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Risk and Potential: An Asset Allocation Framework with Applications to Robo-Advising

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Abstract

We propose a novel dynamic asset allocation framework based on a family of mean-variance-induced utility functions that alleviate the non-monotonicity and time-inconsistency problems of mean-variance optimization. The utility functions are motivated by the equivalence between the mean-variance objective and a quadratic

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Xiang-Yu Cui, Xiao Qiao and Moris S. Strub would like to dedicate this paper to the co-author, late Professor Duan Li, in commeoration of his contribution to optimizaion, financial enginerring, and risk management.

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utility function. Crucially, our framework differs from mean-variance analysis in that we allow different treatment of upside and downside deviations from a target wealth level. This naturally leads to a different characterization of possible investment outcomes below and above a target wealth as risk and potential. Our proposed asset allocation framework retains two attractive features of mean-variance optimization: an intuitive explanation of the investment objective and an easily computed optimal strategy. We establish a semi-analytical solution for the optimal trading strategy in our framework and provide numerical examples to illustrate its behavior. Finally, we discuss applications of this framework to robo-advisors.

Keywords Mean-risk optimization · Mean-variance · Expected utility maximization · Portfolio choice · Risk · Potential · Robo-advising · FinTech

Mathematics Subject Classification 91G10 · 91B05 · 91B16

1 Introduction

Portfolio construction is a central issue in financial economics of interest to both academics and practitioners. There are two main schools of thought on how to select an optimal investment portfolio: the mean-variance framework pioneered by Markowitz [1] and the expected utility maximization framework justified by the von Neumann–Morgenstern utility theorem ([2]). The two approaches differ in that the mean-variance framework seeks an optimal trade-off between risk and return, whereas the expected utility framework searches for the investment mix that maximizes the utility function of a rational decision maker. Despite its mathematical rigor, expected utility maximization has not been embraced by the investment community due to its abstract nature. Because mean-variance optimization is intuitive and easy to implement, it is broadly used in the investment industry. Mean-risk analysis, a generalization of mean-variance that uses other risk measures such as Value-at-Risk (VaR) or Conditional-Value-at-Risk (CVaR) in place of variance, has also been adopted in industry.

Although mean-variance optimization is widely used, this approach suffers from well-known drawbacks. The most notable shortcoming of the mean-variance framework is that the investor preferences are time-inconsistent and non-monotonic. In this paper, we propose an alternative to mean-variance preferences that alleviates these issues. We introduce a parameterized family of utility functions, motivated by Li and Ng [3] showing that a certain quadratic utility function is equivalent to the mean-variance objective. In our approach, we keep two attractive features of the mean-variance framework: *an intuitive explanation of the objective* and *an easily-calculated optimal investment strategy*. A key difference between our approach and the mean-variance framework is that our approach allows investors to treat upside and downside deviations from a pre-determined investment target differently.

The asymmetric treatment of upside and downside deviations is important for investors. When volatility is high, investors are likely to experience either a large upside move or a large downside move. However, investors may not view upside and downside moves as equally risky—upside may be characterized as the potential for gain and downside as undesirable losses. This idea has been recognized in the investments literature at least since Markowitz [4], who proposed using semi-variance as a key measure of risk. Hogan and Warren [5], Bawa and Lindenberg [6], and Harlow and Rao [7] expanded this idea of asymmetric risk into equilibrium asset pricing frameworks. We will refer to risk as the possibility of falling short of an investment target and potential as the possible outcomes exceeding the investment target. Our proposed framework distinguishes between risk and potential, and allows the investor to treat these quantities differently in her portfolio problem.

Our family of utility functions contains three key parameters: target wealth, potential-aversion, and a weight parameter that captures how investors view risk versus potential. We first derive the well-posedness conditions for the problem and provide a semi-analytical solution under general assumptions. We then examine the effects of the parameters on the investor's terminal wealth, illustrating how the potential-aversion parameter affects the terminal wealth in two numerical examples. In the first example, we assume the underlying returns follow a lognormal distribution. In the second example, we investigate the binomial and trinomial models. Our results show that holding expected terminal wealth constant, reducing the potential-aversion parameter leads to a Pareto improvement of both risk and potential.

The risk-potential framework prescribes investment decisions that retain attractive aspects of mean-variance optimization while alleviating its drawbacks. As such, this framework can be applied to a variety of dynamic asset allocation decisions faced by investment managers, trading desks, financial advisors, or individual investors. Our proposed framework has several desirable properties viewed through the lens of asset allocation decisions. First, the framework provides an economically sound decision-making process that encourages diversification in the sense that resulting strategies invest across the full range of available assets. Second, the optimal investment strategy can be calculated in a dynamic setting in a straightforward manner and implemented by standard algorithms. Third, the framework allows for intuitive graphic representations of the relationship between investor decisions and expected outcomes.

We illustrate the usefulness of our approach in an application to robo-advisors. Robo-advisors are digital platforms that provide automated investment advice and financial services. When setting up an account with a robo-advisor, an investor typically answers a series of questions (Faloon and Scherer [8]) designed to elicit her needs and preferences. Based on the answers, the robo-advisor offers investment choices that best suit this particular investor. Once the investment decision has been made, the robo-advisor then handles most future actions such as portfolio rebalancing without much additional oversight on the part of the investor.

All robo-advisors are faced with the problem of optimal asset allocation that take into account specific investor preferences. While the current robo-advising industry standard uses the mean-variance framework for making investment decisions, we argue that our mean-variance induced utility functions constitute a superior decision-making framework compared to mean-variance. In addition to the three desirable properties highlighted above for general asset allocation problems, our framework offers a fourth desirable property important for robo-advisors: investor preferences can be easily elicited through simple questionnaires.

The remainder of this paper is organized as follows. In Sect. 2, we review the discrete-time mean-variance problem and its equivalence to an expected utility maximization problem with a quadratic utility function. We introduce a family of mean-variance induced utility functions and discuss the role of target wealth, potential-aversion, and the weighting parameter in Sect. 3. In Sect. 4, we provide conditions for the well-posedness of the problem and derive the optimal investment strategy. Section 5 provides an illustration of our approach when returns follows a lognormal distribution, and Sect. 6 shows how our approach works in a binomial setting. Section 7 discusses potential robo-advising applications. Section 8 concludes.

2 Discrete-Time Mean-Variance Optimization Revisited

In this section, we introduce a discrete-time model for the financial market and revisit the seminal work of Li and Ng [3] on dynamic mean-variance portfolio selection. There are n risky assets with random total returns and one risk-free asset having a deterministic total return. At time zero, an investor with an initial wealth x_0 joins the market and decides how to allocate her wealth among the $(n+1)$ assets. She can reallocate her wealth among the $(n+1)$ assets at the beginning of each of the following $(T-1)$ periods, where T is the time horizon of the investor. The deterministic total return of the risk-free asset between time t and $t+1$ is denoted by $r_t \geqslant 1$ and the random total return of the risky assets by $\mathbf{e}_t = [e_t^1, \cdots, e_t^n]'$, where e_t^i is the random total return of the i'th asset between time t and $t+1$. The total returns $\mathbf{e}_t$, $t = 0, \cdots, T-1$, are assumed to be time-independent and, for simplicity, have finite moments of all order. The last assumption could be relaxed to square-integrability of the total returns for most of our analysis. Assuming finite moments of all order spares us from tedious case distinctions later on. It will be obvious under which type of preferences integrability assumptions could be relaxed. The vector of excess returns for the time period between t and $t+1$ is defined by

$$\mathbf{P}_t = \left[P_t^1, P_t^2, \cdots, P_t^n\right]' = \left[(e_t^1 - r_t), (e_t^2 - r_t), \cdots, (e_t^n - r_t)\right]'.$$

We assume that $\mathbb{E}\left[\mathbf{P}_t\mathbf{P}_t'\right]$ is strictly positive definite and thus invertible. Our probability space is endowed with the filtration generated by the process of excess returns, $\mathcal{F}_t = \sigma(\mathbf{P}_0, \mathbf{P}_1, \cdots, \mathbf{P}_{t-1})$, and $\mathcal{F}_0$ is the trivial σ-algebra over Ω. Let X_t be the wealth of the investor at time t and u_t^i, $i = 1, 2, \cdots, n$, be the dollar amount the investor chooses to invest in the i'th risky asset for the time period between t and $t+1$. The amount invested in the risk-free asset at the beginning of time period t is then equal to $X_t - \sum_{i=1}^n u_t^i$ by the self-financing condition. To disregard investing with hindsight, trading strategies are assumed to be adapted to the filtration $(\mathcal{F}_t)_{t=0}^T$, but we do not impose any further restrictions on trading. For a given initial wealth x_0 and trading strategy $\mathbf{u} = (\mathbf{u}_t)_{t=0}^{T-1}$, the wealth process is determined by

$$X_{t+1} = r_t X_t + \mathbf{u}_t'\mathbf{P}_t, \qquad t = 0, \cdots, T,$$

and is an adapted stochastic process.

A mean-variance investor seeks to determine a trading strategy which optimizes a trade-off between mean and variance of her terminal wealth. In the discrete-time setting studied in this paper, the mean-variance investment problem can be formulated as follows:

$$\begin{aligned} \sup_{\mathbf{u}=(\mathbf{u}_t)_{t=0}^{T-1}} \quad & \mathbb{E}\left[X_T\right] - \omega \mathrm{Var}\left(X_T\right) \\ \text{s.t.} \quad & X_{t+1} = r_t X_t + \mathbf{u}_t' \mathbf{P}_t, \quad t = 0, 1, \cdots, T-1, \end{aligned} \qquad (MV_0(\omega))$$

where $\omega > 0$ represents the trade-off parameter between mean and variance. Note ω is also called the risk-aversion parameter and varying it in $(0, +\infty)$ traces out the efficient frontier in the mean-variance space. Li and Ng [3] showed that the discrete-time mean-variance problem ($MV_0(\omega)$) is equivalent to the following expected utility maximization problem in the sense that they have the same optimal investment strategies,

$$\begin{aligned} \sup_{\mathbf{u}=(\mathbf{u}_t)_{t=0}^{T-1}} \quad & \mathbb{E}\left[-(X_T - \gamma)^2\right], \\ \text{s.t.} \quad & X_{t+1} = r_t X_t + \mathbf{u}_t' \mathbf{P}_t, \quad t = 0, 1, \cdots, T-1, \end{aligned} \qquad (EU_0(\gamma))$$

where

$$\gamma = \frac{1 + 2\omega \mathbb{E}[X_T^{mv}]}{2\omega}, \tag{1}$$

where X_T^{mv} is the terminal wealth under the optimal strategy. Based on this observation, Li and Ng [3] solved ($MV_0(\omega)$), where dynamic programming cannot be applied because the variance term causes the problem to be time-inconsistent, by invoking the solution of ($EU_0(\gamma)$) which is of a linear-quadratic structure.

***Remark* 1** A separate literature takes a different approach when investigating time-inconsistent preferences by considering an agent who cannot pre-commit and instead seeks to determine a subgame perfect equilibrium, see, e.g., Basak and Chabakauri [9], Wang and Forsyth [10], Czichowsky [11], Björk and Murgoci [12], Björk et al. [13], Chiu and Wong [14], Cong and Oosterlee [15], Van Staden et al. [16], He and Jiang [17, 18], He and Zhou [19].

The variance induced "utility" function in ($EU_0(\gamma)$) has a quadratic form. Since it is not monotone, it does not qualify as a standard utility function. Recall that monotonicity of the utility function is necessary and sufficient for the represented preferences to be consistent with first-order stochastic dominance. Furthermore, due to an interplay between time-inconsistency, non-monotonicity, discontinuous asset prices, and an incomplete financial market, ($MV_0(\omega)$) is not time-consistent in efficiency (Cui et al. [20]). Whereas time inconsistency implies that a global optimal strategy may fail to be locally optimal for the truncated investment problem with the original risk-aversion parameter, time inconsistency in efficiency means that the globally optimal investment strategy could fail to be locally optimal for *any* positive risk-aversion trade-off. When preferences are not time-consistent in efficiency, the investor could exhibit irrational behavior in some scenarios, such as taking a position which minimizes both the variance and the expected return. Cui et al. [20] developed a revised mean-variance policy which dominates the pre-committed optimal mean-variance portfolio policy—achieving the same feasible mean-variance combinations while allowing the investor to receive a free cash flow stream during the investment process, by relaxing the self-financing condition and allowing investors to withdraw capital when the wealth level exceeds a threshold value. Dang and Forsyth [21] considered semi-self-financing mean-variance strategies for continuous time markets by using a Hamilton–Jacobi–Bellman (HJB) equation: When the wealth level exceeds a threshold, switch to the minimum variance policy and deposit the extra money into another separate risk-free account.

***Remark* 2** Maccheroni et al. [22] took an alternative approach to amend the issue of non-monotonicity by considering a portfolio selection problem with monotone mean-variance preferences, represented by the minimal monotone functional coinciding with mean-variance preferences on their domain of monotonicity. Strub and Li [23] showed that the optimal strategies for the monotone and classical mean-variance preferences coincide when asset prices are continuous. In this case, the drawbacks related to the non-monotonicity of mean-variance preferences have no impact on investment decisions.

3 Mean-Variance-Induced Utility Maximization Problem

The shortcomings due to the decreasing part of the quadratic "utility" function in ($EU_0(\gamma)$) motivates us to propose an amelioration to this problem. We propose a family of mean-variance-induced utility functions that alleviates the non-monotonicity issue. Our family of utility functions embeds ($EU_0(\gamma)$) as a special case under parametric restrictions. We formulate the following *mean-variance-induced utility maximization problem*:

$$\begin{aligned} \sup_{\mathbf{u}=(\mathbf{u}_t)_{t=0}^{T-1}} \quad & \mathbb{E}\left[-(|X_T-\gamma|)^\alpha 1_{\{X_T<\gamma\}} - \delta(X_T-\gamma)^\alpha 1_{\{X_T\geqslant\gamma\}}\right] \\ \text{s.t.} \quad & X_{t+1} = r_t X_t + \mathbf{u}_t' \mathbf{P}_t, \quad t = 0, 1, \cdots, T-1. \end{aligned} \qquad (EU_1(\gamma,\delta,\alpha))$$

Let us discuss this formulation in detail. Compared with the original problem ($EU_0(\gamma)$) which was parameterized only by one parameter γ, the new formulation now contains three parameters: γ, δ and α. γ is the *target wealth* of the investor and can be directly mapped to the same parameter in ($EU_0(\gamma)$). An important insight that can be inferred from the equivalence between ($MV_0(\omega)$) and ($EU_0(\gamma)$) is that a mean-variance investor behaves like an investor with a target wealth γ who seeks to minimize deviations from this target. This perspective has been employed successfully in the context of defined contribution pension schemes in Vigna [24] and in the presence of withdrawals by Dang et al. [25]. In the case of mean-variance preferences, (1) gives us an explicit relation between target wealth, risk aversion, and expected terminal wealth. It is clear that any positive risk-aversion parameter leads to $\gamma \geqslant E[X_T^{mv}]$. Therefore, any risk-averse mean-variance investor on average does not achieve the implied target of her preferences. If the risk-aversion parameter goes to infinity, the target wealth decreases to the terminal wealth resulting from only investing in the risk-free asset. We thus impose the following lower bound on the target wealth

$$\gamma \geqslant \prod_{t=0}^{T-1} r_t x_0. \tag{2}$$

***Remark* 3** For the time-consistent mean-variance formulation, Cong and Oosterlee [26] computed an implied target wealth which is implicitly contained in the time-consistent strategy and depends on the time and current wealth level. Hence the interpretation of the mean-variance objective as a minimization problem of deviations from an investment target also applies in the time-consistent formulation.

The mean-variance investor equally dislikes terminal wealth levels below the target and terminal wealth levels above the target, a feature that strikes us as not entirely realistic. Investors typically view investment gains and losses differently, a fact recognized at least since Markowitz [4]. This is exactly the motivation for introducing the parameter δ, which we call *potential-aversion*. By introducing this parameter, we allow the investor to distinguish between upside deviations—potential, from downside deviations—risk.

The mean-variance investor has a potential-aversion of $\delta = 1$ which reflects she is as averse to outcomes above the target wealth level as she is to outcomes that fall short of this level. A lower value of the potential-aversion parameter δ reflects that the investor has a stronger dislike for wealth levels below the target than for those exceeding the target. This feature does not imply the investor actively seeks upside. For δ values between zero and one, the investor dislikes deviations above the target level, just to a lesser extent than her dislike of deviations below. Positive values of potential-aversion are relevant for investors that are most concerned with not falling short of an investment target, for example because the target represents a fixed future obligation. We will show later in Sect. 5 that at least when the distribution of returns is lognormal, the resulting terminal wealth will simultaneously achieve a lower expected square deviation of outcomes falling short of the target and a higher expected square deviation of outcomes exceeding the target than the mean-variance benchmark for any given expected wealth level. At the same time, terminal wealth distributions corresponding to a positive potential-aversion remain close to the efficient frontier in terms of mean-variance trade-off. Such terminal wealth distributions are thus intuitively appealing to a large number of investors. If $\delta = 0$, the investor is solely concerned about her wealth levels not reaching her target. This case corresponds to the below-target semi-variance proposed by Markowitz [4]. In particular, the family of mean-variance induced utility maximization problems covers optimal investment problems in which the investor looks for a minimal lower-partial moment portfolio. If δ becomes negative, potential-aversion turns into potential-seeking behavior, and the investor regards wealth levels above her target as positive outcomes she desires rather than dislikes.

As long as delta remains nonnegative, the problem remains within the realm of convex optimization and is well-posed, i.e., its supremum is finite. The utility function becomes non-symmetric with respect to the target wealth γ and we can effectively flatten out the decreasing part of the quadratic utility function in ($EU_0(\gamma)$) by decreasing $\delta \in (0, 1)$. The case with a negative potential-aversion is more delicate. The investor actively seeks to exceed her target, and this problem may become ill-posed, depending on how strongly her desire to seek potential and what opportunities are being offered by the financial market. We provide conditions for well-posedness of the problem in Sect. 4. In this case, the problem is also non-concave, and solutions are therefore potentially not unique. We refer to Remark 6 for a discussion and to Sect. 6 for an example where the solution is not unique.

Finally, the parameter α, which we assume to be larger than one, is a *weighting parameter* which determines how the investor views deviations of different sizes from the target wealth γ. For larger values of α, the investor places more emphasis on larger deviations from γ. In the case of ($EU_0(\gamma)$), $\alpha = 2$ corresponds to the Euclidean distance. Our framework possesses additional flexibility in allowing the value of α to vary, thus generalizing ($EU_0(\gamma)$) along another dimension. If $\alpha = 1$, the mean-variance-induced utility function becomes piecewise-linear and the objective resembles models of portfolio selection under loss-aversion, cf. Shi et al. [27] or Strub and Li [28]. Figure 1 illustrates the effect of the potential-aversion parameter on the shape of the mean-variance-induced utility function.

***Remark* 4** We already noted that the objective is not globally concave when potential-aversion is negative. A further interpretation of the mean-variance-induced utility function in this case is to regard it as a Friedman–Savage-type utility for a low-income investor, cf. Figure 2 in Friedman and Savage [29]. The inverse S-shape of this utility function leads to risk-seeking behavior for large wealth levels, consistent with the empirical evidence for the house money effect, see, e.g., Thaler and Johnson [30] and Post et al. [31].

There are numerous approaches in the existing literature trying to amend the shortcomings of the mean-variance framework. However, they all focus on revising the mean-variance objective directly, mostly by choosing an alternative risk measure. Contemporaneous with Markowitz [1], the safety-first principle in Roy [32] aimed at minimizing a disaster probability. Several downside risk-measures such as the semi-variance and lower partial moments have been proposed in the literature (see [33] for an overview). Artzner et al. [34] introduced the concept of a coherent risk measure based on a set of axioms—variance does not satisfy any of them. Maccheroni et al. [22] investigated the minimal monotone functional coinciding with the mean-variance preferences on their domain of monotonicity. Our approach is different from those in that we amend the utility function equivalent to the mean-variance objective instead of changing the underlying preferences directly.

Our approach has several advantages, in particular with regard to our main objectives of keeping the intuitive explanation of the mean-variance framework and being able to efficiently compute an optimal investment strategy. For the first objective, it is simple to illustrate the equivalence between the mean-variance objective and the setting in which the investor seeks to minimize deviations from a target wealth. For the second objective, we will illustrate that the optimal strategy has a piecewise-linear structure and can be easily computed. Our tractable approach distinguishes this framework from most other approaches. Jin et al. [35] showed that an optimal solution to mean-semi-variance and general mean-downside-risk problems exists in single period settings, but even in this static setting without time-consistency issues, it is difficult to characterize the solution. In complete markets, Jin et al. [36] showed that mean-downside-risk problems are generally ill-posed. Notable exceptions for mean-risk optimization problems where one can characterize the optimal trading strategy in incomplete discrete-time markets are the safety-first problem as shown in Li et al. [37] and the mean-CVaR optimization problem studied in Strub et al. [38].

The framework of mean-variance-induced utility functions is time-consistent because it is essentially an expected utility maximization problem and the tower property of the conditional expectation holds. It is thus in particular also time-consistent in efficiency. While the represented preferences are not monotone for positive values of the potential-aversion δ, non-monotonicity is alleviated when compared with the mean-variance framework and fully resolved for nonnegative or negative potential-aversions. This is illustrated in Fig. 1.

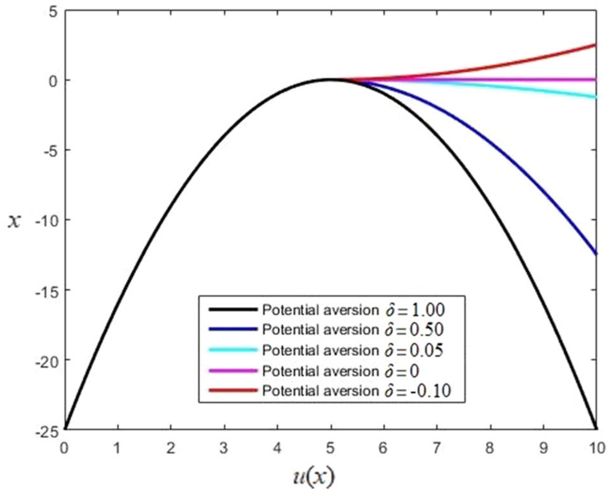


Fig. 1 Mean-variance-induced utility functions for different values of the potential-aversion parameter *Notes* This figure shows the mean-variance-induced utility function with target wealth $\gamma = 5$, weighting parameter $\alpha = 2$ for different values of the potential-aversion parameter

4 Optimal Investment Strategy

We investigate the well-posedness conditions of the mean-variance-induced utility maximization problem ($EU_1(\gamma, \delta, \alpha)$), derive its optimal investment strategy, and discuss the implications of parameter values on trading behavior and the terminal wealth distribution.

We start with a change of variables in order to move the parameter γ from the objective into the initial value. Let

$$Y_t = \rho_t X_t - \gamma,$$

where $\rho_t = \prod_{l=t}^{T-1} r_l$ and $\rho_T = 1$ denotes the terminal value of one dollar invested in the risk-free asset at time t. Then, the new state process $Y = (Y_t)_{t=0}^{T}$ evolves according to

$$Y_{t+1} = \rho_{t+1} X_{t+1} - \gamma = \rho_{t+1} \left(r_t X_t + \mathbf{P}_t' \mathbf{u}_t \right) - \gamma = Y_t + \rho_{t+1} \mathbf{P}_t' \mathbf{u}_t,$$

which resembles the dynamics of the wealth process $X = (X_t)_{t=0}^{T}$. Using this change of variables, we can transform ($EU_1(\gamma, \delta, \alpha)$) into the following equivalent form

$$\begin{aligned} \sup_{\mathbf{u}=(\mathbf{u}_t)_{t=0}^{T-1}} \quad & \mathbb{E}\left[-|Y_T|^{\alpha} \mathbf{1}_{\{Y_T<0\}} - \delta Y_T^{\alpha} \mathbf{1}_{\{Y_T \geqslant 0\}} \right], \\ \text{s.t.} \quad & Y_{t+1} = Y_t + \rho_{t+1} \mathbf{P}_t' \mathbf{u}_t, \quad t = 0, 1, \cdots, T-1, \end{aligned} \qquad (\widetilde{EU}_1(\gamma, \delta, \alpha))$$

which still depends on the target wealth γ because of its inclusion in the initial state $y_0 = \rho_0 x_0 - \gamma$. Note that a negative Y_T corresponds to states in which the terminal wealth X_T falls short of the target wealth, and a positive Y_T corresponds to states in which the terminal wealth exceeds the target wealth.

To build intuition, we first consider the one period optimization problem by setting $T = 1$. The investor chooses $\mathbf{u}_0 \in \mathbb{R}^n$ in order to solve

$$\sup_{\mathbf{u}_0 \in \mathbb{R}^n} \mathbb{E}\left[- \left| y_0 + \mathbf{u}_0' \mathbf{P}_0 \right|^{\alpha} \mathbf{1}_{\{y_0 + \mathbf{u}_0' \mathbf{P}_0 < 0\}} - \delta \left(y_0 + \mathbf{u}_0' \mathbf{P}_0 \right)^{\alpha} \mathbf{1}_{\{y_0 + \mathbf{u}_0' \mathbf{P}_0 \geqslant 0\}} \right]. \qquad (3)$$

Because of (2), $y_0 \leqslant 0$. If this inequality is strict, i.e., the target wealth is strictly larger than the initial wealth invested in the risk-free asset, we can write $\mathbf{u}_0 = y_0 \mathbf{K}_0$ and (3) becomes

$$|y_0|^{\alpha} \sup_{\mathbf{K}_0 \in \mathbb{R}^n} \mathbb{E}\left[- \left(1 + \mathbf{K}_0' \mathbf{P}_0 \right)^{\alpha} \mathbf{1}_{\{1 + \mathbf{K}_0' \mathbf{P}_0 > 0\}} - \delta \left| 1 + \mathbf{K}_0' \mathbf{P}_0 \right|^{\alpha} \mathbf{1}_{\{1 + \mathbf{K}_0' \mathbf{P}_0 \leqslant 0\}} \right]. \qquad (4)$$

It is apparent that the optimization problem over $\mathbf{K}_0 \in \mathbb{R}^n$ now does not depend on y_0 and thus not on the target wealth γ, but only on the potential-aversion δ, the weighting parameter α, and the financial market represented by the distribution of the excess returns $\mathbf{P}_0$. The following proposition gives conditions on the well-posedness of this problem.

Proposition 1 *Assume that*

$$\max_{\|\mathbf{L}\|=1} \mathbb{E}\left[-\left(\mathbf{L}'\mathbf{P}_0\right)^{\alpha} \mathbf{1}_{\{\mathbf{L}'P_0>0\}} - \delta \left|\mathbf{L}'\mathbf{P}_0\right|^{\alpha} \mathbf{1}_{\{\mathbf{L}'P_0\leqslant 0\}}\right] < 0. \tag{5}$$

Then, there is an optimizer $\widetilde{\mathbf{K}}_0$ *for the optimization problem in* (4) *and the optimal value of this problem is finite and negative.*

Proof The objective in (4) can be written as

$$\begin{aligned}
&\sup_{\mathbf{K}_0\in\mathbb{R}^n} \quad \mathbb{E}\left[-\left(1+\mathbf{K}_0'\mathbf{P}_0\right)^{\alpha} \mathbf{1}_{\{1+\mathbf{K}_0'\mathbf{P}_0>0\}} - \delta \left|1+\mathbf{K}_0'\mathbf{P}_0\right|^{\alpha} \mathbf{1}_{\{1+\mathbf{K}_0'\mathbf{P}_0\leqslant 0\}}\right] \\
&= \sup_{k>0}\left(k^{\alpha} \max_{\|\mathbf{L}\|=1} \mathbb{E}\left[-\left(\frac{1}{k}+\mathbf{L}'\mathbf{P}_0\right)^{\alpha} \mathbf{1}_{\{\mathbf{L}'\mathbf{P}_0>\frac{-1}{k}\}} - \delta\left|\frac{1}{k}+\mathbf{L}'\mathbf{P}_0\right|^{\alpha} \mathbf{1}_{\{\mathbf{L}'\mathbf{P}_0\leqslant\frac{-1}{k}\}}\right]\right).
\end{aligned}$$

Note that for any $k \geqslant 1$ and $\mathbf{L} \in \mathbb{R}^n$ with $\|L\| = 1$,

$$\begin{aligned}
&\left|-\left(\frac{1}{k}+\mathbf{L}'\mathbf{P}_0\right)^{\alpha} \mathbf{1}_{\{\mathbf{L}'\mathbf{P}_0>\frac{-1}{k}\}} - \delta\left|\frac{1}{k}+\mathbf{L}'\mathbf{P}_0\right|^{\alpha} \mathbf{1}_{\{\mathbf{L}'\mathbf{P}_0\leqslant\frac{-1}{k}\}}\right| \\
&\leqslant \left(\frac{1}{k}+\mathbf{L}'\mathbf{P}_0\right)^{\alpha} \mathbf{1}_{\{\mathbf{L}'\mathbf{P}_0>\frac{-1}{k}\}} + |\delta|\left|\frac{1}{k}+\mathbf{L}'\mathbf{P}_0\right|^{\alpha} \mathbf{1}_{\{\mathbf{L}'\mathbf{P}_0\leqslant\frac{-1}{k}\}} \\
&\leqslant \left(1+\mathbf{L}'\mathbf{P}_0\right)^{\alpha} \mathbf{1}_{\{\mathbf{L}'\mathbf{P}_0>0\}} + |\delta|\left|\mathbf{L}'\mathbf{P}_0\right|^{\alpha} \mathbf{1}_{\{\mathbf{L}'\mathbf{P}_0\leqslant 0\}} + 1 \\
&\leqslant 1 + (1+|\delta|)\left(1+\sum_{i=1}^{n}|P_t^i|\right)^{\alpha},
\end{aligned}$$

which is integrable because of our assumption that total returns have finite moments of all order. By dominated convergence and (5), the expectation converges to a strictly negative number when k goes to infinity. Hence, we can limit ourselves to a compact set of $\mathbb{R}^n$. Further note that the objective function is continuous in $\mathbf{K}_0$ by another application of dominated convergence with a similar dominating function as above. We can thus conclude that the optimizer exists and the objective value is finite. If $\delta \geqslant 0$, the objective is obviously negative. If $\delta < 0$, the objective value is negative because, for any $k > 0$ and any $\mathbf{L} \in \mathbb{R}^n \setminus \{0\}$,

$$\begin{aligned}
&\mathbb{E}\left[-\left(\frac{1}{k}+\mathbf{L}'\mathbf{P}_0\right)^{\alpha} \mathbf{1}_{\{\mathbf{L}'\mathbf{P}_0>\frac{-1}{k}\}} - \delta\left|\frac{1}{k}+\mathbf{L}'\mathbf{P}_0\right|^{\alpha} \mathbf{1}_{\{\mathbf{L}'\mathbf{P}_0\leqslant\frac{-1}{k}\}}\right] \\
&\leqslant \mathbb{E}\left[-\left(\mathbf{L}'\mathbf{P}_0\right)^{\alpha} \mathbf{1}_{\{\mathbf{L}'\mathbf{P}_0\geqslant 0\}} - \left(\frac{1}{k}+\mathbf{L}'\mathbf{P}_0\right)^{\alpha} \mathbf{1}_{\{0>\mathbf{L}'\mathbf{P}_0>\frac{-1}{k}\}} - \delta\left|\mathbf{L}'\mathbf{P}_0\right|^{\alpha} \mathbf{1}_{\{\mathbf{L}'\mathbf{P}_0\leqslant\frac{-1}{k}\}}\right] \\
&\leqslant \mathbb{E}\left[-\left(\mathbf{L}'\mathbf{P}_0\right)^{\alpha} \mathbf{1}_{\{\mathbf{L}'\mathbf{P}_0\geqslant 0\}} - \delta\left|\mathbf{L}'\mathbf{P}_0\right|^{\alpha} \mathbf{1}_{\{\mathbf{L}'\mathbf{P}_0\leqslant 0\}}\right],
\end{aligned}$$

(5) holds, and the objective is also negative when evaluated at $\mathbf{K}_0 = 0$.

***Remark* 5** Clearly, (5) is satisfied whenever potential-aversion is positive. The problem can only be ill-posed if the investor seeks potential. For the case with only a single risky asset, $n = 1$, (5) simplifies to the following lower bound for the potential-aversion

$$-\delta < \min \left\{ \frac{\mathbb{E}\left[|P_0|^\alpha \mathbf{1}_{\{P_0 \leqslant 0\}}\right]}{\mathbb{E}\left[(P_0)^\alpha \mathbf{1}_{\{P_0 > 0\}}\right]}, \frac{\mathbb{E}\left[(P_0)^\alpha \mathbf{1}_{\{P_0 > 0\}}\right]}{\mathbb{E}\left[|P_0|^\alpha \mathbf{1}_{\{P_0 \leqslant 0\}}\right]} \right\},$$

where the first term corresponds to taking a short position in the risky asset and the second term to a long position in the risky asset. Hence, assumption (5) implies the desire for potential must be limited by the opportunities offered in the market. Otherwise, the investor is tempted to take an infinitely large long or short position in the risky asset.

If we assume (5) and denote the optimizer and optimal value of problem (4) by $\widetilde{\mathbf{K}}_0$ and C_0, respectively, then the optimal investment strategy for the single period problem (3) is given by $\mathbf{u}_0^* = y_0 \widetilde{\mathbf{K}}_0$ and the optimal value of this problem is $C_0 |y_0|^\alpha$. When $y_0 = 0$, assumption (5) immediately yields the optimal investment strategy $u_0^* = \mathbf{0} \in \mathbb{R}^n$ and the optimal value is zero, hence we solve the single-period case completely.

The single-period solution already hints at how we can approach the multi-period problem, as well as the structure of the optimal investment strategy and value function. Indeed, the multi-period case can be solved using dynamic programming. If at time $T-1$ the state variable Y_{T-1} is non-positive, we can proceed exactly as in the single-period setting. The optimal strategy is of the form $Y_{T-1}\widetilde{\mathbf{K}}_{T-1}$ and the value function $C_{T-1}|Y_{T-1}|^\alpha$ for some negative constant C_{T-1}. If the state is positive, $Y_{T-1} > 0$, we can proceed as above by writing $\mathbf{u}_{T-1} = Y_{T-1}\mathbf{K}_{T-1}$ in order to separate the state from the objective. The resulting optimization problem over $\mathbf{K}_{T-1}$ will be similar to (4) except the weights for the positive and negative states of $1 + \mathbf{K}'_{T-1}\mathbf{P}_{T-1}$ will be interchanged. Hence, the optimizer $\widehat{\mathbf{K}}_{T-1}$ specifying the distribution of investment among the risky assets and the optimal value D_{T-1} will be different from the case of a negative state, but the optimal investment strategy is still linear in the state variable, $\mathbf{u}_{T-1} = Y_{T-1}\widehat{\mathbf{K}}_{T-1}$, and the optimal value is given by $D_{T-1}Y_{T-1}^\alpha$. Combining the nonnegative and positive cases, the optimal strategy will be piecewise linear in the following form:

$$\mathbf{u}_{T-1} = Y_{T-1}\widetilde{\mathbf{K}}_{T-1}\mathbf{1}_{\{Y_{T-1} \leqslant 0\}} + Y_{T-1}\widehat{\mathbf{K}}_{T-1}\mathbf{1}_{\{Y_{T-1} > 0\}}$$

and the value-to-go function is a piecewise power function with order α,

$$J_{T-1}(Y_{T-1}) = C_{T-1}|Y_{T-1}|^\alpha \mathbf{1}_{\{Y_{T-1} \leqslant 0\}} + D_{T-1}Y_{T-1}^\alpha \mathbf{1}_{\{Y_{T-1} > 0\}}.$$

Note the value-to-go function is of the same form as the original objective function. We can thus recursively determine the optimal investment proportions, strategies, and value-to-go functions.

Motivated by the above reasoning, we recursively define the following functions, their optimizers, and optimal values. For fixed $\alpha \geqslant 1$ and δ, let $C_T = -1$ and $D_T = -\delta$ and define for $t = T-1, T-2, \cdots, 0$,

$$g_t(\mathbf{K}) := \mathbb{E}\left[C_{t+1}\left(1+\rho_{t+1}\mathbf{K}'\mathbf{P}_t\right)^{\alpha}\mathbf{1}_{\{1+\rho_{t+1}\mathbf{K}'\mathbf{P}_t>0\}} + D_{t+1}\left|1+\rho_{t+1}\mathbf{K}'\mathbf{P}_t\right|^{\alpha}\mathbf{1}_{\{1+\rho_{t+1}\mathbf{K}'\mathbf{P}_t\leqslant 0\}}\right],$$
$$h_t(\mathbf{K}) := \mathbb{E}\left[C_{t+1}\left|1+\rho_{t+1}\mathbf{K}'\mathbf{P}_t\right|^{\alpha}\mathbf{1}_{\{1+\rho_{t+1}\mathbf{K}'\mathbf{P}_t<0\}} + D_{t+1}\left(1+\rho_{t+1}\mathbf{K}'\mathbf{P}_t\right)^{\alpha}\mathbf{1}_{\{1+\rho_{t+1}\mathbf{K}'\mathbf{P}_t\geqslant 0\}}\right], \tag{6}$$

where the constants C_t and D_t are the optimal values of the those functions for $t = T-1, T-2, \cdots, 0$, i.e.,

$$C_t = \sup_{\mathbf{K}\in\mathbb{R}^n} g_t(\mathbf{K}) \quad \text{and} \quad D_t = \sup_{\mathbf{K}\in\mathbb{R}^n} h_t(\mathbf{K}). \tag{7}$$

The optimizers are denoted by

$$\widetilde{\mathbf{K}}_t = \arg\max_{\mathbf{K}\in\mathbb{R}^n} g_t\,(\mathbf{K}) \qquad \text{and} \qquad \widehat{\mathbf{K}}_t = \arg\max_{\mathbf{K}\in\mathbb{R}^n}\, h_t(\mathbf{K}). \tag{8}$$

Proposition 1 immediately extends to the dynamic setting.

Proposition 2 *For $t = T, T-1, \cdots, 1$, we assume that*

$$\max_{\|\mathbf{L}\|=1} \mathbb{E}\left[C_t\left(\mathbf{L}'\mathbf{P}_{t-1}\right)^{\alpha}\mathbf{1}_{\{\mathbf{L}'P_{t-1}>0\}} + D_t\left|\mathbf{L}'\mathbf{P}_{t-1}\right|^{\alpha}\mathbf{1}_{\{\mathbf{L}'P_{t-1}\leqslant 0\}}\right] < 0. \tag{A}$$

Then, the subsequent optimization problems for the functions g_{t-1} and h_{t-1} defined in (6) *are well-posed, the optimizers $\widetilde{\mathbf{K}}_{t-1}$ and $\widehat{\mathbf{K}}_{t-1}$ in* (8) *exist, and the optimal values* (7) *are finite.*

Proof The proof for the problems resulting from optimizing the functions g_t is exactly as in Proposition 1. The statement for the optimization problems related to h_t follows by noting that we can always consider $-\mathbf{L}$ for any given $\mathbf{L}$ with $\|\mathbf{L}\| = 1$ and this reverses the sign of $\mathbf{L}'\mathbf{P}_t$.

It is clear from the recursive construction of $(C_t)_{t=0}^T$ and $(D_t)_{t=0}^T$ that both weights on the utility function are decreasing in time, i.e., $C_t \geqslant C_{t+1}$ and $D_t \geqslant D_{t+1}$ for $t = 0, \cdots, T-1$. Hence, assumption (5) becomes increasingly hard to hold when going backwards in time. While it is difficult to obtain analytical expressions for $(C_t)_{t=0}^T$ and $(D_t)_{t=0}^T$ in general, the intuition provided in Remark 5 still holds: The problem can only be ill-posed if the investor seeks potential, and, in this case, the desire for potential must be limited by the opportunities offered by the market.

***Remark* 6** While Proposition 1 and Proposition 2 establish existence of the solutions to (4) and (8), respectively, obtaining conditions for uniqueness when $\delta < 0$ and the objective is thus non-concave remains an open problem. We refer to Sect. 6 for an example where the solution is not unique in binomial and trinomial models of the financial market. In the case that the solution is not unique, we suggest to select the solution with smallest Euclidean norm.

Summarizing the arguments of this section leads to the following theorem, which characterizes the optimal investment strategy for the mean-variance-induced utility maximization problem.

Theorem 1 *Assume assumption (A) holds for* $t = T, T-1, \cdots, 1$ *and let the recursive constants* $(C_t)_{t=0}^T$ *and* $(D_t)_{t=0}^T$ *be given by (7) and let* $(\widetilde{\mathbf{K}}_t)_{t=0}^{T-1}$ *and* $(\widehat{\mathbf{K}}_t)_{t=0}^{T-1}$ *be given by (8). The optimal investment strategy for* $(EU_1(\gamma, \delta, \alpha))$ *is*

$$\mathbf{u}_t = (\rho_t X_t - \gamma)\,\widetilde{\mathbf{K}}_t \mathbf{1}_{\{\rho_t X_t - \gamma \leqslant 0\}} + (\rho_t X_t - \gamma)\,\widehat{\mathbf{K}}_t \mathbf{1}_{\{\rho_t X_t - \gamma > 0\}}$$

for $t = 0, 1, \cdots, T-1$ *and the value-to-go function of this problem is*

$$J_t(X_t) = C_t\,|\rho_t X_t - \gamma|^\alpha\,\mathbf{1}_{\{Y_t \leqslant 0\}} + D_t\,(\rho_t X_t - \gamma)^\alpha\,\mathbf{1}_{\{Y_t > 0\}}$$

for $t = 0, 1, \cdots, T$.

Remark 7 A similar result of a piecewise linear optimal strategy in the current wealth and a value-to-go function which has two parts with different weights was obtained in the context of discrete-time mean-variance optimization under no-shorting constraints in Cui et al. [39] and later generalized for mean-variance optimization with cone constraints by Cui et al. [40]. Different from our setting, mean-variance problems under constraints are always well-posed because they can be embedded into an expected utility maximization problem with a utility function which is bounded from above.

Mean-variance optimization is a special case of our framework. In the mean-variance case, potential-aversion is $\delta = 1$ and the weighting parameter $\alpha = 2$. Then, $C_T = D_T = -1$ and thus $g_{T-1}(\mathbf{K}) = h_{T-1}(\mathbf{K})$ for all $\mathbf{K} \in \mathbb{R}^n$. It follows by backward induction that $g_t = h_t$, $\widetilde{\mathbf{K}}_t = \widehat{\mathbf{K}}_t$ and $C_t = D_t < 0$ for $t = T-1, \cdots, 0$ and we can solve the optimization problems of $g_t = h_t$ explicitly as they reduce to

$$\min_{\mathbf{K}_t \in \mathbb{R}^n} \mathbb{E}\left[\left(1 + \rho_{t+1}\mathbf{K}_t'\mathbf{P}_t\right)^2\right],$$

whose solution can easily be derived to be

$$\widetilde{\mathbf{K}}_t^{mv} = \widehat{\mathbf{K}}_t^{mv} = -\frac{1}{\rho_{t+1}}\mathbb{E}\left[\mathbf{P}_t\mathbf{P}_t'\right]^{-1}\mathbb{E}[\mathbf{P}_t].$$

We are thus able to recover the optimal strategy for the discrete-time mean-variance problem first derived in Li and Ng [3],

$$\mathbf{u}_t^{mv} = -\,(\rho_t X_t - \gamma)\,\frac{1}{\rho_{t+1}}\mathbb{E}\left[\mathbf{P}_t\mathbf{P}_t'\right]^{-1}\mathbb{E}[\mathbf{P}_t].$$

5 An Example with Lognormal Returns

The previous sections establish the basic properties and theoretical foundations of mean-variance-induced utility functions. In this section, we illustrate the behavior of our proposed asset allocation framework with a numerical example. Assuming lognormal returns is a common assumption in finance, see for example the widely used options pricing model of Black and Scholes [41]. While we do not make use of particular properties of the lognormal distribution other than being absolutely continuous and having finite moments of all order, assuming lognormal returns allows us to explore the merits and pinpoint unique aspects of our framework in a familiar setting.

Let the cumulative return process, $(\mathbf{e}_t)_{t=0}^{T-1}$, follow a multivariate lognormal distribution. For the ease of illustration, we let returns in each period t to be independent and identically distributed. This example serves to illustrate the effect of the model parameters on the optimal terminal wealth distribution for the mean-variance induced utility function. It also demonstrates that the optimal investment strategy can be computed numerically for one of the most popular distributions of asset returns and discusses some potential difficulties for numerical computations of optimal strategies. Our setting follows the example in Li and Ng [3]. This example became a benchmark example studied in numerous follow-up work due to the impact of that paper. An investor with initial wealth x_0 seeks to determine the mean-variance optimal strategy in a multi-period market with $T = 4$. The risk-free asset offers a return of $r_t = 1.04$ each period. There are three risky assets, denoted by A, B and C, whose returns have the following mean and covariance matrix:

$$\mathbb{E}[\mathbf{e}_t] = \mathbb{E}\left[\begin{pmatrix} e_t^A \\ e_t^B \\ e_t^C \end{pmatrix}\right] = \begin{pmatrix} 1.162 \\ 1.246 \\ 1.228 \end{pmatrix} \quad \text{and} \quad \mathrm{Cov}(\mathbf{e}_t) = \begin{pmatrix} 0.0146 & 0.0187 & 0.0145 \\ 0.0187 & 0.0854 & 0.0104 \\ 0.0145 & 0.0104 & 0.0289 \end{pmatrix}. \tag{9}$$

Note that while the mean and the covariance matrix of the excess returns are sufficient to determine the optimal solution for the classical mean-variance problem, the specific distribution matters for the mean-variance-induced utility maximization problem. In order to solve (8) numerically, we simulate 1 000 000 random numbers for lognormally distributed returns given the above mean and covariance. In our analysis, we focus on the impact of the potential-aversion δ, for which we consider the following values: $\delta \in \{1, 0.5, 0.05, 0, -10^{-10}\}$. We fix the weighting parameter at $\alpha = 2$ for an easy comparison to the mean-variance framework. The calculated optimal investment proportions, $(\widetilde{\mathbf{K}}_t)_{t=0}^{T-1}$ and $(\widehat{\mathbf{K}}_t)_{t=0}^{T-1}$, are given in Table 1.

Recall that $\widetilde{\mathbf{K}}_t$ corresponds to the optimal investment proportions among the risky assets when the current wealth is below the discounted target wealth. As δ decreases, the investor prefers investing more in the two riskier assets B and C while keeping the amount invested in the safer asset A roughly constant, when the current wealth does not exceed the discounted target wealth. To better understand this result, we split up

Table 1 Numerically computed optimizers of g_t and h_t

	$t=0$	$t=1$	$t=2$	$t=3$
$\delta=1$	$\widetilde{\mathbf{K}}_0=\begin{pmatrix}-0.34\\-0.56\\-1.98\end{pmatrix}$	$\widetilde{\mathbf{K}}_1=\begin{pmatrix}-0.36\\-0.58\\-2.06\end{pmatrix}$	$\widetilde{\mathbf{K}}_2=\begin{pmatrix}-0.37\\-0.60\\-2.19\end{pmatrix}$	$\widetilde{\mathbf{K}}_3=\begin{pmatrix}-0.39\\-0.62\\-2.22\end{pmatrix}$
	$\widehat{\mathbf{K}}_0=\begin{pmatrix}-0.34\\-0.56\\-1.98\end{pmatrix}$	$\widehat{\mathbf{K}}_1=\begin{pmatrix}-0.36\\-0.58\\-2.06\end{pmatrix}$	$\widehat{\mathbf{K}}_2=\begin{pmatrix}-0.37\\-0.60\\-2.19\end{pmatrix}$	$\widehat{\mathbf{K}}_3=\begin{pmatrix}-0.39\\-0.62\\-2.22\end{pmatrix}$
$\delta=0.5$	$\widetilde{\mathbf{K}}_0=\begin{pmatrix}-0.35\\-0.60\\-2.12\end{pmatrix}$	$\widetilde{\mathbf{K}}_1=\begin{pmatrix}-0.34\\-0.63\\-2.25\end{pmatrix}$	$\widetilde{\mathbf{K}}_2=\begin{pmatrix}-0.38\\-0.67\\-2.36\end{pmatrix}$	$\widetilde{\mathbf{K}}_3=\begin{pmatrix}-0.38\\-0.71\\-2.52\end{pmatrix}$
	$\widehat{\mathbf{K}}_0=\begin{pmatrix}-0.35\\-0.52\\-1.84\end{pmatrix}$	$\widehat{\mathbf{K}}_1=\begin{pmatrix}-0.35\\-0.53\\-1.90\end{pmatrix}$	$\widehat{\mathbf{K}}_2=\begin{pmatrix}-0.39\\-0.54\\-1.93\end{pmatrix}$	$\widehat{\mathbf{K}}_3=\begin{pmatrix}-0.39\\-0.55\\-1.97\end{pmatrix}$
$\delta=0.05$	$\widetilde{\mathbf{K}}_0=\begin{pmatrix}-0.33\\-0.78\\-2.72\end{pmatrix}$	$\widetilde{\mathbf{K}}_1=\begin{pmatrix}-0.32\\-0.85\\-3.00\end{pmatrix}$	$\widetilde{\mathbf{K}}_2=\begin{pmatrix}-0.36\\-0.95\\-3.29\end{pmatrix}$	$\widetilde{\mathbf{K}}_3=\begin{pmatrix}-0.35\\-1.05\\-3.65\end{pmatrix}$
	$\widehat{\mathbf{K}}_0=\begin{pmatrix}-0.35\\-0.40\\-1.46\end{pmatrix}$	$\widehat{\mathbf{K}}_1=\begin{pmatrix}-0.36\\-0.39\\-1.44\end{pmatrix}$	$\widehat{\mathbf{K}}_2=\begin{pmatrix}-0.39\\-0.38\\-1.40\end{pmatrix}$	$\widehat{\mathbf{K}}_3=\begin{pmatrix}-0.39\\-0.37\\-1.37\end{pmatrix}$
$\delta=0$	$\widetilde{\mathbf{K}}_0=\begin{pmatrix}-0.29\\-1.15\\-3.97\end{pmatrix}$	$\widetilde{\mathbf{K}}_1=\begin{pmatrix}-0.29\\-1.19\\-4.12\end{pmatrix}$	$\widetilde{\mathbf{K}}_2=\begin{pmatrix}-0.34\\-1.25\\-4.26\end{pmatrix}$	$\widetilde{\mathbf{K}}_3=\begin{pmatrix}-0.32\\-1.30\\-4.46\end{pmatrix}$
	$\widehat{\mathbf{K}}_0=\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\widehat{\mathbf{K}}_1=\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\widehat{\mathbf{K}}_2=\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\widehat{\mathbf{K}}_3=\begin{pmatrix}0\\0\\0\end{pmatrix}$
$\delta=-10^{-10}$	$\widetilde{\mathbf{K}}_0=\begin{pmatrix}-0.29\\-1.15\\-3.97\end{pmatrix}$	$\widetilde{\mathbf{K}}_1=\begin{pmatrix}-0.29\\-1.19\\-4.12\end{pmatrix}$	$\widetilde{\mathbf{K}}_2=\begin{pmatrix}0.34\\-1.25\\-4.26\end{pmatrix}$	$\widetilde{\mathbf{K}}_3=\begin{pmatrix}-0.32\\-1.30\\-4.46\end{pmatrix}$
	$\widehat{\mathbf{K}}_0=\begin{pmatrix}-0.47\\0.49\\2.45\end{pmatrix}$	$\widehat{\mathbf{K}}_1=\begin{pmatrix}0.26\\1.17\\0.97\end{pmatrix}$	$\widehat{\mathbf{K}}_2=\begin{pmatrix}0.29\\1.21\\0.92\end{pmatrix}$	$\widehat{\mathbf{K}}_3=\begin{pmatrix}0.35\\1.20\\0.64\end{pmatrix}$

Notes the table shows the numerically computed optimizers g_t and h_t defined in (6) based on simulation of 1 000 000 lognormal random numbers. The risk-free return is $r_t = 1.04$ and the mean and covariance of the three risky returns are given in (9)

the mean-variance optimal investment proportions by risky asset,

$$\widetilde{\mathbf{K}}_0^A=\widehat{\mathbf{K}}_0^A=\begin{pmatrix}-0.34\\0\\0\end{pmatrix},\quad \widetilde{\mathbf{K}}_0^B=\widehat{\mathbf{K}}_0^B=\begin{pmatrix}0\\-0.56\\0\end{pmatrix},\quad \widetilde{\mathbf{K}}_0^C=\widehat{\mathbf{K}}_0^C=\begin{pmatrix}0\\0\\-1.98\end{pmatrix}.$$

We also observe similar values for $\widetilde{\mathbf{K}}_t$ and $\widehat{\mathbf{K}}_t$ at $t = 1, 2, 3$. We then simulate trading under these strategies and consider in what percentage of scenarios the terminal wealth exceeds the target wealth and lies in the region of the mean-variance induced utility function influenced by the value of the potential-aversion parameter. If we would trade according to $\widetilde{\mathbf{K}}_t^A$ and $\widehat{\mathbf{K}}_t^A$, none of the 1 000 000 simulated scenarios would exceed the target wealth. When trading according to $\widetilde{\mathbf{K}}_t^B$ and $\widehat{\mathbf{K}}_t^B$, 0.15% of scenarios exceed the target wealth. When trading according to $\widetilde{\mathbf{K}}_t^C$ and $\widehat{\mathbf{K}}_t^C$, 23.31% of scenarios exceed the target wealth. Because lowering potential-aversion δ results in a smaller aversion to scenarios exceeding the target wealth, it is not surprising that the investment in assets B and C increases, while the investment in asset A remains roughly constant.

***Remark* 8** The following observations holds in general: When the return distribution is concentrated around a moderate mean return, then the effect of the potential-aversion parameter on the optimal strategy is negligible. Indeed, when starting from a wealth level below the discounted target wealth, one can only enter the region of the utility function above the target wealth when $1 + \rho_{t+1}\mathbf{K}_t'\mathbf{P}_t < 0$ at some time t, which is very unlikely to happen if the return distribution is concentrated around a moderate mean return.

When the current wealth exceeds the target wealth and so the investor allocations in the risky assets follow $\widehat{\mathbf{K}}_t$, lowering the potential-aversion parameter δ reduces the absolute amount invested in assets B and C. Note that as long as there is positive potential-aversion, the investor shorts the risky assets when the target wealth is exceeded and thus in effect follows a strategy with a negative expected return. Lowering δ thus alleviates this undesirable investment behavior. When there is zero potential-aversion, the investor stops investing in risky assets as soon as her current wealth exceeds the discounted target wealth. This is intuitive: when a terminal wealth with maximal utility can be reached, there is no reason to risk falling back into the region below the target wealth where utility is strictly lower. When potential-aversion is negative, the investor seeks outcomes above the target wealth rather than avoiding them. The optimal strategy achieves a positive expected return independent of whether current wealth is above or below the discounted target wealth.

***Remark* 9** The seeming discontinuity of $\widehat{\mathbf{K}}_t$ as a function of potential-aversion δ around $\delta = 0$ is due to the sensitivity to outliers in the numerical simulation. Moreover, the numerical values for the case $\delta = -10^{-10}$ are not robust and should be taken with caution. The reason for this is the same as for Remark 8: $\widehat{K}_t$ is chosen in maximizing an objective of the following form:

$$h_t(\mathbf{K}) := \mathbb{E}\left[C_{t+1}\left|1 + \rho_{t+1}\mathbf{K}'\mathbf{P}_t\right|^{\alpha} \mathbf{1}_{\{1+\rho_{t+1}\mathbf{K}'\mathbf{P}_t<0\}} + D_{t+1}\left(1 + \rho_{t+1}\mathbf{K}'\mathbf{P}_t\right)^{\alpha} \mathbf{1}_{\{1+\rho_{t+1}\mathbf{K}'\mathbf{P}_t\geqslant 0\}}\right].$$

However, the event $\mathbf{1}_{\{1+\rho_{t+1}\mathbf{K}'\mathbf{P}_t<0\}}$ happens only for a small number of scenarios, in particular when $\|\mathbf{K}\|$ is small. In contrast, $\|C_{t+1}\|$ is large compared to $\|D_{t+1}\|$ when δ is a small negative number such as $\delta = -10^{-10}$. The objective function thus gives a large relative weight to a small number of extreme scenarios, thereby making the optimizer sensitive to outliers.

We next turn our attention to the distribution of the terminal wealth for different target wealth and potential-aversion parameters. In the classical mean-variance framework, there is an explicit relationship between target wealth and expected terminal wealth given in (1). When the potential-aversion parameter is smaller than one, the investor dislikes large wealth levels exceeding the target wealth less than the mean-variance investor and thus generally achieves a higher expected terminal wealth than a mean-variance investor with the same target wealth. In order to better compare two different investors and isolate the effect of potential-aversion, we choose the target wealth for the investor with a lower potential-aversion such that this investor achieves the same expected terminal wealth as the corresponding mean-variance investor. This allows us to compare the resulting terminal wealth distributions from several different perspectives: We consider the trade-offs between mean and standard deviation, mean and lower standard deviation, mean and upper standard deviation, lower and upper standard deviation, mean and risk and mean and potential. The corresponding frontiers are shown in Fig. 2.

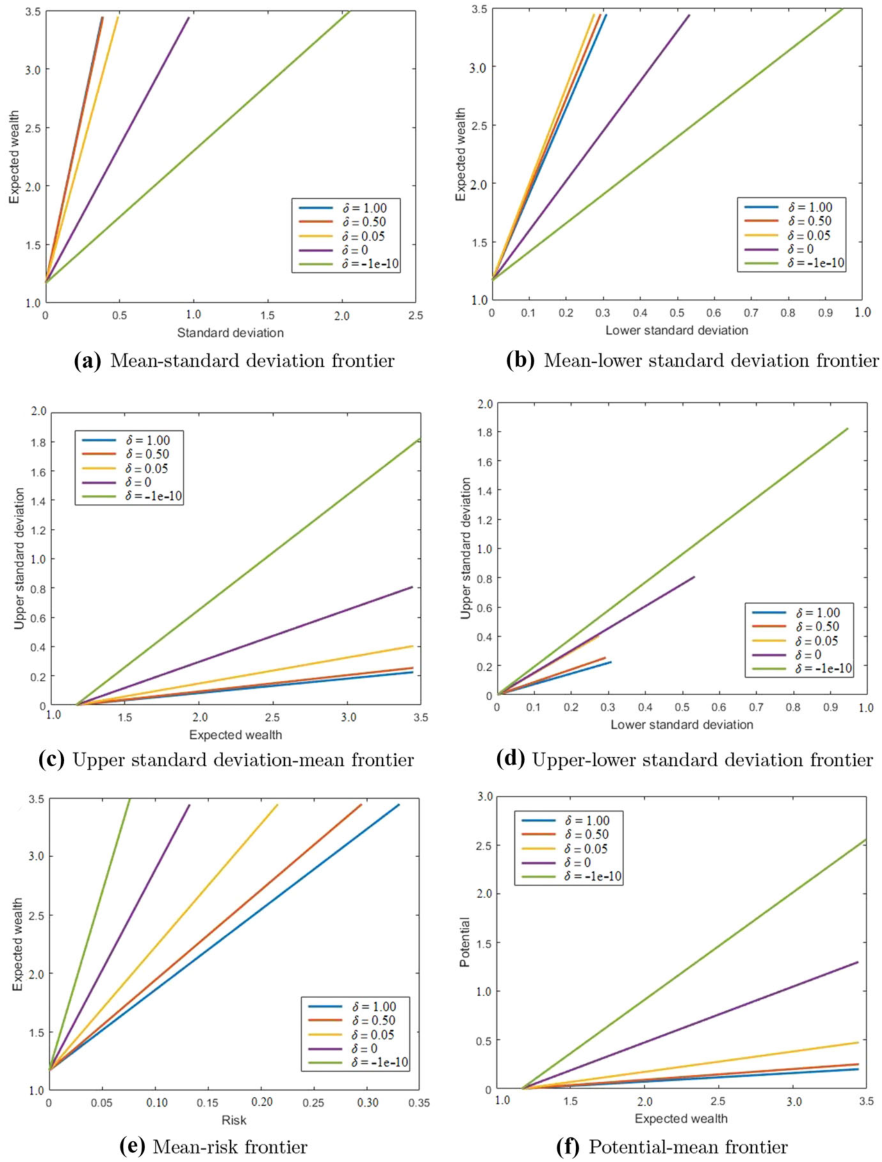
We arrange the coordinates in Fig. 2 such that the further northwest the frontier generated by the strategy optimal for a given potential-aversion δ, the more desirable that particular strategy is. As one would expect, the mean-standard deviation frontier shown in Fig. 2(a) shows that the mean-variance optimal strategy with a potential-aversion of one achieves the best mean-standard deviation trade-off. Lowering the potential-aversion parameter results in a slightly worse mean-standard deviation trade-off as long as potential-aversion remains positive, and a considerably worse trade-off if the potential-aversion parameter is zero or negative. In Fig. 2(b) and 2(c), we split the standard deviation in a lower part and an upper part, corresponding to outcomes below and above the mean. Interestingly, in terms of mean-lower standard deviation trade-off, lowering the aversion to potential improves the performance as long as potential-aversion is positive, whereas a non-positive aversion to potential still results in a markedly larger risk for the same expected wealth. In terms of mean-upper standard deviation, the lower the potential-aversion the better the performance. If an investor regards lower standard deviation as risk and upper standard deviation as potential, lowering the potential-aversion δ results in both less risk and more potential as long as δ remains positive, and a much larger potential at the expense of larger risk when it is zero or negative. In Fig. 2(d), we plot the upper versus lower standard deviation and observe that lowering δ results in a better trade-off between these two measures.

Finally, we plot expected wealth against two new deviation measures, risk $\mathcal{R}_\gamma(X)$ and potential $\mathcal{P}_\gamma(X)$:

$$\mathcal{R}_\gamma(X) := \sqrt{\mathbb{E}\left[(X-\gamma)^2\,\mathbf{1}_{\{X\leqslant\gamma\}}\right]} \quad \text{and} \quad \mathcal{P}_\gamma(X) := \sqrt{\mathbb{E}\left[(X-\gamma)^2\,\mathbf{1}_{\{X\geqslant\gamma\}}\right]}.$$

The two measures compute the square root of the average squared distance of outcomes below and above the target wealth. These two measures can be motivated by the embedding of the mean-variance problem into the expected utility maximization problem ($EU_0(\gamma)$), which reveals that a mean-variance investor seeks to minimize deviations from the target wealth γ. From this perspective of a target wealth, falling short of the target is perceived as risk, while exceeding the target is regarded as potential. Figures 2(e) and 2(f) show that a lower aversion to potential improves the performance in terms of both risk and potential.

The skewness of the terminal distribution does not depend on the target wealth, but only on the potential-aversion. This is not surprising because skewness is normalized by the standard deviation. Table 2 shows that the skewness of the terminal distribution increases from being strongly negative for the mean-variance investor to strongly pos-



(a) Mean-standard deviation frontier

(b) Mean-lower standard deviation frontier

(c) Upper standard deviation-mean frontier

(d) Upper-lower standard deviation frontier

(e) Mean-risk frontier

(f) Potential-mean frontier

Fig. 2 Statistics of terminal wealth distributions optimal for varying potential-aversion parameters. *Notes* This figure shows the statistics of terminal wealth distributions optimal for varying potential-aversion parameters δ based on 1 000 000 sample paths. The risk-free return is $r_t = 1.04$ and the three risky returns are assumed to follow a lognormal distribution with mean and covariance given in (9)

itive for an investor which is potential-seeking as the parameter of potential-aversion decreases. The exception to this observation is $\delta = 0$, which leads to a smaller skewness than $\delta = 0.05$. When $\delta = 0$, it is optimal to refrain from investing in the risky assets once the investor's wealth exceeds the target. Therefore, the investor's portfolio

Table 2 Skewness of the terminal distribution

	$\delta = 1$	$\delta = 0.5$	$\delta = 0.05$	$\delta = 0$	$\delta = -10^{-10}$
Skew $\left(X_T^{\gamma,\delta}\right)$	$-1.750\,1$	$-0.907\,6$	$2.299\,2$	$1.635\,1$	$3.005\,4$

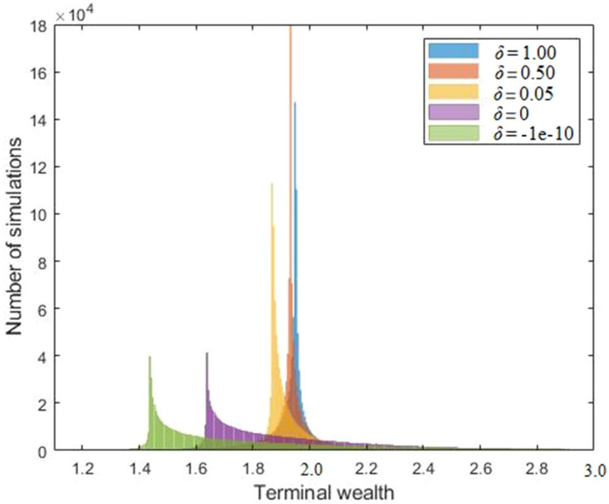


Fig. 3 Histograms of terminal wealth distributions optimal for varying potential-aversion parameters. *Notes* This figure shows the histograms of the terminal wealth distributions optimal for varying potential-aversion parameters δ based on 1 000 000 sample paths. The expected terminal wealth of each distribution is 1.93. The risk-free return is $r_t = 1.04$ and the three risky returns are assumed to follow a lognormal distribution with mean and covariance given in (9)

contains a smaller allocation to risky assets for $\delta = 0$ than those of $\delta = 0.05$ or $\delta = -10^{-10}$.

Figure 3 shows the histograms of terminal wealth based on 1 000 000 sample paths for varying potential-aversion parameters δ. The target wealth γ is selected such that the expected terminal wealth remains constant at 1.93. This figure confirms and illustrates the main intuitions we discussed: Decreasing the potential-aversion parameter δ leads to a higher standard deviation, and this higher standard deviation is mostly driven by a larger upper standard deviation.

***Remark* 10** The findings of this section are robust with respect to both the market parameters and the distribution of the returns, as long as the return distribution is continuous (the discrete case is discussed in the following Section 6) and not concentrated around a moderate mean return, in which case Remark 8 would apply. We refer to Strub [42] for further numerical examples with multivariate lognormal, multivariate normal, and multivariate t-distributions.

6 Binomial and Trinomial Models

The numerical illustration in the previous section, which assumes a lognormal return distribution, is also applicable to other continuous return distributions. We further demonstrate the usage of our proposed asset allocation framework in a setting where the distribution is discrete. Possibly the most popular discrete process used to describe asset return dynamics is the binomial model (Cox and Ross [43], Cox et al. [44]), which assumes that there are only two potential outcomes for a given risky asset. Although conceptually simple, the binomial model can be generalized to approximate complex return distributions by allowing the number of time periods to increase.

In this section, we study the framework of mean-variance induced utility maximization in a binomial model with one risky asset and later extend this setting to a special case of the trinomial model. The simple settings allow us to derive analytical results that can be compared to the observations we made for the lognormal case in Sect. 5. We find that the implications for risk, potential, and asset allocation decisions are qualitatively unchanged whether the underlying return distribution is discrete or continuous.

We assume that $r_t = 1$ for $t = 0, 1, \cdots, T-1$ and consider one risky asset whose return e_t takes either the value $\tilde{u} > 1$ with probability $p \in (0, 1)$ or the value $\tilde{d} < 1$ with probability $1-p$. Let $u = \tilde{u} - 1$ and $d = \tilde{d} - 1$ denote excess returns. As in the previous section, we set the weighting parameter $\alpha = 2$, corresponding to the classical mean-variance case, and we investigate the impact of potential-aversion δ. Given the model parameters, the investor needs to find the optimizers g_t and h_t from (6) and compute the recursively constants $(C_t)_{t=0}^T$ and $(D_t)_{t=0}^T$ in (7) in order to determine the optimal strategy (see Theorem 1). In the binomial case, assumption (A) simplifies to

$$C_{t+1}pu^2 + D_{t+1}(1-p)d^2 < 0 \quad \text{and} \quad D_{t+1}pu^2 + C_{t+1}(1-p)d^2 < 0, \tag{10}$$

for $t = T, T-1, \cdots, 1$, and the functions g_t and h_t take the following forms:

$$\begin{aligned} g_t(K) = {}& p\left(C_{t+1}\,(1+Ku)^2\,\mathbf{1}_{\{1+Ku>0\}} + D_{t+1}\,(1+Ku)^2\,\mathbf{1}_{\{1+Ku\leqslant 0\}}\right) \\ &+ (1-p)\left(C_{t+1}\,(1+Kd)^2\,\mathbf{1}_{\{1+Kd>0\}} + D_{t+1}\,(1+Kd)^2\,\mathbf{1}_{\{1+Kd\leqslant 0\}}\right), \\ h_t(K) = {}& p\left(C_{t+1}\,(1+Ku)^2\,\mathbf{1}_{\{1+Ku<0\}} + D_{t+1}\,(1+Ku)^2\,\mathbf{1}_{\{1+Ku\geqslant 0\}}\right) \\ &+ (1-p)\left(C_{t+1}\,(1+Kd)^2\,\mathbf{1}_{\{1+Kd<0\}} + D_{t+1}\,(1+Kd)^2\,\mathbf{1}_{\{1+Kd\geqslant 0\}}\right). \end{aligned}$$

When $K < -1/u$, the well-posedness condition (10) implies

$$
\begin{aligned}
g_t'(K) &= 2K\left(D_{t+1}pu^2 + C_{t+1}(1-p)d^2\right) + 2\left(D_{t+1}pu + C_{t+1}(1-p)d\right)\\
&> -\frac{2}{u}\left(D_{t+1}pu^2 + C_{t+1}(1-p)d^2\right) + 2\left(D_{t+1}pu + C_{t+1}(1-p)d\right)\\
&= 2C_{t+1}(1-p)d(-\frac{d}{u}+1)\\
&> 0.
\end{aligned}
$$

Hence, no admissible $K < -1/u$ maximizes g_t when (10) holds. One similarly finds that $g_t'(K) < 2C_{t+1}pu(-u/d+1) < 0$ when $K > -1/d$ and hence $\widetilde{K}_t$ must be in $[-1/u, -1/d]$. Solving the first-order condition yields

$$
\widetilde{K}_t^b = -\frac{pu + (1-p)d}{pu^2 + (1-p)d^2} = -\frac{\mathbb{E}[P_t]}{\mathbb{E}[P_t^2]} = \widetilde{K}_t^{b,mv}.
$$

We add here the superscript b to indicate the optimality for the binomial market. Note that because $Y_{t+1} = Y_t(1 + \widetilde{K}_t P_t)$ will remain negative in all states of the world, the investor will trade exactly like a mean-variance investor independently of δ. Hence, in a binomial market, the potential-aversion parameter is only required to determine whether the problem is well-posed, but otherwise has no influence on the optimal asset allocation decision. This result is an extreme version of the observation we made in Remark 8—potential-aversion has little influence on the optimal trading strategy when the distribution is concentrated around a modest average return.

***Remark* 11** Bäuerle and Grether [45] showed that complete markets do not allow free cash flow streams. In the notation of this paper, free cash flow streams appear when the auxiliary state Y_t becomes positive. In this sense, our finding constitutes a special case of their more general result on free cash flows.

In order to study the effect of varying potential-aversion parameter values, we expand the binomial model to a trinomial setting in which there is a small probability of obtaining a large gain. We first fix the parameter values p, u, d for the binomial model and assume that (10) holds as well as that

$$
pu + (1-p)d > 0 \text{ and } D_{t+1}pu^2 + C_{t+1}(1-p)d^2 > C_{t+1}pu^2 + D_{t+1}(1-p)d^2. \tag{11}
$$

The first inequality simply says that the risky asset offers a positive expected excess return and assures that the investor takes a long position in the risky asset when her wealth lies below the target wealth. The second inequality assures that she still takes a long position in the risky asset when D_{t+1} is positive and her current wealth is above the target wealth. When (10) holds and $D_{t+1} > 0$, it is straightforward to show that

$h_t(K)$ has two local maxima given by

$$\widehat{K}_t^{b,1} = -\frac{C_{t+1}pu + D_{t+1}(1-p)d}{C_{t+1}pu^2 + D_{t+1}(1-p)d^2} < -\frac{1}{u},$$
$$\widehat{K}_t^{b,2} = -\frac{D_{t+1}pu + C_{t+1}(1-p)d}{D_{t+1}pu^2 + C_{t+1}(1-p)d^2} > -\frac{1}{d}$$

and that $\widehat{K}_t^b = \widehat{K}_t^{b,2}$ when (11) holds. When $D_{t+1} < 0$, we have

$$\widehat{K}_t^b = -\frac{pu + (1-p)d}{pu^2 + (1-p)d^2} = -\frac{\mathbb{E}[P_t]}{\mathbb{E}[P_t^2]} = \widehat{K}_t^{b,mv},$$

as before. When $D_{t+1} = 0$ the optimizer of h_t is not unique. Following the suggestion in Remark 6, we take $\widehat{K}_t^b = 0$ in this case, which is always optimal.

We extend the binomial model to a trinomial model by assuming that the excess return can take the values w, u and d with probability ϵ, $p(1-\epsilon)$ and $(1-p)(1-\epsilon)$, respectively. The interpretation is that there is a small probability ϵ for a large excess return

$$w > \frac{pu^2 + (1-p)d^2}{pu + (1-p)d}$$

of the risky asset. The probability ϵ is assumed to be small enough such that

$$\widetilde{K}_t(\delta) = -\frac{D_{t+1}\epsilon w + C_{t+1}p(1-\epsilon)u + C_{t+1}(1-p)(1-\epsilon)d}{D_{t+1}\epsilon w^2 + C_{t+1}p(1-\epsilon)u^2 + C_{t+1}(1-p)(1-\epsilon)d^2} \in \left(-\frac{1}{u}, \widetilde{K}_t^b\right),$$
$$\widehat{K}_t^a(\delta) = -\frac{C_{t+1}\epsilon w + D_{t+1}p(1-\epsilon)u + D_{t+1}(1-p)(1-\epsilon)d}{C_{t+1}\epsilon w^2 + D_{t+1}p(1-\epsilon)u^2 + D_{t+1}(1-p)(1-\epsilon)d^2} \in \left(-\frac{1}{u}, -\frac{1}{d}\right),$$

in which case $\widetilde{K}_t$ is the optimizer of g_t and $\widehat{K}_t^a$ is the optimizer of h_t when $D_{t+1} < 0$. If D_{t+1} is zero we again take $\widehat{K}_t = 0$ following Remark 6. When $D_{t+1} > 0$, the optimizer of h_t is given by

$$\widehat{K}_t^s(\delta) = -\frac{D_{t+1}\epsilon w + D_{t+1}p(1-\epsilon)u + C_{t+1}(1-p)(1-\epsilon)d}{D_{t+1}\epsilon w^2 + D_{t+1}p(1-\epsilon)u^2 + C_{t+1}(1-p)(1-\epsilon)d^2} > -\frac{1}{d}.$$

The superscripts a and s indicate potential aversion and potential seeking, respectively, and we wrote $\widetilde{K}_t(\delta)$ and $\widehat{K}_t(\delta)$ in order to emphasize the dependence on δ as a preference parameter in contrast to the dependence on the market parameters ϵ, p and w, u, d.

Let us now consider the case of two trading periods ($T = 2$). We first compute the recursive constants starting from $C_2 = -1$ and $D_2 = -\delta$. We then have

$$C_1 = \left(\delta\epsilon w^2 + p(1-\epsilon)u^2 + (1-p)(1-\epsilon)d^2\right)^{-2} \times \Big(-\epsilon\delta\left(p(1-\epsilon)u(u-w) + (1-p)(1-\epsilon)d(d-w)\right)^2 - p(1-\epsilon)\left(\delta\epsilon w(w-u) + (1-p)(1-\epsilon)d(d-u)\right)^2 - (1-p)(1-\epsilon)\left(\delta\epsilon w(w-d) + p(1-\epsilon)u(u-d)\right)^2 \Big).$$

To compute D_1, we distinguish the cases $\delta > 0$, $\delta = 0$, and $\delta < 0$. When $\delta > 0$, $\widehat{K}_1^a(\delta)$ maximizes h_1 and thus

$$D_1^a = \left(\epsilon w^2 + \delta p(1-\epsilon)u^2 + \delta(1-p)(1-\epsilon)d^2\right)^{-2} \times \Big(-\epsilon\delta^2\left(p(1-\epsilon)u(u-w) + (1-p)(1-\epsilon)d(d-w)\right)^2 - \delta p(1-\epsilon)\left(\epsilon w(w-u) + \delta(1-p)(1-\epsilon)d(d-u)\right)^2 - \delta(1-p)(1-\epsilon)\left(\epsilon w(w-d) + \delta p(1-\epsilon)u(u-d)\right)^2 \Big).$$

When $\delta = 0$, then clearly $D_1 = 0$. Lastly, when $\delta < 0$, $\widehat{K}_1^s(\delta)$ maximizes h_1 and thus

$$D_1^s = \left(\delta\epsilon w^2 + \delta p(1-\epsilon)u^2 + (1-p)(1-\epsilon)d^2\right)^{-2} \times \Big(-\epsilon\delta\left(\delta p(1-\epsilon)u(u-w) + (1-p)(1-\epsilon)d(d-w)\right)^2 - \delta p(1-\epsilon)\left(\delta\epsilon w(w-u) + (1-p)(1-\epsilon)d(d-u)\right)^2 - \delta^2(1-p)(1-\epsilon)\left(\epsilon w(w-d) + p(1-\epsilon)u(u-d)\right)^2 \Big).$$

If we plug C_1 and D_1^a, respectively, D_1^s, back into $\widetilde{K}_0(\delta)$ and $\widehat{K}_0^a(\delta)$, respectively, $\widehat{K}_0^s(\delta)$, we obtain an analytical solution for the optimal trading strategy. The optimal terminal wealth can then be expressed explicitly as

$$X_2(\gamma, \delta) = \gamma + (x_0 - \gamma)S(\delta),$$

where

$$S(\delta) := \left(1 + \widetilde{K}_0(\delta)P_0\right)\left(1 + \left(\widetilde{K}_1(\delta)\mathbf{1}_{\{P_0 \in \{u,d\}\}} + \widehat{K}_1(\delta)\mathbf{1}_{\{P_0 = w\}}\right)P_1\right)$$

denotes the random component of the optimal terminal wealth. Note that $S(\delta)$ is independent of the initial and target wealth levels and only depends on the potential-aversion and market parameters. Given a target wealth γ_1 for a mean-variance investor, we can compute the corresponding target wealth $\gamma(\delta)$ for an investor with lower potential-aversion which achieves the same expected terminal wealth by

$$\gamma(\delta) = \frac{1}{\mathbb{E}[S(\delta)]}\Big(x_0\left(\mathbb{E}[S(1)] - \mathbb{E}[S(\delta)]\right) + \gamma_1\left(1 - \mathbb{E}[S(1)]\right)\Big).$$

In particular, since γ_1 is directly related to the expected terminal wealth of the mean-variance investor via (1), see also equation (26) in Li and Ng [3], we can determine $\gamma(\delta)$ explicitly in order to achieve a target expected wealth. Note that, whereas we can explicitly express the variance of the optimal terminal wealth as $\mathrm{Var}(X_2(\delta)) = (x_0 - \gamma(\delta))^2\,\mathrm{Var}\,(S(\delta))$, the expressions for the upper- and lower-standard deviation depend on a cumbersome case distinction for the market parameters. We thus instead focus on an explicit numerical example to analyze the effect of different degrees of potential-aversion on the distribution of the terminal wealth. In contrast to the lognormal case, all values can now be computed explicitly and do not rely on a random number generator. We consider a market where $w = 0.7$, $u = 0.1$, $d = -0.1$ and $\epsilon = 0.05$, $p = 0.6$.

Figure 4 shows the mean-risk and potential-mean frontiers for varying potential-aversion parameters δ. We consider the same values for δ as in Sect. 5 with the exception of the negative value. Since returns are bounded, we choose a smaller negative δ to better illustrate the results. The figure confirms the finding of the previous section, namely that for a given expected terminal wealth level, lowering the potential-aversion achieves simultaneously a lower risk and higher potential. The performance with respect to the mean-standard deviation trade-off decreases only minimally as long as potential-aversion is nonnegative but is considerable once it becomes negative mainly because of larger upper standard deviation. This is not surprising since there is only one risky stock and two trading periods. We refrain from plotting the mean-standard deviation frontier because the numerical differences are too small and a figure would thus not add any further insight.

7 Applications to Robo-Advising

Robo-advisors are digital wealth management services primarily targeted at retail investors. They represent a rapidly growing part of the wealth management industry with over 290 million investors and over \$1367 billion assets under management as of 2021 ([46]). In China alone, robo-advisors currently manage over \$91 billion of over 170 million investors ([47]).

All robo-advisors face an important problem: how to choose a decision-making framework to manage clients' assets. The current industry practice implicitly assumes that investors are mean-variance optimizers (Beketov et al. [48], D'Acunto et al. [49], Betterment [50], Wealthfront [51]). The mean-variance framework has some distinct advantages relevant for robo-advising applications. The framework has an intuitive

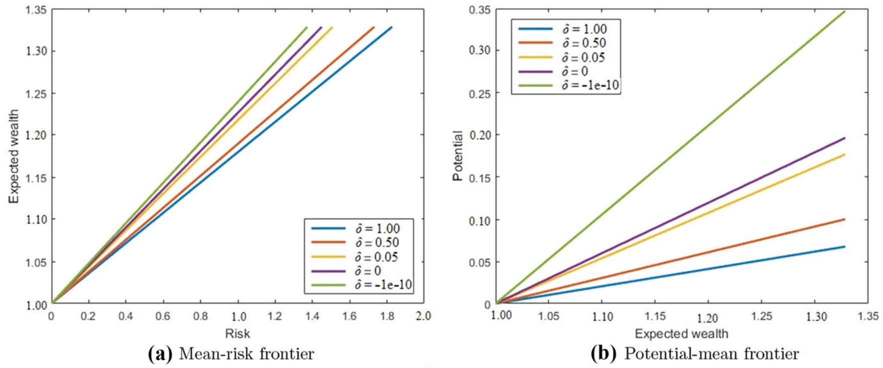


Fig. 4 Mean-risk and potential-mean frontiers of the terminal wealth distributions optimal for varying potential-aversion parameters δ in a trinomial model

appeal and is easy to understand, and the investment performance can be illustrated in terms of mean-variance diagrams. However, time-inconsistency and non-monotonicity of the mean-variance allocations are serious drawbacks. Surprisingly, there are so far only few related papers discussing the asset allocation aspect of robo-advising and proposing alternatives or amendments to the classical mean-variance setting. Capponi et al. [52] incorporated dynamically evolving and stochastic risk preferences of a client which are only communicated to the robo-advisor at specific times into a discrete-time mean-variance objective. They introduced a measure of portfolio personalization and characterize an optimal interaction frequency that balances a trade-off between uncertainty about the risk preferences of the agent and behavioral biases the agent has when communicating with the robo-advisor. Dai et al. [53] considered the mean-variance criterion for log-returns studied in Dai et al. [54] for robo-advising applications in continuous-time models. This framework notably does not suffer from time-inconsistency and does not short a risky asset with positive expected excess returns. Liang et al. [55] proposed to utilize the framework of predictable forward performance processes (Angoshtari et al. [56], Strub and Zhou [57]) to model preferences of clients of robo-advisors and determine an optimal interaction schedule that balances a trade-off between increasing uncertainty about the client's beliefs on the financial market and an interaction cost.

We provide a complementary perspective to existing works studying alternatives to the mean-variance framework in the robo-advising setting. The framework of mean-variance induced utility functions we introduced can be a viable alternative to mean-variance optimization used by robo-advisors. Our earlier analysis show our approach contains several desirable features for dynamic asset allocation problems. First, the framework puts forward an economically sound decision-making process that encourages diversification as the optimal strategies invest across the full range of available assets. The mean-variance induced utility functions overcome the issue of time-inconsistency and alleviate the non-monotonicity that make the mean-variance framework less economically appealing. Second, the optimal investment strategy can be readily computed in a dynamic setting. This feature has been clearly demonstrated in Sects. 5 and 6 for two simple models for the financial market, and it can be extended to more complex settings. Third, our framework allows for simple and intuitive graphic representations, such as Fig. 2, of the relationship between investor preferences and expected investment outcomes. These illustrations can help robo-advisors and their clients better understand the consequences of their decisions.

In order to use our proposed framework for investment decisions, the robo-advisor must elicit investor preferences. Although they are not a perfect tool, questionnaires have been commonly used to elicit investor preferences (Barsky et al. [58], Holt and Laury [59]). They are also primarily used by robo-advising firms to evaluate the risk preference of investors, see Alsabah et al. [60], which is also the first paper to study algorithmic aspects of robo-advising and employ machine learning methods to estimate investor risk preferences. For example, Wealthfront asks how investors would react to significant losses implied by a market decline and whether they are more concerned with maximizing gains, minimizing losses, or both equally.

One further advantage of the framework of mean-variance-induced utility functions is that its key parameters, i.e., the target wealth γ and potential-aversion δ, can be elicited through simple questions once the weighting parameter is set to the benchmark value of $\alpha = 2$. First, the client is asked to communicate her wealth target to the robo-advisor. This step determines γ. Second, the robo-advisor elicits the potential-aversion parameter δ. To facilitate the client's choice of the potential-aversion parameter, the robo-advisor illustrates its effect on resulting wealth distributions by exposing the client to the six diagrams in Fig. 2 and the histogram in Fig. 3. A larger potential-aversion reflects a client that cares less about overshooting the target wealth and instead places more importance on not falling short of it. This may be appropriate in cases where the client has a fixed future obligation such as a tuition payment or a down payment on a house. In other cases, for example for general long-term investing, the client might prefer positively skewed distributions that offer high potential. In such cases, a smaller or even negative potential-aversion parameter might be more appropriate.

8 Conclusion

In this paper, we introduce a family of mean-variance-induced utility functions which retain two attractive features of the mean-variance framework—an intuitive explanation of the objective and an easily-calculated optimal investment strategy. The utility functions are parameterized by a target wealth, a potential-aversion parameter, and a weighting parameter. Unlike the classical mean-variance objective, this framework naturally leads to time-consistent optimal strategies. Furthermore, issues related to non-monotonicity are alleviated with decreasing potential-aversion and completely resolved for non-positive values of potential-aversion. This framework naturally leads to two measures of deviation from a target wealth level: risk—the average weighted outcomes below the target wealth, and potential—the average weighted outcomes above the target wealth. Our numerical examples in a lognormal setting and the binomial model setting show that a lower potential-aversion parameter is associated with better portfolio performance in risk and potential, holding expected terminal wealth constant. We also illustrate how our framework is applicable to robo-advisors, and we develop an algorithm to elicit investor preferences.

In exploring our framework through the lognormal and binomial examples, we deliberately construct two contrived settings to clearly show the implications of our approach. One potentially interesting future research direction is to investigate our framework under more general assumptions about the underlying returns process. For example, it would be intriguing to see an asset allocation exercise involving a more complex returns process that incorporate empirical regularities such as time-series predictability or common factors. Richer asset price dynamics may allow for a more in-depth analysis of our framework compared to alternative approaches.

Our proposed framework can be applied to other investments or portfolio allocation settings. It may be fruitful to explore a dynamic multi-asset portfolio optimization problems using our family of asymmetric utility functions, which can provide a complementary perspective to existing papers that derive asset pricing equilibrium models using asymmetric preferences. Another potentially interesting idea is to calibrate our model using market data. The calibrated model parameters may reveal valuable information about investor behavior. We leave these questions to future research.

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