = r'_j$, then the magnitude of r_j does not influence the comparison between $\boldsymbol{r}$ and $\boldsymbol{r}'$.
- **The Pigou-Dalton Transfer Principle**: If $r_i < r_j$ and $r'_i + r_j = r'_j + r_i$ for a pair $(i,j) \in [N] \times [N]$ and $r_k = r'_k$ for all $k \neq i, j \in [N]$, then $\boldsymbol{r}' \succ \boldsymbol{r}$. This condition implies that, all else being equal, a social welfare function should favor allocations that are more equitable, reflecting a preference for balancing the rewards between individuals i and j.
- **Translation Independence**: If $\boldsymbol{r} \succ \boldsymbol{r}'$, then $\boldsymbol{r} + c \succ \boldsymbol{r}' + c$ for $c \in \mathbb{R}^N$.
- **Continuity**: In the context of social choice with a continuous preference scale, continuity means that small changes in the individual preferences should not lead to abrupt changes in the collective decision.

Equation (4.2) and its monotonically increasing transformation is only reward aggregation that satisfying the above six axioms. In (Zhong et al., 2024), the consider *Scale Independence* rather than *Translation Independence*, which is defined as follows:

- **Scale Independence**: If $\boldsymbol{r} \succ \boldsymbol{r}'$, then $\lambda \cdot \boldsymbol{r} \succ \lambda \cdot \boldsymbol{r}'$ for $\lambda > 0$.

In this case, the reward aggregations that satisfying six axioms are

$$\mathrm{Agg}_\alpha(\boldsymbol{r}) = \begin{cases} \frac{1}{N\alpha}\sum_{i\in[N]} r_i^\alpha & \alpha \neq 0 \\ \prod_{i\in[N]} r_i & \alpha = 0 \end{cases}$$

for $\alpha \in [-\infty, \infty]$.

I.2 Deferred Statement of Lower Bound for the Sub-Optimality Gap of Aggregation

Theorem I.1. (Lower Bound for the Sub-Optimality Gap of Aggregation). *For any $k > 6, N_p \geq Ck\Lambda^2, \Lambda \geq 2$, and $\alpha \in \mathbb{R}$ there exists a representation function $\phi(\cdot)$ so that*

$$\inf_{\widehat{\boldsymbol{\pi}}} \sup_{Q\in \mathrm{CB}(\Lambda)} \left(\max_{\pi^*\in\Pi} J(\pi^*; Agg_\alpha(\boldsymbol{r}_{\omega,\boldsymbol{\theta}})) - J(\widehat{\pi}; Agg_\alpha(\boldsymbol{r}_{\omega,\boldsymbol{\theta}}))\right) \geq C\Lambda \cdot \sqrt{\frac{k}{N_p}},$$

where

$$\mathrm{CB}(\Lambda) := \left\{ Q := \left(\{\mu_0, \mu_1\}, \{\tau_{i,0}^{(j)}, \tau_{i,1}^{(j)}\}_{i\in[N], j\in[N_p]}, \omega, \boldsymbol{\theta}\right) \mid C_{\boldsymbol{r}}'(\mathcal{G}_{\boldsymbol{r}}, \pi^\star, \mu_1, i) \leq \Lambda \text{ for all } i \in [N]\right\}$$

is the family of MDP with N reward functions and $H = 1$ instances. $C_{\boldsymbol{r}}'$ is defined in Equation (G.10).

I.3 Proof of Theorem 4.1

Theorem 4.1. (Expected Value Function Gap). *Suppose Assumptions 1, 2, 3, and 4 hold. For any $\delta \in (0, 1]$, all $i \in [N]$ and $\lambda > 0$, with probability at least $1 - \delta$, the output $\widehat{\pi}$ of Algorithm 4 satisfies*

$$\begin{aligned} &J(\pi_{tar}; Agg_\alpha(\boldsymbol{r}^\star)) - J(\widehat{\pi}; Agg_\alpha(\boldsymbol{r}^\star)) \\ &\leq \sqrt{c_\alpha C_{\boldsymbol{r}}(\mathcal{G}_{\boldsymbol{r}}, \pi_{tar}, \mu_{ref})^2 \left(\frac{k\kappa^2 \log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/(NN_p))/(\delta/N))}{NN_p} + \frac{\xi^2(k + \log(N/\delta))}{\eta^2 N_p} + \lambda B^2\right)} \end{aligned}$$

where $c_\alpha > 0$ is a constant depending on α, and other constants are defined in Section 3.1.1.

Proof. Define $C_\alpha := \max_{x,y,z,w\in[-R_{\max}, R_{\max}]} \frac{|(\exp(\alpha x) - \exp(\alpha y)) - (\exp(\alpha z) - \exp(\alpha w))|}{\alpha|(x-y)-(z-w)|}$ for $\alpha \neq 0$ and $C_\alpha = 1$ for $\alpha = 0$. Then we know that $C_\alpha < \infty$. Now, in the same way of proof of Theorem 3.1, we have

$$\begin{aligned} &J(\pi_{\mathrm{tar}}; \mathrm{Agg}_\alpha(r_1^\star, \dots, r_N^\star)) - J(\widehat{\pi}; \mathrm{Agg}_\alpha(r_1^\star, \dots, r_N^\star)) \\ &\leq C_{\boldsymbol{r}}(\mathcal{G}_{\boldsymbol{r}}, \pi_{\mathrm{tar}}, \mu_{\mathrm{ref}}) \sqrt{\mathbb{E}_{\mu_0,\mu_1}\left[\left| (\mathrm{Agg}_\alpha(\boldsymbol{r}^\star(\tau_1)) - \mathrm{Agg}_\alpha(\boldsymbol{r}^\star(\tau_0))) - (\mathrm{Agg}_\alpha(\boldsymbol{r}_{\pi_{\mathrm{tar}}}^{\inf}(\tau_1)) - \mathrm{Agg}_\alpha(\boldsymbol{r}_{\pi_{\mathrm{tar}}}^{\inf}(\tau_0)))\right|^2\right]} \\ &\leq C_{\boldsymbol{r}}(\mathcal{G}_{\boldsymbol{r}}, \pi_{\mathrm{tar}}, \mu_{\mathrm{ref}}) \sqrt{C_\alpha^2 \mathbb{E}_{\mu_0,\mu_1}\left[\frac{1}{N}\sum_{i\in[N]} \left| (r_i^\star(\tau_1) - r_i^\star(\tau_0)) - (r_{\pi_{\mathrm{tar}}}^{\inf}(\tau_1) - r_{\pi_{\mathrm{tar}}}^{\inf}(\tau_0))\right|^2\right]} \\ &\leq \sqrt{c_\alpha \left(\frac{k\kappa^2 \log(\mathcal{N}_{\mathcal{G}_{\boldsymbol{r}}}(1/(NN_p))/(\delta/N))}{NN_p^2} + \frac{\xi^2(k+\log(N/\delta))}{\eta^2 N_p} + \lambda B^2\right)}. \end{aligned}$$

where the last line is from Lemma 5, which conclude the proof. $\square$

I.4 Proof of Theorem I.1

Theorem I.1. (Lower Bound for the Sub-Optimality Gap of Aggregation). *For any $k > 6, N_p \geq Ck\Lambda^2, \Lambda \geq 2$, and $\alpha \in \mathbb{R}$ there exists a representation function $\phi(\cdot)$ so that*

$$\inf_{\widehat{\boldsymbol{\pi}}} \sup_{Q\in \mathrm{CB}(\Lambda)} \left(\max_{\pi^*\in\Pi} J(\pi^*; Agg_\alpha(\boldsymbol{r}_{\omega,\boldsymbol{\theta}})) - J(\widehat{\pi}; Agg_\alpha(\boldsymbol{r}_{\omega,\boldsymbol{\theta}}))\right) \geq C\Lambda \cdot \sqrt{\frac{k}{N_p}},$$

where

$$\mathrm{CB}(\Lambda) := \left\{ Q := \left(\{\mu_0, \mu_1\}, \{\tau_{i,0}^{(j)}, \tau_{i,1}^{(j)}\}_{i\in[N], j\in[N_p]}, \omega, \boldsymbol{\theta}\right) \mid C_{\boldsymbol{r}}'(\mathcal{G}_{\boldsymbol{r}}, \pi^\star, \mu_1, i) \leq \Lambda \text{ for all } i \in [N]\right\}$$

is the family of MDP with N reward functions and $H = 1$ instances. $C_{\boldsymbol{r}}'$ is defined in Equation (G.10).

Proof. We start with the same setting and the same instances that achieve the lower bounds with Theorem 3.2. Since

$$\mathbb{E}_s[\mathrm{Agg}_\alpha(\boldsymbol{r})(s,\pi^\star) - \mathrm{Agg}_\alpha(\boldsymbol{r})(s,\pi')] \geq \Omega\left(\mathbb{E}_s[\sum_{i\in[N]}(r_i(s,\pi^\star) - r_i(s,\pi'))]\right) \geq \Omega\left(C\Lambda\cdot\sqrt{\frac{k}{N_p}}\right)$$

We can finish the proof for all $\alpha \in \mathbb{R}$. The first inequality holds by definition when $\alpha = 0$. When $\alpha \neq 0$, for any $i \in [N]$, we have $\exp(r_i(s,\pi^\star)) - \exp(r_i(s,\pi')) \geq \exp(-R_{\max})\,|r_i(s,\pi^\star) - r_i(s,\pi')| \geq \Omega\left(C\Lambda\cdot\sqrt{\frac{k}{N_p}}\right)$. □

I.5 Remark on Aggregation of Probabilistic Opinion (Equation (4.3))

Remark 4. *The case where $\alpha = 0$ is referred to as the geometric pooling function (McConway, 1978). This function is known for preserving unanimity and not being eventwise independent, while it does satisfy external Bayesianity (Madansky, 1964; Dietrich and List, 2016). External Bayesianity mandates that updating the probabilities with new information should yield consistent results regardless of whether the update occurs before or after the aggregation process (Genest, 1984).*

I.6 Proof of Theorem 4.2

Theorem 4.2. (Relationship between Reward Aggregation and Preference Aggregation). *Suppose human preferences are modeled by the PL model, and all human labelers share a common lower bound on their reward functions. Let $(R_i(a))_{a\in\mathcal{A}}$ represent the reward function associated with action $a \in \mathcal{A}$ and $P_i \in \Delta(\mathcal{A})$ denote the corresponding probabilistic opinion for individual $i \in [N]$. Then, the preference aggregation Agg-p$_\alpha(\mathbf{P})$, is equivalent to the preference derived under the PL model with the aggregated rewards $(\mathrm{Agg}_\alpha(\mathbf{R}(a)))_{a\in\mathcal{A}}$ for any $\alpha \in [-\infty, \infty]$.*

Proof. By the PL modeling, we have

$$P_i(a) = \frac{\exp(R_i(a))}{\sum_{a'\in\mathcal{A}}\exp(R_i(a'))}. \tag{I.1}$$

We divide Equation (I.1) by $P_i(a_{\text{fix}})$, we have

$$R_i(a) = \log P_i(a) - (\log P_i(s, a_{\text{fix}}) - R_i(a_{\text{fix}})) := \log P_i(a) - C_i \tag{I.2}$$

where $C_i := \log P_i(a_{\text{fix}}) - R_i(a_{\text{fix}})$. Since $R_i(a)$ have upper bound as C_i, and we assumed that every reward $R_i(a)$ have the same upper bound, we can assume $C_i = C$ for every i. Therefore, plugging Equation (I.2) provides the equivalence between $\mathrm{Agg}_\alpha(\boldsymbol{R})$ and Agg-p$_\alpha(\boldsymbol{P})$. □

I.7 Deferred Algorithm for Human Feedback with Probabilistic Opinions

Now, we provide an algorithm that uses the feedback in the form of probabilistic opinions (Algorithm 7). The only difference from the DPO algorithm (Rafailov et al., 2024) is to change the deterministic answer a_i to the a_i sampled based on the probabilistic opinion pooling, which is in the second line in the for loop of Algorithm 7.

I.8 Relationship between KL divergence and variant of α-Renyi divergence.

By L'Hôpital's rule, we have

$$\lim_{\alpha\to 1}\frac{1}{1-\alpha}\left(1 - \sum_{j\in\mathcal{A}} p_{ij}\left(\frac{q_{ij}}{p_{ij}}\right)^{1-\alpha}\right) = \sum_{j\in\mathcal{A}}\lim_{\beta\to 0}\left(-p_{ij}\log\left(\frac{q_{ij}}{p_{ij}}\right)\left(\frac{q_{ij}}{p_{ij}}\right)^{\beta}\right) = \mathtt{KL}(p,q).$$

Algorithm 7 Probabilistic Opinion Pooling DPO (POP-DPO)

Input: Dataset $\widehat{\mathcal{D}} = \cup_{i\in[N]}\widehat{\mathcal{D}}_i$ where $\widehat{\mathcal{D}}_i = \{q_i^{(j)}(s_i^{(j)}), s^{(j)}, i)\}_{j\in[N_p]}$ is the probabilistic opinion dataset for the ith individual, $q_i^{(j)} \in \Delta(\mathcal{A})$ with $|\mathcal{A}| = 2$, β is a parameter for DPO, α is a parameter for aggregation

for every epoch **do**

For every question $s^{(j)}$ where j is in the batch, $q^{(j)} := \text{Agg-p}_\alpha(\boldsymbol{q}^{(j)})$.

Sample $a_0^{(j)} \sim \text{Multinomial}(q^{(j)})$ and define $a_1^{(j)}$ as non-selected answer.

Run a few steps of optimization to update π (for example, gradient ascent or Adam) to maximize

$$\sum_{j\in\text{batch}} \log\sigma\left(\beta\log\frac{\pi(a_0^{(j)} \mid s^{(j)})}{\pi^{\text{old}}(a_0^{(j)} \mid s^{(j)})} - \beta\log\frac{\pi(a_1^{(j)} \mid s^{(j)})}{\pi^{\text{old}}(a_1^{(j)} \mid s^{(j)})}\right)$$

end for

Output: π

I.9 Deferred Explanation of Mechanism Design for RLHF

In this setup, we will first prove the existence of a cost function $c_i : \Delta(\mathcal{A})^N \to \mathbb{R}$ for all $i \in [N]$ that induces truthful reporting of probabilistic opinions from human labelers. Here, the input of c_i is the probabilistic opinion of every human labeler. This is also called the dominant strategy incentive-compatible (DSIC) mechanism (Nisan and Ronen, 1999; Börgers, 2015; Roughgarden, 2010). Then, we prove that there exists an aggregation rule and cost function that induce DSIC, and also maximize social welfare. We denote each human labeler's underlying (true) probabilistic opinion as $p_i\left(s^{(j)}\right)$ for each question $s^{(j)}$. Accounting for such cost, we define the *utility function* of individual i for question $s^{(j)}$ as

$$u_i^{(j)}\left(p_i\left(s^{(j)}\right), \left(P_i\left(s^{(j)}\right)\right)_{i\in[N]}\right) = -d\left(p_i\left(s^{(j)}\right), \text{Agg-p}\left(\left(P_i\left(s^{(j)}\right)\right)_{i\in[N]}\right)\right) - c_i\left(\left(P_i\left(s^{(j)}\right)\right)_{i\in[N]}\right).$$

Here, $d : \Delta(\mathcal{A}) \times \Delta(\mathcal{A}) \to \mathbb{R}$ represents the distance between the underlying true probabilistic opinion and the aggregated preference. Moreover, we define the *welfare function* of individual i from addressing question $s^{(j)}$ as $\text{Wel}_i^{(j)}(O) = -d(p_i(s^{(j)}), O)$ for any $O \in \Delta(\mathcal{A})$.

Remark 5 (Examples of Distance Function d)**.** *We can instantiate $d(p, q)$ as the KL-divergence. Also, we may instantiate $d_\alpha(p, q) = sgn(\alpha)\frac{1}{1-\alpha}\sum_{j\in\mathcal{A}}\left(1 - p_j^\alpha q_j^{1-\alpha}\right)$, which is a variant of the α-Renyi divergence for $\alpha \neq 0$. One can easily check that $d_\alpha(p, q) \geq 0$. In fact, one can also prove that $\lim_{\alpha\to1} d_\alpha(p, q) = d(p, q)$ with $d(p, q)$ being the KL-divergence (Appendix I.8).*

I.9.1 Mechanism and Guarantees

We design a mechanism inspired by the Vickery-Clarke-Groves mechanism (Vickrey, 1961; Clarke, 1971; Groves, 1973), as defined below.

Definition I.1 (VCG Mechanism)**.** *Assume that there are n strategic agents and a finite set X of outcome, and each individual i has a private valuation v_i for each outcome $x \in X$. The bidding $\boldsymbol{b} = (b_1, \ldots, b_N)^\intercal \in (\mathbb{R}^{|X|})^N$ where $b_i \in \mathbb{R}^{|X|}$ is bidding for all outcome of individual $i \in [N]$. Define their utility function as $v_i(\boldsymbol{x}(\boldsymbol{b})) - c_i(\boldsymbol{b})$, where $\boldsymbol{x} : (\mathbb{R}^{|X|})^N \to X$ is the allocation rule and $c_i : (\mathbb{R}^{|X|})^N \to \mathbb{R}$ is the cost function. The summation of welfare function of all agents is defined as Wel$(x) = \sum_{i\in[N]} v_i(x)$ for all $x \in X$. The goal is to design $\boldsymbol{x}$ and $(c_i)_{i\in[N]}$ functions to make a DSIC and welfare-maximizing mechanism. The following $\boldsymbol{x}$ and c_i for $i \in [N]$ is DSIC welfare maximizing mechanism:*

$$\boldsymbol{x}(\boldsymbol{b}) = \arg\max_{x\in X}\sum_{i\in[N]} b_i(x), \qquad c_i(\boldsymbol{b}) = \max_{x\in X}\sum_{j\neq i} b_j(x) - \sum_{j\neq i} b_j(\boldsymbol{x}(\boldsymbol{b}))\textit{ for all } i \in [N].$$

Unfortunately, the classical VCG mechanism presents certain limitations such as it cannot be solved in polynomial time in general (Nisan and Ronen, 1999; Börgers, 2015; Roughgarden, 2010). We here adopt certain forms of allocation rule (which corresponds to the aggregation rule in our RLHF setting) and cost functions as follows, which allow the outcome set to be a simplex (with infinitely many outcomes):

$$\text{Agg-p}(\boldsymbol{P}) = \arg\min_{p \in \Delta(\mathcal{A})} \sum_{i \in [N]} d(\boldsymbol{P}, p), \qquad c_i(\boldsymbol{P}) = \sum_{j \neq i} d(P_i, \text{Agg-p}(\boldsymbol{P})) - \min_{p \in \Delta(\mathcal{A})} \sum_{j \neq i} d(P_i, p). \tag{I.3}$$

Theorem I.2. (DSIC Welfare-Maximizing Mechanism). *The aggregation rule and the cost function as in Equation* (I.3) *provide a DSIC welfare-maximizing mechanism.*

Due to the modeling, we have an advantage compared to the original VCG mechanism. The minimization in the aggregation function can be achieved using a simple optimization method such as gradient descent, which makes our aggregation rule and cost function computation easy, which is in contrast with the original VCG mechanism.

Now, we connect our mechanism design with pre-defined preference aggregation function (Agg-p$_\alpha$ in Equation (4.3)). Theorem I.3 implies that Equation (4.3) is maximizing social welfare and also we are available to construct the cost function to make human feedback truthful.

Theorem I.3. *If we set d as a variant of the α-Renyi distance for $\alpha \neq 0$ (Remark 5) and define d as KL-divergence for $\alpha = 0$, the DSIC welfare-maximizing aggregation rule is Equation* (4.3)*. Therefore, aggregation rule Equation* (4.3) *is also welfare-maximizing with appropriate cost function.*

If we assume the relationship between reward and preference follows the PL model (Definition 4.1), then Equation (4.1) implies a welfare-maximizing aggregation rule, which connects reward aggregation and mechanism design. We defer all proofs for the results in Appendix I.9.1 to Appendix I.8.

I.9.2 Proof of Appendix I.9.1

The proof of Theorem I.2 is exactly the same as the proof of the fact that the VCG mechanism is DSIC welfare-maximizing. The difference with the proof of the original VCG mechanism's property is the parametrization of bidding, which will be explained in this section.

Theorem I.2. (DSIC Welfare-Maximizing Mechanism). *The aggregation rule and the cost function as in Equation* (I.3) *provide a DSIC welfare-maximizing mechanism.*

Proof. The aggregated result space $\Delta(\mathcal{A})$ corresponds to the output space X of Definition I.1. We can interpret the bidding part, $b_j(x)$, of Definition I.1 as $-d(P_j, p)$. So, instead of bidding on every output without any rule, we can interpret the bidding as the minus distance function between their own probabilistic opinion and aggregated probabilistic opinion. The underlying value function therefore corresponds to $-d(p_j, p)$. This interpretation provides the same line of proof of the VCG mechanism's property. $\square$

By good parametrization of the VCG mechanism, we can also achieve the computational efficiency of our cost function computation.

Theorem I.3. *If we set d as a variant of the α-Renyi distance for $\alpha \neq 0$ (Remark 5) and define d as KL-divergence for $\alpha = 0$, the DSIC welfare-maximizing aggregation rule is Equation* (4.3)*. Therefore, aggregation rule Equation* (4.3) *is also welfare-maximizing with appropriate cost function.*

Proof. We solve the optimization problem as follows:

$$\arg\min_{p \in \Delta(\mathcal{A})} \sum_{i \in [N]} d_\alpha(P_i, p) \tag{I.4}$$

where $d_\alpha(p, q) = \text{sgn}(\alpha) \frac{1}{1-\alpha} \sum_{j \in \mathcal{A}} \left(1 - p_j^\alpha q_j^{1-\alpha}\right)$. We can check that $d_\alpha(p, q)$ is a convex function with respect to q, as

$$\frac{d^2}{dq_j^2} d_\alpha(p, q) = \alpha \text{sgn}(\alpha) q_j^{-\alpha-1} \geq 0.$$

Therefore, Equation (I.4) can be solved with first-order condition:

$$\sum_{i\in[N]} \left(\frac{P_{ij}}{p_j}\right)^{\alpha} = \lambda \qquad \text{for all } j \in \mathcal{A}$$

which provides Equation (4.3). □

J Experiment Details

For the Reddit TL;DR summarization dataset, (Stiennon et al., 2020) filtered the TL;DR summarization dataset (Völske et al., 2017) to ensure quality. The Reddit TL;DR human feedback dataset is constructed with two components: `comparison` and `axes evals`. The `comparison` component contains labeled comparisons between pairs of summaries with workers identified by unique IDs, while the `axes evals` component contains ratings of summaries along three axes: accuracy, coverage, and coherence.

In Section 6.1, we fine-tuned the personalized reward model with Algorithm 1 and Algorithm 2, without pessimism. We ranked workers based on the number of annotated comparisons in the training split of the dataset and included the top 5 workers for training. To balance the number of samples for each worker, we took the worker with the fewest samples among the top 5 as the baseline. We then randomly sampled the same number of comparisons from the other workers so that each worker had 5,373 comparison samples, resulting in a total of 26,865 samples for training. Similarly, for the validation set, we applied the same method. We randomly sampled the same number of comparisons as the worker with the fewest samples from the top 5 workers used in training. Each worker had 1,238 samples for validation, resulting in a total of 6,190 samples for validation. In Section 6.2, we fine-tuned the personalized reward model using Algorithm 4, without incorporating pessimism. We considered three types of reward functions: accuracy-reward, coverage-reward, and coherence-reward. Since this dataset is only publicly available for the validation set (with 8,585 samples) and the test set (with 6,313 samples), we used the validation set for fine-tuning the training set of our model and validated it with the samples in the test set.

For reward model training, we used the AdamW optimizer (Loshchilov and Hutter, 2018) with a learning rate of 1e-6 and a batch size of 8 for 1 epoch. The learning rate was linearly warmed up from 0 to 1e-6 over 150 steps. For fine-tuning the language model with the trained reward model, we used the AdamW optimizer with a learning rate of 5e-6 and a batch size of 4. We employed Proximal Policy Optimization (PPO) (Schulman et al., 2017) with 128 rollouts, which is the default setting in the TRLX library (Havrilla et al., 2023). For SFT, for the GPT-J 6B model, we initialized a personalized language model using an open-source SFT by CarperAI (Havrilla et al., 2023), which fine-tuned the GPT-J 6B model (Wang and Komatsuzaki, 2021) with the Reddit TL;DR summarization dataset using the TRLX library (Havrilla et al., 2023). For the LLaMA 3, as there is no fine-tuned open model for the Reddit TL;DR summarization, we trained it with (Stiennon et al., 2020)'s dataset. We used NVIDIA A100 80GB for every experiment.

For the reward model structure of the general representation function, we froze the first 70% of the language model's layers, using the outputs of these layers as the representation. For the linear representation function, we froze the entire language model and only trained the additional final layer.

J.1 Detailed Aggregated Results in Section 6.2

We provide the first 5 examples in the test set, and compare the results. The examples might have undesirable vocabulary or topics.

Prompt 1

```
SUBREDDIT: r/AskReddit
TITLE: How do you get someone out of your head?
POST: Hi,
I'm 22, and I have been with my girlfriend for 5 years now.  We recently
moved together.  We've always loved each other intensely.

Problem, I recently started to have feelings for an other person (a friend).
This person has had a boyfriend for now 3 years, and has absolutely no
ideas.  Those feelings were so strong, it was hard to hide them.  After
2 months of me being distant and really sad, my girlfriend forced me to say
what was bothering me.  I'm not a good liar, and now she knows.

We decided to give us a week alone, I went to my parents.

Now, I'm completely lost.  I keep on thinking about this person, and I hate
that.  I would like for those feelings to go away, to leave me alone.  But
I can't.

What do I do?  It's been 3 months now, and I'm just desperate.
TL;DR:
```

Aggregation with $\alpha = -\infty$

```
I'm in love with my girlfriend, but recently started to have feelings for
another person (a friend).  How do you get someone out of your head?
```

Aggregation with $\alpha = -1$

```
I don't know how to get out of my head, to get over my feelings for someone
else, without cheating, without breaking up.  Help.
```

Aggregation with $\alpha = 0$

```
Have in couple for 5 years, moved in together.  It has the tendency to fall
in love with other random people.  How can I get those feelings to leave me
alone?
```

Aggregation with $\alpha = 1$

```
I have a girlfriend I love, and also a friend that I have feelings for.  I
don't know how to get rid of those feelings.  Please help.
```

Aggregation with $\alpha = \infty$

```
I'm with my girlfriend for 5 years, I have feelings for an other person and
I need to get those feelings gone.
```

Prompt 2

```
SUBREDDIT: r/pettyrevenge
TITLE: So, my mom woke me up with a loud TV.
POST: She was in her living room, watching TV. This was at about 8:30 in
the morning, and she was exercising.  She turned the TV up extra loud to
hear it over her excercycle, and woke me up.  I went in there asking for
her to turn it down.  She said she didn't have to; I explained that I
always used headphones so she didn't have to deal with my noise and that
she should give me a little more respect, given that I paid rent at the
time.

She disagreed.  I went back to my room, rather pissed off at the lack of
equality.  I had no lock on my door; but I had a dresser right next to it,
so I pulled one of the drawers out enough so that it caused the door to not
be openable.  Then, I turned my speakers up really loud and blasted Gangnam
Style on repeat, with the bass cranked up as high as it could go.

If you hate Gangnam Style for being overplayed, you will see why I chose
that particular song.  I personally don't mind it.  But here's the thing
about my bass; it vibrates the walls, making one hell of a lot of noise.
Needless to say, my mom was not pleased and shut off the internet.  But it
was oh so worth it.
TL;DR:
```

Aggregation with $\alpha = -\infty$

```
I make a racket when I sleep, then I play Gangnam Style at full volume to
"wake me up" and piss my mom off.
```

Aggregation with $\alpha = -1$

```
Moms wake me up by watching me play video games.  She calls it disrespect
for her having to listen to my music.  I play Gangnam Style on repeat until
she shuts off the Internet.
```

Aggregation with $\alpha = 0$

```
Mom was awake early and didn't have to deal with my noise, so I played
Gangnam Style and left it at that.
```

Aggregation with $\alpha = 1$

```
Mom wakes me up to background noise even when she does not have headphones
on.  I crank up Gangnam style on repeat enough to cause some structural
damage to her house.
```

Aggregation with $\alpha = \infty$

```
So, my mom wakes me up with my TV; I pay rent, but don't always have to bow
to her wishes.  Fuck yes.
```

Prompt 3

```
SUBREDDIT: r/relationships
TITLE: My girlfriend (20f) of two years cheated on me (20m) by kissing two
guys at a Halloween party.
POST: Lately her and I have been having a few problems, and these problems
have been brought up before a few times.  One problem being that I don't
show enough affection.  I don't tell her she's pretty very often or don't
compliment her much.  I feel terrible about it, but this time I was really
trying to change for her.

For Halloween she went to visit her step brother at a college and I got
drunk with my friends and watched movies.  Last night (11/1) we got in a
huge fight about me not changing and how our relationship won't work out
and basically broke up over the phone.  So in an effort to try and fix it
I drove to her house.  She told me how at the parties she went to that two
guys kissed her.  The first one she pushed away, but the second one I asked
her if she kissed him back and she said yes and that she did it because
it made her feel wanted, which I guess I haven't been making her feel that
way lately.  We cried, we talked about everything, we had great sex, and I
stayed over at her house just to sleep with her and then snuck out in the
morning so her parents wouldn't know.

We both obviously want to work things out but aren't sure if we should.
I love this girl, but the more I think about it, all I can think about is
her cheating on me, and more importantly, liking it.  It makes me sick to
my stomach.  Should I even try to fix it or would I be better off cutting
all ties.
TL;DR:
```

Aggregation with $\alpha = -\infty$

```
Girlfriend cheated on me for a feeling of closeness, should I even try to
fix things or am I just getting screwed?
```

Aggregation with $\alpha = -1$

```
girlfriend cheated on me by kissing two guys last night.  I love her and
don't want to break up, but don't know what to do.  Is cheating even worth
fighting for?
```

Aggregation with $\alpha = 0$

```
My girlfriend of two years, whom I love dearly, recently kissed two guys
at a Halloween party.  Should I try to salvage the relationship or cut
everything off?
```

Aggregation with $\alpha = 1$

```
Girlfriend of two years cheated on me by kissing two guys, I don't know if
I should even try to fix it.
```

Aggregation with $\alpha = \infty$

```
Girlfriend of 2 years cheated on me with two guys last night.  I feel sick
to my stomach.  I do still love her, should I work things out or just cut
all ties?
```

Prompt 4

```
SUBREDDIT: r/running
TITLE: One year post-pregnancy, aches and pains in hips and knees, can't
even comfortably do day 1 of C25k.  Help?
POST: I'll try to keep this short!

*Background**

I've always been an on again/off again (very casual!)  jogger, typically
doing 3 - 5 k
My knees have always been finicky, and I went to a physio who thought I had
"runner's knee"
Pre-pregnancy, my "runner's knee" would flare up when I got to the 8 - 10 k
distance range, even if I had a decent base (doing a C210k type program)

*Current Problem**

I had my baby a year ago, so all in all I haven't run for about 1.5
years.  I'm quite slim and have been doing aerobics-style classes for
the past year, so I'm not totally out of shape.  Body weight exercises,
aerobics, bikes and ellipticals are all fine. However, when I run even the
tiniest bit, or even go on a long walk or a hike, my pelvis gets very sore
and tight, and my knees start hurting very quickly.  I already am doing
general squats/lunges/stretching type things.

I'm starting to feel like running just isn't for me anymore.  Which is a
bummer, because I really enjoy running!

Has anyone had something similar?  Can anyone recommend some stretches
or exercises that might help?  Should I see a Dr?  Or should I just see a
physio?  Not quite sure how to proceed.

Thanks!
TL;DR:
```

Aggregation with $\alpha = -\infty$

```
Have always been a runner, always had knee troubles, no longer pregnant,
aches and pains from running now.  Help?
```

Aggregation with $\alpha = -1$

```
Was a fan of running.  Has your body ached?  What happened when you started
running again?  How did you go about it?  :)
```

Aggregation with $\alpha = 0$

```
Had runner's knee, started jogging again immediately post-pregnancy,
getting back to where I was at pre-pregnancy.  Now pelvis/knee pain after
even tiny runs.  What do?
```

Aggregation with $\alpha = 1$

```
running post-pregnancy, pelvis/knees are very unforgiving.  How best to
proceed (stretches?  exercises?  etc.)?  And what can I do to make it
better?
```

Aggregation with $\alpha = \infty$

```
1.5 year post-pregnancy, hip/knee pain while "running" even a few feet, no
injuries/illnesses/pain beforehand, not out of shape.  Advice?
```

Prompt 5

SUBREDDIT: r/tifu
TITLE: TIFU by getting scared for no reason and peeing on myself
POST: (Pre-apology for formatting errors since i'm on my phone)

Well, this was embarrassing and happened a few minutes ago.
I work at a small law office which shares a building with a few other businesses, a dentist's office and a military recruitment center. There is no drinking fountain in the building, so I bring a giant 2L water bottle which I sip on all day to keep hydrated and counteract all the copious amounts of alcohol i drink (after work, of course). Unfortunately, I have been cursed with a small bladder, and have to go pee sometimes 8-10 times a day. Luckily the back door of my office opens up right near the bathroom, so I can usually drain the main vein without embarrassing myself by being frequently seen there.

As I was wrapping up work for the day, I realized I would have to pee one more time before leaving. It was slightly after 5, so when I walked to the bathroom, I wasn't expecting to see anyone in there (it is the kind of bathroom with a urinal and a stall).

Fuckup begins when I open the door and there is a dude who had just finished shitting and was leaving the stall. For whatever reason this terrified me for a split second and I jumped and gasped.

The guy was amused and chuckled and was like, did I scare you? I muttered something like "haha, yeah, a little bit" and I kept my head down, beelined to the urinal and unleashed the krakan.

I was super embarrassed and very preoccupied so I didn't notice that the bottom part of my untucked, button-down shirt had slipped into my urine stream and was getting soaked. I didn't realize it until it began to sort of soak back onto my pant leg and getting my thigh wet.

Well, fuck, i thought. This sucks. Then i remembered my fiancee was coming to pick me up straight from work so we could go grocery shopping. I cleaned up as best as i could, but its still very obvious. Luckily I have a P coat which is long enough to cover my pee-soaked netherregion, so i may get through this unscathed. My girl didn't
TL;DR:

Aggregation with $\alpha = -\infty$

was scared for no reason, and pissed on myself when doing so. Not sure if I'm going to be able to avoid a major wardrobe change before tonight...

Aggregation with $\alpha = -1$

I was startled by someone, lost my bladder control and pee'd all over myself, after I finished up and was getting ready to leave for my fiancees house. Fortunately for me, it looks like i just peed myself.

Aggregation with $\alpha = 0$

I got scared for no reason and peed on myself, then covered it up with my fiancee as we're about to leave work, but it's still visible.

Aggregation with $\alpha = 1$

work didn't have a toilet, had to use bathroom at my office. Sharted in the bathroom stall and got my shirt stuck to me, but may have gotten rid of all evidence of the incident by bringing a P coat

Aggregation with $\alpha = \infty$

`I jumped, freaked out, urinated, and forgot to change my wet shirt. When I got home, I had to change the dress so my fiancee wouldn't catch a glimpse.`