On suitable modules and G-perfect ideals

A. A. Gerko

A module K over a commutative Noetherian local ring R is said to be *suitable* (see [1]) if $\mathrm{Hom}_R(K,K) \cong R$ and $\mathrm{Ext}^i_R(K,K) = 0$ for $i > 0$. Trivial examples are a free module of rank 1 and a canonical module. We recall that the type of a module M (denoted $\operatorname{type} M$) is by definition the dimension, over the residue field $k \cong R/\mathfrak{m}$ of R, of the first non-zero $\mathrm{Ext}^i_R(k,M)$. The type of a suitable module satisfies the condition

$$\operatorname{type} K * \dim_k(K/\mathfrak{m}K) = \operatorname{type} R. \qquad (1)$$

In [2] (where such modules are called semidualizing) an example of a ring of type 2^i is constructed with suitable modules of all types permissible by condition (1) for every natural number i.

Let $S_1,\dots,S_n$ be finite local algebras over a local ring R. We denote by $\mathfrak{m}_i$ the maximal ideal in S_i and by $\mathfrak{m}$ the maximal ideal of R. We consider the R-algebra $S = S_1 \otimes_R S_2 \otimes_R \cdots \otimes_R S_n$.

For any $P \subset \{1,\dots,n\}$ we put $S_P = \bigotimes_R S_i$, where i goes through P.

Proposition 1. *For every i let S_i be free as an R-module, and let $S_i/\mathfrak{m}_i \cong R/\mathfrak{m}$. Then the algebra S is local, and for any $P \subset \{1,\dots,n\}$ the S-module $K_P = \mathrm{Hom}_{S_P}(S, S_P)$ is suitable and*

$$\operatorname{type} K_P = \operatorname{type} R * \prod_{i\in P} \dim_{S_i/\mathfrak{m}_i} \mathrm{Hom}_{S_i/\mathfrak{m}S_i}(S_i/\mathfrak{m}_i, S_i/\mathfrak{m}S_i).$$

Moreover, if all the rings $S_i/\mathfrak{m}S_i$ are not Gorenstein, then the modules K_P are mutually non-isomorphic.

Let us represent the natural number n as a product of primes: $n = \prod_{i=1}^m p_i$. We consider the ring $T_n = \bigotimes_{i=1}^m k \ltimes k^{p_i}$, where k is a field. The type of T_n is equal to n, and by what has been proved, for any divisor a of n there exists a suitable T_n-module K of type a. We also note that if under the hypotheses of Proposition 1 some of the rings S_i are isomorphic to each other, then in general the modules K_P cannot be distinguished by invariants like the Bass or Betti numbers.

It is interesting to ask whether this construction is universal, at least in the case of finite-dimensional algebras over fields. All examples known to the author follow this scheme.

Proposition 2. *Let R be a complete local ring, and let x be an R-regular sequence. There is a one-to-one correspondence between the isomorphism classes of suitable modules over the rings R and $R/(x)$.*

The suitable modules are closely connected with G-dimension, the invariant characterizing Gorenstein rings in the same sense that projective dimension characterizes regular rings. We recall its definition.

Definition 3 [3]. We put $\operatorname{G-dim}_R P = 0$ if the natural homomorphism

$$P \to \mathrm{Hom}_R(\mathrm{Hom}_R(P,R),R)$$

is bijective and $\mathrm{Ext}^i_R(P,R) = 0 = \mathrm{Ext}^i_R(P^*,R)$ for any i. In the general case

$$\operatorname{G-dim}_R M = \inf\bigl\{n \mid \text{there exists an exact sequence } 0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0 \text{ with } \operatorname{G-dim}_R P_i = 0\bigr\}.$$

We denote by $\operatorname{grade} I$ the length of a maximal R-regular sequence in an ideal I. It is known that $\operatorname{grade} I = \inf\{i \mid \mathrm{Ext}^i_R(R/I,R) \neq 0\}$.

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Proposition 4 [1]. *The following two conditions on an ideal I are equivalent*:

1) $\text{G-dim}_R R/I = \operatorname{grade} I$ (*in particular*, $\text{G-dim}_R R/I < \infty$);
2) $\operatorname{Ext}_R^k(R/I,R) = 0$ *for* $k \neq \operatorname{grade} I$, *and* $\operatorname{Ext}_R^{\operatorname{grade} I}(R/I,R)$ *is a suitable R/I-module.*

Such ideals are said to be G-*perfect.* If, in addition, $\operatorname{Ext}_R^{\operatorname{grade} I}(R/I,R) \simeq R/I$, then the ideal is said to be G-*Gorenstein.*

The following result is known from [1].

Lemma 5 [1]. *If $\mathfrak{a}$ is a* G-*Gorenstein ideal*, $\mathfrak{a} \subset I$, *and* $\operatorname{grade} I = \operatorname{grade} \mathfrak{a}$, *then*

$$\operatorname{Ext}_R^{i+\operatorname{grade} I}(R/I,R) \cong \operatorname{Ext}_{R/\mathfrak{a}}^i(R/I,R/\mathfrak{a})$$

for every $i > 0$. In particular, $\operatorname{Ext}_R^{\operatorname{grade} I}(R/I,R) \cong (\mathfrak{a} : I)/\mathfrak{a}$.

The ideals I and J are said to be directly G-*connected* if there exists a Gorenstein ideal $\mathfrak{a}$ such that $I = (\mathfrak{a} : J)$ and $J = (\mathfrak{a} : I)$. In this case $\operatorname{grade} I = \operatorname{grade} \mathfrak{a} = \operatorname{grade} J$.

It follows from Lemma 5 that the ideals I and J are directly G-connected by a G-Gorenstein ideal $\mathfrak{a}$ if and only if $\operatorname{Ext}_R^{\operatorname{grade} J}(R/J,R) \cong I/\mathfrak{a}$ and $\operatorname{Ext}_R^{\operatorname{grade} I}(R/I,R) \cong J/\mathfrak{a}$.

Theorem 6. *The following condition on an ideal I is equivalent to conditions* 1)–2):

3) *There exists and ideal J such that I and J are directly* G-*connected*, $\operatorname{Ext}_R^{\operatorname{grade} I}(R/I,R)$ *is a suitable R/I-module, and* $\operatorname{Ext}_R^{\operatorname{grade} J}(R/J,R)$ *is a suitable R/I-module.*

Proof. 1)⇒3). As the ideal $\mathfrak{a}$ we can take an ideal generated by a maximal regular sequence in I. 3)⇒1). Let $\mathfrak{a}$ be the corresponding G-Gorenstein ideal. Since condition 1) is satisfied simultaneously in the rings R and $R/\mathfrak{a}$, we can pass to the quotient ring $R/\mathfrak{a}$. In particular, we can assume that $\operatorname{grade} I = 0$. Then condition 3) assumes the following form: $\operatorname{Ann} I = J$, $\operatorname{Ann} J = I$, I is a suitable R/J-module, and J is a suitable R/I-module. Under these conditions, the exact sequences $0 \to I \to R \to R/I \to 0$ and $0 \to J \to R \to R/J \to 0$ are dual to each other, and for $i > 1$ we obtain the following isomorphisms: $\operatorname{Ext}_R^i(R/I,R) \cong \operatorname{Ext}_R^{i-1}(I,R)$, $\operatorname{Ext}_R^i(R/J,R) \cong \operatorname{Ext}_R^{i-1}(J,R)$, and $\operatorname{Ext}_R^1(R/I,R) = 0$, $\operatorname{Ext}_R^1(R/J,R) = 0$. It is now sufficient to show that $\operatorname{Ext}_R^i(R/I,R) = 0 = \operatorname{Ext}_R^i(R/J,R)$ for $i > 0$. The base of the induction is proved; let $k \geq 1$ and let the assertion be true for $i \leq k$. We consider the spectral sequences of the exchange of rings

$$\operatorname{Ext}_{R/I}^i(J,\operatorname{Ext}_R^j(R/I,R)) \Rightarrow \operatorname{Ext}_R^{i+j}(J,R) \quad \text{and} \quad \operatorname{Ext}_{R/J}^i(I,\operatorname{Ext}_R^j(R/J,R)) \Rightarrow \operatorname{Ext}_R^{i+j}(I,R).$$

We have $\operatorname{Ext}_{R/I}^i(J,\operatorname{Ext}_R^{k-i}(R/I,R)) = 0$ for $i \geq 0$ by the induction hypothesis and by the condition of suitability of the R/I-module J. Consequently, $\operatorname{Ext}_R^{k+1}(R/J,R) \simeq \operatorname{Ext}_R^k(J,R) = 0$. Similarly, $\operatorname{Ext}_R^{k+1}(R/I,R) = 0$.

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Moscow State University
E-mail: gerko@mccme.ru

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