Phase transitions in the simplicial Ising model on hypergraphs

Keywords: hypergraph, Ising model, simplicial rule, phase transition, Landau theory

Extended Abstract

Pairwise interactions have been widely assumed in modeling the dynamics of complex systems including contagion and synchronization; however, this assumption may fail when group or higher-order interactions (HOIs) are relevant [1]. Despite active research on nonequilibrium dynamics in HOI networks, the Ising model, a canonical equilibrium system in statistical physics, has not been thoroughly explored in the context of HOIs. From statistical physics to network science, the Ising model has been a foundational framework for understanding the role of structural properties in phase transitions and critical phenomena [2]. Therefore, extending the Ising model to HOI networks may pave the way for integrating recent advances in HOIs.

An archetypal type of higher-order interaction is the so-called “simplicial rule” [3] based on the unanimity condition, which was introduced in contagion model and shown to give rise to discontinuous transitions. Recently, an Ising model with the Hamiltonian formulated under the simplicial rule, named the simplicial Ising model (SIM), was introduced in Ref. [4]. The Hamiltonian of SIM is given by $\mathscr{H} = -\sum_{e\in\mathscr{E}} J_{|e|}\delta_{\{S_i\}_{i\in e}} - H\sum_{i\in\mathscr{V}} S_i$, where S_i is the Ising spin variable, $\mathscr{V} = \{i\}$ and $\mathscr{E} = \{e\}$ are the vertex and hyperedge set, respectively, $J_{|e|} > 0$ the ferromagnetic coupling constant dependent on the hyperedge cardinality $|e|$, δ is the higher-order generalization of Kronecker delta, and H the external field. In SIM, the pairwise coupling between two spins is extended to the higher-order coupling with exclusive preference for unanimity among multiple spins (see Fig. 1a), while retaining the $\mathbb{Z}_2$ symmetry of the original Ising model. Similar unanimous decision making may also play a role in social dynamics such as jury decisions. However, many of SIM’s properties including phase transitions remain to be understood.

In this presentation [5], we explore the phase transitions in the SIM. We show that within the mean-field theory the simplicial HOIs yield diverse scenarios of phase transitions depending on the sizes of hyperedges. We achieve this by constructing the Landau free energy density for the SIM Hamiltonian, obtained as $f(m) = f(0) - Hm + \sum_{n=1}^{\infty} C_{2n}m^{2n}$ with $C_{2n} = -\sum_{q\geq 2n}^{\infty} \frac{\rho_q J_q}{2^{q-1}}\binom{q}{2n} + \frac{1}{2n(2n-1)}T$, where $\rho_q = |\mathscr{E}_q|/|\mathscr{V}|$ and T the temperature. This means that the presence of size-q hyperedges modifies all the even order terms (that is, m^{2n}-terms up to $2n \leq q$) of the Landau free energy density. By analyzing the resulting $f(m)$ we are able to investigate how the phase transition properties change with the hyperedge size distribution systematically.

To be specific, we first consider q-uniform hypergraphs (in which all hyperedges have the same size q) and study how the phase transition nature changes with the hyperedge size q. To this end we obtain the phase diagram for q-uniform hypergraphs in (q,T)-plane (Fig. 1b). Notably, we found that discontinuous transitions between para- and ferromagnetic phases occur when $q > 4$ with the tricritical point at $q = 4$. This result is to be compared with previous studies on synchronization and simplicial contagion, in which discontinuous transitions appear as early as $q = 3$. Moreover, the transition temperatures as a function of q can be nonmonotonic due to the ambivalent influence of group size q on the lower-order energetic terms in the free energy.

Next we consider the $(2,q)$ "bi-uniform" hypergraphs (in which the pairwise and size-q HOIs coexist). We compute the phase diagram for the $(2,q)$-hypergraphs in the (q,r)-plane (Fig. 1c), where r is the propensity parameter for the HOIs given by $\rho_q J_q = r$ and $\rho_2 J_2 = 1-r$. Notably, there appears a new regime that we call the "double-transition" regime, when the size q of HOIs is $q > 8$ and r takes intermediate value. In the double-transition regime, a continuous and a discontinuous jump in magnetization occur consecutively as the temperature changes. We explain how the interplay between the pairwise and HOIs gives rise to the double transition in the framework of Landau theory and further show that they play distinct but synergistic roles in the emergence of such double transitions. We corroborate these predictions by Monte Carlo simulations and more detailed calculations using the Bethe–Peierls method.

In this study, we have performed a systematic statistical mechanical investigation of the simplicial Ising model (SIM) and have shown that diverse phase transition scenarios arise due to the effective lower-order couplings in the Hamiltonian induced by the simplicial HOIs. Moreover, such group interactions exhibit an ambivalent effect on the transition temperatures, leading to their optimality. Finally, we found that the double transition occurs also in higher-order synchronization and simplicial contagion dynamics on $(2,q)$ bi-uniform hypergraphs with sufficiently large q, suggesting that the phenomenon is generic.

References

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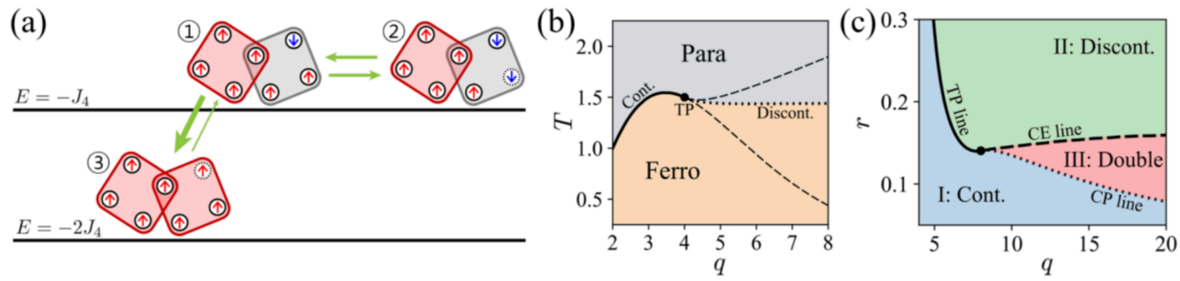

Figure 1: **The simplicial Ising model (SIM).** (a) Schematic illustration of the SIM. Three configurations are displayed along the energy level based on the Hamiltonian. (b) Phase diagram for q-uniform hypergraphs in (q,T)-space. The solid (dotted) line corresponds to continuous (discontinuous) transitions at the thermodynamic transition temperature T_c. The circle indicates the tricritical point (TP) at $q=4$. The two dashed lines represent the limits of metastability on heating and cooling. (c) Phase diagram for $(2,q)$-hypergraphs on the (q,r)-space. Regimes I, II, and III are divided by the tricritical point (TP), critical point (CP), and critical endpoint (CE) lines. The circle indicates the "special" TP at $q=8$. In Regime I (II), a continuous (discontinuous) transition occurs. In Regime III, a continuous-discontinuous double transition occurs.