Sbornik: *Mathematics* **192**:8 1165–1179

Matematicheskiĭ Sbornik **192**:8 79–94 DOI 10.1070/SM2001v192n08ABEH000587

On homological dimensions

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Abstract. For finite modules over a local ring the general problem is considered of finding an extension of the class of modules of finite projective dimension preserving various properties. In the first section the concept of a suitable complex is introduced, which is a generalization of both a dualizing complex and a suitable module. Several properties of the dimension of modules with respect to such complexes are established. In particular, a generalization of Golod's theorem on the behaviour of G_K-dimension with respect to a suitable module K under factorization by ideals of a special kind is obtained and a new form of the Avramov–Foxby conjecture on the transitivity of G-dimension is suggested. In the second section a class of modules containing modules of finite CI-dimension is considered, which has some additional properties. A dimension constructed in the third section characterizes the Cohen–Macaulay rings in precisely the same way as the class of modules of finite projective dimension characterizes regular rings and the class of modules of finite CI-dimension characterizes complete intersections.

Bibliography: 19 titles.

Introduction

In this paper we consider local rings and, unless otherwise stated, finitely generated modules over them. It is well known that the class of modules of finite projective dimension over a ring R characterizes regular rings in the following precise sense:

$$R \text{ is regular} \quad \Leftrightarrow \quad \operatorname{pd} M < \infty \text{ for each } M \quad \Leftrightarrow \quad \operatorname{pd} k < \infty, \tag{1}$$

where $k \simeq R/\mathfrak{m}$ is the residue field of the ring R.

Moreover, there are reasons to believe that modules of finite projective dimension behave in a certain sense similarly to modules over regular rings (see, for instance, the introduction in [1]). A reasonable question arising in this connection is whether it is possible to extend in a natural way the class of modules of finite projective dimension so that this extension would characterize, in the sense of (1), other classes of rings important for algebraic geometry, namely, local complete intersections, Gorenstein and Cohen–Macaulay rings. Here by a natural extension we mean an extension preserving various properties of the modules of finite projective

This research was carried out with the partial support of the Russian Foundation for Basic Research (grant no. 99-01-01144).

AMS 2000 *Mathematics Subject Classification.* Primary 13D05, 13C15.

dimension. Such classes previously appeared in various problems of commutative algebra.

For Gorenstein rings the corresponding class has been considered by Auslander and Bridger in [2]. One says that $\text{G-dim}\, M = 0$ if the natural homomorphism $M \to \operatorname{Hom}(\operatorname{Hom}(M, R), R)$ is an isomorphism and $\operatorname{Ext}^i_R(M, R) = 0 = \operatorname{Ext}^i_R(M^*, R)$ for $i > 0$. Consider next left resolutions of a module M by modules P such that $\text{G-dim}\, P = 0$. The G-dimension of M is by definition the infimum of the lengths of such resolutions. This approach was further developed in [3] and [4], where the authors considered the so-called G_K dimension with respect to a *suitable* module K (see Definition 1.1) characterizing pairs of the form (a Cohen–Macaulay ring, the canonical module over it).

For complete intersections the corresponding class of modules, which were called modules of finite virtual projective dimension (vpd) was introduced by Avramov (see [5]) in the context of the study of the properties of Betti numbers for modules of infinite projective dimension; $\operatorname{vpd}_R M$ is set to be finite if there exists a surjective homomorphism of rings $S \to \widehat{R}$, where $\widehat{R}$ is the $\mathfrak{m}$-adic completion of R, such that its kernel is generated by a regular sequence and $\operatorname{pd}_S(M \otimes_R \widehat{R}) < \infty$. For the (possibly) broader class of modules of finite CI-dimension (see Definition 2.2) introduced later in [6] and also characterizing complete intersections several results have been established, for which it is not yet known if they hold for modules of finite virtual projective dimension; for example, this class behaves nicely under localization. Besides, modules of finite CI-dimension actually demonstrate in some problems a behaviour similar to that of modules over complete intersections. As examples we can cite the papers [7], [8], and [9], where the so-called 'depth formula' is generalized from modules over complete intersections to modules of finite CI-dimension, and [8], where Auslander's 'freeness criterion' is generalized in the same direction.

We also mention the important implication

$$\operatorname{pd}_R M < \infty \Rightarrow \text{CI-dim}_R\, M < \infty \Rightarrow \text{G-dim}_R\, M < \infty,$$

which in the case $M = k$ reduces to the following well-known result:

$$R \text{ is regular} \Rightarrow R \text{ is a complete intersection} \Rightarrow R \text{ is Gorenstein.}$$

We say that *a generalized homological dimension* is defined if for each ring R we have a class of modules H_R and a map H-dim_R from H_R into $\mathbb{Z}$. Of course, such a concept is too general to be interesting, therefore we write down a list of natural conditions:

(I) If $M \in \mathrm{H}_R$, then $\text{H-dim}_R\, M + \operatorname{depth} M = \operatorname{depth} R$.

(II) $k = R/\mathfrak{m} \in \mathrm{H}_R$ if and only if $M \in \mathrm{H}_R$ for all R modules.

(III) Let x be an R- and M-regular element. If $M \in \mathrm{H}_R$, then $M/xM \in \mathrm{H}_{R/xR}$ and $\text{H-dim}_R\, M = \text{H-dim}_{R/xR}\, M/xM$.

(IV) If $M \in \mathrm{H}_R$, then $M_{\mathfrak{p}} \in \mathrm{H}_{R_{\mathfrak{p}}}$ and $\text{H-dim}_R\, M \geqslant \text{H-dim}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$.

(V) If a sequence of modules $0 \to M \to N \to K \to 0$ is exact and any two of these modules belong to H_R, then the third module also has this property. If this sequence is split exact and $N \in \mathrm{H}_R$, then $M \in \mathrm{H}_R$ and $K \in \mathrm{H}_R$.

In general we do not require $\text{H-dim}_R\, M$ to be non-negative.

Note that all these conditions are satisfied by projective dimension and G-dimension. Properties (I) and (II) are known to hold for virtual projective dimension and properties (I)–(IV) hold for CI-dimension.

The definition of G_K-dimension does not completely fit into this scheme because of the additional parameter, but we still have perfect analogues of properties I–V.

In the first part of this paper we introduce homological dimension with respect to a *suitable* complex, which is a generalization of the concepts of *suitable* module and dualizing complex. Among other things this approach allows us to obtain a simple proof of Proposition 5 in [4] along with a generalization of it. Recently this author became aware that these complexes had been previously studied by Christensen [10] and most results of the first part were already known.

In the second part we consider an alternative approach to the definition of the class of modules characterizing complete intersections, which uses resolutions by modules of dimension zero. The resulting class is an extension of the class of modules of finite CI-dimension; it satisfies condition (V). We also give a simpler proof than in [11] of the following result: the class of rings of local complete intersection localizes.

In the third part we introduce a dimension characterizing Cohen–Macaulay rings. Using the scheme proposed in Definition 2.2 one can consider several definitions of such a dimension. The one proposed and studied here has properties (I)–(IV), and notably, the class of modules of finite CM-dimension contains all modules of finite G-dimension.

The author would like to thank his scientific advisor E. S. Golod for valuable discussions concerning the material presented in this paper.

1. $\mathrm{G}_{\mathbf{I}}$-dimension

We use the notation $\mathcal{C}(R)$ for the category of R-complexes. The differential of a complex $\mathbf{X}$ acting from X^n into X^{n+1} is denoted by δ^n. The following quantities are associated with each complex $\mathbf{I}$:

$$\sup(\mathbf{I}) = \sup\{n : \mathrm{H}^n(\mathbf{I}) \neq 0\}, \qquad \inf(\mathbf{I}) = \inf\{n : \mathrm{H}^n(\mathbf{I}) \neq 0\},$$
$$\mathrm{amp}(\mathbf{I}) = \sup(\mathbf{I}) - \inf(\mathbf{I}).$$

The notation $\mathcal{C}^+(R)$ ($\mathcal{C}^-(R)$, $\mathcal{C}^b(R)$) will be used for the full subcategories of complexes in $\mathcal{C}(R)$ with $\inf(\mathbf{I}) > -\infty$ ($\sup(\mathbf{I}) < \infty$, $\mathrm{amp}(\mathbf{I}) < \infty$, respectively). We shall denote by $\mathcal{C}_f(R)$ ($\mathcal{C}_f^-(R)$, $\mathcal{C}_f^+(R)$, $\mathcal{C}_f^b(R)$) the full subcategory of $\mathcal{C}(R)$ ($\mathcal{C}^-(R)$, $\mathcal{C}^+(R)$, $\mathcal{C}^b(R)$, respectively) whose objects are complexes with finitely generated homology modules.

First, we present a list of basic facts about G_K-dimension and G_K-perfect modules required in what follows. We fix a module K. For a module P we set $P^* = \mathrm{Hom}_R(P, K)$. The module P is said to be *K-reflexive* if the canonical biduality homomorphism $P \to P^{**}$ is bijective.

Definition 1.1 [4]**.** If $\operatorname{Ext}^i_R(P,K)=0=\operatorname{Ext}^i_R(P^*,K)$ for a K-reflexive module P and all $i>0$, then we set

$$\mathrm{G}_K\text{-}\dim_R P = 0,$$
$$\mathrm{G}_K\text{-}\dim_R M = \inf\{n : \text{there exists an exact sequence } 0\to P_n\to P_{n-1}\to\cdots\to P_0\to M\to 0, \text{ where } \mathrm{G}_K\text{-}\dim_R P_i=0\}.$$

If $\mathrm{G}_K\text{-}\dim M$ is finite, then its value can be expressed in the following way.

Proposition 1.2 [4]. *If* $\mathrm{G}_K\text{-}\dim M<\infty$, *then*

$$\mathrm{G}_K\text{-}\dim M = \sup\{n : \operatorname{Ext}^n_R(M,K)\neq 0\}.$$

If $\mathrm{G}_K\text{-}\dim_R R=0$, then K is said to be *suitable*. In other words, K is suitable if and only if $\operatorname{Hom}_R(K,K)\simeq R$ and $\operatorname{Ext}^i_R(K,K)=0$ for all $i>0$. Trivial examples are the free module of rank one (the corresponding dimension will be denoted $\mathrm{G}\text{-}\dim M$) and the dualizing module. An analogue of the Auslander–Buchsbaum formula holds for G_K-dimension with respect to a suitable module, and we also have the following result.

Proposition 1.3. *The following assertions are equivalent*:

(1) *K is a dualizing module*;
(2) $\mathrm{G}_K\text{-}\dim M<\infty$ *for all R-modules M*;
(3) $\mathrm{G}_K\text{-}\dim k<\infty$.

Recall that $\operatorname{grade} M=\inf\{i : \operatorname{Ext}^i_R(M,R)\neq 0\}$. If I is an ideal in R, then one usually writes $\operatorname{grade} I$ instead of $\operatorname{grade} R/I$ (the reason is that in this notation we have $\operatorname{grade} I=\{$the length of a maximal R-regular sequence in $I\}$).

It can be easily seen that $\operatorname{grade} M\leqslant \mathrm{G}_K\text{-}\dim M$. If $\operatorname{grade} M=\mathrm{G}_K\text{-}\dim M$, then M is said to be G_K-*perfect*. An ideal I is said to be G_K-*perfect* if R/I is a G_K-perfect module over R. The meaning of this concept is revealed by the following result.

Theorem 1.4 ([4], Proposition 5). *Let I be a G_K-perfect ideal, and K a suitable R-module. Then $\operatorname{Ext}^{\operatorname{grade} I}_R(R/I,K)$ is a suitable R/I-module, and the following implication holds for all R/I-modules M*:

$$\mathrm{G}_K\text{-}\dim_R M<\infty \Leftrightarrow \mathrm{G}_{K'}\text{-}\dim_{R/I} M<\infty,$$

where $K'=\operatorname{Ext}^{\operatorname{grade} I}_R(R/I,K)$. *Finally, if these dimensions are finite, then*

$$\mathrm{G}_K\text{-}\dim_R M = \operatorname{grade} I + \mathrm{G}_{K'}\text{-}\dim_{R/I} M.$$

We give a new proof of this theorem below (Remark 1.16), which is different from the proof given in [4].

Definition 1.5. Under the assumptions of Theorem 1.4 let $K'\simeq R/I$. Then the ideal I is said to be G_K-*Gorenstein*.

We proceed now to the main definitions.

Definition 1.6. Let $\mathbf{I} \in \mathcal{C}_f^b(R)$ be an injective left-bounded complex. Then it is said to be *suitable* if the natural biduality morphism $R \xrightarrow{\alpha_R} \mathrm{Hom}_R(\mathrm{Hom}_R(R,\mathbf{I}),\mathbf{I})$ is a quasi-isomorphism.

Note that if we impose an additional condition on $\mathbf{I}$, namely, if we require that it be quasi-isomorphic to a finite complex of injective modules, then we obtain the definition of a *dualizing* complex (cf., for example, [12]). Another example of a suitable complex is $\mathbf{I} = \mathbf{I}(K)$, an injective resolution of a suitable module K.

We set for brevity $\mathbf{M}^{**} = \mathrm{Hom}_R(\mathrm{Hom}_R(M,\mathbf{I}),\mathbf{I})$.

Definition 1.7. Let $M \xrightarrow{\alpha_M} \mathbf{M}^{**}$ be a quasi-isomorphism. Then we set

$$\mathrm{G_I}\text{-}\dim M = \sup(\mathrm{Hom}(M,\mathbf{I})) - \sup(\mathbf{I}).$$

It can be easily seen that the $\mathrm{G_I}$-dimension of a module does not change if we replace $\mathbf{I}$ by another injective complex $\mathbf{I}'$ quasi-isomorphic to $\mathbf{I}$. We shall now explain the relation between Definitions 1.1 and 1.7.

Lemma 1.8. *If a sequence of modules* $0 \to M \to N \to K \to 0$ *is exact and two of these modules have finite* $\mathrm{G_I}$*-dimension, then the third module also has this property.*

Proof. Everything follows from the consideration of the exact sequence of complexes $0 \to \mathrm{Hom}(K,\mathbf{I}) \to \mathrm{Hom}(N,\mathbf{I}) \to \mathrm{Hom}(M,\mathbf{I}) \to 0$ and the following commutative diagram of complexes with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & K & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \alpha_M} & & \Big\downarrow{\scriptstyle \alpha_N} & & \Big\downarrow{\scriptstyle \alpha_K} & & \\ 0 & \longrightarrow & \mathbf{M}^{**} & \longrightarrow & \mathbf{N}^{**} & \longrightarrow & \mathbf{K}^{**} & \longrightarrow & 0 \end{array} .$$

Lemma 1.9. *Let* $0 \to M \to N \to K \to 0$ *be an exact sequence of modules, and assume that* $\mathrm{G_I}\text{-}\dim N{=}0$ *and* $0{<}\mathrm{G_I}\text{-}\dim K{<}\infty$. *Then* $\mathrm{G_I}\text{-}\dim M{=}\mathrm{G_I}\text{-}\dim K - 1$.

Proof. This is similar to the previous lemma.

Theorem 1.10. *Let* $\mathbf{I}$ *be an injective resolution of a suitable module* K. *Then* $\mathrm{G_I}\text{-}\dim M = \mathrm{G}_K\text{-}\dim M$ *for all modules* M.

Proof. We carry out induction on one of the above dimensions, provided that it is finite. It is easy to see that $\mathrm{G}_K\text{-}\dim M = 0 \Leftrightarrow \mathrm{G_I}\text{-}\dim M = 0$. For if one of these dimensions is zero, then we have $\mathrm{Ext}_R^n(M,K) = 0$ for all $n > 0$. In this case

$$\begin{aligned} & M \xrightarrow{\alpha_M} \mathbf{M}^{**} \text{ is a quasi-isomorphism} \\ \Leftrightarrow\ & \text{the canonical homomorphism } M \to M^{**} \text{ is an isomorphism,} \end{aligned}$$

as required. Now, if one of the dimensions is finite and greater than zero, then we consider a projective cover of the module M and apply Lemma 1.9 and the inductive hypothesis.

Remark 1.11. From Lemma 1.8 we can see, in particular, that if $\mathrm{pd}\, M < \infty$, then $\mathrm{G_I}\text{-}\dim M < \infty$ for each suitable complex $\mathbf{I}$.

Suitable complexes that are not resolutions of suitable modules can be encountered only over ‘bad’ rings. More precisely, the following result holds.

Proposition 1.12. *If* $\mathbf{I}$ *is a suitable complex over* R *and* $\operatorname{amp}(\mathbf{I}) > 0$, *then* R *is not Cohen–Macaulay.*

Proof. Assume that R is a Cohen–Macaulay ring. If x is an R-regular element, then we see from the exact sequence $0 \to \operatorname{Hom}_R(R/xR, \mathbf{I}) \to \mathbf{I} \xrightarrow{x} \mathbf{I} \to 0$ that $\operatorname{amp}(\operatorname{Hom}_R(R/xR, \mathbf{I})) \geqslant \operatorname{amp}(\mathbf{I})$. Using induction on depth we can content ourselves with the case of an Artin ring. The complex $\operatorname{Hom}(\mathbf{I}, \mathbf{I})$ is quasi-isomorphic to $\operatorname{Hom}(\mathbf{P}(\mathbf{I}), \mathbf{I})$. We have

$$0 = \mathrm{H}^{n - \operatorname{amp}(\mathbf{I})}(\mathbf{P}(\mathbf{I}), \mathbf{I}) = \operatorname{Hom}(\mathrm{H}^{n \sup(\mathbf{I})}(\mathbf{I}), \mathrm{H}^{n \inf(\mathbf{I})}(\mathbf{I})),$$

and $\operatorname{amp}(\mathbf{I}) \neq 0$. This is a contradiction since $\operatorname{Hom}(M, N) \neq 0$ over Artin rings.

We shall repeatedly require the following technical result.

Lemma 1.13 ([12], Lemma 3.4(ii)). *Let* $\mathbf{I} \in \mathcal{C}_f(R)$ *be a left-bounded injective complex and let* M *be a finitely generated module. Then* $\operatorname{Hom}(M, \mathbf{I}) \in \mathcal{C}_f(R)$.

Proposition 1.14. *Let* $\mathbf{I}$ *be a suitable complex over* S, *and* R *a finite (=finitely generated as an* S*-module)* S*-algebra. Let* $\mathbf{I}' = \operatorname{Hom}_S(R, \mathbf{I})$. *Then*

$$\mathrm{G_I}\text{-}\dim_S R < \infty \ \Leftrightarrow\ \mathbf{I}' \textit{ is a suitable complex over } R.$$

Proof. Assume that $\mathrm{G_I}\text{-}\dim_S R < \infty$. In this case $\mathbf{I}'$ has only finitely many nonzero homology modules (by the definition of $\mathrm{G_I}\text{-}\dim_S R$), all of which are finitely generated in view of Lemma 1.13. The injectivity of $\mathbf{I}'$ is well known. The functor $\operatorname{Hom}_R(\,\cdot\,, \operatorname{Hom}_S(R, \mathbf{I}))$ from $\mathcal{C}(R)$ into itself is isomorphic to $\operatorname{Hom}_S(\,\cdot\,, \mathbf{I})$. Combining this with the quasi-isomorphism $R \to \operatorname{Hom}_S(\operatorname{Hom}_S(R, \mathbf{I}), \mathbf{I})$ we see that $\mathbf{I}'$ is R-suitable. The converse is proved in the same manner.

It is easy to see that for an R-module M in the above situation $\mathrm{G_I}\text{-}\dim_S M < \infty$ if and only if $\mathrm{G_{I'}} \dim_R M < \infty$. Moreover, the following result holds.

Theorem 1.15. *If* M *is an* R*-module, then* $\mathrm{G_I}\text{-}\dim_S M < \infty$ *if and only if* $\mathrm{G_{I'}}\text{-}\dim_{\mathrm{R}} M < \infty$ *In addition,* $\mathrm{G_I}\text{-}\dim_S M = \mathrm{G_{I'}}\text{-}\dim_{\mathrm{R}} M + \mathrm{G_I}\text{-}\dim_S R$.

Proof. By definition, $\sup(\mathbf{I}) - \sup(\mathbf{I}') = \mathrm{G_I}\text{-}\dim_S R$. Furthermore, we have the equality $\sup(\operatorname{Hom}_S(M, \mathbf{I})) = \sup(\operatorname{Hom}_R(M, \mathbf{I}'))$. Subtracting it from the first equality we obtain the required result.

Remark 1.16. Let $\mathbf{I} = \mathbf{I}(K)$ be an injective resolution of a suitable R-module K and let $\mathfrak{a}$ be a G_K-perfect ideal. Then it is easy to see that $\mathbf{I}' = \mathbf{I}(K')$ is an injective resolution of the $R/\mathfrak{a}$-suitable module K'. Applying Theorem 1.15 we obtain the result of Theorem 1.4.

We now describe suitable complexes $\mathbf{I}$ such that for all R-modules M we have $\mathrm{G_I}\text{-}\dim M < \infty$. Let $\mu^i(\mathfrak{p}, \mathbf{I}) = \dim_{k(\mathfrak{p})}\ (\mathrm{H}^i(\operatorname{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), \mathbf{I}_\mathfrak{p})))$, where $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$, be an analogue of the Bass numbers. We shall require a slight modification of Lemma 3.1 in [13].

Lemma 1.17. *Let* $\mathfrak{q}$ *and* $\mathfrak{p}$, $\mathfrak{q} \subset \mathfrak{p}$, *be distinct prime ideals such that there is no prime ideal between them. If* $\mathbf{I} \in \mathcal{C}_f(R)$ *is a left-bounded injective complex, then*

$$\mu^i(\mathfrak{q}, \mathbf{I}) > 0 \Rightarrow \mu^{i+1}(\mathfrak{p}, \mathbf{I}) > 0.$$

Proof. Localizing we can reduce the problem to the case $R = R_\mathfrak{p}$. For $x \in \mathfrak{p}$, $x \notin \mathfrak{q}$, we set $S = R/\mathfrak{q}$ and $T = S/xS = R/(x, \mathfrak{q})$. The module T has finite length and x is S-regular. From the exact sequence $0 \to S \xrightarrow{x} S \to T \to 0$ we see that the sequence

$$\mathrm{H}^i(\mathrm{Hom}_R(S, \mathbf{I})) \xrightarrow{x} \mathrm{H}^i(\mathrm{Hom}_R(S, \mathbf{I})) \to \mathrm{H}^{i+1}(\mathrm{Hom}_R(T, \mathbf{I}))$$

is exact. Since $\mathrm{H}^i(\mathrm{Hom}_R(S, \mathbf{I})_\mathfrak{q}) \neq 0$ and $\mathrm{H}^i(\mathrm{Hom}_R(S, \mathbf{I}))$ is finitely generated, we can apply Nakayama's lemma and obtain

$$\mathrm{H}^{i+1}(\mathrm{Hom}_R(T, \mathbf{I})) \neq 0.$$

Conversely, assuming that $\mathrm{H}^{i+1}(\mathrm{Hom}_R(R/\mathfrak{p}, \mathbf{I})) = 0$ and using the half-exactness of $\mathrm{H}^{i+1}(\mathrm{Hom}_R(\,\cdot\,, \mathbf{I}))$ we can show by induction that

$$\mathrm{H}^{i+1}(\mathrm{Hom}_R(T, \mathbf{I})) = 0,$$

which is a contradiction.

Theorem 1.18. *If* $\mathbf{I}$ *is a suitable complex over a ring* R *and* $\mathrm{G_I}\text{-}\dim_R k < \infty$, *then* $\mathbf{I}$ *is a dualizing complex and* $\mathrm{G_I}\text{-}\dim_R M < \infty$ *for all* R*-modules* M.

Proof. By Proposition 1.14 $\mathrm{Hom}_R(R/\mathfrak{m}, \mathbf{I})$ is a suitable complex over the vector space k, therefore there exists a unique integer $i = i_0$ such that $\mu^i(\mathfrak{m}, \mathbf{I}) \neq 0$. From Lemma 1.17 we now see that $\mu^i(\mathfrak{p}, \mathbf{I}) = 0$ for all $i \leqslant i_0$. Hence $\mathbf{I}$ is the direct sum of an acyclic and a bounded injective complex, and therefore it is dualizing.

Injective modules are generally not finitely generated, which makes it more complicated to analyse the behaviour of G_I-dimension under localization. The following technical lemma is helpful here.

Lemma 1.19. *Let* $\mathbf{I} \in \mathcal{C}(R)$ *be an injective left-bounded complex and let*

$$\xi\colon\ \mathrm{Hom}_R(\,\cdot\,, \mathbf{I}) \otimes R_\mathfrak{p} \to \mathrm{Hom}_{R_\mathfrak{p}}(\,\cdot\, \otimes R_\mathfrak{p}, \mathbf{I} \otimes R_\mathfrak{p})$$

be the natural morphism of functors from $\mathcal{C}(R)$ *to* $\mathcal{C}(R_\mathfrak{p})$. *If* $\mathbf{X} \in \mathcal{C}_f^-(R)$, *then* $\xi(\mathbf{X})$ *is a quasi-isomorphism.*

Proof. Note that this is true when $\mathbf{X}$ is either acyclic or has only one homology distinct from zero. Let us introduce the following notation:

$\sigma_{>n}(\mathbf{X})$ is the complex $\cdots \to 0 \to \operatorname{im} d^n_\mathbf{X} \to X^{n+1} \to X^{n+2} \to \cdots$;
$\sigma'_{\geqslant n}(\mathbf{X})$ is the complex $\cdots \to 0 \to \operatorname{coker} d^{n-1}_\mathbf{X} \to X^{n+1} \to X^{n+2} \to \cdots$.

We have the exact sequence of complexes

$$0 \to \{-n\}\mathrm{H}^n(\mathbf{X}) \to \sigma'_{\geqslant n}(\mathbf{X}) \to \sigma_{>n}(\mathbf{X}) \to 0.$$

We shall prove that $\xi(\sigma_{>n}(\mathbf{X}))$ is an isomorphism by descending induction on n. This is obvious for $n \gg 0$ since $\xi(\sigma_{>n}(\mathbf{X}))$ is acyclic in this case. Setting for brevity $\mathrm{F}(\,\cdot\,) = \mathrm{Hom}_R(\,\cdot\,, \mathbf{I}) \otimes R_{\mathfrak{p}}$ and $\mathrm{G}(\,\cdot\,) = \mathrm{Hom}_{R_{\mathfrak{p}}}(\,\cdot \otimes R_{\mathfrak{p}}, \mathbf{I} \otimes R_{\mathfrak{p}})$ we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{F}(\sigma_{>n}(\mathbf{X})) & \longrightarrow & \mathrm{F}(\sigma'_{\geqslant n}(\mathbf{X})) & \longrightarrow & \mathrm{F}(\{-n\}\mathrm{H}^n(\mathbf{X})) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \xi(\sigma_{>n}(\mathbf{X}))} & & \big\downarrow{\scriptstyle \xi(\sigma'_{\geqslant n}(\mathbf{X}))} & & \big\downarrow{\scriptstyle \xi(\{-n\}\mathrm{H}^n(\mathbf{X}))} & & \\ 0 & \longrightarrow & \mathrm{G}(\sigma_{>n}(\mathbf{X})) & \longrightarrow & \mathrm{G}(\sigma'_{\geqslant n}(\mathbf{X})) & \longrightarrow & \mathrm{G}(\{-n\}\mathrm{H}^n(\mathbf{X})) & \longrightarrow & 0 \end{array},$$

in which the rows are exact. We know that $\xi(\sigma_{>n}(\mathbf{X}))$ is a quasi-isomorphism by the inductive hypothesis, and $\xi(\{-n\}\mathrm{H}^n(\mathbf{X}))$ is a quasi-isomorphism since $\mathrm{H}^n(\mathbf{X})$ is finitely generated. Using now the five lemma for morphisms of the corresponding long homology sequences we see that $\xi(\sigma'_{\geqslant n}(\mathbf{X}))$ is also a quasi-isomorphism. However, the complex $\sigma'_{\geqslant n}(\mathbf{X})$ is quasi-isomorphic to $\sigma_{>n-1}(\mathbf{X})$, therefore $\xi(\sigma_{>n}(\mathbf{X}))$ is a quasi-isomorphism for each n. Hence the lemma holds in the case when $\mathbf{X}$ is bounded from the left because then $\sigma_{>n}(\mathbf{X}) \simeq \mathbf{X}$ for $n \ll 0$. Consider now the case of an arbitrary complex $\mathbf{X}$. Note that the complex $\mathrm{F}(\mathbf{X})$ (respectively, $\mathrm{G}(\mathbf{X})$) can be represented as the union of the complexes $\mathrm{F}(\sigma_{>n}(\mathbf{X}))$ (respectively, of the $\mathrm{G}(\sigma_{>n}(\mathbf{X}))$), and by the above $\xi(\sigma_{>n}(\mathbf{X}))$ is a quasi-isomorphism for each n.

We claim that the map in the homology induced by $\xi(\mathbf{X})$ is an isomorphism.

Injectivity. Let x be an arbitrary cycle in $\mathrm{F}(\mathbf{X})$ such that its image is a boundary y in $\mathrm{G}(\mathbf{X})$. Then y is already a boundary in $\mathrm{G}(\sigma_{>n}(\mathbf{X}))$ for $n < n_0$ and x belongs to the complexes $\mathrm{F}(\sigma_{>m}(\mathbf{X}))$ for $m < m_0$. Hence $\xi(\sigma_{>k}(\mathbf{X}))$ is not a quasi-isomorphism for $k < \min(n_0, m_0)$. The *surjectivity* is obvious.

The next result is a generalization of Lemma 1.13.

Proposition 1.20. *If* $\mathbf{I} \in \mathcal{C}_f(R)$ *is an injective left-bounded complex and* $\mathbf{X} \in \mathcal{C}_f^-(R)$, *then* $\mathrm{Hom}(\mathbf{X}, \mathbf{I}) \in \mathcal{C}_f^+(R)$.

Proof. This is similar to the proof of the previous lemma. We use the complexes $\sigma_{>n}(\mathbf{X})$, descending induction, and the fact that if in an exact sequence $N' \to N \to N''$ the modules N' and N'' are finitely generated, then N too is finitely generated.

Theorem 1.21. *Let* $\mathbf{I}$ *be a suitable complex over* R, *and* M *an* R-*module. Then* $\mathbf{I}_{\mathfrak{p}}$ *is suitable over* $R_{\mathfrak{p}}$ *and the finiteness of* $\mathrm{G}_{\mathrm{I}}\text{-}\dim_R M$ *ensures the finiteness of* $\mathrm{G}_{\mathrm{I}_{\mathfrak{p}}}\text{-}\dim_{\mathrm{R}_{\mathfrak{p}}} \mathrm{M}_{\mathfrak{p}}$.

Proof. We merely need to demonstrate the properties relating to reflexivity. We see from Lemma 1.19 applied to $\mathbf{X} = \mathbf{I}$ that $\mathbf{I}_{\mathfrak{p}}$ is $R_{\mathfrak{p}}$-suitable. Applying the same lemma to $\mathbf{X} = \mathrm{Hom}(M, \mathbf{I})$ we see that $M_{\mathfrak{p}} \to \mathrm{Hom}_{R_{\mathfrak{p}}}(\mathrm{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, \mathbf{I}_{\mathfrak{p}}), \mathbf{I}_{\mathfrak{p}})$ is a quasi-isomorphism as a localization of the quasi-isomorphism $M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, \mathbf{I}), \mathbf{I})$.

We shall now prove an analogue of the Auslander–Buchsbaum formula for G_{I}-dimension.

Theorem 1.22. *If* $\mathbf{I}$ *is a suitable complex over* R *and* $\mathrm{G}_{\mathrm{I}}\text{-}\dim_R M < \infty$, *then* $\mathrm{G}_{\mathrm{I}}\text{-}\dim_R M + \operatorname{depth} M = \operatorname{depth} R$. *In particular, the value of* $\mathrm{G}_{\mathrm{I}}\text{-}\dim M$, *provided that it is finite, does not depend on the choice of the suitable complex* $\mathbf{I}$.

Proof. We have

$$\begin{aligned}\operatorname{depth} M &= \inf(\operatorname{Hom}(\mathbf{P}(k), M)) = \inf(\operatorname{Hom}(\mathbf{P}(k), \operatorname{Hom}(\operatorname{Hom}(M, \mathbf{I}), \mathbf{I}))) \\ &= \inf(\operatorname{Hom}(\mathbf{P}(k) \otimes \operatorname{Hom}(M, \mathbf{I}), \mathbf{I})) = \inf(\operatorname{Hom}(k \otimes \mathbf{P}(\operatorname{Hom}(M, \mathbf{I})), \mathbf{I})).\end{aligned}$$

The complex $k \otimes \mathbf{P}(\operatorname{Hom}(M, \mathbf{I}))$ is a complex of vector spaces over the field k; moreover, we have $\sup(k \otimes \mathbf{P}(\operatorname{Hom}(M, \mathbf{I}))) = \sup(\operatorname{Hom}(M, \mathbf{I}))$. Hence we obtain $\operatorname{depth} M = \inf(\operatorname{Hom}(k \otimes \mathbf{P}(\operatorname{Hom}(M, \mathbf{I})), \mathbf{I})) = \inf(\operatorname{Hom}(k, \mathbf{I})) - \sup(\operatorname{Hom}(M, \mathbf{I}))$. In a similar way, $\operatorname{depth} R = \inf(\operatorname{Hom}(k, \mathbf{I})) - \sup(\mathbf{I})$. The last two equalities give us the required result

$$\operatorname{depth} R - \operatorname{depth} M = \mathrm{G}_{\mathrm{I}}\text{-}\dim M.$$

Consider now the following situation: R is a finite S-algebra, $\operatorname{pd}_S R$ is finite, and M is an R-module of finite R-projective dimension. Then it is easy to see that $\operatorname{pd}_S M$ is also finite and, moreover, $\operatorname{pd}_S M = \operatorname{pd}_R M + \operatorname{pd}_S R$. The following question has been posed in [14], Remark 4.8: does an analogue of this result hold for G-dimension? Using Theorem 1.15 we can put forward the following, possibly more general, formulation of this conjecture:

Conjecture 1.23. *Let* $\mathbf{I}$ *be a suitable complex over* R. *Then*

$$\mathrm{G}_{\mathrm{I}}\text{-}\dim M \leqslant \mathrm{G}\text{-}\dim M$$

and this relation becomes an equality if the right-hand side is finite.

We now show how a confirmation of this conjecture in the case of suitable complexes of a certain special form brings us to the required result.

Corollary 1.24. *Let* R *be a finite* S*-algebra, assume that* $\mathrm{G}\text{-}\dim_S R < \infty$, *and let* M *be an* R*-module. If Conjecture* 1.23 *holds for the ring* R, *then the following implication also holds*: $\mathrm{G}\text{-}\dim_R M < \infty \Rightarrow \mathrm{G}\text{-}\dim_S M < \infty$; *moreover,* $\mathrm{G}\text{-}\dim_S M = \mathrm{G}\text{-}\dim_R M + \mathrm{G}\text{-}\dim_S R$ *if these dimensions are finite.*

Proof. Consider the complex $\mathbf{I}' = \operatorname{Hom}_S(R, \mathbf{I})$, where $\mathbf{I}$ is an injective resolution of S as a module over itself. From Proposition 1.14 we see that $\mathbf{I}'$ is a suitable complex over R, therefore if Conjecture 1.23 is true, then

$$\mathrm{G}_{\mathrm{I}'} \dim_{\mathrm{R}} M < \infty.$$

Applying 1.15 we see that $\mathrm{G}_{\mathrm{I}}\text{-}\dim_S M < \infty$. The equality

$$\mathrm{G}_{\mathrm{I}}\text{-}\dim_S M = \mathrm{G}_{\mathrm{I}'}\text{-}\dim_{\mathrm{R}} M + \mathrm{G}_{\mathrm{I}}\text{-}\dim_S R$$

now follows from Theorem 1.22.

The following question is also interesting.

Question 1.25. *Is it true that all suitable complexes over* R *have the following form*: $\mathbf{I} = \operatorname{Hom}_S(R, \mathbf{I}(S))$, *where* R *is a quotient ring of the ring* S?

In § 3 we show that the answer is affirmative in the case when $\mathbf{I}$ is an injective resolution of a suitable module.

2. CI-dimension

Definition 2.1 [6]**.** A *quasi-deformation* of a ring R is a diagram of local homomorphisms $R \to R' \leftarrow Q$, where $R \to R'$ is a flat extension and $R' \leftarrow Q$ is a *deformation*, that is, a surjective homomorphism with kernel generated by a regular sequence.

Definition 2.2 [6]**.** $\text{CI-dim}_R M = \inf\{\text{pd}_Q(M \otimes_R R') - \text{pd}_Q R' : R \to R' \leftarrow Q$ is a quasi-deformation$\}$.

Definition 2.3. For a module M over a field R we say that $\text{PCI-dim}_R M = 0$ if

$$\text{G-dim}_R M = 0$$

and the Betti numbers $\beta_n^R(M)$ of M have an estimate that is polynomial in n. For arbitrary modules we set

$$\begin{aligned}\text{PCI-dim}_R M = \inf\{n : &\text{ there exists an exact sequence}\\ &0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0, \text{ where } \text{PCI-dim}_R P_i = 0\}.\end{aligned}$$

The next result is well known [15], but we give a simpler proof.

Proposition 2.4. *If R is a local complete intersection, then for each R-module M the Betti numbers $\beta_n^R(M)$ have an estimate that is polynomial in n.*

Proof. We reduce the problem to the case $\operatorname{depth} M = \operatorname{depth} R$ first. We set $n = \operatorname{depth} R - \operatorname{depth} M$ and denote by $\text{Syz}_n^R(M)$ the cokernel of the map δ_{n+1}, where $(\mathbf{F}, \delta)$ is a minimal free resolution of M. Then $\beta_i^R(\text{Syz}_n^R(M)) = \beta_{i+n}^R(M)$ for $i \gg 0$. On the other hand $\text{G-dim}_R M = \operatorname{depth} R - \operatorname{depth} M$, therefore $\text{G-dim}_R \text{Syz}_n^R(M) = 0$, so that $\operatorname{depth} \text{Syz}_n^R(M) = \operatorname{depth} R$. If $\operatorname{depth} M = \operatorname{depth} R$, then we select an R- and M-regular sequence $(x) = (x_1, x_2, \dots, x_{\operatorname{depth} R})$. Since $\text{Tor}_i^R(R/(x), M) = 0$, it follows that $\beta_i^R(M) = \beta_i^{R/(x)}(M/(x))$. Thus, our problem reduces to the case when R is Artin. The proof proceeds by induction on the length of M. For the residue field k this is a classical result [16]. The inductive step is an easy consequence of the exact sequence $0 \to k \to M \to M/k \to 0$.

Proposition 2.5. *If R is a complete intersection, then* $\text{PCI-dim}_R M < \infty$ *for each R-module M. Conversely, if* $\text{PCI-dim}_R k < \infty$, *then R is a complete intersection.*

Proof. Let R be a complete intersection. For an R-module M we shall construct its resolution from modules of PCI-dimension zero. We set $n = \operatorname{depth} R - \operatorname{depth} M$. Let $\text{Syz}_n^R(M) = \operatorname{coker} \delta_{n+1}$, where $(\mathbf{F}, \delta)$ is a minimal free resolution of M over R. The ring R is Gorenstein, therefore $\text{G-dim} M$ is finite and $\text{G-dim}_R \text{Syz}_n^R(M) = 0$. For $i \gg 0$ we have $\beta_i^R(\text{Syz}_n^R(M)) = \beta_{i+n}^R(M)$. Hence using the fact that the Betti numbers of an arbitrary module over a complete intersection have a polynomial estimate (Proposition 2.4) we obtain the equality $\text{PCI-dim}_R \text{Syz}_n^R(M) = 0$.

Conversely, if $\text{PCI-dim}_R k < \infty$, then the $\beta_i^R(k)$ have a polynomial estimate, and therefore R is a complete intersection [17].

Proposition 2.6. *The inequality* PCI-dim$_R M \leqslant$ CI-dim$_R M$ *holds, with equality in the case when the right-hand side is finite.*

Proof. If CI-dim$_R M < \infty$ then G-dim$_R M < \infty$ ([6], Theorem 1.4). We now set $n = \operatorname{depth} R - \operatorname{depth} M$. Let $\mathrm{Syz}_n^R(M) = \operatorname{coker} \delta_{n+1}$, where $(\mathbf{F}, \delta)$ is a minimal free resolution of M over R. Since G-dim M is finite, it follows that G-dim$_R \mathrm{Syz}_n^R(M)$=0. For $i \gg 0$ we have $\beta_i^R(\mathrm{Syz}_n^R(M)) = \beta_{i+n}^R(M)$. Hence, using the fact that the Betti numbers of a module of finite CI-dimension are bounded by a polynomial (cf. [6], Lemma 1.5) we obtain

$$\text{PCI-dim}_R \mathrm{Syz}_n^R(M) = 0.$$

Proposition 2.7. *If* PCI-dim $M < \infty$, *then* PCI-dim M + depth M = depth R.

This is trivial since under the conditions imposed

$$\text{PCI-dim}\, M = \text{G-dim}\, M,$$

and the required formula holds for G-dimension.

In the same manner we can prove some other properties of PCI-dimension similar to properties of CI-dimension. The main point here is that if in a short exact sequence there is a polynomial bound on the growth of the Betti numbers of two of the modules, then the same holds for the third module. Moreover, the following result clearly follows from the properties of G-dimension.

Proposition 2.8. *If two modules in a short exact sequence have finite* PCI-*dimension, then the third module also has this property.*

It is unknown, however, if a similar result holds for CI-dimension.

As shown by Proposition 2.6, finite CI-dimension ensures finite PCI-dimension, and there arises a natural question: is it true that the corresponding classes of modules are the same? The answer is negative as shown by Veliche [18], who proved the following result.

Proposition 2.9 [18]**.** *If a local ring Q contains a field and* depth $Q \geqslant 4$, *then there exist a perfect ideal $I \subset Q$ such that* grade $I = 4$ *and a module M over $R = Q/I$ such that*

$$0 = \text{PCI-dim}_R M < \text{CI-dim}_R M = \infty.$$

We prove next that PCI-dimension localizes.

Proposition 2.10. *The inequality $\beta_i^{R_\mathfrak{p}}(M_\mathfrak{p}) \leqslant \beta_i^R(M)$ holds for all M and $\mathfrak{p}$. In particular, a polynomial estimate of the right-hand side ensures a polynomial estimate of the left-hand side.*

Proof. We consider the minimal free resolution of M over R and its tensor product with the (R-flat) module $R_\mathfrak{p}$. The resulting complex consists of free $R_\mathfrak{p}$-modules, and is a direct sum of the minimal resolution of $M_\mathfrak{p}$ over $R_\mathfrak{p}$ and several complexes of the form $0 \to R_\mathfrak{p} \to R_\mathfrak{p} \to 0$. Since the ith Betti number is the rank of the ith module in the minimal resolution, the proof is complete.

Proposition 2.11. PCI-$\dim_{R_\mathfrak{p}} M_\mathfrak{p} \leqslant$ PCI-$\dim_R M$.

This is clear from the corresponding property of G-dimension and Proposition 2.10.

Proposition 2.12 [11]. *If R is a complete intersection, then $R_\mathfrak{p}$ is also a complete intersection.*

Proof. If R is a complete intersection, then the $\beta_n^R(R/\mathfrak{p})$ are bounded by a polynomial (see Proposition 2.4). Using Proposition 2.10 we obtain that the quantities $\beta_i^{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})$ are also bounded by a polynomial. However, $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is a residue field of $R_\mathfrak{p}$, and a polynomial bound on its Betti numbers ensures [17] that $R_\mathfrak{p}$ is a complete intersection.

3. CM-dimension

Definition 3.1. A G-*quasi-deformation* of a ring R is a diagram of local homomorphisms $R \to R' \leftarrow Q$, where $R \to R'$ is a flat extension and $R' \leftarrow Q$ is a G-*deformation*, that is, a surjective homomorphism whose kernel I is a G-perfect ideal.

Definition 3.2. CM-$\dim_R M = \inf\{$G-$\dim_Q(M \otimes_R R')-$G-$\dim_Q R' : R \to R' \leftarrow Q$ is a G-*quasi-deformation*$\}$.

We prove first that the finiteness of G_K-$\dim_R M$ with respect to some suitable module K ensures the finiteness of CM-$\dim_R M$.

The following construction was considered in a similar context in [19] in the case when K was the dualizing module. Let K be a suitable module over a ring R. We define multiplication on the module $S = R \oplus K$ in the following way: $(a_1, r_1) * (a_2, r_2) = B(a_1 * a_2, a_1 * r_2 + a_2 * r_1)$. It is easy to see that this definition endows S with the structure of a ring. Note that we have a surjective homomorphism from S to R with kernel K, and therefore R can be regarded as an S-module.

Lemma 3.3. *R is reflexive as an S-module.*

Proof. Since $\mathrm{Hom}_R(K, K) = R$, it follows that $\mathrm{Ann}_R(K) = 0$, and therefore $\mathrm{Ann}_S(K) = K$. Hence $\mathrm{Hom}_S(R, S) \simeq K$ because the identity of R is taken to an element annihilated by the S-ideal K.

Similarly, $\mathrm{Hom}_S(K, S) \simeq R$. Thus, the natural homomorphism of S-modules $R \to R^{**}$ is an isomorphism.

Lemma 3.4. $\mathrm{Ext}_S^1(R, S) = 0$.

Proof. We have an exact sequence of S-modules: $0 \to K \to S \to R \to 0$. It suffices to prove that each homomorphism from K into S can be extended to a homomorphism from S into S. Since the range of each homomorphism from K into S is annihilated by the ideal K, it lies in $\mathrm{Ann}_S(K) = K$. We know, however, that each element of $\mathrm{Hom}_R(K, K)$ is in fact multiplication by an element $x \in R$. Such a homomorphism can be extended to S as multiplication by $(x, 0)$.

Remark 3.5. It is obvious from the short exact sequence $0 \to K \to S \to R \to 0$ of S-modules that

$$\operatorname{Ext}_S^{i+1}(R,S) \simeq \operatorname{Ext}_S^i(K,S)$$

for all $i > 0$.

Lemma 3.6. $\operatorname{G-dim}_S R = 0$.

Proof. S-reflexivity is already established, therefore we only need to prove that $\operatorname{Ext}_S^i(R,S) = 0$ and $\operatorname{Ext}_S^i(R^*,S) \simeq \operatorname{Ext}_S^i(K,S) = 0$ for all $i > 0$. In view of Remark 3.5, it suffices to show that $\operatorname{Ext}_S^i(R,S) = 0$ for all $i > 0$. The proof proceeds by induction on i. We take for its basis the result of Lemma 3.4. Assume now that $\operatorname{Ext}_S^i(R,S) = 0$ for $0 < i \leqslant n$ (respectively, $\operatorname{Ext}_S^i(K,S) = 0$ for $0 < i \leqslant n-1$). We consider the following change-of-rings spectral sequence:

$$\mathrm{E}_2^{pq} = \operatorname{Ext}_R^p(K, \operatorname{Ext}_S^q(R,S)) \;\Rightarrow\; \operatorname{Ext}_S^{p+q}(K,S).$$

By the induction hypothesis we obtain $\mathrm{E}_2^{pq} = 0$ for $p+q = n$, $q > 0$. Moreover, $\mathrm{E}_2^{n,0} = 0$ since $\operatorname{Ext}_R^n(K,K) = 0$ for $n > 0$. Hence $\operatorname{Ext}_S^n(K,S) = 0$ and $\operatorname{Ext}_S^{n+1}(R,S) = 0$.

Theorem 3.7. *If* $\mathrm{G}_K\text{-dim}\, M < \infty$ *for a suitable module* K, *then* $\operatorname{CM-dim} M < \infty$.

Proof. Consider the following G-quasi-deformation: $R \to R \leftarrow S$, where S is the ring $R \oplus K$ considered above. Then the equality $\mathrm{G}_K\text{-dim}_R M = \operatorname{G-dim}_S M$ holds by Theorem 1.4.

The following definition of CM-dimension is equivalent to Definition 3.2, but is more convenient from all points of view.

Definition 3.2′. $\operatorname{CM-dim}_R M = \inf\{\mathrm{G}_K\text{-dim}_{R'}(M \otimes_R R') : R \to R'$ is a local flat extension and K is a suitable R'-module$\}$.

In particular, it is now clear that $\operatorname{CM-dim} M \geqslant 0$. We shall use this definition to prove that CM-dimension actually characterizes Cohen–Macaulay rings.

Theorem 3.8. *If* $\operatorname{CM-dim}_R M < \infty$, *then* $\operatorname{CM-dim}_R M + \operatorname{depth} M = \operatorname{depth} R$.

Proof. This follows from the corresponding equality for G-dimension:

$$\begin{aligned}
\operatorname{CM-dim}_R M &= \operatorname{G-dim}_Q M' - \operatorname{G-dim}_Q R' \\
&= (\operatorname{depth} Q - \operatorname{depth}_Q M') - (\operatorname{depth} Q - \operatorname{depth}_Q R') \\
&= \operatorname{depth}_Q R' - \operatorname{depth}_Q M' = \operatorname{depth} R' - \operatorname{depth}_{R'}(M \otimes_R R') \\
&= \operatorname{depth} R - \operatorname{depth} M.
\end{aligned}$$

Theorem 3.9. *If* R *is a Cohen–Macaulay ring, then* $\operatorname{CM-dim}_R M < \infty$ *for each* R*-module* M. *Conversely, if* $\operatorname{CM-dim}_R k < \infty$, *then* R *is a Cohen–Macaulay ring.*

Proof. If R is Cohen–Macaulay, then its completion R' can be represented as a quotient ring of a regular ring S modulo a G-perfect ideal, and $\operatorname{CM-dim}_R M$ is finite because all modules over regular rings have finite G-dimension (and even finite projective dimension). Conversely, if $\text{CM-dim}_R\, k < \infty$, then let $R \to R'$ be a corresponding flat extension and K an R'-suitable module. Let $(\mathbf{x}) = (x_1, \dots, x_{\operatorname{depth} R})$ be a maximal R-regular sequence. Then $\text{CM-dim}_{R/(\mathbf{x})}\, k < \infty$. For since $R \to R'$ is a flat extension, $(\mathbf{x})$ is an R'-regular sequence and $R/(\mathbf{x}) \to R'/(\mathbf{x})$ is also a flat extension. By Theorem 1.4 the $R'/(\mathbf{x})$-module $K/(\mathbf{x})K$ is suitable and

$$\mathrm{G}_{K/(\mathbf{x})K} \dim_{R'/(\mathbf{x})}(k \otimes R'/(\mathbf{x})) = \mathrm{G}_K\text{-}\dim_{R'}(k \otimes R') - \operatorname{depth} R.$$

We can thus assume without loss of generality that $\operatorname{depth} R = 0$. We claim that R is Artin. For otherwise $\mathfrak{m}^n$ is non-zero for each n. For each n we also have an injection $0 \to \operatorname{Hom}(k, \mathfrak{m}^n) \to \operatorname{Hom}(k, R)$. Since $\bigcap \mathfrak{m}^n = 0$, it follows that $\operatorname{Hom}(k, \mathfrak{m}^n) = 0$ for all $n \gg 0$; hence $\operatorname{depth} \mathfrak{m}^n \neq 0$.

On the other hand, using Lemma 1.8 and induction on length we can prove that for an R-module M of finite length we have $\mathrm{G}_K\text{-}\dim_{R'} M \otimes R' < \infty$. Consider now the exact sequence

$$0 \to \mathfrak{m}^n \otimes_R R' \to R' \to R/\mathfrak{m}^n \otimes_R R' \to 0.$$

The length of the module $R/\mathfrak{m}^n$ is finite, therefore $\mathrm{G}_K\text{-}\dim_{R'} R/\mathfrak{m}^n \otimes R' < \infty$, and by Lemma 1.8 we obtain $\mathrm{G}_K\text{-}\dim_{R'} \mathfrak{m}^n \otimes_R R' < \infty$, so that $\text{CM-dim}_R\, \mathfrak{m}^n < \infty$. This is in contradiction with Theorem 3.8:

$$0 < \operatorname{depth} \mathfrak{m}^n + \text{CM-dim}_R\, \mathfrak{m}^n = \operatorname{depth} R = 0.$$

Proposition 3.10. $\text{CM-dim}_{R_\mathfrak{p}} M_\mathfrak{p} \leqslant \text{CM-dim}_R M$.

Proof. Obviously, we can assume that $\text{CM-dim}_R M$ is finite. Let $R \to R' \leftarrow Q$ be a corresponding G-quasi-deformation. Since $R \to R'$ is a flat extension, there exists $\mathfrak{p}' \subset R'$ such that $R \cap \mathfrak{p}' = \mathfrak{p}$. Let $\mathfrak{q} \subset Q$ be the inverse image of $\mathfrak{p}'$. It easy to see that the diagram $R_\mathfrak{p} \to R'_{\mathfrak{p}'} \leftarrow Q_\mathfrak{q}$ is a G-quasi-deformation. We have

$$\begin{gathered}\text{G-dim}_Q M \otimes_R R' \geqslant \text{G-dim}_{Q_\mathfrak{q}} (M \otimes_R R')_\mathfrak{q} = \text{G-dim}_{Q_\mathfrak{q}} M_\mathfrak{p} \otimes_{R_\mathfrak{p}} R'_{\mathfrak{p}'}, \\ \text{G-dim}_Q R' = \text{G-dim}_{Q_\mathfrak{q}} R'_{\mathfrak{p}'},\end{gathered}$$

as required.

Remark 3.11. From Theorem 3.8 and Proposition 3.10 we see that if a module M has a finite CM-dimension, then $\operatorname{depth} R - \operatorname{depth} M \geqslant \operatorname{depth} R_\mathfrak{p} - \operatorname{depth} M_\mathfrak{p}$ for each prime ideal $\mathfrak{p}$. Such an assumption about a module M was used, for example, in [9]; in particular, the authors note in Remark 5 of [9] that it holds in the case of $\text{G-dim}\, M < \infty$. We have thus obtained a certain extension of the class of modules for which the assumptions of [9], Corollary 4 are known to be satisfied.

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Received 24/AUG/00
Translated by A. GERKO