Processing Symbolic Numbers: The Example of Distance and Size Effects

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1 Introduction

In the last few decades, the most prevalent model of numerical cognition supposed that an evolutionarily ancient, simple representation is the basis of human numerical understanding. The model accounted for many phenomena described in the numerical cognition literature. Here, we present an alternative account, that proposes that, while nonsymbolic numerosities may be processed by this evolutionarily old system, the processing of symbolic numbers is supported by an architecture that is entirely different from the classic proposal and which alternative representation is more similar to mental conceptual networks or to the mental lexicon. With the example of distance and size effects in number comparison tasks we present several recently described phenomena supporting this alternative account.

Discussing these topics, the first section of this chapter summarizes the main features of the classic account and some of the relevant phenomena on which the model is based. Then, the second section introduces our alternative account for the same phenomena. Finally, the third section presents recently discovered phenomena that demonstrate why symbolic distance and size effects cannot be accounted for by the classic model, and how our alternative model can explain these effects.

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2 The Approximate Number System Account

In 1967, Moyer and Landauer published a relatively simple experiment in *Nature* that, despite its simplicity, profoundly changed how cognitive scientists thought about mathematical abilities. Before Moyer and Landauer's work, it was supposed that mathematics is a complicated subject for most people and that, from a cognitive perspective, numerical cognition is a high-level, human-specific, culture-dependent capability, which relies on complex mental processes. The work of Moyer and Landauer (1967) changed this view radically; they proposed that even symbolic number tasks may rely on a mechanism as simple as the perceptual representations described in psychophysics. In their experiment, it was found that, when participants compare single-digit numbers (e.g., "which one is larger, 5 or 8?"), the their performance depended on the ratio of the two numbers, with a behavioral pattern resembling the performance that the well-known Weber's law would generate. This meant that understanding even symbolic mathematics, which had been believed to be a high-level and culture-dependent cognitive process, may be supported by a very simple, evolutionarily old representation, which was later named the Approximate Number System (ANS).

In line with this discovery, many later studies demonstrated that the functioning of the ANS can be observed in young children or even infants (e.g., Izard, Sann, Spelke, & Streri, 2009), as well as in non-human animals (e.g., Hauser, 2000). Many further works proposed that the ANS played pivotal roles in many phenomena, such as the interference of numerical and spatial information (Dehaene, Bossini, & Giraux, 1993), approximate mathematical operations (Dehaene, Spelke, Pinel, Stanescu, & Tsivkin, 1999), understanding the cardinality principle (Piazza, 2010), developmental dyscalculia (Piazza et al., 2010), and math achievements (Halberda, Mazzocco, & Feigenson, 2008).

From the viewpoint of the present review, an important feature of the ANS is that while it is—as its name suggests—imprecise, it may also be the root of precise symbolic number processing, as demonstrated for example, in the original work of Moyer and Landauer (1967). Importantly, while the model assumes that the same kind of mechanisms support both imprecise nonsymbolic (such as arrays of dots, series of sounds, and baskets of marbles) processing and precise symbolic (such as Indo-Arabic digits, number words, and Roman numbers) processing, this does not necessarily mean that there is a single ANS behind such numerical tasks. It is possible that there is a noisier mechanism processing imprecise nonsymbolic stimuli, in addition to a less noisy mechanism processing precise symbolic stimuli (Dehaene, 1997, 2007; and see a model that is considered to be a possible implementation in Verguts & Fias, 2004).

With a simple implementation of the ANS, one may imagine that numbers are stored on a continuum, where the representation of the numbers are noisy (see Fig. 1). The noisy signal can be described as a Gaussian distribution, where the mean of that distribution is the to-be-represented value, and the dispersion of the distribution is proportional to the value (i.e., the larger the number, the noisier its representation is).

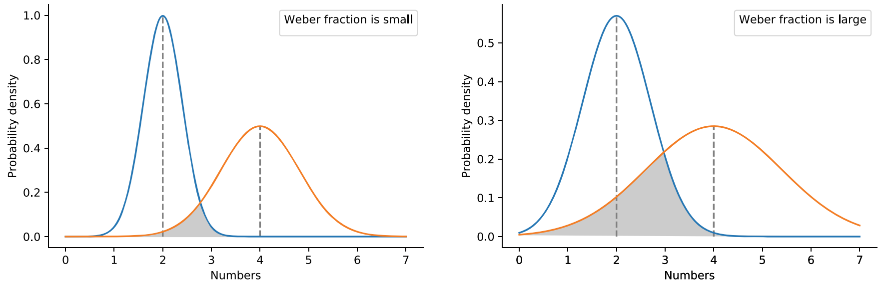

Fig. 1 Possible implementation of the ANS model, displaying the representation of number 2 and number 4 with different Weber fractions (i.e., different sensitivity; see the two panels)

This simple implementation can account for the ratio effect, which is observed in, for example, the number comparison task: The difficulty of the task depends on the overlap of the two numbers' noisy representation (see the more mathematical details of this explanation for example in Dehaene, 2007).

In this simple implementation of the ANS, the dispersion of the Gaussian distribution depends not only on the value to be represented (technically, this is the mean of the distribution) but also on another parameter—the sensitivity of the system—which is expressed as the Weber fraction of the representation (Dehaene, 2007) (see the left and right panels of Fig. 1 for representations with different Weber fractions). Technically, the standard deviation of the distribution is the product of the to-be-represented value (i.e., the mean of the distribution) and the Weber fraction. The Weber fraction can be responsible either for the individual differences of the system, which differences may partially explain, for example later math achievements (Halberda et al., 2008), or for the different sensitivity of the ANS for processing symbolic and nonsymbolic stimuli.

In the last few decades, the ANS has become the most frequently investigated representation that supports number processing and has become dominant in many aspects of numerical cognition, such as bases of adult numerical understanding, infant cognition, education, impairments of numerical abilities, and even animal cognition (Dehaene, 1992, 2007; Halberda et al., 2008; Piazza, 2010).

3 An Alternative Account: Discrete Semantic System

While the ANS model is an elegant and parsimonious solution that offers explanations for a series of phenomena, in the present section, we outline an alternative explanation. We hypothesize that, while the ANS may have a role in processing nonsymbolic numerosities, parallel symbolic numerical phenomena are handled by an entirely different architecture (Krajcsi, Lengyel, & Kojouharova, 2016).

Before we outline this alternative account, a technical detail should first be clarified. While the psychophysical model formulates the performance of a comparison

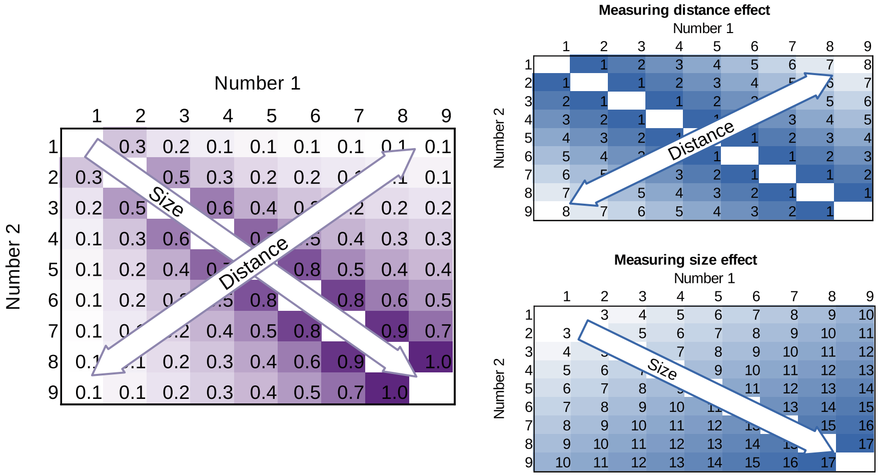

Fig. 2 Left panel. The ANS model predicted performance for all combinations of numbers in a single-digit comparison task. Values are measured on an arbitrary scale. More difficult number pairs are denoted by darker shades. Right panel. Distance and size effect regressors for measuring those effects in number comparison tasks

task as the ratio effect, technically, many numerical cognition studies have measured the distance effect (better behavioral performance with a larger distance between the two numbers) or, occasionally, the size effect (better behavioral performance with smaller numbers). Importantly, the distance and size effects are considered to be alternative approximate measurements of the ratio effect. To understand more accurately how the distance and size effects may reflect the underlying ratio effect, the behavioral pattern shown in the left panel of Fig. 2 must be considered. The figure displays the performance of a comparison task predicted by the ANS model, where the rows and columns denote the two numbers to be compared, and the values and corresponding shadings of the cells display the difficulty of the task (darker cells denoting worse performance). The figure shows a diagonal necktie-shaped pattern, which reflects the ratio effect, which effect is based on the psychophysical model. In this context, many studies measure the distance and size effects based on the regressors displayed in the right panel of Fig. 2, where data cells with the same regressor values are collapsed and where performance change is measured as a function of distance or size. In the ANS model, the distance and size effects are no more than two additional ways to measure the ratio effect; no matter which one is measured, they reflect the same phenomena. The relation of the distance and size effects with the ratio effect is important in not only the ANS model but also our alternative explanation.

Turning to a possible alternative explanation of the relevant symbolic numerical effects, it is important to highlight that, whenever the distance effect can be observed in a numerical task, the literature assumes that it is a sign of the ANS. In other words, it is supposed that it is only the ANS that could generate a distance effect in a numerical task. However, this may not always be true. For example, a semantic distance-based effect was also observed in a linguistic task: In a picture-naming task, the size of the priming effect of the previous picture was proportional to the semantic distance between the priming and target pictures (Vigliocco, Vinson, Damian, & Levelt, 2002). This semantic distance effect is conceptually and technically very similar to the numerical distance effect, because the numerical distance effect is, in fact, a gradual effect, where the effect size depends on the semantic distance between the stimuli (i.e., on the numerical distance). Importantly, in the linguistic task, it is most likely not a single (or several) continuous dimension that generates the performance pattern, like in the case of the ANS, but rather it may be a network of nodes (words, concepts, etc.) with specific connections. Overall, in contrast with the presumption of the current view in the numerical cognition literature, a continuous representation, such as the ANS, is not the only representation that can be the source of a numerical distance effect, where some other architectures can also produce the effect, for example a network of nodes.

Based on the idea of this potential alternative generator, our research group proposed a comprehensive alternative account for symbolic number processing, which tries to account for all the phenomena that the ANS accounts for (Krajcsi et al., 2016). We once more point out that this alternative explanation deals only with symbolic numbers and not with nonsymbolic numerosities. We discuss below more details of the representation in this alternative framework that supports nonsymbolic numerosities.

In the Discrete Semantic System (DSS) model, symbolic numbers are represented by nodes and their connections (see the lower part of Fig. 3). (Note that the word ‘semantic’ in the name of the model refers to the fact that the model is similar to other conceptual or linguistic models accounting for semantic representations; it does not necessarily mean that this is the system that holds the meaning of the numbers: The effects that the model accounts for may be simply rooted in simple statistical properties of the numerical stimuli. See examples of the stimulus statistics-based phenomena below.) Single digits, number words, and special multi-digit numbers could be nodes in the network. Connections may depend on the semantic or statistical

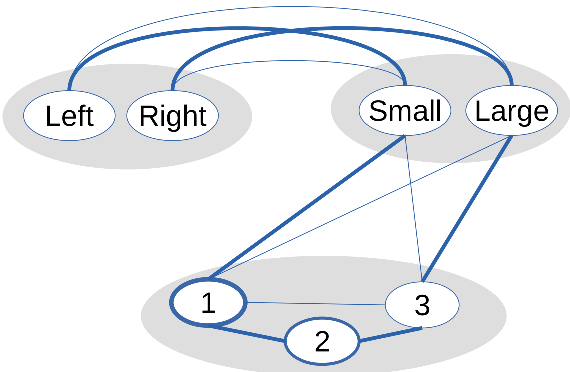

Fig. 3 Schematic example of the Discrete Semantic System. Width of the lines between the nodes reflect the strength of the connections. Width of the node outlines reflect the frequency of those symbols

(in the sense of environmental occurrence) properties of the nodes. For example, connection strength may depend on semantic relations (e.g., numbers with a smaller distance are connected more strongly, or even numbers have stronger connections with each other), while some of the semantic relations may also be based on non-mathematical properties (e.g., lucky numbers). Some other relations may depend on statistical correlations of the stimuli (e.g., if two numbers are observed together more frequently compared to other number pairs, their relation may become stronger). This statistical property can be related to semantic properties (e.g., numbers close to each other are more likely to be mentioned frequently together, such as in a counting list), but it may also be independent of semantic properties (e.g., numbers learned by roulette or darts players based on the locations of different values on the roulette wheel or on the dartboard). The network does not necessarily include only numbers; it may also include other related concepts, such as “small”, “large”, “even”, and “odd” (see the upper part of Fig. 3). (In a different conceptualization, one may imagine that the whole network is only a part of the conceptual system or the mental lexicon.) This architecture is simply the architecture of various conceptual networks or mental lexicon models, which are applied to symbolic numbers.

We propose that this simple architecture can account for all the phenomena in symbolic numbers that the ANS is supposed to account for. Because the ratio effect is the most frequently referenced defining feature of the ANS, we first discuss how the apparent ratio effect can be accounted for in the DSS model. In this model, comparison distance and size effects are two independent effects with two independent sources, and they form an illusionary ratio effect, which, in fact, has nothing to do with the psychophysical ratio effect. First, in the DSS model, the size effect is a frequency effect. In everyday life, smaller numbers are more frequent than larger numbers (Dehaene & Mehler, 1992). Both in linguistic models and in many other areas, stimulus frequency heavily influences the processing speed; for example, word frequency is one of the strongest predictors of word reading time. Similarly, it is possible that smaller numbers are easier to process than larger numbers, because they are more common, which produces the size effect. Second, the distance effect may be related to the value or relative order of the numbers. According to one possibility, smaller digits are more strongly associated with the “small” node than larger digits, and larger digits are more strongly associated with the “large” node than smaller digits (see Fig. 3). In this configuration, when two numbers are compared, numbers with a larger distance are easier to compare, because they have more differing associations with the “large” and “small” nodes than number pairs with a smaller distance. Alternatively, one may imagine that numbers with a smaller numerical distance have a stronger connection than numbers with a larger distance (e.g., 3 has a stronger connection with 4 than with 7; see also Fig. 3). In this explanation, the distance effect could be the result of the spreading activation between the nodes: Numbers with a stronger connection (and with a smaller distance) between them may cause more interference, which produces the distance effect. No matter which explanation could be the appropriate one, the important point is that the DSS architecture can offer a mechanism to account for the distance effect.

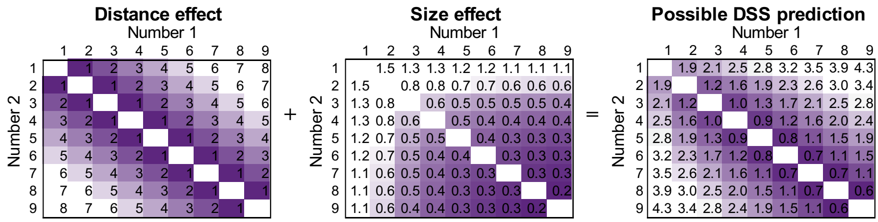

Fig. 4 Predicted symbolic comparison task distance effect performance (left) and size effect performance (middle) and their sum (right) in the DSS model. Values are measured on an arbitrary scale. See more technical details of the quantitative description in Krajcsi et al. (2016)

A final step to account for the apparent ratio effect in symbolic number comparisons is to combine the size and distance effects in the DSS model. Adding the two effects together results in a very similar pattern to the one seen in the ANS model (see the distance and size effect components and their sum in Fig. 4, and contrast this with the DSS description of the effect with the ANS description displayed in the left panel of Fig. 2). (See more details in Krajcsi et al., 2016 about how hypothetical quantitative descriptions of the DSS could be formed.) The similarity of the DSS prediction and the ANS prediction can be captured in several ways. For example, the correlation of the cells in the two models in one-digit comparison tasks is high: The exact value depends on the exact formulation of the model predictions, and it is $r = 0.89$ in the versions shown in Figs. 2 and 4. Another way to demonstrate the similarities of the two models is to contrast them as predictors of empirical behavioral performance in comparison task. In such a study, we found that it is practically impossible to find which model predicts behavioral data better, because the difference of the models is smaller than typical noise in the measured data (Krajcsi et al., 2016, Experiment 1).

While, in the present review, we focus on the essential comparison distance and size effects, the DSS model can explain several other phenomena that have been attributed to the ANS model. For example, the interference of numerical and spatial information was originally explained by the ANS model, where the spatial property of the ANS interferes with spatial representation (Dehaene et al., 1993), but it can also emerge in the DSS model, where "small"–"large" nodes are connected to the "left"– "right" nodes, respectively (see the upper part of Fig. 3) (Krajcsi, Lengyel, & Laczkó, 2018). (Note that this latter explanation is in line with other linguistic based accounts of the numerical interference effects, but those models were not intended to account for a broader range of numerical effects.) See additional examples of how the DSS can account for symbolic numerical effects in Krajcsi et al. (2016).

As it was mentioned above, in this framework, we hypothesize that nonsymbolic stimuli are still processed by the ANS (see Fig. 5 in the next section). The difference between the classic ANS account and the alternative DSS account can be emphasized by highlighting that, while the classic view believes that both symbolic and nonsymbolic stimuli are processed by the ANS (which we term the pure ANS account), our alternative explanation supposes that it is only the nonsymbolic stimuli that are processed by the ANS and that symbolic stimuli are handled by the DSS (which we

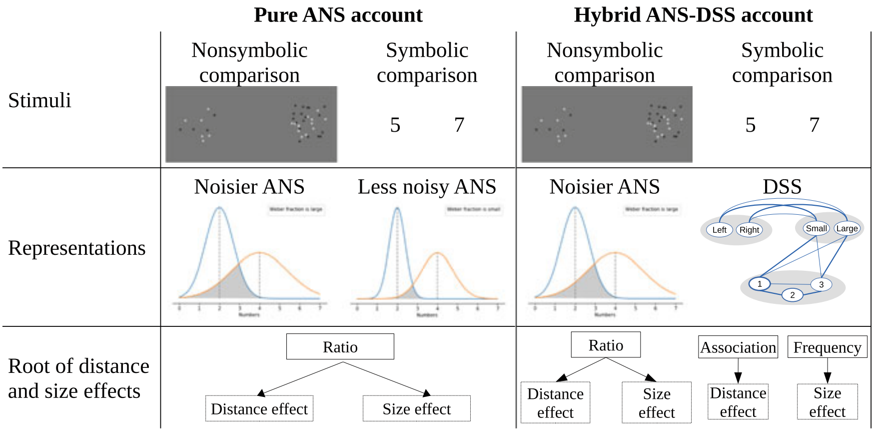

Fig. 5 Summary of the pure ANS and the hybrid ANS-DSS accounts for comparison tasks

term the hybrid ANS–DSS account). Note that the pure ANS account supposes that, while both symbolic and nonsymbolic numerical information is processed by the ANS, the account does not propose a single ANS, but two instantiations of the ANS, and the two instantiations have different Weber fractions (see Fig. 5).

In summary, we propose an alternative account for symbolic number processing phenomena, which were formerly attributed to the ANS. We suggest that a simple architecture comprising a network of nodes can account for those phenomena both qualitatively and quantitatively, as demonstrated in the example of the comparison numerical distance and size effects. Overall, the DSS model offers as an appropriate prediction for the numerical effects as the ANS model.

4 Contrasting the Two Accounts with New Phenomena

The previous section explained that the DSS model can account for symbolic number processing as appropriately as can the ANS model. More generally, we found that the two models produce the same predictions for almost all of the known phenomena. While this explanatory equality of the models means that the DSS model is a viable option, it also introduces a new challenge: It is not possible to evaluate the two models and determine which serves as a better account of those phenomena.

To overcome this issue, we designed several new tests to contrast the two models. These studies were designed to reveal phenomena for which the two models have different predictions. In this section, we focus on recent results that contrast the predictions of the two models in a comparison task measuring the distance and size effects. Testing the comparison distance and size effects is essential, because, in the ANS model, the ratio effect reflects the defining feature of the ANS (i.e., the functioning according to Weber's law). Most additional pieces of evidence that seemingly support the ANS model are either other examples of the ratio effect or direct consequences of Weber's law-based functioning. To summarize the main results of our tests in advance, all of them are so far only in line with the DSS model, and they reveal new details of the comparison task that cannot be reconciled with the ANS model.

Figure 5 summarizes the two accounts to be contrasted and their main assumptions about the comparison distance and size effects in symbolic and nonsymbolic comparison tasks. While the pure ANS account supposes that both symbolic and nonsymbolic comparisons are handled by the ANS (although with different Weber fractions), the hybrid ANS–DSS account supposes that nonsymbolic comparison is handled by the ANS, and that symbolic comparisons are processed by the DSS. In addition, the ANS model presumes that the distance and size effects are no more than two different ways to measure the ratio effect, while the DSS model proposes that the size effect depends on the frequency of the symbols and that the distance effect depends on the associations of the symbols.

In a series of empirical works (see a summary of the findings in Table 1), first, it was demonstrated that, in a symbolic comparison task, the size effect is a frequency effect (Kojouharova & Krajcsi, 2019; Krajcsi et al., 2016) (see the first row of Table 1). In these two studies, the frequency of the stimuli was manipulated, and the size

Table 1 Properties of the distance and size effects in symbolic and nonsymbolic comparison tasks. Flexibility of the nonsymbolic effects has not yet been tested, as noted by the question mark. See Fig. 5 for the predictions of the pure ANS and the hybrid ANS-DSS accounts

	Symbolic		Nonsymbolic	
	Distance effect	Size effect	Distance effect	Size effect
Source of the effect	Large-small association of the numbers (Kojouharova & Krajcsi, 2018; Krajcsi & Kojouharova, 2017)	Frequency of the symbols (Kojouharova & Krajcsi, 2019; Krajcsi et al., 2016)	Psychophysical model: Ratio effect according to Weber's law (Krajcsi, Lengyel, & Kojouharova, 2018)	
Independence (Kojouharova & Krajcsi, 2018, 2019; Krajcsi & Kojouharova, 2017; Krajcsi et al., 2016)	Dissociation of the effects		No dissociation of the effects is observed	
Correlation of the slopes (Krajcsi, 2017)	Independent		Strongly correlate	
Flexibility for the statistics of the stimuli	Highly flexible (Kojouharova & Krajcsi, 2018)	Moderately flexible (Kojouharova & Krajcsi, 2019)	Rigid?	

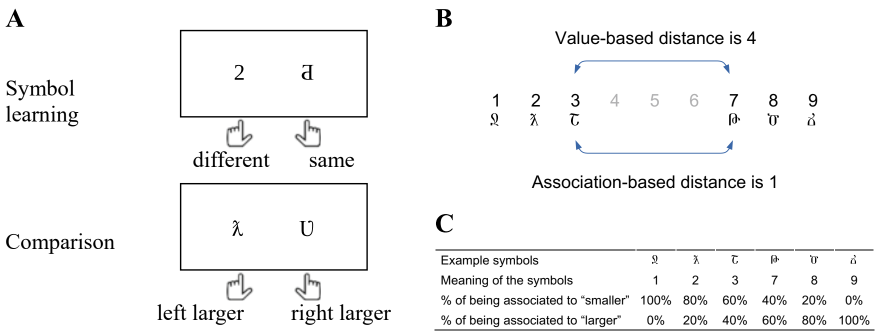

Fig. 6 **A** Two tasks of the new symbol comparison paradigm. After seeing a list of new symbols and their meaning in Indo–Arabic numbers, participants practiced understanding the symbols in the symbol learning task and then compared numbers in the new symbol notation. **B** Numbers between 1 and 3 and 7 and 9 (i.e., a gap between 3 and 7) were used in the comparison task. If the distance effect is based on the values of the numbers, then the gap should be four-units long. If the distance effect is based on the symbol frequency (see Panel C), then the gap should be one-unit long. **C** When numbers with a gap are presented with equal frequencies in a comparison task, the proportion of being a number "smaller" and "larger" is shown. The proportions are continuously increasing/decreasing across the gap

effect changed accordingly. In the first study, the participants learned new symbols and, after learning their meaning, they performed comparison tasks (see the summary of the paradigm in Fig. 6A). It was supposed that, if the new symbols do not take the frequency information of the Indo–Arabic numbers (or another symbolic notation), which precondition proved to be true, then the size effect may reflect purely the frequency of the stimuli of the session. The results confirmed this supposition: Number stimuli following an everyday number frequency (Dehaene & Mehler, 1992) caused a regular size effect, and a uniform number frequency removed the size effect (Krajcsi et al., 2016). This result is in line with the DSS model, but it is in contrast with the ANS model, which proposes that the size effect is rooted in the ratio of the values, independent of the frequencies of the stimuli. In a subsequent study, the same manipulation of stimulus frequency was applied to Indo–Arabic numbers. It was found that, extending the first findings, stimulus frequency also changed the size effect in Indo–Arabic numbers, although the frequency manipulation did not entirely change the size effect; rather, the observed size effect was a combination of the participants' previous experience with the Indo–Arabic numbers and the actual session statistics (Kojouharova & Krajcsi, 2019).

Second, it was demonstrated that, in a symbolic comparison task, the distance effect follows the association of the digits with the "large" and "small" properties (see the first raw of Table 1) (Kojouharova & Krajcsi, 2018; Krajcsi & Kojouharova, 2017). To contrast whether the distance effect is rooted in the value of the digits (prediction of the ANS) or in the association of the digits with "large" and "small" properties (prediction of the DSS model), participants compared single-digit numbers, where numbers 4, 5, and 6 were omitted, creating a gap in the range (see Fig. 6B). If the distance effect depends on the values of the numbers, then the distance between the two sides of the gap measured empirically should show a three-unit distance. However, if the distance effect depends on the association of the digits and the "large"– "small" properties, then in a comparison task where all number pairs are presented uniformly, the distance between the two sides of the gap measured empirically should show a one-unit distance (see the explanation in Fig. 6B, C). It was found both in new artificial and Indo–Arabic notations that the distance between the two sides of the gap was one-unit long, supporting the DSS model (Kojouharova & Krajcsi, 2018; Krajcsi & Kojouharova, 2017). In addition, in the Indo–Arabic notation, the distance effect entirely followed the association statistics of the session, and it was not a combination of the current session statistic and the previous experience, as it was observed in the frequency-based size effect. These results further confirm that the symbolic number comparison distance effect cannot be accounted for by the ANS model, because the distance effect is not driven by the values of the numbers, but by the associations between the digits and the "large"– "small" properties.

Third, in a different type of study, the psychophysical model was tested to determine whether it can describe symbolic and nonsymbolic comparisons equally well (Krajcsi, Lengyel, & Kojouharova, 2018). While this model has been applied to both symbolic and nonsymbolic comparison tasks, and it has repeatedly been found to be appropriate for both types of notations (e.g., Dehaene, 2007; Moyer & Landauer, 1967), our study included more extensive tests. One such test investigated whether the psychophysical model describes the drift rates of the comparisons correctly in the diffusion model framework. The diffusion model supposes that, in a trial, evidence is accumulated until a prespecified threshold is reached (Ratcliff & McKoon, 2008; Smith & Ratcliff, 2004). Drift rate is the average amount of evidence that is accumulated in a time unit, and it can be considered as the difficulty of the task or the efficiency of the appropriate representation that is responsible for the property on which the decision is based. See an introduction for more details about the diffusion model in Ratcliff and McKoon (2008) and Smith and Ratcliff (2004). This model successfully explains many phenomena related to the reaction time and error rate distribution within a participant (Ratcliff & McKoon, 2008; Smith & Ratcliff, 2004). In the ANS model, as in similar psychophysical models, it is assumed that, as the two to-be-compared values get closer (in terms of ratio), the drift rate approaches zero (Dehaene, 2007; Palmer, Huk, & Shadlen, 2005). For example, in a nonsymbolic dot comparison task, as the two values become more and more similar, the participant becomes unsure which array has more items. Our analysis revealed that, in the nonsymbolic comparison task, the drift rate indeed approached zero as the task became more difficult. However, in the symbolic comparison task, the drift rate approached a non-zero value. This latter result is in line with the commonsense observation that, even if comparing two symbolic numbers gets harder as the distance (or ratio) decreases, it never becomes impossible to solve (unlike for a nonsymbolic comparison task). The results further reveal that, while nonsymbolic comparison works according to the psychophysical ANS model, symbolic comparison works differently, and the psychophysical model cannot describe it correctly (see the first row of Table 1).

Fourth, related to the previous properties and findings, we may state that, in the symbolic comparison task, the distance and size effects dissociate (see the second row of Table 1). In the studies described above, where size effects were manipulated by utilizing different frequencies of the numbers, the size effect changed according to the stimulus frequency, while the distance effect was not influenced (Kojouharova & Krajcsi, 2019; Krajcsi et al., 2016). Similarly, in the studies where the distance effect was manipulated by presenting a number range with a gap, the size effect still depended on the frequencies of the digit (Kojouharova & Krajcsi, 2018; Krajcsi & Kojouharova, 2017). Thus, unlike the prediction of the ANS model, distance and size effects can be modified and manipulated independently.

Fifth, related to the independence and dissociation of the distance and size effects described in the previous point, the two accounts have different predictions for the correlation between the slopes of the distance and size effects (see the third row of Table 1). In the ANS model, the slopes of the distance and size effects are influenced only by the ratio effect slope, which is, in turn, influenced only by the Weber fraction (Dehaene, 2007). (This is true even if the slopes of the ratio, distance, and size effects show some non-monotonic relation with the Weber fraction, see Chesney, 2018; and Krajcsi, 2020.) In other words, in the ANS model, if the distance and size effects are measured independently, then the same construct is actually measured in two different ways, and the two indexes should correlate perfectly (supposing perfect reliability of the measurements). In contrast, according to the hybrid ANS–DSS account, this perfect correlation is only expected in nonsymbolic comparison, while symbolic comparison distance and size effects may be independent (see again Fig. 5). In a study measuring the correlation between the slope of the distance effect and the slope of the size effect in nonsymbolic and symbolic comparison tasks, it was found that, in nonsymbolic comparison after correcting for the reliability-caused attenuation of correlation (i.e., calculating the correlation as if the reliability was perfect), the correlation was practically 1, on the other hand, in symbolic comparison, the same correlation was not different from 0 (Krajcsi, 2017) (where the original uncorrected correlations were around 0.9 in the nonsymbolic, and 0.1 in the symbolic comparison tasks, and the reliabilities of the indexes were between 0.71 and 0.94: therefore, utilizing the corrections was reasonable). This result further makes the case for different types of mechanisms behind symbolic and nonsymbolic comparison in line with the hybrid ANS–DSS account.

Sixth, the aforementioned flexibility of the symbolic distance and size effects is also informative in the present investigation. As described above, the symbolic distance effect proved to be highly flexible, almost entirely relying on the stimulus statistics of the session and ignoring former experience (Kojouharova & Krajcsi, 2018), while the symbolic size effect is less flexible, combining the statistics of the actual session and former experience (Kojouharova & Krajcsi, 2019). In the pure ANS account the two effects should show the same flexibility (and they should not be modified by those stimulus statistics in the first place), which further suggests that symbolic numbers are compared by the DSS.

Overall, these results show that (a) the properties of the symbolic comparison and the nonsymbolic comparison differ, and (b) while the properties of the distance and the size effects are similar within a nonsymbolic comparison task, the properties of the two effects are different within a symbolic comparison task (see Table 1). Both of these findings are in line only with the hybrid ANS–DSS account and not with the pure ANS account.

5 Conclusions

In the last few decades, the ANS model has been the dominant account in explaining many phenomena related to numerical understanding. Here, we briefly described an alternative model (DSS) with an entirely different architecture, which is also capable of explaining the symbolic phenomena that has been attributed to the ANS. The current review summarized several recently described effects in the number comparison task, which reveals a defining feature of the ANS. None of these effects can be explained by the classic pure ANS account, while all of them can be explained by our alternative hybrid ANS–DSS account.

It is essential to highlight that, while the ANS representation was believed to be one of the main sources of numerical meaning, the DSS representation can be considered a simple associative network, where the critical effects discussed in the literature simply reflect statistical features or correlations of the stimuli without referring to the meaning of the numerical information. In other words, the DSS may not be the primary source of numerical meaning. This property has at least two important consequences. First, many of the effects that the literature has been investigating in recent decades, such as the distance and size effects or the spatial-numerical effect, in fact, do not reflect a meaningful understanding and semantic processing of the numerical information. Second, and as a consequence, a large part of the meaning and semantic processing should be handled by other representations, which should be investigated via other effects and phenomena. Recognizing that the symbolic comparison distance and size effects, and related phenomena are not processed by the ANS, but by the DSS, opens the possibility to look for other phenomena and representations that may account for meaningful mathematical understanding.

This alternative account is in line with several similar alternative models (e.g., Leth-Steensen, Lucas, & Petrusic, 2011; Pinhas & Tzelgov, 2012; Proctor & Cho, 2006; Tzelgov, Ganor-Stern, & Maymon-Schreiber, 2009; Verguts & Van Opstal, 2014; Verguts, Fias, & Stevens, 2005), although the scope of these models is less comprehensive than the intended scope of the DSS model. Also, the present model extends and specifies the ideas of many recent works suggesting that symbolic and nonsymbolic numerical information is processed differently (e.g., Leibovich & Ansari, 2016; Lyons, Ansari, & Beilock, 2015; Noël & Rousselle, 2011; Sasanguie, Defever, Maertens, & Reynvoet, 2014; Schneider et al., 2017). Our proposal, together with the increasing number of publications discussing the limitations of the ANS model and offering solutions for some of the issues, can form a new framework of numerical cognition, which framework can integrate the results from recent years more successfully than the current dominant view of number processing and can open the way to discover additional essential representations that support mathematical understanding.

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