Optimal Regret and Hard Violation for Constrained Markov Decision Processes with Adversarial Losses and Constraints

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Abstract

We investigate online learning in finite-horizon episodic Constrained Markov Decision Processes (CMDPs) under the most demanding setting: adversarial losses and constraints, bandit feedback, and unknown transitions. The most popular approaches, like primal-dual or linear programming, either rely on Slater's condition (yielding occasionally vacuous bounds) or require solving a complex optimization problem every round. Inspired by the groundbreaking work of Sinha & Vaze (2024) in Constrained Online Convex Optimization (COCO), we map the CMDP instances to a corresponding COCO problem. Thus, creating simple and elegant algorithms that require only a single Euclidean projection per episode. Our algorithm first attains $\widetilde{\mathcal{O}}(\sqrt{T})$ regret and $\widetilde{\mathcal{O}}(\sqrt{T})$ hard cumulative constraint violation for adversarial losses and constraints, unknown transition dynamics, bandit feedback, without Slater's condition and also without access to a strictly feasible policy. We achieve $\mathcal{O}(\sqrt{T})$ regret and $\widetilde{\mathcal{O}}(\sqrt{T})$ hard violation for known transitions. Additionally, we study the remaining three permutations of known-unknown transitions and full-bandit feedback, again achieving optimal regret and hard violation bounds in each case. Besides closing several gaps in the literature, our simple construction of biased estimators for the sub-gradient could be of independent interest for didactic purposes.

1 Introduction

The arrival of AlphaGo (Silver et al., 2017) ignited an unprecedented curiosity about the capabilities of Reinforcement Learning (RL) (Sutton & Barto, 2018) among researchers. Numerous works highlight that RL is remarkably effective across multiple domains, including games (Jaderberg et al., 2019; Mathieu et al., 2023), robotic locomotion (Smith et al., 2024), control (Hegde et al., 2024; Du et al., 2023), and Large Language Models (LLMs) such as GPT-4 (OpenAI et al., 2024) and DeepSeek-V3 (DeepSeek-AI et al., 2024). Quite naturally, a comprehensive understanding of Markov Decision Processes (MDPs) (Puterman, 2014) is essential, as they lie at the core of any RL problem. In other words, RL seeks to address a sequential decision-making problem by learning an optimal policy; thus, MDPs are utilized for modeling any RL task. The final goal in vanilla RL is to discover a policy that maximizes the expected cumulative sum of rewards. However, in many real-world scenarios, such as self-driving cars and recommender systems, the agent is often required to satisfy both safety and budget constraints in addition to maximizing reward. For instance, autonomous vehicles should not meet with an accident or crash (Wen et al., 2020), and bidding parties in an auction cannot exceed a budget (He et al., 2021). To address such scenarios, the Constrained Markov Decision Process (CMDP) (Altman, 1999) serves as an excellent tool, as it naturally incorporates constraints within the classical MDP framework. In contrast to MDPs, the objective in CMDPs is to learn a policy that maximizes the expected cumulative reward, subject to satisfying the constraints.

Online learning in finite-horizon episodic CMDPs, a topic that has long piqued the interest of the community (Wei et al., 2018; Efroni et al., 2020; Müller et al., 2024), is the central theme of our work. This setting necessitates that the learner's objective be to minimize both the *regret* and the *cumulative constraint violation* (also referred to as *violation* for brevity). The regret quantifies the difference between the cumulative loss of the learner and that of the optimal policy. To be specific, the optimal policy is the best-in-hindsight policy that satisfies the constraints during the learning process. On the other hand, the cumulative constraint violation tracks the total sum of constraint violations across all episodes. Both the regret and the cumulative violation should ideally be sublinear in T, i.e., the total number of episodes. We mention specific directions from the vast literature of online learning in CMDPs (see Section 2 for detailed related works) that have been instrumental in motivating this paper:

1. Hard/Soft Violations: Many works on CMDPs are bothered with *soft constraint violations* (Efroni et al., 2020; Qiu et al., 2020), in which the effect of the positive violations is nullified (or diminished) by the negative ones across the whole learning process (Ghosh et al., 2022; Wei et al., 2023). Such nullifications are absolutely impractical in real-world environments. On the contrary, *hard constraint violations* (Stradi et al., 2025b) are a significantly stronger and practical constraint violation condition that solely cares about the positive violations. An example: let a CMDP model a clinical trial for a newly discovered drug, where each episode represents treating a patient. The aim is to minimize disease symptoms, and the constraint is to keep the probability of causing a severity below 1%. Say, in the first episode, the drug causes a hemorrhage to the patient, incurring a massive constraint violation of $+0.99$ above the threshold. In the second episode, imagine the drug works safely on the patient, receiving a negative violation of -0.01. The cumulative soft violation over these two episodes is $0.99 + (-0.01) = 0.98$, which seems to improve from the first episode. However, the hemorrhage caused in the first episode is irreversible and catastrophic. In contrast, hard violation would have only counted the positive violations: $0.99 + 0 = 0.99$. Thus, correctly identifying that the drug was unsafe for the patient, and the harm caused in an episode can never be compensated for by good performance in subsequent episodes.

2. Adversarial/Stochastic Loss and Constraints: A critical aspect of online learning in CMDPs is the factor of how the losses (or rewards) and constraints are chosen in each episode – stochastically or adversarially? If the choice is made stochastically, then the losses and/or constraints are selected by sampling from an unknown and stationary probability distribution. In the adversarial case, there is no statistical assumption on the selection, and the adversary has complete freedom. Hence, it is widely acknowledged that CMDPs with adversarial losses and constraints are much more complex to solve than their stochastic counterparts. There exists a plethora of seminal works in the literature that deal with stochastic losses and constraints (Zheng & Ratliff, 2020; Efroni et al., 2020), adversarial losses and stochastic constraints (Wei et al., 2018; Qiu et al., 2020). The works of Germano et al. (2023) and Stradi et al. (2024b) were among the first ones to provide regret and violation bounds for adversarial constraints, but with a dependence on the Slater condition.

3. Bandit/Full Feedback: The feedback received at the end of an episode for the losses and constraints is another crucial component for online learning in CMDPs. In the *full feedback* case (Wei et al., 2018; Qiu et al., 2020), the loss and constraint costs for all the possible state-action pairs are revealed to the learner when an episode ends. While in *bandit feedback* (Müller et al., 2023; Müller et al., 2024), the loss and constraint costs for only those state-action pairs are given that the learner had visited on that specific episode. It is naturally understood that working with bandit feedback is significantly more challenging than working with full feedback. Moreover, such settings can naturally capture the whole essence of numerous real-life problems, e.g., recommender systems and budget depletion in online bidding.

From the above points 1, 2, and 3, we highlight some gaps that are omnipresent in the literature of online learning in CMDPs. We discuss them one-by-one: (G1) Several approaches have been employed to bound the regret and violation for online learning in CMDPs, e.g., linear programming (Efroni et al., 2020), upper confidence (Zheng & Ratliff, 2020), and primal-dual (Stradi et al., 2024a;b; 2025a; Müller et al., 2024). Primal-dual-based algorithms have arguably gained the most prominence over the years. However, these methods rely on Slater's condition, which assumes the existence of a policy satisfying all constraints with at least $\xi > 0$ slackness (Stradi et al., 2025b; Germano et al., 2023). The guarantees of such algorithms scale with $\frac{1}{\xi}$, leading to vacuous bounds (i.e., huge sub-optimal bounds), if ξ is very small. Moreover, assuming Slater's condition is highly impractical because it requires prior knowledge of a strictly feasible policy or its slackness parameter, an information that is rarely available in real-world problems; (G2) A large portion of

Table 1: Detailed comparison of our theoretical results with the current best state-of-the-art methods. The symbol $\perp$ marks those works that consider the easier setup of stochastic losses (or rewards) and constraints. $\top$ denotes the work with adversarial losses and stochastic constraints. Zhu et al. (2025) is marked by ‡ to denote that it deals with bandit feedback for stochastic losses and full feedback for adversarial constraints. All the works reported in the table deal with hard violations. "`F/B`" is a shorthand for "`Full/Bandit`".

State-of-the-art	Transition	Feedback	Regret	Violation	With Slater
Kitamura et al. (2024)	`Known`	`F/B`	✗/✗	✗/✗	NA
	`Unknown`	`F/B`	$\widetilde{\mathcal{O}}(T^{6/7})^{\perp}$/✗	$\widetilde{\mathcal{O}}(T^{6/7})^{\perp}$/✗	✓
Müller et al. (2024)	`Known`	`F/B`	✗/✗	✗/✗	NA
	`Unknown`	`F/B`	✗/$\widetilde{\mathcal{O}}(T^{0.93})^{\perp}$	✗/$\widetilde{\mathcal{O}}(T^{0.93})^{\perp}$	✓
Zhu et al. (2025)	`Known`	`F/B`	✗/✗	✗/✗	NA
	`Unknown`	`F/B`	✗/$\widetilde{\mathcal{O}}(\sqrt{T})^{\ddagger}$	$\widetilde{\mathcal{O}}(\sqrt{T})^{\ddagger}$/✗	✗
Stradi et al. (2025a)	`Known`	`F/B`	✗/✗	✗/✗	NA
	`Unknown`	`F/B`	✗/$\widetilde{\mathcal{O}}(\sqrt{T})^{\perp}$	✗/$\widetilde{\mathcal{O}}(\sqrt{T})^{\perp}$	✓
Stradi et al. (2025b)	`Known`	`F/B`	✗/✗	✗/✗	NA
	`Unknown`	`F/B`	✗/$\widetilde{\mathcal{O}}(\sqrt{T})^{\top}$	✗/$\widetilde{\mathcal{O}}(\sqrt{T})^{\top}$	✓
This Work	**`Known`**	**`F/B`**	$\mathcal{O}(\sqrt{T})/\mathcal{O}(\sqrt{T})$	$\widetilde{\mathcal{O}}(\sqrt{T})/\widetilde{\mathcal{O}}(\sqrt{T})$	✗
	`Unknown`	**`F/B`**	$\widetilde{\mathcal{O}}(\sqrt{T})/\widetilde{\mathcal{O}}(\sqrt{T})$	$\widetilde{\mathcal{O}}(\sqrt{T})/\widetilde{\mathcal{O}}(\sqrt{T})$	✗

the works focus on stochastic loss and/or constraints (Efroni et al., 2020; Bai et al., 2023; Liu et al., 2021; Stradi et al., 2025a), while the ones for adversarial losses/constraints (Stradi et al., 2025a; Germano et al., 2023) are relatively less. The reason for this trend is the inherent difficulty associated with adversarial cases; (G3) Notably, the most challenging and non-trivial setup remains scarcely addressed in the literature: online learning in CMDPs with an unknown transition function and adversarial losses and constraints.

Sinha & Vaze (2024) obtained $\mathcal{O}(\sqrt{T})$ regret and $\widetilde{\mathcal{O}}(\sqrt{T})$ cumulative constraint violation (hard) in the domain of Constrained Online Convex Optimization (COCO) for the first time. The proposed first-order algorithm was efficient and straightforward, requiring only one projection per round. Most recently, Zhu et al. (2025) gave the `Optimistic Mirror Descent Primal-Dual (OMDPD)` algorithm, achieving the optimal $\widetilde{\mathcal{O}}(\sqrt{T})$ regret and $\widetilde{\mathcal{O}}(\sqrt{T})$ hard violation for online learning in finite-horizon episodic CMDPs. Employing some tools from Sinha & Vaze (2024) and optimizing dual variables, `OMDPD` was the first algorithm of its kind to derive optimal regret and violation bounds with adversarial constraints, without any need for Slater's condition. However, we elaborate on two critical gaps in `OMDPD` (Zhu et al., 2025): (G4) The losses were stochastic, i.e., sampled from a distribution, for all episodes; (G5) Full feedback was assumed (instead of the more realistic bandit feedback) while considering adversarial constraints.

Our Contributions: To the best of our knowledge, this work is the first to pose and tackle the following question for online learning in finite-horizon episodic CMDPs: (CQ) *"With no reliance on Slater's condition, with no access to a strictly feasible policy, for adversarial losses and constraints, with unknown transition function and bandit feedback, can an algorithm be designed with $\widetilde{\mathcal{O}}(\sqrt{T})$ regret and $\widetilde{\mathcal{O}}(\sqrt{T})$ hard cumulative constraint violation?"*. We formally describe our contributions below:

- Although `OMDPD` borrowed elements from Sinha & Vaze (2024), they did not capitalize on the potential of using COCO to solve the setting described in (CQ). However, our work achieves this by mapping the CMDP problem to a corresponding COCO instance and employing techniques from Sinha & Vaze (2024) to frame an elegant analysis for deriving optimal regret and hard violation bounds.

- Our proposed algorithms are also efficient, because only one Euclidean projection onto a simple polytope is performed per episode. Unlike primal-dual and linear programming-based approaches, our algorithms are simple to understand. The simplicity and elegance of our framework make it a valuable didactic resource, especially for those interested in the connection between online learning in CMDPs and COCO.

- Considering adversarial losses and constraints, we solve four cases: (1) known transition function and full feedback; (2) known transition function and bandit feedback; (3) unknown transition function and full feedback; (4) unknown transition function and bandit feedback (the solution to CQ). Thus, we not only answer CQ in the resounding affirmative but also solve all possible combinations that could occur with adversarial losses and constraints with known/unknown transitions. To the best of our knowledge, an exhaustive case analysis of this nature is not present in the literature, nor does it rely on or assume Slater's condition.
- We derive optimal regret and cumulative constraint violation (hard) bounds in each case, i.e., $\mathcal{O}(\sqrt{T})$ regret and $\widetilde{\mathcal{O}}(\sqrt{T})$ violation for (1) and (2), and $\widetilde{\mathcal{O}}(\sqrt{T})$ regret and $\widetilde{\mathcal{O}}(\sqrt{T})$ violation for (3) and (4). Also, we construct biased estimators for the sub-gradient while solving (2) and (4), which might be of independent interest for didactic uses. Together with the earlier points, responding positively to (CQ) automatically resolves the gaps G1, G2, G3, G4, and G5. Table 1 compares our theoretical results with numerous state-of-the-art methods.
- Unlike Müller et al. (2023), we do not require access to a strictly feasible policy. We operate under the standard assumption that at least one feasible policy exists, but none of our algorithms need to know what that policy is. This particular feasibility assumption is almost ubiquitous in the COCO literature (Yi et al., 2021; 2023).

The rest of this paper is structured as follows: In Section 2, we survey the related works in online learning for MDPs, CMDPs, and constrained online optimization, highlighting both classical results and recent advances. Section 3 provides the necessary background, including the formal setup of CMDPs, occupancy measures, and COCO. Section 4 develops our algorithms and theoretical guarantees under known transition dynamics, analyzing both full and bandit feedback settings. Then, in Section 5, we extend to the more challenging regime of unknown transitions, again addressing full and bandit feedback. A brief yet insightful discussion on the optimality of our derived bounds is in Section 6. Finally, in Section 7, we state the concluding remarks.

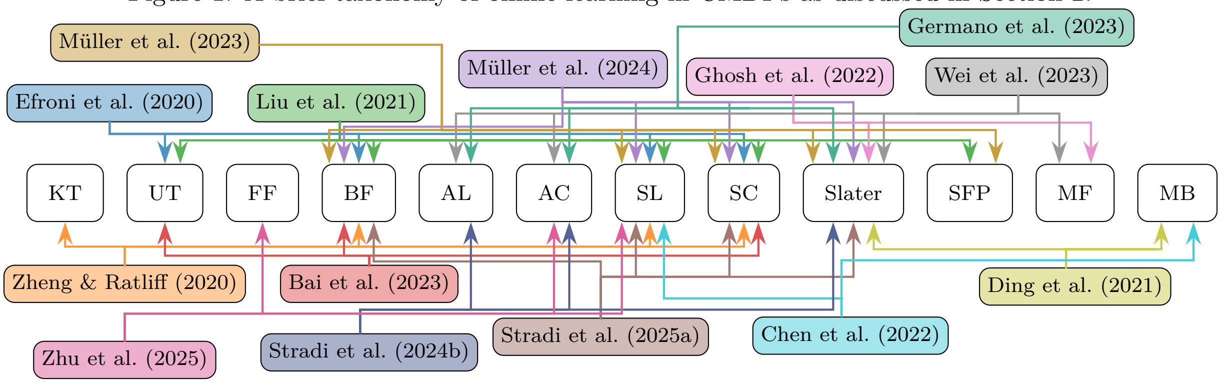
2 Related Works

We categorize the prior works into three groups. Firstly, we survey some of the interesting works involving the application of online learning in traditional MDPs over the years. Secondly, we discuss the related works in the literature of online learning in CMDPs. Lastly, we briefly examined some critical works in the classical online learning problem (Cesa-Bianchi & Lugosi, 2006) with constraints.

Online Learning in MDPs: The `UCRL2` algorithm (Jaksch et al., 2010) is one of the seminal works in this domain that proved $\widetilde{\mathcal{O}}(\sqrt{T})$ regret for undiscounted MDPs. Neu et al. (2010) showed a $\widetilde{\mathcal{O}}(T^{2/3})$ bound on the regret for undiscounted MDPs where an oblivious adversary chose the loss function. The work of Rosenberg & Mansour (2019b) used entropic regularization to establish $\widetilde{\mathcal{O}}(\sqrt{T})$ regret of episodic MDPs with unknown transitions, adversarial losses, and full feedback. An identical setting with bandit feedback has been dealt with by Rosenberg & Mansour (2019a) with $\widetilde{\mathcal{O}}(T^{3/4})$ regret. The elegant `UOB-REPS` algorithm (Jin et al., 2020) was the first to achieve $\widetilde{\mathcal{O}}(\sqrt{T})$ regret upper bound in the same problem setup as of Rosenberg & Mansour (2019a). Lee et al. (2020) obtained data-dependent high probability $\widetilde{\mathcal{O}}(\sqrt{T})$ regret bounds with an adaptive adversary and bandit feedback. It used standard unbiased estimators and a simple learning rate schedule. Furthermore, works like Bacchiocchi et al. (2024) provided off-policy regret bounds for adversarial MDPs while Maran et al. (2024) studied online configuration of MDPs with stochastic losses, bandit feedback, and continuous decision spaces.

Online Learning in CMDPs: Many works in this area emphasized stochastic losses and constraints. Under bandit feedback, stochastic losses and constraints, and unknown transitions, Efroni et al. (2020) employed linear programming and primal-dual methods for tackling exploration-exploitation in episodic CMDPs. Sublinear regret and cumulative constraints violation were assured. Zheng & Ratliff (2020) concentrated on

Figure 1: A brief taxonomy of online learning in CMDPs as discussed in Section 2.



LEGEND: KT = Known Transition; UT = Unknown Transition; FF = Full Feedback; BF = Bandit Feedback; AL = Adversarial Loss; AC = Adversarial Constraint; SL = Stochastic Loss; SC = Stochastic Constraint; Slater = Using Slater's condition; SFP = Access to a strictly feasible policy; MF = Model Free; MB = Model Based;

fully-stochastic episodic CMDPs, under bandit feedback and known transitions, achieving $\widetilde{\mathcal{O}}(T^{3/4})$ regret. At the same time, its violation was shown to be below a threshold with a given probability. The seminal work of Bai et al. (2023) provided sublinear regret in the presence of peak stochastic constraints, unknown transitions, and deterministic rewards.

Focusing only on stochastic losses, numerous works (Liu et al., 2021; Müller et al., 2024; Stradi et al., 2025a) obtain sublinear bounds for hard violations of stochastic constraints. Various model-free (Ghosh et al., 2022; Wei et al., 2023) and model-based (Ding et al., 2021; Chen et al., 2022) works have also studied soft violation in CMDPs. Also, the work of Stradi et al. (2024b) gave bounds for soft constraint violations, but the losses were adversarial. With a reliance on the Slater's slackness parameter, Stradi et al. (2025a) dealt with hard constraint violation for the stochastic loss and constraints. The best-of-both-worlds regret and violation were established in Germano et al. (2023) where the loss and constraints could be both stochastic and adversarial. Although the results of Ding & Lavaei (2023); Wei et al. (2023), and Stradi et al. (2024c) do not work for adversarial losses, they establish regret and violation guarantees by considering non-stationary losses and constraints. Additionally, these works also assume a bound on the variance of the losses and constraints. Very recently, the `OMDPD` algorithm (Zhu et al., 2025) tackled adversarial constraints and obtained $\widetilde{\mathcal{O}}(\sqrt{T})$ regret and $\widetilde{\mathcal{O}}(\sqrt{T})$ violation without Slater's condition. Having access to a strictly feasible policy, and with stochastic losses and constraints, Müller et al. (2023) utilized an augmented Lagrangian approach to obtain optimal hard violation. Figure 1 contains a schematic categorization of the works as mentioned above.

Online Learning with Constraints: Liakopoulos et al. (2019) examined adversarially chosen long-term budget constraints. However, their regret was defined with respect to a comparator satisfying the budget over a fixed window. Castiglioni et al. (2022a) and Castiglioni et al. (2022b) supplied the first best-of-both-worlds algorithm with long-term constraints. Hard constraint violations have also been studied in simple stochastic settings (Pacchiano et al., 2021; Bernasconi et al., 2022), in Online Convex Optimization (OCO) (Guo et al., 2022b), and in Constrained OCO (COCO) (Sinha & Vaze, 2024). Also, Sinha & Vaze (2024) first showed that it is possible to design an online policy in COCO without extra assumptions that achieves $\mathcal{O}(\sqrt{T})$ regret and $\widetilde{\mathcal{O}}(\sqrt{T})$ violation. The recent work of Lekeufack & Jordan (2024) considered a setup where the predictions of the loss and constraints are accessible. By utilizing the tools from Sinha & Vaze (2024), they (Lekeufack & Jordan, 2024) slightly improved upon the $\mathcal{O}(\sqrt{T})$ regret and $\widetilde{\mathcal{O}}(\sqrt{T})$ violation bounds.

3 Preliminaries

For any $n \in \mathbb{N}_{>0}$ and $z \in \mathbb{R}$, we define the notations $[n] \equiv \{1, 2, \ldots, n\}$, $[n]^{-1} \equiv \{0, 1, \ldots, n-1\}$, and $(z)^+$ (or z^+) $\equiv \max(0, z)$. We use the notation $\|\cdot\|$ to denote the L^2-norm throughout the document. Also, unless mentioned otherwise, we denote by ∇r the sub-gradient of an arbitrary convex function r.

Algorithm 1 Interaction between the learner and the CMDP

for $t = 1, \ldots, T$ **do**
  The learner chooses a policy $\pi_t : \mathcal{S} \times \mathcal{A} \to [0, 1]$.
  The adversary decides the loss and constraint vectors, i.e., $\boldsymbol{\ell}_t$ and $\boldsymbol{c}_t$.
  The learner starts from the fixed initial state s_0.
  for $h = 0, \ldots, H - 1$ **do**
    The learner plays the action $a_h \sim \pi_t(\cdot \mid s_h)$.
    A new state $s_{h+1} \sim \mathcal{P}(\cdot \mid s_h, a_h)$ is reached.
    The learner observes the new state s_{h+1}.
  end for
  The adversary reveals $\boldsymbol{\ell}_t$ and $\boldsymbol{c}_t$ to the learner in *full* or *bandit* feedback.
end for

3.1 Constrained Markov Decision Process

A finite episodic Constrained Markov Decision Process (CMDP) (Altman, 1999), is defined as the tuple $\mathcal{M} = (T, H, \mathcal{S}, \mathcal{A}, \mathcal{P}, \{\boldsymbol{\ell}_t\}_{t=1}^T, \{\boldsymbol{c}_t\}_{t=1}^T)$, where: T is the total number of episodes; H is the length of each episode; $\mathcal{S}$ and $\mathcal{A}$ are a finite state and action space with $|\mathcal{S}| = S$ and $|\mathcal{A}| = A$; $\mathcal{P} : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \to [0, 1]$ is a transition probability function; $\{\boldsymbol{\ell}_t\}_{t=1}^T$ and $\{\boldsymbol{c}_t\}_{t=1}^T$ are the sequence of loss and constraint vectors respectively. For a fixed t and for all $h \in [H]^{-1}$, the vector $\boldsymbol{\ell}_t \in [0, 1]^{S \times A \times H}$ constitutes of the loss $\ell_{t,h} : \mathcal{S} \times \mathcal{A} \to [0, 1]$, suffered by the learner for playing action $a \in \mathcal{A}$ in state $s \in \mathcal{S}$ at the h-th step in the t-th episode. Similarly, for a fixed t and for all $h \in [H]^{-1}$, the components $c_{t,h} : \mathcal{S} \times \mathcal{A} \to [-1, 1]$ of the vector $\boldsymbol{c}_t \in [-1, 1]^{S \times A \times H}$, encode the cost of the constraint incurred by the learner on taking action $a \in \mathcal{A}$ in state $s \in \mathcal{S}$. Note that for each state-action pair, multiple constraints can be replaced by a single constraint, which is the point-wise maximum of the given constraints. Therefore, in this work, we assume that the learner is presented with only one constraint. Without loss of generality, we consider $\mathcal{M}$ to be *loop-free*, i.e., we assume that $\mathcal{S}$ is partitioned into $H + 1$ layers $\mathcal{S}_0, \ldots, \mathcal{S}_H$, such that $\mathcal{S}_0 = \{s_0\}$ and $\mathcal{S}_H = \{s_H\}$. Here, s_0 and s_H are the fixed initial and terminal states, respectively. For all $s \notin \mathcal{S}_H$, when playing action a in state s, $\mathcal{P}(\cdot \mid s, a)$ is the distribution of the next state. We assume that $\mathcal{P}(s' \mid s, a) \neq 0$ only when $s \in \mathcal{S}_h$ and $s' \in \mathcal{S}_{h+1}$ for some $h < H$.

Online learning in CMDPs with adversarial losses and constraints is conducted over T episodes, where each episode consists of H steps. In each episode $t \in [T]$, the learner chooses a stochastic policy $\pi_t : \mathcal{S} \times \mathcal{A} \to [0, 1]$, where $\pi_t(a \mid s)$ is the probability of selecting the action $a \in \mathcal{A}$ in the state $s \in \mathcal{S}$. The adversary also selects the loss vector $\boldsymbol{\ell}_t$ and the constraint vector $\boldsymbol{c}_t$ at the beginning of an episode $t \in [T]$. Starting from s_0, the learner executes π_t for H steps and observes the trajectory $\big\{(s_h, a_h, \ell_{t,h}(s_h, a_h), c_{t,h}(s_h, a_h))\big\}_{h=0}^{H-1}$ (where the action $a_h \sim \pi_t(\cdot \mid s_h)$, and the next state $s_{h+1} \sim \mathcal{P}(\cdot \mid s_h, a_h)$) before reaching s_H. It is only when the t-th episode ends that the adversary reveals $\boldsymbol{\ell}_t$ and $\boldsymbol{c}_t$ to the learner, either in *full* or *bandit* feedback. The loss and constraint costs for every state-action pair are disclosed to the learner in the full feedback case. In contrast, for bandit feedback, the loss and constraint costs for only the observed state-action pairs (in a trajectory) are revealed to the learner. We consider an episodic setting where the policy remains fixed within each episode and is updated only at the end of the episode. Algorithm 1 formally describes how the learner communicates with the CMDP. For an episode $t \in [T]$, a policy π_t, and a loss vector $\boldsymbol{\ell}_t \in [0, 1]^{S \times A \times H}$, we call the *episodic loss* the expected total loss of the learner in that episode. It is defined as:

$$V^{\pi_t}(s_0; \boldsymbol{\ell}_t) \coloneqq \mathbb{E}\left[\sum_{h=0}^{H-1} \ell_{t,h}(s_h, a_h) \,\middle|\, a_h \sim \pi_t(\cdot \mid s_h), s_{h+1} \sim \mathcal{P}(\cdot \mid s_h, a_h)\right], \tag{1}$$

where the learner starts from the initial state s_0 and follows π_t subsequently. It is clear from the definition above that $V^{\pi_t}(s_H; \boldsymbol{\ell}_t) = 0$. The episodic loss can be generalized to start from any state s, with an arbitrary loss vector $\boldsymbol{\ell}$, and following π afterwards as: $V^{\pi}(s; \boldsymbol{\ell}) \coloneqq \mathbb{E}_{a \sim \pi(\cdot \mid s)}\big[Q^{\pi}(s, a; \boldsymbol{\ell})\big]$, where $Q^{\pi}(s, a; \boldsymbol{\ell}) \coloneqq \ell(s, a) + \mathbb{1}_{s \notin \mathcal{S}_H} \mathbb{E}_{s' \sim \mathcal{P}(\cdot \mid s, a)}\big[V^{\pi}(s'; \boldsymbol{\ell})\big]$ (where $\ell(s, a)$ is a component of the vector $\boldsymbol{\ell}$) is the Bellman equation denoting the expected loss starting from s, taking action a, and following π afterward. Similar to the episodic loss

$V^{\pi_t}(s_0;\boldsymbol{\ell}_t)$, we define $V^{\pi_t}(s_0;\boldsymbol{c}_t)$ for computing the expected violation of the constraints in an episode as:

$$V^{\pi_t}(s_0;\boldsymbol{c}_t) \coloneqq \mathbb{E}\left[\sum_{h=0}^{H-1} c_{t,h}(s_h,a_h) \;\middle|\; a_h \sim \pi_t(\cdot \mid s_h), s_{h+1} \sim \mathcal{P}(\cdot \mid s_h,a_h)\right]. \tag{2}$$

We term $V^{\pi_t}(s_0;\boldsymbol{c}_t)$ as the *episodic constraint violation* which can also be generalized to start from any state s, with an arbitrary constraint vector $\boldsymbol{c}$, and following π afterwards as: $V^{\pi}(s;\boldsymbol{c}) \coloneqq \mathbb{E}_{a\sim\pi(\cdot|s)}\big[Q^{\pi}(s,a;\boldsymbol{c})\big]$, where the Bellman equation $Q^{\pi}(s,a;\boldsymbol{c}) \coloneqq c(s,a) + \mathbb{1}_{s\notin\mathcal{S}_H}\mathbb{E}_{s'\sim\mathcal{P}(\cdot|s,a)}\big[V^{\pi}(s';\boldsymbol{c})\big]$ (where $c(s,a)$ is a component of the vector $\boldsymbol{c}$) denotes the expected constraint violations starting from s, taking action a, and following π afterward. For a known transition function $\mathcal{P}$, the expectations in Eqn. 1 and Eqn. 2 will only be taken on the randomness in sampling the actions. One could simply write $V^{\pi_t}(\boldsymbol{\ell}_t)$ and $V^{\pi_t}(\boldsymbol{c}_t)$ when the starting state is clear from the context.

Let us assume $\pi^\star \in \arg\min_{\pi\in\Pi}\sum_{t=1}^{T} V^{\pi_t}(s_0;\boldsymbol{\ell}_t)$ to be a fixed optimal policy in hindsight, where Π is the class of all stochastic policies. The final objective of the learner is to learn a policy that jointly minimizes the expected regret and the expected cumulative constraint violation over all the episodes:

$$\mathbb{E}\big[\mathcal{R}_T\big] \coloneqq \mathbb{E}\left[\sum_{t=1}^{T} V^{\pi_t}(s_0;\boldsymbol{\ell}_t)\right] - \sum_{t=1}^{T} V^{\pi^\star}(s_0;\boldsymbol{\ell}_t), \text{ and} \tag{3}$$

$$\mathbb{E}\big[\mathcal{Z}_T\big] \coloneqq \mathbb{E}\left[\sum_{t=1}^{T} \max\big(0, V^{\pi_t}(s_0;\boldsymbol{c}_t)\big)\right] = \mathbb{E}\left[\sum_{t=1}^{T}\big(V^{\pi_t}(s_0;\boldsymbol{c}_t)\big)^+\right]. \tag{4}$$

For the bandit feedback setting, the expectations in the above equations are taken over the randomness in choosing π_t at the beginning of each episode. In the full feedback case, there is no stochasticity in the policy, so the expectations will not be present in Eqn. 3 and Eqn. 4.

3.2 Occupancy Measures

It is well known that any policy π and a transition probability function $\mathcal{P}$ induce an *occupancy measure* $\rho^{\mathcal{P},\pi} : \mathcal{S}\times\mathcal{A} \to [0,1]$ (Altman, 1999; Rosenberg & Mansour, 2019b), where $\rho^{\mathcal{P},\pi}(s,a)$ is the probability of visiting the state-action pair (s,a) when the learner starts from the initial state and acts according to π. Consider the following definition, which formalizes the notion of occupancy measures.

Definition 1 (Occupancy Measure)**.** *For every $s \in \mathcal{S}$ and $a \in \mathcal{A}$ the occupancy measure $\rho^{\mathcal{P},\pi} : \mathcal{S}\times\mathcal{A} \to [0,1]$ induced by a policy π and a transition function $\mathcal{P}$ is the probability of visiting the pair (s,a) when the agent begins from s_0 and then follows π in an episode. Therefore, the probability of visiting a state $s \in \mathcal{S}$ in an episode will be:*

$$\rho^{\mathcal{P},\pi}(s) = \sum_{a\in\mathcal{A}} \rho^{\mathcal{P},\pi}(s,a). \tag{5}$$

From now on, we omit writing $\mathcal{P}$ in $\rho^{\mathcal{P},\pi}$ for simplicity (unless absolutely required). Let $\Omega = \{\rho^{\pi} \mid \pi \in \Pi\}$ be the set of all *valid occupancy measures*. From the work of Luo et al. (2021), we have an alternative characterization for Ω that is widely used in the literature, and it is elucidated in the following definition.

Definition 2 (Valid Occupancy Measures)**.** *We have the following equivalent definition of Ω:*

$$\Omega = \left\{\rho \in [0,1]^{S\times A\times H} \;\middle|\; \rho(s_0) = 1; \rho(s') = \sum_{s\in\mathcal{S}_h}\sum_{a\in\mathcal{A}} \rho(s,a)\mathcal{P}(s' \mid s,a), \forall s' \in \mathcal{S}_{h+1} \text{ and } \forall h \in [H]^{-1}\right\}. \tag{6}$$

Any $\rho \in \Omega$ corresponds to the occupancy measure induced by the policy π^{ρ} with $\pi^{\rho}(a \mid s) = \frac{\rho(s,a)}{\rho(s)}$, i.e., $\pi^{\rho}(a \mid s) \propto \rho(s,a)$. It is evident from Eqn. 1 and Eqn. 2 that $V^{\pi_t}(s_0;\boldsymbol{\ell}_t)$ and $V^{\pi_t}(s_0;\boldsymbol{c}_t)$ are non-convex in π_t. It is important to note that ρ^{π_t}, $\boldsymbol{\ell}_t$, and $\boldsymbol{c}_t$ are vectors of dimension $S \times A \times H$. Thus, being equipped with Definition 1, the episodic loss $V^{\pi_t}(s_0;\boldsymbol{\ell}_t)$ and the episodic constraint violation $V^{\pi_t}(s_0;\boldsymbol{c}_t)$ can be re-written as $\langle \rho^{\pi_t}, \boldsymbol{\ell}_t \rangle$ and $\langle \rho^{\pi_t}, \boldsymbol{c}_t \rangle$ respectively, thereby, making $V^{\pi_t}(s_0; \boldsymbol{\ell}_t)$ and $V^{\pi_t}(s_0; \boldsymbol{c}_t)$ linear in the occupancy measure ρ^{π_t}. Consequently, the expected regret in Eqn. 3 and the expected cumulative constraint violation in Eqn. 4 can be equivalently expressed as:

$$\mathbb{E}\big[\mathcal{R}_T\big] \coloneqq \mathbb{E}\left[\sum_{t=1}^{T} \langle \rho^{\pi_t} - \rho^{\pi^\star}, \boldsymbol{\ell}_t \rangle\right], \text{ and} \tag{7}$$

$$\mathbb{E}\big[\mathcal{Z}_T\big] \coloneqq \mathbb{E}\left[\sum_{t=1}^{T} \max\big(0, \langle \rho^{\pi_t}, \boldsymbol{c}_t \rangle\big)\right] = \mathbb{E}\left[\sum_{t=1}^{T} \langle \rho^{\pi_t}, \boldsymbol{c}_t \rangle^+\right]. \tag{8}$$

As before, the expectations in Eqn. 7 and Eqn. 8 will not be present in the full feedback case. From now on, we will employ the shorthand ρ_t and $\rho^\star$ instead of ρ^{π_t} and $\rho^{\pi^\star}$ respectively. Also, note that Eqn. 8 and Eqn. 4 naturally encapsulate the notion of *hard constraint violation*.

3.3 Constrained Online Convex Optimization

Online Convex Optimization (OCO) (Hazan, 2016; Orabona, 2025) provides a valuable arsenal for tackling online decision-making problems. The framework of Constrained Online Convex Optimization (COCO) (Guo et al., 2022a; Sinha & Vaze, 2024) is a generalization of OCO that involves a round-based game between an online policy and an adversary. At each round $t \in [T]$, the online policy selects an action $x_t \in \mathcal{X}$, where $\mathcal{X}$ is called the *admissible set*. Then, a convex cost function $\mu_t : \mathcal{X} \to \mathbb{R}$ and a convex constraint function $\nu_t : \mathcal{X} \to \mathbb{R}$ are chosen by the adversary. To be specific, on playing the action x_t, the online policy suffers a cost $\mu_t(x_t)$ and a constraint violation $\nu_t(x_t) \leq 0$.

Let $\mathcal{X}^\star$ be the set of all admissible actions satisfying the constraint on every round, i.e., $\mathcal{X}^\star = \{x \in \mathcal{X} \mid \nu_t(x) \leq 0, \forall t \geq 1\}$. The set $\mathcal{X}^\star$ is called the *feasible set* in the standard COCO literature. The end goal of any COCO problem is to build an online policy that jointly minimizes regret and cumulative constraint violation, which are defined as:

$$\text{Regret}_T \coloneqq \sum_{t=1}^{T} \mu_t(x_t) - \sup_{x^\star \in \mathcal{X}} \sum_{t=1}^{T} \mu_t(x^\star), \text{ and} \tag{9}$$

$$\text{CCV}_T \coloneqq \sum_{t=1}^{T} \max\big(0, \nu_t(x_t)\big) = \sum_{t=1}^{T} \nu_t(x_t)^+. \tag{10}$$

We state three standard assumptions prevalent in the COCO literature (Yi et al., 2021; Guo et al., 2022a; Yi et al., 2023). The first one, i.e., Assumption 1, is on the convexity of the admissible set $\mathcal{X}$, while Assumption 2 describes the Lipschitz continuity of $\{\mu_t\}_{t=1}^T$ and $\{\nu_t\}_{t=1}^T$. The direct implication of this assumption is that the L^2-norm of $\{\nabla \mu_t\}_{t=1}^T$ and $\{\nabla \nu_t\}_{t=1}^T$ is uniformly upper bounded by the Lipschitz constant. Assumption 3 states that the feasible set $\mathcal{X}^\star$ is non-empty.

Assumption 1 (Convexity)**.** *The admissible set $\mathcal{X} \subseteq \mathbb{R}^d$ is closed and convex and has a finite Euclidean diameter of D. For all $t \in [T]$, the cost functions $\{\mu_t\}_{t=1}^T$ and the constraint functions $\{\nu_t\}_{t=1}^T$ are convex.*

Assumption 2 (Lipschitzness)**.** *All the costs $\{\mu_t\}_{t=1}^T$ and constraints $\{\nu_t\}_{t=1}^T$ are L-Lipschitz. Thus, for all $a, b \in \mathcal{X}$ and for every $t \in [T]$, we have:*

$$|\mu_t(a) - \mu_t(b)| \leq L \cdot \|a - b\|, \; |\nu_t(a) - \nu_t(b)| \leq L \cdot \|a - b\|. \tag{11}$$

Assumption 3 (Feasibility)**.** *The feasible set is non-empty, i.e., $\mathcal{X}^\star \neq \emptyset$, as there always exists an $x^\star \in \mathcal{X}$ for which $\nu_t(x^\star) \leq 0$, for all $t \in [T]$.*

It is essential to recognize that the objective in COCO and online learning in CMDPs is the same, namely, minimizing regret and cumulative constraint violation. This fact makes solving CMDPs possible with algorithms for COCO, after proper reductions. Inspired by Sinha & Vaze (2024), we utilize a Lyapunov potential function to regulate the growth of violations and construct a surrogate loss by linearly combining an upper bound on the change of the Lyapunov function with the cost function.

3.4 Reduction from CMDP to COCO - a simple toy example

We provide a toy example to illustrate the reduction that is central in the upcoming sections. Let us consider a CMDP $\mathcal{G} = (T, H, \mathcal{S}, \mathcal{A}, \mathcal{P}, \{\boldsymbol{\ell}_t\}_{t=1}^T, \{\boldsymbol{c}_t\}_{t=1}^T)$ with $|\mathcal{S}| = S$, $|\mathcal{A}| = A$, and with horizon length of two, i.e., let $H = 2$. Assume that the transition function $\mathcal{P}$ is known. Since $\mathcal{G}$ is loop-free, the finite state space $\mathcal{S}$ can be written as: $\mathcal{S} = \bigcup_{h=0}^{2} \mathcal{S}_h = \mathcal{S}_0 \bigcup \mathcal{S}_1 \bigcup \mathcal{S}_2$ and $\mathcal{S}_k \bigcap \mathcal{S}_l = \emptyset$ for $k \neq l$. By the definition given in Section 3.1, the first and last layer only contain the fixed initial and terminal state respectively, i.e., $\mathcal{S}_0 = \{s_0\}$ and $\mathcal{S}_2 = \{s_2\}$. Let the intermediate state layer be $\mathcal{S}_1 = \{x, y\}$ and the finite action space be $\mathcal{A} = \{0, 1\}$. In this case, the occupancy vector is:

$$\rho = [\rho_0(s_0, 0), \rho_0(s_0, 1), \rho_1(x, 0), \rho_1(x, 1), \rho_1(y, 0), \rho_1(y, 1)]\,. \tag{12}$$

Moreover, the valid set Ω will contain any $\rho \in [0, 1]^{S \times A \times H}$ satisfying the following constraints:

1. For $h = 0$: $\rho_0(s_0, 0) + \rho_0(s_0, 1) = 1$.
2. For $h = 1$: $\forall s' \in \{x, y\}$, $\rho_1(s', 0) + \rho_1(s', 1) = \sum_{a \in \mathcal{A}} \rho_0(s_0, a)\mathcal{P}(s' \mid s_0, a)$.

Any ρ satisfying the above constraints is realizable by the policy: $\pi_\rho(a \mid s) = \frac{\rho_h(s,a)}{\sum_{a'} \rho_h(s,a')}$, whenever $\sum_{a'} \rho_h(s, a') > 0$. For episode $t \in [T]$, with losses $\ell_{t,h}(s, a)$ and constraints $c_{t,h}(s, a)$, we have the following definitions for the cost function μ_t and the constraint function ν_t:

$$\mu_t(\rho) = \sum_{(s,a,h)} \rho_h(s, a) \cdot \ell_{t,h}(s, a), \text{ and} \tag{13}$$

$$\nu_t(\rho) = \sum_{(s,a,h)} \rho_h(s, a) \cdot c_{t,h}(s, a), \tag{14}$$

which are linear (and hence convex) in ρ. Thus, one CMDP episode is equivalent to one round in the COCO problem with the decision $\rho_t \in \Omega$.

4 Known Transition Function

When the transition function $\mathcal{P}$ is known for the CMDP $\mathcal{M}$, then there would be no randomness linked with the next-state s_{h+1} in an episode $t \in [T]$. Therefore, we rewrite the episodic loss and episodic constraint violation in Eqn. 1 and Eqn. 2 as follows:

$$V^{\pi_t}(s_0; \boldsymbol{\ell}_t) \coloneqq \mathbb{E}\left[\sum_{h=0}^{H-1} \ell_{t,h}(s_h, a_h) \,\middle|\, a_h \sim \pi_t(\cdot \mid s_h)\right], \text{ and} \tag{15}$$

$$V^{\pi_t}(s_0; \boldsymbol{c}_t) \coloneqq \mathbb{E}\left[\sum_{h=0}^{H-1} c_{t,h}(s_h, a_h) \,\middle|\, a_h \sim \pi_t(\cdot \mid s_h)\right]. \tag{16}$$

Throughout this section, we will use Eqn. 15 and Eqn. 16 as the definition of the episodic loss and episodic constraint violation, respectively.

4.1 Full Feedback and Known Transition

In addition to the transition function being known, the entire loss vector $\boldsymbol{\ell}_t$ and the constraint vector $\boldsymbol{c}_t$ are revealed to the learner at the end of an episode. Consequently, the regret $\mathcal{R}_T$ and the cumulative constraint violation $\mathcal{Z}_T$ to be minimized in this scenario are given as:

$$\mathcal{R}_T \coloneqq \sum_{t=1}^{T} \langle \rho_t - \rho^\star, \boldsymbol{\ell}_t \rangle, \text{ and} \tag{17}$$

$$\mathcal{Z}_T \coloneqq \sum_{t=1}^{T} \langle \rho_t, \boldsymbol{c}_t \rangle^+. \tag{18}$$

Algorithm 2 `Full AdaGrad with Known Transition (FAG-K)`

Require: L, D, Euclidean projection operator $\Pi_\Omega(\cdot)$ on Ω.

Set the parameters $\omega = \frac{1}{2LD}$, $\theta = \frac{1}{2\sqrt{T}}$, and choose $\varphi(\zeta_t) = \exp(\theta\zeta_t) - 1$, $\forall t \geq 1$.

Intialize $\rho_1 \in \Omega$ arbitrarily (e.g., uniformly) and set $\zeta_0 = 0$.

for $t = 1, \ldots, T$ **do**

Extract the policy π_t such that $\pi_t(a \mid s) \propto \rho_t(s, a)$, $\forall (s, a) \in \mathcal{S} \times \mathcal{A}$.

The adversary decides $\boldsymbol{\ell}_t$ and $\boldsymbol{c}_t$.

for $h = 0, \ldots, H-1$ **do**

The learner plays $a_h \sim \pi_t(\cdot \mid s_h)$.

The learner reaches new state $s_{h+1} \sim \mathcal{P}(\cdot \mid s_h, a_h)$ and observes s_{h+1}.

end for

The adversary reveals $\boldsymbol{\ell}_t$ and $\boldsymbol{c}_t$ in *full* feedback.

Define $\mu_t(\rho_t) = \langle \rho_t, \boldsymbol{\ell}_t \rangle$, and $\nu_t(\rho_t) = \langle \rho_t, \boldsymbol{c}_t \rangle$.

Compute $\tilde{\mu_t} \leftarrow \omega\mu_t$, and $\tilde{\nu_t} \leftarrow \omega(\nu_t)^+$.

Compute $\zeta_t = \zeta_{t-1} + \tilde{\nu_t}(\rho_t)$ and $\hat{\mu}_t(\rho_t) \coloneqq \tilde{\mu_t}(\rho_t) + \varphi'(\zeta_t)\tilde{\nu_t}(\rho_t)$.

According to Eqn. 25, compute the sub-gradient $\nabla_t = \nabla\hat{\mu}_t(\rho_t)$.

Update $\rho_{t+1} = \Pi_\Omega(\rho_t - \eta_t\nabla_t)$, where $\eta_t = \frac{\sqrt{2}D}{2\sqrt{\sum_{\tau=1}^{t} \|\nabla_\tau\|^2}}$.

end for

return ρ_T and π_T.

Owing to the above definitions, our optimization problem involves searching for an occupancy measure from the space of all valid occupancy measures, i.e., looking for $\rho_t \in \Omega$ for all $t \in [T]$. We will jointly minimize Eqn. 17 and Eqn. 18 by mapping our problem to a corresponding instance of the COCO problem. As already described in Section 3.3, COCO proceeds as a game of T rounds between an online policy and an adversary. Quite clearly, one round in COCO corresponds to one episode of length H in the CMDP. For every $t \in [T]$, we define the cost function $\mu_t : \Omega \to \mathbb{R}$ and constraint function $\nu_t : \Omega \to \mathbb{R}$ as:

$$\mu_t(\rho_t) = \sum_{h=0}^{H-1} \rho_t(s_h, a_h) \cdot \ell_{t,h}(s_h, a_h) = \langle \rho_t, \boldsymbol{\ell}_t \rangle, \text{ and} \tag{19}$$

$$\nu_t(\rho_t) = \sum_{h=0}^{H-1} \rho_t(s_h, a_h) \cdot c_{t,h}(s_h, a_h) = \langle \rho_t, \boldsymbol{c}_t \rangle. \tag{20}$$

It is clear from Eqn. 19 and Eqn. 20 that μ_t and ν_t are linear in ρ_t (thus, convex). Hence, μ_t and ν_t are indeed Lipschitz continuous with respect to ρ_t. The gradients of $\mu_t(\rho_t)$ and $\nu_t(\rho_t)$ are: $\nabla\mu_t(\rho_t) = \boldsymbol{\ell}_t$ and $\nabla\nu_t(\rho_t) = \boldsymbol{c}_t$. It is easy to see that the maximum L^2-norm of $\boldsymbol{\ell}_t$ and $\boldsymbol{c}_t$ are $\|\boldsymbol{\ell}_t\| = \|\boldsymbol{c}_t\| \leq \sqrt{SHA}$. Therefore, the upper bound on the value of the Lipschitz constant L for Eqn. 19 and Eqn. 20 directly follows from the gradient norms, i.e., $L \leq \sqrt{SHA}$.

Definition 2 necessitates that Ω should be a simple polytope with $\mathcal{O}(S)$-many linear constraints, implying Ω is closed and convex. Since $\Omega \subset [0,1]^{S\times A\times H}$, the largest possible Euclidean distance between any two points $\rho_1^\pi, \rho_2^\pi \in \Omega$ is the diagonal distance of the hypercube $[0,1]^{S\times A\times H}$, which is simply equal to $\sqrt{S \times A \times H}$. Therefore, we have the Euclidean diameter of Ω as: $D \coloneqq \sup_{\rho_1^\pi, \rho_2^\pi \in \Omega} \|\rho_1^\pi - \rho_2^\pi\| = \sqrt{S \times A \times H} = \sqrt{SHA}$.

At this juncture, we can now define the regret and the cumulative constraint violation of the corresponding COCO problem as follows:

$$\text{Regret}_T \coloneqq \sum_{t=1}^{T} \mu_t(\rho_t) - \sum_{t=1}^{T} \mu_t(\rho^\star), \text{ and} \tag{21}$$

$$\text{CCV}_T \coloneqq \sum_{t=1}^{T} \nu_t(\rho_t)^+. \tag{22}$$

Algorithm 3 `Online AdaGrad policy with adaptive step-sizes`

Require: A closed convex set $\mathcal{Y}$ with Euclidean diameter D, positive step sizes $\{\eta_t\}_{t=1}^T$, convex cost functions $\{\mu_t\}_{t=1}^T$, projection operator $\mathcal{P}_\mathcal{Y}(\cdot)$.
  Set $y_1 \in \mathcal{Y}$ arbitrarily.
  for $t = 1, \ldots, T$ **do**
    Execute y_t and observe μ_t.
    Suffer a cost of $\mu_t(y_t)$.
    Compute sub-gradient $\nabla_t \equiv \nabla \mu_t(y_t)$.
    Update $y_{t+1} = \mathcal{P}_\mathcal{Y}(y_t - \eta_t \nabla_t)$.
  end for

In each episode $t \in [T]$, we perform the scaling: $\tilde{\mu_t} \leftarrow \omega\mu_t$, $\tilde{\nu_t} \leftarrow \omega(\nu_t)^+$, where $\omega > 0$. The scaled cost function $\tilde{\mu_t}$ and the scaled constraint function $\tilde{\nu_t}$ are both ωL-Lipschitz for all $t \geq 1$. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be any non-decreasing, differentiable, and convex Lyapunov function such that $\varphi(0) = 0$. Also, let ζ_t be the cumulative constraint violation for the scaled constraint function till the t-th episode, where $\zeta_t = \zeta_{t-1} + \tilde{\nu_t}(\rho_t)$, $t \geq 1$ (with $\zeta_0 = 0$). It follows from the convexity of $\varphi(\cdot)$:

$$\begin{aligned}\varphi(\zeta_{t-1}) \geq \varphi(\zeta_t) + \varphi'(\zeta_t)(\zeta_{t-1} - \zeta_t) &\implies \varphi(\zeta_t) \leq \varphi(\zeta_{t-1}) + \varphi'(\zeta_t)(\zeta_t - \zeta_{t-1}) \\ &\implies \varphi(\zeta_t) - \varphi(\zeta_{t-1}) \leq \varphi'(\zeta_t)\tilde{\nu_t}(\rho_t). \end{aligned} \tag{23}$$

From the stochastic drift-plus-penalty framework of Neely (2010), we define the surrogate loss as (taking the penalty to be 1):

$$\hat{\mu}_t(\rho_t) \coloneqq \tilde{\mu_t}(\rho_t) + \varphi'(\zeta_t)\tilde{\nu_t}(\rho_t),\ \forall t \geq 1. \tag{24}$$

The subgradient of $\hat{\mu}_t$ is computed as follows:

$$\begin{aligned}\nabla_t = \nabla\hat{\mu}_t(\rho_t) = \nabla\tilde{\mu_t}(\rho_t) + \nabla\varphi'(\zeta_t)\tilde{\nu_t}(\rho_t) &= \nabla\langle \rho_t, \omega\boldsymbol{\ell}_t\rangle + \varphi'(\zeta_t)\nabla\langle\rho_t, \omega\boldsymbol{c}_t\rangle^+ \\ \implies \nabla_t &= \begin{cases}\omega\boldsymbol{\ell}_t + \varphi'(\zeta_t)\omega\boldsymbol{c}_t, & \text{if } \langle\rho_t, \omega\boldsymbol{c}_t\rangle > 0, \\ \omega\boldsymbol{\ell}_t, & \text{if } \langle\rho_t, \omega\boldsymbol{c}_t\rangle \leq 0.\end{cases}\end{aligned} \tag{25}$$

We can upper bound $\|\nabla_t\|$ as:

$$\|\nabla_t\| = \|\nabla\hat{\mu}_t(\rho_t)\| = \|\nabla\tilde{\mu_t}(\rho_t)\| + \varphi'(\zeta_t)\,\|\nabla\tilde{\nu_t}(\rho_t)\| \leq \omega L\big(1 + \varphi'(\zeta_t)\big). \tag{26}$$

By the feasibility condition, we have $\nu_\tau(\rho^\star) \leq 0$ (for all $\tau \geq 1$), which implies that $\tilde{\nu_\tau}(\rho^\star) = 0$. Consequently, the following observation is easily made:

$$\hat{\mu}_\tau(\rho^\star) = \tilde{\mu_\tau}(\rho^\star) + \varphi'(\zeta_\tau)\tilde{\nu_\tau}(\rho^\star) \implies \hat{\mu}_\tau(\rho^\star) = \tilde{\mu_\tau}(\rho^\star),\ \forall\tau \geq 1. \tag{27}$$

For any $\tau \geq 1$, using Eqn. 27 and Eqn. 24 in Eqn. 23, we have:

$$\begin{aligned}&\varphi(\zeta_\tau) - \varphi(\zeta_{\tau-1}) \leq \varphi'(\zeta_\tau)\tilde{\nu_\tau}(\rho_\tau) \\ \implies &\varphi(\zeta_\tau) - \varphi(\zeta_{\tau-1}) \leq \varphi'(\zeta_\tau)\frac{\hat{\mu}_\tau(\rho_\tau) - \tilde{\mu_\tau}(\rho_\tau)}{\varphi'(\zeta_\tau)} \\ \implies &\varphi(\zeta_\tau) - \varphi(\zeta_{\tau-1}) \leq \hat{\mu}_\tau(\rho_\tau) - \tilde{\mu_\tau}(\rho_\tau) \\ \implies &\varphi(\zeta_\tau) - \varphi(\zeta_{\tau-1}) - \hat{\mu}_\tau(\rho^\star) \leq \hat{\mu}_\tau(\rho_\tau) - \tilde{\mu_\tau}(\rho_\tau) - \hat{\mu}_\tau(\rho^\star) \\ \implies &\varphi(\zeta_\tau) - \varphi(\zeta_{\tau-1}) + \tilde{\mu_\tau}(\rho_\tau) - \hat{\mu}_\tau(\rho^\star) \leq \hat{\mu}_\tau(\rho_\tau) - \hat{\mu}_\tau(\rho^\star).\end{aligned}$$

Summing the above inequality for $1 \leq \tau \leq t$ and using $\varphi(0) = 0$, we get:

$$\begin{aligned}\sum_{\tau=1}^t \varphi(\zeta_\tau) - \varphi(\zeta_{\tau-1}) + \sum_{\tau=1}^t \tilde{\mu_\tau}(\rho_\tau) - \tilde{\mu}_\tau(\rho^\star) &\leq \sum_{\tau=1}^t \hat{\mu}_\tau(\rho_\tau) - \hat{\mu}_\tau(\rho^\star) \\ \implies \varphi(\zeta_t) + \mathrm{Regret}_t(\rho^\star) &\leq \mathrm{Regret}'_t(\rho^\star),\end{aligned} \tag{28}$$

where Regret_t on the LHS and Regret'_t on the RHS of Eqn. 28 refer to the regret for learning the pre-processed cost functions $\{\tilde{\mu}_t\}_{t\geq 1}$ and the surrogate loss functions $\{\hat{\mu}_t\}_{t\geq 1}$ respectively.

We utilize the online `AdaGrad` policy (Zinkevich, 2003) with adaptive step sizes (Duchi et al., 2011) as a subroutine, described in Algorithm 3, to minimize the surrogate regret $\text{Regret}'_t(\rho^\star)$. Let us recall an important theorem below (given as Theorem 1) from Orabona (2025) and Duchi et al. (2011) that gives the adaptive regret bound attained by the online AdaGrad policy.

Theorem 1. *Given a sequence of convex cost functions $\{\mu_t\}_{t=1}^T$, the adaptive step size schedule for all $t \geq 1$: $\eta_t = \frac{\sqrt{2}D}{2\sqrt{\sum_{\tau=1}^t \|\nabla_\tau\|^2}}$ (D is the diameter of $\mathcal{Y}$), and $\|\nabla_t\|$. Hence, the regret of Algorithm 3 is given by:*

$$\text{Regret}_T \leq \sqrt{2}D\sqrt{\sum_{t=1}^T \|\nabla_t\|^2}. \tag{29}$$

We name our algorithm in this scenario as **F**ull **A**da**G**rad with **K**nown Transition (`FAG-K`), and it is formally presented in Algorithm 2. Using Eqn. 29 from Theorem 1, we can upper bound the surrogate regret as (see Appendix A.1 for the detailed calculation):

$$\text{Regret}'_t(\rho^\star) \leq 2D\omega L\sqrt{t}\left(1 + \varphi'(\zeta_t)\right). \tag{30}$$

Putting $\omega = \frac{1}{2LD}$, choosing $\varphi(\zeta_t) = \exp(\theta\zeta_t) - 1$, $\forall t \geq 1$, and substituting Eqn. 30 into the regret decomposition inequality of Eqn. 28, we have:

$$\begin{aligned}
\varphi(\zeta_t) + \text{Regret}_t(\rho^\star) &\leq \text{Regret}'_t(\rho^\star) \\
\implies \exp(\theta\zeta_t) - 1 + \text{Regret}_t(\rho^\star) &\leq 2D\omega L\sqrt{t}\left(1 + \theta\exp(\theta\zeta_t)\right) \\
\implies \text{Regret}_t(\rho^\star) &\leq 2D\omega L\sqrt{t}\left(1 + \theta\exp(\theta\zeta_t)\right) + 1 - \exp(\theta\zeta_t) \\
\implies \text{Regret}_t(\rho^\star) &\leq \sqrt{t} + \theta\sqrt{t}\exp(\theta\zeta_t) + 1 - \exp(\theta\zeta_t) \\
\implies \text{Regret}_t(\rho^\star) &\leq \exp(\theta\zeta_t)\left(\theta\sqrt{t} - 1\right) + \sqrt{t} + 1.
\end{aligned} \tag{31}$$

Setting any $\theta \leq \frac{1}{\sqrt{T}}$ for all $t \geq 1$, the term $\exp(\theta\zeta_t)\left(\theta\sqrt{t} - 1\right)$ in the above inequality, becomes non-positive for any $t \in [T]$. Therefore, we obtain the following upper bound on $\text{Regret}_t(\rho^\star)$ for all $t \in [T]$:

$$\text{Regret}_t(\rho^\star) \leq \sqrt{t} + 1. \tag{32}$$

Owing to the functions $\{\tilde{\mu}_t\}_{t\geq 1}$ being $\frac{1}{2D}$-Lipschitz, it is easy to realize that $\text{Regret}_t(\rho^\star) = \sum_{\tau=1}^t \tilde{\mu}_\tau(\rho_\tau) - \tilde{\mu}_\tau(\rho^\star) \geq -\frac{t}{2}$. For any $t \in [T]$ and $\theta < \frac{1}{\sqrt{T}}$, we write this lower bound along with Eqn. 31 to get:

$$\begin{aligned}
\exp(\theta\zeta_t)\left(\theta\sqrt{t} - 1\right) + \sqrt{t} + 1 &\geq -\frac{t}{2} \\
\implies \exp(\theta\zeta_t)\left(1 - \theta\sqrt{t}\right) &\leq \sqrt{t} + 1 + \frac{t}{2} \\
\implies \exp(\theta\zeta_t)\left(1 - \theta\sqrt{t}\right) &\leq \frac{2\sqrt{t} + 2 + 2t}{2} \\
\implies \exp(\theta\zeta_t) &\leq \frac{2\sqrt{t} + 2 + 2t}{2\left(1 - \theta\sqrt{t}\right)} \\
\implies \zeta_t &\leq \frac{1}{\theta}\ln\frac{2\sqrt{t} + 2 + 2t}{2\left(1 - \theta\sqrt{t}\right)} \\
\implies \zeta_T &\leq 2\sqrt{T}\ln\left(2\sqrt{T} + 2 + 2T\right),
\end{aligned} \tag{33}$$

where the last line is obtained by setting $\theta = \frac{1}{2\sqrt{T}}$. By multiplying $\frac{1}{\omega}$ to Eqn. 32 and Eqn. 33, we get the bounds for Eqn. 21 and Eqn. 22. It is straightforward to realize that minimizing Eqn. 21 and Eqn. 22 is equivalent to minimizing Eqn. 17 and Eqn. 18. Therefore, we formally state the bounds on Eqn. 17 and Eqn. 18 in the theorem below.

Theorem 2. *Having $\omega = \frac{1}{2LD}$, $L \leq \sqrt{SHA}$, $D = \sqrt{SHA}$, $\varphi(\zeta_T) = \exp(\theta\zeta_T) - 1$, $\theta = \frac{1}{2\sqrt{T}}$, with adversarial loss and constraints, under full feedback, and known transition, the regret and cumulative constraint violation (hard) of* `FAG-K` *(in Algorithm 2) is bounded, $\forall t \in [T]$ as:*

$$\mathcal{R}_t \leq 2SHA\left(\sqrt{t} + 1\right) \text{ and } \mathcal{Z}_T \leq 4SHA\sqrt{T}\ln\left(2\sqrt{T} + 2 + 2T\right). \tag{34}$$

For all the upcoming sections and subsections and for all $t \geq 1$, the definitions of the cost function μ_t, the constraint function ν_t, and the surrogate function $\hat{\mu}_t$ will be the same as those of Eqn. 19, Eqn. 20, and Eqn. 24 respectively. As a result, the regret decomposition inequality in Eqn. 28 will remain unchanged for all cases and will come in handy in every situation. The online `AdaGrad` policy (as in Algorithm 3) with suitably tailored sub-gradient vectors is used to minimize the surrogate regret in the subsequent cases.

4.2 Bandit Feedback and Known Transition

Here, in this subsection, the loss and constraint costs for only the observed state-action pairs (i.e., only the corresponding entries of $\boldsymbol{\ell}_t$ and $\boldsymbol{c}_t$) are revealed to the learner at the end of an episode. The expected regret $\mathbb{E}[\mathcal{R}_T]$ and the expected cumulative constraint violation $\mathbb{E}[\mathcal{Z}_T]$ to be minimized in this case are:

$$\mathbb{E}\big[\mathcal{R}_T\big] \coloneqq \mathbb{E}\left[\sum_{t=1}^{T}\langle \rho_t - \rho^\star, \boldsymbol{\ell}_t\rangle\right], \text{ and} \tag{35}$$

$$\mathbb{E}\big[\mathcal{Z}_T\big] \coloneqq \mathbb{E}\left[\sum_{t=1}^{T}\langle \rho_t, \boldsymbol{c}_t\rangle^+\right]. \tag{36}$$

The learner only observes the values for H state-action pairs for the vectors $\boldsymbol{\ell}_t$ and $\boldsymbol{c}_t$. We employ the widely popular technique of *implicit exploration* (Kocák et al., 2014; Neu, 2015), i.e., a small value is added to the importance weight, to construct biased estimators $\forall t \in [T]$ and $\forall h \in [H]^{-1}$:

$$\hat{\ell}_{t,h}(s,a) = \frac{\ell_{t,h}(s,a)}{\rho_t(s,a) + \Lambda_t}\mathbf{1}_t(s,a), \text{ and } \hat{c}_{t,h}(s,a) = \frac{c_{t,h}(s,a)}{\rho_t(s,a) + \Lambda_t}\mathbf{1}_t(s,a), \tag{37}$$

where $\Lambda_t > 0$ is an appropriately chosen parameter (to be fixed later) and $\mathbf{1}_t(s,a)$ is 1 if (s,a) is visited during episode t and 0 otherwise. The estimated loss and constraint-cost vectors are respectively defined as $\hat{\boldsymbol{\ell}}_t$ and $\hat{\boldsymbol{c}}_t$, having entries of the form $\hat{\ell}_{t,h}$ and $\hat{c}_{t,h}$ for all $t \in [T]$ and $h \in [H]^{-1}$. Clearly, $\hat{\boldsymbol{\ell}}_t$ and $\hat{\boldsymbol{c}}_t$ both have at most H non-zero entries. The term Λ_t enforces a minimal exploration in the learner, induces a small bias, and ensures that the variance of the estimator remains bounded (Kocák et al., 2014; Neu, 2015). This trick is essential for keeping the regret and the violation terms under control. We state two useful lemmas below.

Lemma 1. *The estimators defined in Eqn. 37 satisfy $\mathbb{E}_t[\hat{\ell}_{t,h}(s,a)] = \frac{\ell_{t,h}(s,a)}{\rho_t(s,a)+\Lambda_t}\rho_t(s,a)$, $\mathbb{E}_t[\hat{c}_{t,h}(s,a)] = \frac{c_{t,h}(s,a)}{\rho_t(s,a)+\Lambda_t}\rho_t(s,a)$, $\mathbb{E}_t[\hat{\ell}_{t,h}(s,a)^2] \leq \frac{1}{\rho_t(s,a)+\Lambda_t}$, and $\mathbb{E}_t[\hat{c}_{t,h}(s,a)^2] \leq \frac{1}{\rho_t(s,a)+\Lambda_t}$.*

Proof. See Appendix A.2. □

Lemma 2. *Show that $0 \leq \ell_{t,h}(s,a) - \mathbb{E}_t[\hat{\ell}_{t,h}(s,a)] \leq \frac{\Lambda\ell_{t,h}(s,a)}{\rho_t(s,a)}$ and $0 \leq c_{t,h}(s,a) - \mathbb{E}_t[\hat{c}_{t,h}(s,a)] \leq \frac{\Lambda c_{t,h}(s,a)}{\rho_t(s,a)}$.*

Proof. See Appendix A.3. □

Again, for this subsection, the regret and the cumulative constraint violation (hard) of the equivalent COCO problem can be naturally defined as in Eqn. 21 and Eqn. 22. It is not possible to compute the actual sub-gradient of the surrogate loss under bandit feedback, unlike the full feedback case. However, we can define a biased estimate of the true sub-gradient ∇_t (as given in Eqn. 25) of the surrogate loss as follows:

$$\widehat{\nabla}_t = \begin{cases} \omega\hat{\boldsymbol{\ell}}_t + \varphi'(\zeta_t)\omega\hat{\boldsymbol{c}}_t, & \text{if } \mathcal{C}_t > 0, \\ \omega\hat{\boldsymbol{\ell}}_t, & \text{if } \mathcal{C}_t \leq 0, \end{cases} \tag{38}$$

Algorithm 4 `Bandit AdaGrad with Known Transition (BAG-K)`

Require: L, D, Euclidean projection operator $\Pi_\Omega(\cdot)$ on Ω.

Set the parameters $\omega = \frac{1}{2LD}$, $\theta = \frac{D+\frac{1}{2}}{3\sqrt{T}(1+D)^2}$, $\Lambda_t = \omega\sqrt{H}$, and choose $\varphi(\zeta_t) = \exp(\theta\zeta_t) - 1$, $\forall t \geq 1$.
Intialize $\rho_1 \in \Omega$ arbitrarily (e.g., uniformly) and set $\zeta_0 = 0$.
for $t = 1, \ldots, T$ **do**
  Extract the policy π_t such that $\pi_t(a \mid s) \propto \rho_t(s, a)$, $\forall (s, a) \in \mathcal{S} \times \mathcal{A}$.
  The adversary decides $\boldsymbol{\ell}_t$ and $\boldsymbol{c}_t$.
  Set $\mathcal{C}_t \leftarrow 0$
  for $h = 0, \ldots, H-1$ **do**
    The learner plays $a_h \sim \pi_t(\cdot \mid s_h)$.
    The learner reaches new state $s_{h+1} \sim \mathcal{P}(\cdot \mid s_h, a_h)$ and observes s_{h+1}.
  end for
  The adversary reveals $\boldsymbol{\ell}_t$ and $\boldsymbol{c}_t$ in *bandit* feedback.
  Compute $\mathcal{C}_t = \sum_{h=0}^{H-1} c_{t,h}(s_h, a_h)$ for the observed state-action pairs.
  Define $\mu_t(\rho_t) = \langle \rho_t, \boldsymbol{\ell}_t \rangle$, and $\nu_t(\rho_t) = \langle \rho_t, \boldsymbol{c}_t \rangle$.
  Compute $\tilde{\mu_t} \leftarrow \omega\mu_t$, and $\tilde{\nu_t} \leftarrow \omega(\nu_t)^+$.
  Construct estimators $\hat{\ell}_{t,h}(s, a)$ and $\hat{c}_{t,h}(s, a)$ according to Eqn. 37.
  Compute $\zeta_t = \zeta_{t-1} + \tilde{\nu_t}(\rho_t)$ and $\hat{\mu}_t(\rho_t) \coloneqq \tilde{\mu_t}(\rho_t) + \varphi'(\zeta_t)\tilde{\nu_t}(\rho_t)$.
  Compute $\widehat{\nabla}_t$ by Eqn. 38.
  Update $\rho_{t+1} = \Pi_\Omega(\rho_t - \eta_t\widehat{\nabla}_t)$, where $\eta_t = \frac{\sqrt{2}D}{2\sqrt{\sum_{\tau=1}^{t}\left\|\widehat{\nabla}_\tau\right\|^2}}$.
end for
return ρ_T and π_T.

where $\mathcal{C}_t = \sum_{h=0}^{H-1} c_{t,h}(s_h, a_h)$ is the observed constraint violation in the t-th episode. Let $\boldsymbol{b}_t$ denote the bias vector for $\widehat{\nabla}_t$ given as: $\boldsymbol{b}_t = \mathbb{E}_t[\widehat{\nabla}_t] - \nabla_t$. We can upper bound the L^2-norm of $\boldsymbol{b}_t$ as: $\|\boldsymbol{b}_t\| \leq \omega L + \omega\varphi'(\zeta_t)\big(L + \sqrt{H}/\Lambda_t\big)$ (see Appendix A.4 for detailed calculations). Additionally, it is easy to see that the upper bound on the L^2-norm of $\widehat{\nabla}_t$ is: $\left\|\widehat{\nabla}_t\right\| \leq \frac{\omega\sqrt{H}}{\Lambda_t}\big(1 + \varphi'(\zeta_t)\big)$. By the triangle inequality for norms:

$$\left\|\mathbb{E}_t[\widehat{\nabla}_t]\right\| \leq \|\boldsymbol{b}_t\| + \|\nabla_t\| \leq \omega L + \omega\varphi'(\zeta_t)\big(L + \sqrt{H}/\Lambda_t\big) + \omega L\big(1 + \varphi'(\zeta_t)\big). \tag{39}$$

Our proposed algorithm for this section, **B**andit **A**da**G**rad with **K**nown Transition (`BAG-K`), is described in Algorithm 2. We will use $\widehat{\nabla}_t$ (as given by Eqn. 38) in the online `AdaGrad` policy (described in Algorithm 3) for minimizing the surrogate regret $\text{Regret}'_t(\rho^\star)$. By the convexity of $\hat{\mu}_\tau$, (for all $\tau \geq 1$), the surrogate regret $\text{Regret}'_t(\rho^\star)$ could be decomposed as:

$$\begin{aligned}
\text{Regret}'_t(\rho^\star) &= \sum_{\tau=1}^{t} \hat{\mu}_\tau(\rho_\tau) - \hat{\mu}_\tau(\rho^\star) \leq \sum_{\tau=1}^{t} \langle \rho_\tau - \rho^\star, \nabla_\tau \rangle \\
&= \sum_{\tau=1}^{t} \langle \rho_\tau - \rho^\star, \mathbb{E}_\tau[\widehat{\nabla}_\tau] \rangle + \sum_{\tau=1}^{t} \langle \rho_\tau - \rho^\star, \nabla_\tau - \mathbb{E}_\tau[\widehat{\nabla}_\tau] \rangle \\
&= \sum_{\tau=1}^{t} \langle \rho_\tau - \rho^\star, \mathbb{E}_\tau[\widehat{\nabla}_\tau] \rangle + \sum_{\tau=1}^{t} \langle \rho_\tau - \rho^\star, -\boldsymbol{b}_\tau \rangle \\
&= \overbrace{\sum_{\tau=1}^{t} \langle \rho_\tau - \rho^\star, \mathbb{E}_\tau[\widehat{\nabla}_\tau] \rangle}^{T_1} - \overbrace{\sum_{\tau=1}^{t} \langle \rho_\tau - \rho^\star, \boldsymbol{b}_\tau \rangle}^{T_2},
\end{aligned} \tag{40}$$

where T_1 is simply the regret from Eqn. 29, with $\left\|\mathbb{E}_\tau[\widehat{\nabla}_\tau]\right\|$ being used instead of $\|\nabla_\tau\|$, and T_2 is the bias term. The computations for upper bounding T_1 and T_2, are deffered to Appendix A.5. Setting $\Lambda_t = \omega\sqrt{H}$

for all $t \geq 1$, and from Eqn. 75 and Eqn. 76 of Appendix A.5, we have:

$$\text{Regret}'_t(\rho^\star) \leq \sqrt{12t} \cdot \varphi'(\zeta_t) + \frac{\sqrt{6t}}{2} + \frac{\sqrt{12t}}{2} + D\sqrt{12t} \cdot \varphi'(\zeta_t) - \frac{t}{2} - \frac{t}{2} \cdot \varphi'(\zeta_t) - Dt \cdot \varphi'(\zeta_t). \tag{41}$$

Choosing $\varphi(\zeta_t) = \exp(\theta\zeta_t) - 1$, $\forall t \geq 1$, and putting Eqn. 41 into the regret decomposition inequality of Eqn. 28, we observe:

$$\begin{aligned}
&\varphi(\zeta_t) + \text{Regret}_t(\rho^\star) \leq \text{Regret}'_t(\rho^\star) \\
\implies &\exp(\theta\zeta_t) - 1 + \text{Regret}_t(\rho^\star) \leq \sqrt{12t} \cdot \theta \exp(\theta\zeta_t) + \frac{\sqrt{6t}}{2} + \frac{\sqrt{12t}}{2} + D\sqrt{12t} \cdot \theta \exp(\theta\zeta_t) \\
&\qquad - \frac{t}{2} - \frac{t}{2} \cdot \theta \exp(\theta\zeta_t) - Dt \cdot \theta \exp(\theta\zeta_t) \\
\implies &\text{Regret}_t(\rho^\star) \leq \sqrt{12t} \cdot \theta \exp(\theta\zeta_t) + D\sqrt{12t} \cdot \theta \exp(\theta\zeta_t) - \frac{t}{2} \cdot \theta \exp(\theta\zeta_t) \\
&\qquad - Dt \cdot \theta \exp(\theta\zeta_t) - \exp(\theta\zeta_t) + 1 + \frac{\sqrt{6t}}{2} + \frac{\sqrt{12t}}{2} - \frac{t}{2} \\
\implies &\text{Regret}_t(\rho^\star) \leq \exp(\theta\zeta_t) \left(\theta\sqrt{12t} + \theta D\sqrt{12t} - \frac{\theta t}{2} - \theta Dt - 1 \right) \\
&\qquad + 1 + \frac{\sqrt{6t}}{2} + \frac{\sqrt{12t}}{2}.
\end{aligned} \tag{42}$$

Let $k(t) = \sqrt{12t} + D\sqrt{12t} - \frac{t}{2} - Dt$ be a function for any $t \geq 1$. The maximum of $k(t)$ occurs at $t^* = \frac{3(1+D)^2}{(D+\frac{1}{2})^2}$ and the maximum value is $k(t^*) = \frac{3(1+D)^2}{D+\frac{1}{2}}$. We express Eqn. 42 as: $\text{Regret}_t(\rho^\star) \leq \exp(\theta\zeta_t)\left(\theta k(t) - 1\right) + 1 + \frac{\sqrt{6t}}{2} + \frac{\sqrt{12t}}{2}$. With $\theta = \frac{D+\frac{1}{2}}{3(1+D)^2}$ for all $t \geq 1$, the term $\theta k(t) - 1 \leq 0$, so: $\exp(\theta\zeta_t)\left(\theta k(t) - 1\right) \leq 0$. Therefore, by choosing any $\theta \leq \frac{D+\frac{1}{2}}{3(1+D)^2}$, we can bound the regret as:

$$\text{Regret}_t(\rho^\star) \leq 1 + \frac{\sqrt{6t}}{2} + \frac{\sqrt{12t}}{2}, \forall t \in [T]. \tag{43}$$

For any $t \in [T]$, any $\theta < \frac{D+\frac{1}{2}}{3(1+D)^2}$, and utilizing the fact that $\text{Regret}_t(\rho^\star) \geq -\frac{t}{2}$ along with Eqn. 42 we obtain an upper bound on ζ_t:

$$\begin{aligned}
&\exp(\theta\zeta_t)\left(\theta k(t) - 1\right) + 1 + \frac{\sqrt{6t}}{2} + \frac{\sqrt{12t}}{2} - \frac{t}{2} \geq -\frac{t}{2} \\
\implies &\exp(\theta\zeta_t)\left(1 - \theta k(t)\right) \leq 1 + \frac{\sqrt{6t}}{2} + \frac{\sqrt{12t}}{2} \\
\implies &\exp(\theta\zeta_t) \leq \frac{1 + \frac{\sqrt{6t}}{2} + \frac{\sqrt{12t}}{2}}{1 - \theta k(t)} \\
\implies &\zeta_t \leq \frac{1}{\theta} \ln \frac{1 + \frac{\sqrt{6t}}{2} + \frac{\sqrt{12t}}{2}}{1 - \theta\sqrt{12t} + \theta D\sqrt{12t} - \frac{\theta t}{2} - \theta Dt} \\
\implies &\zeta_T \leq \frac{6\sqrt{T}(1+D)^2}{2D+1} \ln \frac{1 + \frac{\sqrt{6T}}{2} + \frac{\sqrt{12T}}{2}}{1 - \frac{1}{\sqrt{T}}},
\end{aligned} \tag{44}$$

where the last line is obtained by setting $\theta = \frac{D+\frac{1}{2}}{3\sqrt{T}(1+D)^2}$. We multiply $\frac{1}{\omega}$ to Eqn. 43 and Eqn. 44 to obtain the bounds for Eqn. 21 and Eqn. 22. In this scenario, minimizing Eqn. 21 and Eqn. 22 leads to an upper bound of Eqn. 35 and Eqn. 36, and we formalize the final bounds in the following theorem.

Theorem 3. *Having $\omega = \frac{1}{2LD}$, $L \leq \sqrt{SHA}$, $D = \sqrt{SHA}$, $\varphi(\zeta_T) = \exp(\theta\zeta_T) - 1$, $\theta = \frac{D+\frac{1}{2}}{3\sqrt{T}(1+D)^2}$, with adversarial loss and constraints, under bandit feedback, and known transition, the expected regret and expected* *cumulative constraint violation (hard) of* `BAG-K` *(in Algorithm 4) is bounded,* $\forall t \in [T]$ *as:*

$$\mathbb{E}[\mathcal{R}_t] \leq 2SHA\left(1+\frac{\sqrt{6t}}{2}+\frac{\sqrt{12t}}{2}\right), \textit{and } \mathbb{E}[\mathcal{Z}_T] \leq \frac{12\sqrt{TSHA}\left(1+\sqrt{SHA}\right)^2}{2\sqrt{SHA}+1}\ln\frac{1+\frac{\sqrt{6T}}{2}+\frac{\sqrt{12T}}{2}}{1-\frac{1}{\sqrt{T}}}. \tag{45}$$

5 Unknown Transition Function

An unknown transition function for the CMDP $\mathcal{M}$ presents two significant challenges. Firstly, there would be a randomness linked with the next-state s_{h+1} in an episode $t \in [T]$. Therefore, the episodic loss in Eqn. 1 and episodic constraint violation in Eqn. 2 would be applicable throughout this section. We re-mention them below for the sake of convenience:

$$V^{\pi_t}(s_0;\boldsymbol{\ell}_t) := \mathbb{E}\left[\sum_{h=0}^{H-1}\ell_{t,h}(s_h,a_h)\;\middle|\; a_h\sim\pi_t(\cdot\mid s_h), s_{h+1}\sim\mathcal{P}(\cdot\mid s_h,a_h)\right], \text{ and}$$
$$V^{\pi_t}(s_0;\boldsymbol{c}_t) := \mathbb{E}\left[\sum_{h=0}^{H-1}c_{t,h}(s_h,a_h)\;\middle|\; a_h\sim\pi_t(\cdot\mid s_h), s_{h+1}\sim\mathcal{P}(\cdot\mid s_h,a_h)\right].$$

Secondly, the decision space Ω is not known in advance owing to the unknown $\mathcal{P}$. The occupancy measure of π_t, i.e., ρ_t, is also unknown. We denote by $\Omega_{\mathcal{P}_i} \subset \Omega$ the set of occupancy measures whose induced transition function belongs to a set of transition functions $\mathcal{P}_i$.

To tackle both the aforementioned challenges, we resort to maintaining a *confidence set* for the unknown transition function $\mathcal{P}$ (Burnetas & Katehakis, 1997) and an epoch-doubling strategy (Jin et al., 2020). Let $X_i(s,a)$ and $Y_i(s'\mid s,a)$ denote the total number of visits made by the algorithm to the pair (s,a) and the triplet (s,a,s') before the epoch $i>1$. For any i and any $h \in [H]^{-1}$, if we have $X_i(s_h,a_h) \geq \max\{1, 2X_{i-1}(s_h,a_h)\}$, then we increment the epoch index i by 1. We define the empirical transition function for the i-th epoch as:

$$\bar{\mathcal{P}}_i(s'\mid s,a) = \frac{Y_i(s'\mid s,a)}{\max\{1, X_i(s,a)\}}. \tag{46}$$

For any $\delta \in (0,1)$, let $\epsilon_i(s'\mid s,a)$ be given by (Jin et al., 2020):

$$\epsilon_i(s'\mid s,a) := 2\sqrt{\frac{\bar{\mathcal{P}}_i(s'\mid s,a)\ln\left(\frac{SAT}{\delta}\right)}{\max\{1,X_i(s,a)-1\}}} + \frac{14\ln\left(\frac{SAT}{\delta}\right)}{3\max\{1,X_i(s,a)-1\}}. \tag{47}$$

Similarly to Jin et al. (2020), for each triple (s,a,s'), we build a confidence set containing all transitions with $\epsilon_i(s'\mid s,a)$ distance from $\bar{\mathcal{P}}_i(s'\mid s,a)$ as given below:

$$\mathcal{P}_i = \left\{\hat{\mathcal{P}} : \left|\hat{\mathcal{P}}(s'\mid s,a)-\bar{\mathcal{P}}_i(s'\mid s,a)\right| \leq \epsilon_i(s'\mid s,a), \forall (s,a,s') \in \mathcal{S}_h\times\mathcal{A}\times\mathcal{S}_{h+1}, h=0,\ldots,H-1\right\}. \tag{48}$$

It is naturally understood that, for $i=1$, $\mathcal{P}_i$ is the set of all transitions such that $\Omega_{\mathcal{P}_i}=\Omega$. In any episode $t\in[T]$, we maintain an occupancy measure $\hat{\rho}_t$ and execute the induced policy $\pi_t = \pi^{\hat{\rho}_t}$, because ρ_t is unknown. Again, from Jin et al. (2020), we have: The true transition function $\mathcal{P}$ is present in the confidence set $\mathcal{P}_i$, i.e., $\mathcal{P}\in\mathcal{P}_i, \forall i$, with probability at least $1-4\delta$.

5.1 Full Feedback and Unknown Transition

Because of full feedback, we get to know every component of the vectors $\boldsymbol{\ell}_t$ and $\boldsymbol{c}_t$ at the end of an episode. The regret $\mathcal{R}_T$ and the hard violation $\mathcal{Z}_T$ to be minimized are respectively given by Eqn. 17 and Eqn. 18. However, since ρ_t is unknown, we cannot compute ∇_t (as in Eqn. 25) like we did in the full feedback case of Section 4.1. We slightly tweak ∇_t from Eqn. 25 to obtain an estimated sub-gradient of $\hat{\mu}_t(\rho_t)$ as:

$$\nabla_t = \begin{cases}\omega\boldsymbol{\ell}_t + \varphi'(\zeta_t)\omega\boldsymbol{c}_t, & \text{if } \langle\hat{\rho}_t, \omega\boldsymbol{c}_t\rangle > 0,\\ \omega\boldsymbol{\ell}_t, & \text{if } \langle\hat{\rho}_t, \omega\boldsymbol{c}_t\rangle \leq 0.\end{cases} \tag{49}$$

Algorithm 5 `Full AdaGrad with Unknown Transition (FAG-U)`

Require: L, D, Euclidean projection operator $\Pi_{\Omega_{\mathcal{P}_i}}(\cdot)$ on the decision set $\Omega_{\mathcal{P}_i}$, $\delta \in (0,1)$.
Set the parameters $\omega = \frac{1}{2LD}$, $\theta = \frac{1}{2k(T)}$, and choose $\varphi(\zeta_t) = \exp(\theta\zeta_t) - 1$, $\forall t \geq 1$.
Initialize epoch index $i = 1$ and set $\zeta_0 = 0$.
Initialize $\mathcal{P}_1$ to be the set of all transition functions.
for $h = 0, \ldots, H-1$ and $\forall (s, a, s') \in \mathcal{S}_h \times \mathcal{A} \times \mathcal{S}_{h+1}$ **do**
  Initialize counters: $X_0(s,a) = X_1(s,a) = Y_0(s' \mid s,a) = Y_1(s' \mid s,a) = 0$.
  Initialize occupancy measure $\hat{\rho}_1(s,a) = \frac{1}{|\mathcal{S}_h| \times |\mathcal{A}| \times |\mathcal{S}_{h+1}|}$.
end for
Initialize policy $\pi_1 = \pi^{\hat{\rho}_1}$.
for $t = 1, \ldots, T$ **do**
  The adversary decides $\boldsymbol{\ell}_t$ and $\boldsymbol{c}_t$.
  for $h = 0, \ldots, H-1$ **do**
    The learner plays $a_h \sim \pi_t(\cdot \mid s_h)$.
    The learner reaches new state $s_{h+1} \sim \mathcal{P}(\cdot \mid s_h, a_h)$ and observes s_{h+1}.
    $X_i(s_h, a_h) \leftarrow X_i(s_h, a_h) + 1$.
    $Y_i(s_{h+1} \mid s_h, a_h) \leftarrow Y_i(s_{h+1} \mid s_h, a_h) + 1$.
    if $X_i(s_h, a_h) \geq \max\{1, 2X_{i-1}(s_h, a_h)\}$ **then**
      $i \leftarrow i + 1$.
      Initialize new counters $\forall (s, a, s') : X_i(s,a) = X_{i-1}(s,a), Y_i(s' \mid s,a) = Y_{i-1}(s' \mid s,a)$.
      Update the confidence set $\mathcal{P}_i$ based on Eqn. 48.
    end if
  end for
  The adversary reveals $\boldsymbol{\ell}_t$ and $\boldsymbol{c}_t$ in *full* feedback.
  Define $\mu_t(\rho_t) = \langle \rho_t, \boldsymbol{\ell}_t \rangle$, and $\nu_t(\rho_t) = \langle \rho_t, \boldsymbol{c}_t \rangle$.
  Compute $\tilde{\mu_t} \leftarrow \omega\mu_t$, and $\tilde{\nu_t} \leftarrow \omega(\nu_t)^+$.
  Compute $\zeta_t = \zeta_{t-1} + \tilde{\nu_t}(\rho_t)$ and $\hat{\mu}_t(\rho_t) \coloneqq \tilde{\mu_t}(\rho_t) + \varphi'(\zeta_t)\tilde{\nu_t}(\rho_t)$.
  According to Eqn. 49, compute the subgradient ∇_t.
  Update $\hat{\rho}_{t+1} = \Pi_{\Omega_{\mathcal{P}_i}}(\hat{\rho}_t - \eta_t \nabla_t)$, where $\eta_t = \frac{\sqrt{2}D}{2\sqrt{\sum_{\tau=1}^{t} \|\nabla_\tau\|^2}}$.
  Update policy $\pi_{t+1} = \pi^{\hat{\rho}_{t+1}}$.
end for
return ρ_T and π_T.

Instead of ρ_t, we here use $\hat{\rho}_t$ for sign determination, which is perfectly doable. The norm of ∇_t (as given in Eqn. 49) has the same upper bound as given in Eqn. 26, i.e., $\|\nabla_t\| \leq \omega L\big(1 + \varphi'(\zeta_t)\big)$. The algorithm we propose for this section, named **F**ull **A**da**G**rad with **U**nknown Transition (`FAG-U`), is fully described above in Algorithm 5.

Lemma 3. *Given a collection of transition functions $\{\mathcal{P}_t^s\}_{s \in \mathcal{S}}$ such that $\mathcal{P}_t^s \in \mathcal{P}_{i_t}$. Here, we use i_t to denote the index of the epoch to which episode t belongs. Let $n_t = \big\{(s_h, a_h, \ell_{t,h}(s_h,a_h), c_{t,h}(s_h,a_h))\big\}_{h=0}^{H-1}$ be the observation of the learner in episode t, and $\mathcal{F}_t$ be the σ-algebra generated by the observations $(n_1, \ldots, n_{t-1})$. Then, with probability at least $1 - 6\delta$, the following holds:*

$$\sum_{t=1}^{T} \sum_{s \in \mathcal{S}, a \in \mathcal{A}} \left| \rho^{\mathcal{P}_t^s, \pi_t}(s,a) - \rho_t(s,a) \right| = \mathcal{O}\left(HS\sqrt{AT \ln\left(\frac{SAT}{\delta}\right)} \right),$$

where $\mathcal{P}_{i_t}$ and $\hat{\rho}_t$ are both $\mathcal{F}_t$-measurable.

When we write $\mathbb{E}_t[\cdot]$, it actually means the conditional expectation $\mathbb{E}[\cdot \mid \mathcal{F}_t]$, that is adapted to the filtration $\mathcal{F}_t$. In Lemma 3, we recall a vital lemma from Jin et al. (2020) regarding how the size of the confidence set $\mathcal{P}_i$ gets smaller with time. This lemma plays a pivotal role in bounding a key term in the decomposition of the surrogate regret.

Again, by the convexity of $\hat{\mu}_\tau$, (for all $\tau \geq 1$), we could decompose the surrogate regret $\text{Regret}'_t(\rho^\star)$ into two terms, as given below:

$$\text{Regret}'_t(\rho^\star) = \sum_{\tau=1}^{t} \hat{\mu}_\tau(\rho_\tau) - \hat{\mu}_\tau(\rho^\star) \leq \overbrace{\sum_{\tau=1}^{t} \langle \hat{\rho}_\tau - \rho^\star, \nabla_\tau \rangle}^{\text{Reg}} + \overbrace{\sum_{\tau=1}^{t} \langle \rho_\tau - \hat{\rho}_\tau, \nabla_\tau \rangle}^{\text{Error}}, \tag{50}$$

where the first term "Reg" is bounded by the regret of AdaGrad used with ∇_t (as given in Eqn. 49), and the second term "Error" quantifies the error of using $\hat{\rho}_t$ to approximate ρ_t. The detailed derivation of the upper bound on "Error" is in Appendix A.6. We can upper bound $\text{Regret}'_t(\rho^\star)$ as (see Appendix A.7 for details):

$$\text{Regret}'_t(\rho^\star) \leq \left(1 + \varphi'(\zeta_t)\right) \left(2D\omega L\sqrt{t} + \omega LHS\sqrt{At \ln\left(\frac{SAt}{\delta}\right)}\right). \tag{51}$$

Choosing $\varphi(\zeta_t) = \exp(\theta\zeta_t) - 1$, putting $\omega = \frac{1}{2LD}$, $D = \sqrt{SHA}$, $L \leq \sqrt{SHA}$, and substituting Eqn. 51 into the regret decomposition inequality of Eqn. 28, we get:

$$\begin{aligned}
\exp(\theta\zeta_t) - 1 + \text{Regret}_t(\rho^\star) &\leq \left(1 + \theta\exp(\theta\zeta_t)\right) \left(2D\omega L\sqrt{t} + \omega LHS\sqrt{At \ln\left(\frac{SAt}{\delta}\right)}\right) \\
\implies \text{Regret}_t(\rho^\star) &\leq \left(1 + \theta\exp(\theta\zeta_t)\right) \left(2D\omega L\sqrt{t} + \omega LHS\sqrt{At \ln\left(\frac{SAt}{\delta}\right)}\right) + 1 - \exp(\theta\zeta_t) \\
\implies \text{Regret}_t(\rho^\star) &\leq \left(1 + \theta\exp(\theta\zeta_t)\right) \left(\sqrt{t} + \frac{1}{2}\sqrt{\frac{SH}{A}}\sqrt{At \ln\left(\frac{SAt}{\delta}\right)}\right) + 1 - \exp(\theta\zeta_t) \\
\implies \text{Regret}_t(\rho^\star) &\leq \left(1 + \theta\exp(\theta\zeta_t)\right) \left(\sqrt{t} + \sqrt{\frac{SHt}{4} \ln\left(\frac{SAt}{\delta}\right)}\right) + 1 - \exp(\theta\zeta_t) \\
\implies \text{Regret}_t(\rho^\star) &\leq 1 + k(t) + \exp(\theta\zeta_t)\left(\theta k(t) - 1\right),
\end{aligned} \tag{52}$$

where $k(t) = \sqrt{t} + \sqrt{\frac{SHt}{4} \ln\left(\frac{SAt}{\delta}\right)}$. For upper bounding $\text{Regret}_t(\rho^\star)$ in Eqn. 52, we need to choose θ such that the co-efficient of $\exp(\theta\zeta_t)$ is non-positive. In other words, we require $\theta k(t) - 1 \leq 0 \implies \theta \leq \frac{1}{k(t)}$. Therefore, for any θ less than or equal to $\frac{1}{k(T)}$, we can bound the regret as:

$$\text{Regret}_t(\rho^\star) \leq 1 + \sqrt{t} + \sqrt{\frac{SHt}{4} \ln\left(\frac{SAt}{\delta}\right)}, \forall t \in [T]. \tag{53}$$

Choosing any $\theta < \frac{1}{k(T)}$, and combining $\text{Regret}_t(\rho^\star) \geq -\frac{t}{2}$ with Eqn. 52, for any $t \in [T]$, we obtain:

$$\begin{aligned}
1 + k(t) + \exp(\theta\zeta_t)\left(\theta k(t) - 1\right) &\geq -\frac{t}{2} \\
\implies \exp(\theta\zeta_t)\left(1 - \theta k(t)\right) &\leq 1 + k(t) + \frac{t}{2} \\
\implies \exp(\theta\zeta_t) &\leq \frac{1 + k(t) + \frac{t}{2}}{1 - \theta k(t)} \\
\implies \zeta_t &\leq \frac{1}{\theta} \ln \frac{1 + k(t) + \frac{t}{2}}{1 - \theta k(t)} \\
\implies \zeta_T &\leq \left(2\sqrt{T} + 2\sqrt{\frac{SHT}{4} \ln\left(\frac{SAT}{\delta}\right)}\right) \\
&\quad \times \ln\left(2 + 2T + \sqrt{T} + \sqrt{\frac{SHT}{4} \ln\left(\frac{SAT}{\delta}\right)}\right).
\end{aligned} \tag{54}$$

The last line is obtained by selecting $\theta = \frac{1}{2k(T)} = \frac{1}{2\sqrt{T}+2\sqrt{\frac{SHT}{4}\ln\left(\frac{SAT}{\delta}\right)}}$. On multiplying ω^{-1} to Eqn. 53 and Eqn. 54, we get the bounds for Eqn. 21 and Eqn. 22. In this scenario, minimizing Eqn. 21 and Eqn. 22 leads to an upper bound of Eqn. 17 and Eqn. 18, and we formalize the final bounds of `FAG-U` in the following theorem.

Theorem 4. *Set the parameters $\omega = \frac{1}{2LD}$, $L \leq \sqrt{SHA}$, $D = \sqrt{SHA}$, $\theta = \frac{1}{2k(T)} = \frac{1}{2\sqrt{T}+2\sqrt{\frac{SHT}{4}\ln\left(\frac{SAT}{\delta}\right)}}$, and choose $\varphi(\zeta_T) = \exp(\theta\zeta_T) - 1$. Under adversarial loss and constraints, with full feedback, and unknown transition, the regret $\mathcal{R}_T$ and cumulative hard violation $\mathcal{Z}_T$ of `FAG-U` (in Algorithm 5) are bounded, $\forall t \in [T]$, with probability at least $1-\delta$ as:*

$$\begin{aligned}
&\mathcal{R}_t \leq 2SHA\left(1 + \sqrt{t} + \sqrt{\frac{SHt}{4}\ln\left(\frac{SAt}{\delta}\right)}\right), \textit{ and} \\
&\mathcal{Z}_T \leq 2SHA\left(2\sqrt{T} + 2\sqrt{\frac{SHT}{4}\ln\left(\frac{SAT}{\delta}\right)}\right)\ln\left(2 + 2T + \sqrt{T} + \sqrt{\frac{SHT}{4}\ln\left(\frac{SAT}{\delta}\right)}\right).
\end{aligned} \tag{55}$$

5.2 Bandit Feedback and Unknown Transition

In this case, the expected regret $\mathbb{E}[\mathcal{R}_T]$ and the expected hard cumulative constraint violation $\mathbb{E}[\mathcal{Z}_T]$ to be minimized are respectively given by Eqn. 35 and Eqn. 36. Due to the occupancy measure ρ_t being unknown, estimators cannot be constructed by following Eqn. 37. Inspired by Jin et al. (2020), we replace $\rho_t(s,a)$ with an *upper occupancy bound* given by:

$$u_t(s,a) = \max_{\hat{\mathcal{P}}\in\mathcal{P}_i} \rho^{\hat{\mathcal{P}},\pi_t}(s,a). \tag{56}$$

Thus, we can now have the following estimators:

$$\hat{\ell}_{t,h}(s,a) = \frac{\ell_{t,h}(s,a)}{u_t(s,a)+\Lambda_t}\mathbf{1}_t(s,a), \text{ and } \hat{c}_{t,h}(s,a) = \frac{c_{t,h}(s,a)}{u_t(s,a)+\Lambda_t}\mathbf{1}_t(s,a), \tag{57}$$

where $\Lambda_t > 0$ is an appropriately chosen parameter (to be fixed later) and $\mathbf{1}_t(s,a)$ is 1 if (s,a) is visited during episode t and 0 otherwise. The estimated loss and constraint-cost vectors are respectively defined as $\hat{\boldsymbol{\ell}}_t$ and $\hat{\boldsymbol{c}}_t$, having entries of the form $\hat{\ell}_{t,h}$ and $\hat{c}_{t,h}$ for all $t \in [T]$ and $h \in [H]^{-1}$. Clearly, $\hat{\boldsymbol{\ell}}_t$ and $\hat{\boldsymbol{c}}_t$ both have at most H non-zero entries. Unlike Eqn. 49, we cannot fully compute the sub-gradient. Hence, we resort to a biased estimate as follows:

$$\widehat{\nabla}_t = \begin{cases} \omega\hat{\boldsymbol{\ell}}_t + \varphi'(\zeta_t)\omega\hat{\boldsymbol{c}}_t, & \text{if } \mathcal{C}_t > 0, \\ \omega\hat{\boldsymbol{\ell}}_t, & \text{if } \mathcal{C}_t \leq 0, \end{cases} \tag{58}$$

where $\mathcal{C}_t = \sum_{h=0}^{H-1} c_{t,h}(s_h,a_h)$ is the observed constraint violation in the t-th episode. Let $\boldsymbol{b}_t$ denote the bias vector of $\widehat{\nabla}_t$ which is given by: $\boldsymbol{b}_t = \mathbb{E}_t[\widehat{\nabla}_t] - \nabla_t$. Performing similar calculations as in Appendix A.4, it can be shown for $\boldsymbol{b}_t$ and $\widehat{\nabla}_t$ (as given in Eqn. 58) that,

$$\|\boldsymbol{b}_t\| \leq \omega L + \omega\varphi'(\zeta_t)\big(L + \sqrt{H}/\Lambda_t\big), \text{ and } \left\|\widehat{\nabla}_t\right\| \leq \frac{\omega\sqrt{H}}{\Lambda_t}\big(1+\varphi'(\zeta_t)\big).$$

Thus, implying by the triangle inequality for norms:

$$\begin{aligned}
\left\|\mathbb{E}_t[\widehat{\nabla}_t]\right\| &\leq \|\boldsymbol{b}_t\| + \|\nabla_t\| \\
&\leq \omega L + \omega\varphi'(\zeta_t)\big(L + \sqrt{H}/\Lambda_t\big) + \omega L\big(1+\varphi'(\zeta_t)\big).
\end{aligned} \tag{59}$$

We chalked the **B**andit **A**da**G**rad with **U**nknown Transition (`BAG-U`) algorithm for this section. It is formally depicted in Algorithm 6, and the `COMP-UOB` method is as given in Algorithm 3 of Jin et al. (2020). By the

Algorithm 6 `Bandit AdaGrad with Unknown Transition (BAG-U)`

Require: L, D, Euclidean projection operator $\Pi_{\Omega_{\mathcal{P}_i}}(\cdot)$ on the decision set $\Omega_{\mathcal{P}_i}$, $\delta \in (0,1)$.

Set the parameters $\omega = \frac{1}{2LD}$, $\theta = \frac{1}{2m(T)}$, $\Lambda_t = \omega\sqrt{H}$, and choose $\varphi(\zeta_t) = \exp(\theta\zeta_t) - 1$, $\forall t \geq 1$.
Initialize epoch index $i = 1$ and set $\zeta_0 = 0$.
Initialize $\mathcal{P}_1$ to be the set of all transition functions.
for $h = 0, \ldots, H-1$ and $\forall (s, a, s') \in \mathcal{S}_h \times \mathcal{A} \times \mathcal{S}_{h+1}$ **do**
  Initialize counters: $X_0(s,a) = X_1(s,a) = Y_0(s' \mid s,a) = Y_1(s' \mid s,a) = 0$.
  Initialize occupancy measure $\hat{\rho}_1(s,a) = \frac{1}{|\mathcal{S}_h| \times |\mathcal{A}| \times |\mathcal{S}_{h+1}|}$.
end for
Initialize policy $\pi_1 = \pi^{\hat{\rho}_1}$.
for $t = 1, \ldots, T$ **do**
  The adversary decides $\boldsymbol{\ell}_t$ and $\boldsymbol{c}_t$.
  Set $\mathcal{C}_t \leftarrow 0$
  for $h = 0, \ldots, H-1$ **do**
    The learner plays $a_h \sim \pi_t(\cdot \mid s_h)$.
    The learner reaches new state $s_{h+1} \sim \mathcal{P}(\cdot \mid s_h, a_h)$ and observes s_{h+1}.
    $X_i(s_h, a_h) \leftarrow X_i(s_h, a_h) + 1$.
    $Y_i(s_{h+1} \mid s_h, a_h) \leftarrow Y_i(s_{h+1} \mid s_h, a_h) + 1$.
    Compute $u_t(s_h, a_h) = \texttt{COMP-UOB}(\pi_t, s_h, a_h, \mathcal{P}_i)$.
    if $X_i(s_h, a_h) \geq \max\{1, 2X_{i-1}(s_h, a_h)\}$ **then**
      $i \leftarrow i + 1$.
      Initialize new counters $\forall (s, a, s') : X_i(s,a) = X_{i-1}(s,a), Y_i(s' \mid s,a) = Y_{i-1}(s' \mid s,a)$.
      Update the confidence set $\mathcal{P}_i$ based on Eqn. 48.
    end if
  end for
  The adversary reveals $\boldsymbol{\ell}_t$ and $\boldsymbol{c}_t$ in *bandit* feedback.
  Compute $\mathcal{C}_t = \sum_{h=0}^{H-1} c_{t,h}(s_h, a_h)$ for the observed state-action pairs.
  Define $\mu_t(\rho_t) = \langle \rho_t, \boldsymbol{\ell}_t \rangle$, and $\nu_t(\rho_t) = \langle \rho_t, \boldsymbol{c}_t \rangle$.
  Compute $\tilde{\mu_t} \leftarrow \omega\mu_t$, and $\tilde{\nu_t} \leftarrow \omega(\nu_t)^+$.
  Construct estimators $\hat{\ell}_{t,h}(s,a)$ and $\hat{c}_{t,h}(s,a)$ according to Eqn. 57.
  Compute $\zeta_t = \zeta_{t-1} + \tilde{\nu_t}(\rho_t)$ and $\hat{\mu}_t(\rho_t) \coloneqq \tilde{\mu_t}(\rho_t) + \varphi'(\zeta_t)\tilde{\nu_t}(\rho_t)$.
  Compute $\widehat{\nabla}_t$ by Eqn. 58.
  Update $\hat{\rho}_{t+1} = \Pi_{\Omega_{\mathcal{P}_i}}(\hat{\rho}_t - \eta_t \widehat{\nabla}_t)$, where $\eta_t = \frac{\sqrt{2}D}{2\sqrt{\sum_{\tau=1}^t \|\widehat{\nabla}_\tau\|^2}}$.
  Update policy $\pi_{t+1} = \pi^{\hat{\rho}_{t+1}}$.
end for
return ρ_T and π_T.

convexity of $\hat{\mu}_\tau$, (for all $\tau \geq 1$), the surrogate regret $\text{Regret}'_t(\rho^\star)$ could be decomposed into four terms as:

$$
\begin{aligned}
\text{Regret}'_t(\rho^\star) &= \sum_{\tau=1}^t \hat{\mu}_\tau(\rho_\tau) - \hat{\mu}_\tau(\rho^\star) \\
&\leq \sum_{\tau=1}^t \langle \rho_\tau - \rho^\star, \nabla_\tau \rangle \\
&\leq \sum_{\tau=1}^t \langle \hat{\rho}_\tau - \rho^\star, \nabla_\tau \rangle + \sum_{\tau=1}^t \langle \rho_\tau - \hat{\rho}_\tau, \nabla_\tau \rangle \\
&\leq \overbrace{\sum_{\tau=1}^t \langle \hat{\rho}_\tau - \rho^\star, \widehat{\nabla}_\tau \rangle}^{\text{Reg}} + \overbrace{\sum_{\tau=1}^t \langle \rho_\tau - \hat{\rho}_\tau, \nabla_\tau \rangle}^{\text{Error}} + \overbrace{\sum_{\tau=1}^t \langle \hat{\rho}_\tau, \nabla_\tau - \widehat{\nabla}_\tau \rangle}^{\text{Bias1}} + \overbrace{\sum_{\tau=1}^t \langle \rho^\star, \widehat{\nabla}_\tau - \nabla_\tau \rangle}^{\text{Bias2}}, \qquad (60)
\end{aligned}
$$

where "Reg" is simply bounded by the regret of AdaGrad used with $\widehat{\nabla}_t$ (as presented in Eqn. 58), "Error" is the error of using $\hat{\rho}_t$ to approximate ρ_t, "Bias1" measures how much $\widehat{\nabla}_\tau$ underestimates ∇_τ weighted by $\hat{\rho}_\tau$, and "Bias2" measures the error of $\widehat{\nabla}_\tau$ relative to ∇_τ when weighted by $\rho^\star$.

With probability $1-\delta$ and with $\Lambda_t = \omega\sqrt{H}$, we have the following upper bound on $\mathrm{Regret}'_t(\rho^\star)$ (see Appendix A.8 for detailed calculations):

$$\begin{aligned}\mathrm{Regret}'_t(\rho^\star) \le\; & 2D\sqrt{t}\,(1+\varphi'(\zeta_t)) + \omega LHS\sqrt{At\ln\left(\frac{SAt}{\delta}\right)}\cdot\big(1+\varphi'(\zeta_t)\big)\\ &+ \frac{2\big(1+\varphi'(\zeta_t)\big)}{\sqrt{H}}\sqrt{2t\ln(2/\delta)} - \omega L - \omega Lt\cdot\varphi'(\zeta_t) - t\cdot\varphi'(\zeta_t)\\ &+ \omega H\ln\frac{H}{\delta} + \omega\varphi'(\zeta_t)\cdot H\ln\frac{H}{\delta} - \omega t\cdot\varphi'(\zeta_t).\end{aligned} \tag{61}$$

Substituting Eqn. 61 into the regret decomposition inequality of Eqn. 28, and choosing $\varphi(\zeta_t) = \exp(\theta\zeta_t)-1$, we have:

$$\begin{aligned}\exp(\theta\zeta_t) - 1 + \mathrm{Regret}_t(\rho^\star) \le\; & 2D\sqrt{t}\,(1+\theta\exp(\theta\zeta_t)) + \omega LHS\sqrt{At\ln\left(\frac{SAt}{\delta}\right)}\cdot\big(1+\theta\exp(\theta\zeta_t)\big)\\ &+ \frac{2\big(1+\theta\exp(\theta\zeta_t)\big)}{\sqrt{H}}\sqrt{2t\ln(2/\delta)} - \omega L - \omega Lt\cdot\theta\exp(\theta\zeta_t)\\ &- t\cdot\theta\exp(\theta\zeta_t) + \omega H\ln\frac{H}{\delta} + \omega\theta\exp(\theta\zeta_t)\cdot H\ln\frac{H}{\delta} - \omega t\cdot\theta\exp(\theta\zeta_t).\end{aligned}$$

Grouping all the terms involving $\exp(\theta\zeta_t)$ into one side in the above expression,

$$\begin{aligned}\mathrm{Regret}_t(\rho^\star) \le\; & \exp(\theta\zeta_t)\Bigg(\theta 2D\sqrt{t} + \theta\omega LHS\sqrt{At\ln\left(\frac{SAt}{\delta}\right)} + \frac{2\theta}{\sqrt{H}}\sqrt{2t\ln(2/\delta)}\\ &- \theta\omega Lt - \theta t + \theta\omega H\ln\frac{H}{\delta} - \theta\omega t - 1\Bigg) + 2D\sqrt{t} + \omega LHS\sqrt{At\ln\left(\frac{SAt}{\delta}\right)}\\ &+ \frac{2}{\sqrt{H}}\sqrt{2t\ln(2/\delta)} - \omega L + \omega H\ln\frac{H}{\delta} + 1\\ \implies \mathrm{Regret}_t(\rho^\star) \le\; & 2D\sqrt{t} + \omega LHS\sqrt{At\ln\left(\frac{SAt}{\delta}\right)} + \frac{2}{\sqrt{H}}\sqrt{2t\ln(2/\delta)} - \omega L + \omega H\ln\frac{H}{\delta} + 1\\ &+ \exp(\theta\zeta_t)\Bigg(\theta 2D\sqrt{t} + \theta\omega LHS\sqrt{At\ln\left(\frac{SAt}{\delta}\right)} + \frac{2\theta}{\sqrt{H}}\sqrt{2t\ln(2/\delta)}\\ &+ \theta\omega H\ln\frac{H}{\delta} - 1\Bigg).\end{aligned} \tag{62}$$

Let $m(t) = 2D\sqrt{t} + \omega LHS\sqrt{At\ln\left(\frac{SAt}{\delta}\right)} + \frac{2}{\sqrt{H}}\sqrt{2t\ln(2/\delta)} + \omega H\ln\frac{H}{\delta}$, for all $t\in[T]$. We can rewrite the regret in Eqn. 62 as: $\mathrm{Regret}_t(\rho^\star) \le 2D\sqrt{t} + \omega LHS\sqrt{At\ln\left(\frac{SAt}{\delta}\right)} + \frac{2}{\sqrt{H}}\sqrt{2t\ln(2/\delta)} - \omega L + \omega H\ln\frac{H}{\delta} + 1 + \exp(\theta\zeta_t)\,(\theta m(t) - 1)$. Thus, having any $\theta \le \frac{1}{m(T)}$, we can ensure that the regret is nicely bounded with probability at least $1-\delta$, as given below:

$$\mathrm{Regret}_t(\rho^\star) \le 2D\sqrt{t} + \omega LHS\sqrt{At\ln\left(\frac{SAt}{\delta}\right)} + \frac{2}{\sqrt{H}}\sqrt{2t\ln(2/\delta)} - \omega L + \omega H\ln\frac{H}{\delta} + 1,\ \forall t\in[T]. \tag{63}$$

Selecting any $\theta < \frac{1}{m(T)}$, and combining $\text{Regret}_t(\rho^\star) \geq -\frac{t}{2}$ with Eqn. 62, we obtain an upper bound on ζ_t, for any $t \in [T]$, with probability at least $1-\delta$ as follows:

$$
\begin{aligned}
&2D\sqrt{t} + \omega LHS\sqrt{At\ln\left(\frac{SAt}{\delta}\right)} + \frac{2}{\sqrt{H}}\sqrt{2t\ln(2/\delta)} - \omega L + \omega H\ln\frac{H}{\delta} + 1 + \exp(\theta\zeta_t)\left(\theta m(t) - 1\right) \geq -\frac{t}{2} \\
&\implies \exp(\theta\zeta_t)\left(1 - \theta m(t)\right) \leq 1 + 2D\sqrt{t} + \omega LHS\sqrt{At\ln\left(\frac{SAt}{\delta}\right)} + \frac{2}{\sqrt{H}}\sqrt{2t\ln(2/\delta)} - \omega L + \omega H\ln\frac{H}{\delta} + \frac{t}{2} \\
&\implies \exp(\theta\zeta_t) \leq \frac{1 + 2D\sqrt{t} + \omega LHS\sqrt{At\ln\left(\frac{SAt}{\delta}\right)} + \frac{2}{\sqrt{H}}\sqrt{2t\ln(2/\delta)} - \omega L + \omega H\ln\frac{H}{\delta} + \frac{t}{2}}{1 - \theta m(t)} \\
&\implies \zeta_t \leq \frac{1}{\theta}\ln\frac{1 + 2D\sqrt{t} + \omega LHS\sqrt{At\ln\left(\frac{SAt}{\delta}\right)} + \frac{2}{\sqrt{H}}\sqrt{2t\ln(2/\delta)} - \omega L + \omega H\ln\frac{H}{\delta} + \frac{t}{2}}{1 - \theta m(t)} \\
&\implies \zeta_T \leq 4D\sqrt{T} + 2\omega LHS\sqrt{AT\ln\left(\frac{SAT}{\delta}\right)} + \frac{4}{\sqrt{H}}\sqrt{2T\ln(2/\delta)} + 2\omega H\ln\frac{H}{\delta} \\
&\times\ln\left(2 + 4D\sqrt{T} + 2\omega LHS\sqrt{AT\ln\left(\frac{SAT}{\delta}\right)} + \frac{4}{\sqrt{H}}\sqrt{2T\ln(2/\delta)} - 2\omega L + 2\omega H\ln\frac{H}{\delta} + T\right). \quad (64)
\end{aligned}
$$

where the last line is obtained by choosing $\theta = \frac{1}{2m(T)}$. Putting $\omega = \frac{1}{2LD}$, $L \leq \sqrt{SHA}$, and $D = \sqrt{SHA}$ into Eqn. 63 we have $\forall t \in [T]$:

$$
\begin{aligned}
&\text{Regret}_t(\rho^\star) \leq 2\sqrt{SHAt} + \frac{1}{2}\sqrt{SHt\ln\left(\frac{SAt}{\delta}\right)} + \frac{2}{\sqrt{H}}\sqrt{2t\ln(2/\delta)} - \frac{1}{2\sqrt{SHA}} + \frac{H}{2SHA}\ln\frac{H}{\delta} + 1 \\
&\implies \text{Regret}_t(\rho^\star) \leq \mathcal{O}\left(\sqrt{SHAt} + \sqrt{SHt\ln\left(\frac{SAt}{\delta}\right)} + \sqrt{\frac{t\ln(2/\delta)}{H}}\right). \quad (65)
\end{aligned}
$$

Putting $\omega = \frac{1}{2LD}$, $L \leq \sqrt{SHA}$, and $D = \sqrt{SHA}$ into Eqn. 64 we obtain:

$$
\begin{aligned}
&\zeta_T \leq 4\sqrt{SHAT} + \sqrt{SHT\ln\left(\frac{SAT}{\delta}\right)} + \frac{4}{\sqrt{H}}\sqrt{2T\ln(2/\delta)} + \frac{1}{SA}\ln\frac{H}{\delta} \\
&\quad\times\ln\left(2 + 4\sqrt{SHAT} + \sqrt{SHT\ln\left(\frac{SAT}{\delta}\right)} + \frac{4}{\sqrt{H}}\sqrt{2T\ln(2/\delta)} - \frac{1}{\sqrt{SHA}} + \frac{1}{SA}\ln\frac{H}{\delta} + T\right) \\
&\implies \zeta_T \leq \mathcal{O}\Bigg(\sqrt{SHAT} + \sqrt{SHT\ln\left(\frac{SAT}{\delta}\right)} + \sqrt{\frac{T\ln(2/\delta)}{H}} \\
&\quad\times\ln\left(\sqrt{SHAT} + \sqrt{SHT\ln\left(\frac{SAT}{\delta}\right)} + \sqrt{\frac{T\ln(2/\delta)}{H}} + T\right)\Bigg). \quad (66)
\end{aligned}
$$

Scaling back Eqn. 65 and Eqn. 66 by a factor of $\frac{1}{\omega}$ respectively attains an upper bound for Eqn. 35 and Eqn. 36. We formally state the final bounds in the theorem below.

Theorem 5. *Set the parameters $\delta \in (0,1)$, $\theta = \frac{1}{2D\sqrt{T} + \omega LHS\sqrt{AT\ln\left(\frac{SAT}{\delta}\right)} + \frac{2}{\sqrt{H}}\sqrt{2T\ln(2/\delta)} + \omega H\ln\frac{H}{\delta}}$, $\omega = \frac{1}{2LD}$, $L \leq \sqrt{SHA}$, $D = \sqrt{SHA}$, and choose $\varphi(\zeta_T) = \exp(\theta\zeta_T) - 1$. Having adversarial loss and constraints, under bandit feedback, and unknown transition, the expected regret and the expected cumulative constraint violation*

Table 2: Listing the regret and hard cumulative constraint violation bounds for all four known-unknown transition and full-bandit feedback settings. The bounds have been written using the very standard $\widetilde{\mathcal{O}}(\cdot)$ notation, which ignores all the logarithmic factors.

Algortihm	Transition	Feedback	Regret Bound	Hard Violation Bound
`FAG-K`	`Known`	`Full`	$\mathcal{O}(SHA\sqrt{T})$	$\widetilde{\mathcal{O}}(SHA\sqrt{T})$
`BAG-K`	`Known`	`Bandit`	$\mathcal{O}(SHA\sqrt{T})$	$\widetilde{\mathcal{O}}(SHA\sqrt{T})$
`FAG-U`	`Unknown`	`Full`	$\widetilde{\mathcal{O}}(SHA\sqrt{T})$	$\widetilde{\mathcal{O}}(SHA\sqrt{T})$
`BAG-U`	`Unknown`	`Bandit`	$\widetilde{\mathcal{O}}(S^{\frac{3}{2}}H^{\frac{3}{2}}A^{\frac{3}{2}}\sqrt{T})$	$\widetilde{\mathcal{O}}(S^{\frac{3}{2}}H^{\frac{3}{2}}A^{\frac{3}{2}}\sqrt{T})$

(hard) of `BAG-U` *(in Algorithm 6) are bounded,* $\forall t \in [T]$, *with probability at least* $1-\delta$ *as:*

$$\mathbb{E}[\mathcal{R}_t] \leq \mathcal{O}\left((SHA)^{\frac{3}{2}}\sqrt{t} + SHA\sqrt{SHt\ln\left(\frac{SAt}{\delta}\right)} + SHA\sqrt{\frac{t\ln(2/\delta)}{H}}\right), \ \textit{and} \tag{67}$$

$$\begin{aligned}\mathbb{E}[\mathcal{Z}_T] \leq \mathcal{O}\Bigg(&(SHA)^{\frac{3}{2}}\sqrt{T} + SHA\sqrt{SHT\ln\left(\frac{SAT}{\delta}\right)} + SHA\sqrt{\frac{T\ln(2/\delta)}{H}} \\ &\times \ln\left((SHA)^{\frac{3}{2}}\sqrt{T} + SHA\sqrt{SHT\ln\left(\frac{SAT}{\delta}\right)} + SHA\sqrt{\frac{T\ln(2/\delta)}{H}} + T\right)\Bigg).\end{aligned} \tag{68}$$

All of our proposed algorithms, i.e., `FAG-K`, `BAG-K`, `FAG-U`, and `BAG-U`, perform only one Euclidean projection onto Ω per episode. Since Ω is a simple polytope (as given in Definition 2), the projection amounts to solving a sparse quadratic program with linear flow constraints. In contrast, primal-dual methods (Stradi et al., 2024a;b; 2025a; Müller et al., 2024) must maintain dual variables and update them at each step, which requires two expensive coupled updates (e.g., adding regularizers and using approximations to the Lagrangian). Hence, the computational cost of our updates is lower: one first-order gradient step followed by a single projection, without dependence on Slater-type conditions or instance-dependent feasible policies.

6 Optimality of the bounds

Minimax Optimality: It is stated in Jin et al. (2018) and Jin et al. (2020) that the regret of any algorithm for solving episodic unconstrained adversarial MDPs with full feedback should be at least $\Omega(\sqrt{H^2SAT})$. To the best of our knowledge, no regret and violation lower bounds are known for episodic adversarial CMDPs. For COCO with adversarial constraints, a lower bound of $\Omega(\sqrt{T})$ exists for both regret and hard constraint violation (Sinha & Vaze, 2024). Owing to all the aforementioned results from different settings, we believe that the $\widetilde{\mathcal{O}}(\sqrt{T})$ regret and violation bounds in our adversarial CMDPs ($\mathcal{O}(\sqrt{T})$ regret for known transitions) are tight and cannot be improved in the minimax sense. This optimality holds across all four feedback/transition settings we address, making ours the first comprehensive set of minimax optimal algorithms for adversarial CMDPs for hard cumulative constraint violation, without Slater's condition, and without access to a strictly feasible policy.

Constant Factors: Like any other well-known algorithm in the vast expanse of online learning in finite-horizon episodic CMDPs, the effect of the constants (i.e., every variable apart from T) can matter in practice. In Table 2, we re-state all our derived bounds as given in Theorem 2, Theorem 3, Theorem 4, and Theorem 5. The results of Germano et al. (2023) and Stradi et al. (2024b) are not directly comparable with ours because, although they consider adversarial loss and constraints, their $\widetilde{\mathcal{O}}(\sqrt{T})$ bounds are reliant on the slackness parameter of Slater's condition. However, for the sake of a loose comparison, we mention that both works have a SH^2A factor in their bounds. As stated in Theorem 5.1 of Zhu et al. (2025), constant factors of S^2AH^3 and $H^{\frac{3}{2}}\sqrt{SA}$ are present both in the regret and violation bounds. Given our challenging problem setup, the gaps we close, and the optimal bounds we derive without assumptions, we argue that the constants of SHA and $S^{\frac{3}{2}}H^{\frac{3}{2}}A^{\frac{3}{2}}$ in our attained results might not be optimal, but are not too bad either. In the light of this statement, we leave an intriguing open problem as a future work: improving the SHA and $S^{\frac{3}{2}}H^{\frac{3}{2}}A^{\frac{3}{2}}$ dependence, respectively, for known and unknown transitions in fully adversarial CMDPs.

7 Conclusion

This is the first work to tackle and solve the hallowed problem of online learning in finite-horizon episodic CMDPs under adversarial losses and constraints, bandit feedback, and unknown transition dynamics. By leveraging a reduction to COCO and building on the techniques introduced by the seminal work of Sinha & Vaze (2024), we developed simple and efficient algorithms that require only a single Euclidean projection per episode. Our approach achieves optimal regret and hard cumulative constraint violation bounds across all four combinations of known-unknown transitions and full-bandit feedback settings – without relying on Slater's condition and without having any knowledge about a strictly feasible policy. In other words, we make no additional assumptions except for the standard assumptions in the COCO literature.

Our results not only close several theoretical gaps in the literature but also offer a unified and pedagogically valuable framework for understanding the connections between online learning in CMDPs and COCO. The construction of biased estimators for bandit feedback settings may also be of independent interest for future research and educational purposes. This work lays a foundation for more practical and robust constrained reinforcement learning systems, opening up new avenues for exploring the interplay between online learning, constrained convex optimization, and adversarial CMDPs.

Broader Impact Statement

We propose efficient algorithms for constrained online learning in CMDPs that achieve optimal regret and hard violation bounds in adversarial environments. Thus, this strengthens the theoretical foundations of safe decision-making in CMDPs. One can apply our algorithms to domains such as healthcare, autonomous driving, and resource allocation, where respecting safety and budget constraints is critical.

Like any progress in the adversarial learning settings, these methods could be misused in settings such as manipulative recommendation systems or exploitative bidding strategies. The contributions of this work are primarily theoretical and not intended for direct deployment in safety-critical systems without multiple layers of safeguards. Responsible application requires rigorous testing, domain-specific validation, and ethical oversight.

References

E. Altman. *Constrained Markov Decision Processes*. Chapman and Hall, 1999.

Francesco Bacchiocchi, Francesco Emanuele Stradi, Matteo Papini, Alberto Maria Metelli, and Nicola Gatti. Online learning with off-policy feedback in adversarial mdps. In *Proceedings of the Thirty-Third International Joint Conference on Artificial Intelligence*, IJCAI '24, 2024. ISBN 978-1-956792-04-1. doi: 10.24963/ijcai.2024/409. URL `https://doi.org/10.24963/ijcai.2024/409`.

Qinbo Bai, Vaneet Aggarwal, and Ather Gattami. Provably sample-efficient model-free algorithm for mdps with peak constraints. *J. Mach. Learn. Res.*, 24(1), January 2023. ISSN 1532-4435.

Martino Bernasconi, Federico Cacciamani, Matteo Castiglioni, Alberto Marchesi, Nicola Gatti, and Francesco Trovò. Safe learning in tree-form sequential decision making: Handling hard and soft constraints. In *International Conference on Machine Learning*, pp. 1854–1873. PMLR, 2022.

Apostolos N. Burnetas and Michael N. Katehakis. Optimal adaptive policies for markov decision processes. *Math. Oper. Res.*, 22(1):222–255, February 1997. ISSN 0364-765X. doi: 10.1287/moor.22.1.222. URL `https://doi.org/10.1287/moor.22.1.222`.

Matteo Castiglioni, Andrea Celli, and Christian Kroer. Online learning with knapsacks: the best of both worlds. In Kamalika Chaudhuri, Stefanie Jegelka, Le Song, Csaba Szepesvari, Gang Niu, and Sivan Sabato (eds.), *Proceedings of the 39th International Conference on Machine Learning*, volume 162 of *Proceedings of Machine Learning Research*, pp. 2767–2783. PMLR, 17–23 Jul 2022a. URL https://proceedings.mlr.press/v162/castiglioni22a.html.

Matteo Castiglioni, Andrea Celli, Alberto Marchesi, Giulia Romano, and Nicola Gatti. A unifying framework for online optimization with long-term constraints. In *Proceedings of the 36th International Conference on Neural Information Processing Systems*, NIPS '22, Red Hook, NY, USA, 2022b. Curran Associates Inc. ISBN 9781713871088.

Nicolo Cesa-Bianchi and Gabor Lugosi. *Prediction, Learning, and Games.* Cambridge University Press, USA, 2006. ISBN 0521841089.

Liyu Chen, Rahul Jain, and Haipeng Luo. Learning infinite-horizon average-reward markov decision process with constraints. In *International Conference on Machine Learning*, pp. 3246–3270. PMLR, 2022.

DeepSeek-AI, Aixin Liu, Bei Feng, Bing Xue, Bingxuan Wang, Bochao Wu, Chengda Lu, Chenggang Zhao, Chengqi Deng, Chenyu Zhang, Chong Ruan, Damai Dai, Daya Guo, Dejian Yang, Deli Chen, Dongjie Ji, Erhang Li, Fangyun Lin, Fucong Dai, Fuli Luo, Guangbo Hao, Guanting Chen, Guowei Li, H. Zhang, Han Bao, Hanwei Xu, Haocheng Wang, Haowei Zhang, Honghui Ding, Huajian Xin, Huazuo Gao, Hui Li, Hui Qu, J. L. Cai, Jian Liang, Jianzhong Guo, Jiaqi Ni, Jiashi Li, Jiawei Wang, Jin Chen, Jingchang Chen, Jingyang Yuan, Junjie Qiu, Junlong Li, Junxiao Song, Kai Dong, Kai Hu, Kaige Gao, Kang Guan, Kexin Huang, Kuai Yu, Lean Wang, Lecong Zhang, Lei Xu, Leyi Xia, Liang Zhao, Litong Wang, Liyue Zhang, Meng Li, Miaojun Wang, Mingchuan Zhang, Minghua Zhang, Minghui Tang, Mingming Li, Ning Tian, Panpan Huang, Peiyi Wang, Peng Zhang, Qiancheng Wang, Qihao Zhu, Qinyu Chen, Qiushi Du, R. J. Chen, R. L. Jin, Ruiqi Ge, Ruisong Zhang, Ruizhe Pan, Runji Wang, Runxin Xu, Ruoyu Zhang, Ruyi Chen, S. S. Li, Shanghao Lu, Shangyan Zhou, Shanhuang Chen, Shaoqing Wu, Shengfeng Ye, Shengfeng Ye, Shirong Ma, Shiyu Wang, Shuang Zhou, Shuiping Yu, Shunfeng Zhou, Shuting Pan, T. Wang, Tao Yun, Tian Pei, Tianyu Sun, W. L. Xiao, Wangding Zeng, Wanjia Zhao, Wei An, Wen Liu, Wenfeng Liang, Wenjun Gao, Wenqin Yu, Wentao Zhang, X. Q. Li, Xiangyue Jin, Xianzu Wang, Xiao Bi, Xiaodong Liu, Xiaohan Wang, Xiaojin Shen, Xiaokang Chen, Xiaokang Zhang, Xiaosha Chen, Xiaotao Nie, Xiaowen Sun, Xiaoxiang Wang, Xin Cheng, Xin Liu, Xin Xie, Xingchao Liu, Xingkai Yu, Xinnan Song, Xinxia Shan, Xinyi Zhou, Xinyu Yang, Xinyuan Li, Xuecheng Su, Xuheng Lin, Y. K. Li, Y. Q. Wang, Y. X. Wei, Y. X. Zhu, Yang Zhang, Yanhong Xu, Yanhong Xu, Yanping Huang, Yao Li, Yao Zhao, Yaofeng Sun, Yaohui Li, Yaohui Wang, Yi Yu, Yi Zheng, Yichao Zhang, Yifan Shi, Yiliang Xiong, Ying He, Ying Tang, Yishi Piao, Yisong Wang, Yixuan Tan, Yiyang Ma, Yiyuan Liu, Yongqiang Guo, Yu Wu, Yuan Ou, Yuchen Zhu, Yuduan Wang, Yue Gong, Yuheng Zou, Yujia He, Yukun Zha, Yunfan Xiong, Yunxian Ma, Yuting Yan, Yuxiang Luo, Yuxiang You, Yuxuan Liu, Yuyang Zhou, Z. F. Wu, Z. Z. Ren, Zehui Ren, Zhangli Sha, Zhe Fu, Zhean Xu, Zhen Huang, Zhen Zhang, Zhenda Xie, Zhengyan Zhang, Zhewen Hao, Zhibin Gou, Zhicheng Ma, Zhigang Yan, Zhihong Shao, Zhipeng Xu, Zhiyu Wu, Zhongyu Zhang, Zhuoshu Li, Zihui Gu, Zijia Zhu, Zijun Liu, Zilin Li, Ziwei Xie, Ziyang Song, Ziyi Gao, and Zizheng Pan. Deepseek-v3 technical report, 2024. URL https://arxiv.org/abs/2412.19437.

Dongsheng Ding, Xiaohan Wei, Zhuoran Yang, Zhaoran Wang, and Mihailo Jovanovic. Provably efficient safe exploration via primal-dual policy optimization. In *International conference on artificial intelligence and statistics*, pp. 3304–3312. PMLR, 2021.

Yuhao Ding and Javad Lavaei. Provably efficient primal-dual reinforcement learning for cmdps with non-stationary objectives and constraints. In *Proceedings of the Thirty-Seventh AAAI Conference on Artificial Intelligence and Thirty-Fifth Conference on Innovative Applications of Artificial Intelligence and Thirteenth Symposium on Educational Advances in Artificial Intelligence*, AAAI'23/IAAI'23/EAAI'23. AAAI Press, 2023. ISBN 978-1-57735-880-0. doi: 10.1609/aaai.v37i6.25900. URL https://doi.org/10.1609/aaai.v37i6.25900.

Desong Du, Shaohang Han, Naiming Qi, Haitham Bou Ammar, Jun Wang, and Wei Pan. Reinforcement learning for safe robot control using control lyapunov barrier functions. In *2023 IEEE International Conference on Robotics and Automation (ICRA)*, pp. 9442–9448. IEEE, 2023.

John Duchi, Elad Hazan, and Yoram Singer. Adaptive subgradient methods for online learning and stochastic optimization. *J. Mach. Learn. Res.*, 12(null):2121–2159, July 2011. ISSN 1532-4435.

Yonathan Efroni, Shie Mannor, and Matteo Pirotta. Exploration-exploitation in constrained mdps. *arXiv preprint arXiv:2003.02189*, 2020.

Jacopo Germano, Francesco Emanuele Stradi, Gianmarco Genalti, Matteo Castiglioni, Alberto Marchesi, and Nicola Gatti. A best-of-both-worlds algorithm for constrained mdps with long-term constraints. *arXiv e-prints*, pp. arXiv–2304, 2023.

Arnob Ghosh, Xingyu Zhou, and Ness Shroff. Provably efficient model-free constrained rl with linear function approximation. In *Proceedings of the 36th International Conference on Neural Information Processing Systems*, NIPS '22, Red Hook, NY, USA, 2022. Curran Associates Inc. ISBN 9781713871088.

Hengquan Guo, Honghao Wei, Xin Liu, and Lei Ying. Online convex optimization with hard constraints: towards the best of two worlds and beyond. In *Proceedings of the 36th International Conference on Neural Information Processing Systems*, NIPS '22, Red Hook, NY, USA, 2022a. Curran Associates Inc. ISBN 9781713871088.

Hengquan Guo, Honghao Wei, Xin Liu, and Lei Ying. Online convex optimization with hard constraints: towards the best of two worlds and beyond. In *Proceedings of the 36th International Conference on Neural Information Processing Systems*, NIPS '22, Red Hook, NY, USA, 2022b. Curran Associates Inc. ISBN 9781713871088.

Elad Hazan. Introduction to online convex optimization. *Found. Trends Optim.*, 2:157–325, 2016. URL `https://api.semanticscholar.org/CorpusID:30482768`.

Yue He, Xiujun Chen, Di Wu, Junwei Pan, Qing Tan, Chuan Yu, Jian Xu, and Xiaoqiang Zhu. A unified solution to constrained bidding in online display advertising. In *Proceedings of the 27th ACM SIGKDD Conference on Knowledge Discovery & Data Mining*, KDD '21, pp. 2993–3001, New York, NY, USA, 2021. Association for Computing Machinery. ISBN 9781450383325. doi: 10.1145/3447548.3467199. URL `https://doi.org/10.1145/3447548.3467199`.

Shashank Hegde, Zhehui Huang, and Gaurav S Sukhatme. Hyperppo: A scalable method for finding small policies for robotic control. In *2024 IEEE International Conference on Robotics and Automation (ICRA)*, pp. 10821–10828. IEEE, 2024.

Max Jaderberg, Wojciech M Czarnecki, Iain Dunning, Luke Marris, Guy Lever, Antonio Garcia Castaneda, Charles Beattie, Neil C Rabinowitz, Ari S Morcos, Avraham Ruderman, et al. Human-level performance in 3d multiplayer games with population-based reinforcement learning. *Science*, 364(6443):859–865, 2019.

Thomas Jaksch, Ronald Ortner, and Peter Auer. Near-optimal regret bounds for reinforcement learning. *J. Mach. Learn. Res.*, 11:1563–1600, August 2010. ISSN 1532-4435.

Chi Jin, Zeyuan Allen-Zhu, Sebastien Bubeck, and Michael I Jordan. Is q-learning provably efficient? *Advances in neural information processing systems*, 31, 2018.

Chi Jin, Tiancheng Jin, Haipeng Luo, Suvrit Sra, and Tiancheng Yu. Learning adversarial markov decision processes with bandit feedback and unknown transition. In *International Conference on Machine Learning*, 2020. URL `https://api.semanticscholar.org/CorpusID:208624190`.

Toshinori Kitamura, Tadashi Kozuno, Masahiro Kato, YUKI ICHIHARA, Soichiro Nishimori, Akiyoshi Sannai, Sho Sonoda, Wataru Kumagai, and Yutaka Matsuo. A policy gradient primal-dual algorithm for constrained mdps with uniform pac guarantees. In *First Reinforcement Learning Safety Workshop*, 2024.

Tomáš Kocák, Gergely Neu, Michal Valko, and Rémi Munos. Efficient learning by implicit exploration in bandit problems with side observations. In *Proceedings of the 28th International Conference on Neural Information Processing Systems - Volume 1*, NIPS'14, pp. 613–621, Cambridge, MA, USA, 2014. MIT Press.

Chung-Wei Lee, Haipeng Luo, Chen-Yu Wei, and Mengxiao Zhang. Bias no more: high-probability data-dependent regret bounds for adversarial bandits and mdps. In *Proceedings of the 34th International Conference on Neural Information Processing Systems*, NIPS '20, Red Hook, NY, USA, 2020. Curran Associates Inc. ISBN 9781713829546.

Jordan Lekeufack and Michael I Jordan. An optimistic algorithm for online convex optimization with adversarial constraints. *arXiv preprint arXiv:2412.08060*, 2024.

Nikolaos Liakopoulos, Apostolos Destounis, Georgios Paschos, Thrasyvoulos Spyropoulos, and Panayotis Mertikopoulos. Cautious regret minimization: Online optimization with long-term budget constraints. In Kamalika Chaudhuri and Ruslan Salakhutdinov (eds.), *Proceedings of the 36th International Conference on Machine Learning*, volume 97 of *Proceedings of Machine Learning Research*, pp. 3944–3952. PMLR, 09–15 Jun 2019. URL https://proceedings.mlr.press/v97/liakopoulos19a.html.

Tao Liu, Ruida Zhou, Dileep Kalathil, P. R. Kumar, and Chao Tian. Learning policies with zero or bounded constraint violation for constrained mdps. In *Proceedings of the 35th International Conference on Neural Information Processing Systems*, NIPS '21, Red Hook, NY, USA, 2021. Curran Associates Inc. ISBN 9781713845393.

Haipeng Luo, Chen-Yu Wei, and Chung-Wei Lee. Policy optimization in adversarial mdps: Improved exploration via dilated bonuses. *Advances in Neural Information Processing Systems*, 34:22931–22942, 2021.

Davide Maran, Pierriccardo Olivieri, Francesco Emanuele Stradi, Giuseppe Urso, Nicola Gatti, and Marcello Restelli. Online markov decision processes configuration with continuous decision space. *Proceedings of the AAAI Conference on Artificial Intelligence*, 38(13):14315–14322, Mar. 2024. doi: 10.1609/aaai.v38i13.29344. URL https://ojs.aaai.org/index.php/AAAI/article/view/29344.

Michaël Mathieu, Sherjil Ozair, Srivatsan Srinivasan, Caglar Gulcehre, Shangtong Zhang, Ray Jiang, Tom Le Paine, Richard Powell, Konrad Żołna, Julian Schrittwieser, et al. Alphastar unplugged: Large-scale offline reinforcement learning. *arXiv preprint arXiv:2308.03526*, 2023.

Adrian Müller, Pragnya Alatur, Giorgia Ramponi, and Niao He. Cancellation-free regret bounds for lagrangian approaches in constrained markov decision processes. In *Sixteenth European Workshop on Reinforcement Learning*. OpenReview, 2023.

Adrian Müller, Pragnya Alatur, Volkan Cevher, Giorgia Ramponi, and Niao He. Truly no-regret learning in constrained mdps. In *Proceedings of the 41st International Conference on Machine Learning*, ICML'24. JMLR.org, 2024.

Michael J. Neely. *Stochastic Network Optimization with Application to Communication and Queueing Systems*. Morgan and Claypool Publishers, 2010. ISBN 160845455X.

Gergely Neu. Explore no more: improved high-probability regret bounds for non-stochastic bandits. In *Proceedings of the 29th International Conference on Neural Information Processing Systems - Volume 2*, NIPS'15, pp. 3168–3176, Cambridge, MA, USA, 2015. MIT Press.

Gergely Neu, András György, Csaba Szepesvári, and András Antos. Online markov decision processes under bandit feedback. In *Proceedings of the 24th International Conference on Neural Information Processing Systems - Volume 2*, NIPS'10, pp. 1804–1812, Red Hook, NY, USA, 2010. Curran Associates Inc.

OpenAI, Josh Achiam, Steven Adler, Sandhini Agarwal, Lama Ahmad, Ilge Akkaya, Florencia Leoni Aleman, Diogo Almeida, Janko Altenschmidt, Sam Altman, Shyamal Anadkat, Red Avila, Igor Babuschkin, Suchir Balaji, Valerie Balcom, Paul Baltescu, Haiming Bao, Mohammad Bavarian, Jeff Belgum, Irwan Bello, Jake Berdine, Gabriel Bernadett-Shapiro, Christopher Berner, Lenny Bogdonoff, Oleg Boiko, Madelaine

Boyd, Anna-Luisa Brakman, Greg Brockman, Tim Brooks, Miles Brundage, Kevin Button, Trevor Cai, Rosie Campbell, Andrew Cann, Brittany Carey, Chelsea Carlson, Rory Carmichael, Brooke Chan, Che Chang, Fotis Chantzis, Derek Chen, Sully Chen, Ruby Chen, Jason Chen, Mark Chen, Ben Chess, Chester Cho, Casey Chu, Hyung Won Chung, Dave Cummings, Jeremiah Currier, Yunxing Dai, Cory Decareaux, Thomas Degry, Noah Deutsch, Damien Deville, Arka Dhar, David Dohan, Steve Dowling, Sheila Dunning, Adrien Ecoffet, Atty Eleti, Tyna Eloundou, David Farhi, Liam Fedus, Niko Felix, Simón Posada Fishman, Juston Forte, Isabella Fulford, Leo Gao, Elie Georges, Christian Gibson, Vik Goel, Tarun Gogineni, Gabriel Goh, Rapha Gontijo-Lopes, Jonathan Gordon, Morgan Grafstein, Scott Gray, Ryan Greene, Joshua Gross, Shixiang Shane Gu, Yufei Guo, Chris Hallacy, Jesse Han, Jeff Harris, Yuchen He, Mike Heaton, Johannes Heidecke, Chris Hesse, Alan Hickey, Wade Hickey, Peter Hoeschele, Brandon Houghton, Kenny Hsu, Shengli Hu, Xin Hu, Joost Huizinga, Shantanu Jain, Shawn Jain, Joanne Jang, Angela Jiang, Roger Jiang, Haozhun Jin, Denny Jin, Shino Jomoto, Billie Jonn, Heewoo Jun, Tomer Kaftan, Łukasz Kaiser, Ali Kamali, Ingmar Kanitscheider, Nitish Shirish Keskar, Tabarak Khan, Logan Kilpatrick, Jong Wook Kim, Christina Kim, Yongjik Kim, Jan Hendrik Kirchner, Jamie Kiros, Matt Knight, Daniel Kokotajlo, Łukasz Kondraciuk, Andrew Kondrich, Aris Konstantinidis, Kyle Kosic, Gretchen Krueger, Vishal Kuo, Michael Lampe, Ikai Lan, Teddy Lee, Jan Leike, Jade Leung, Daniel Levy, Chak Ming Li, Rachel Lim, Molly Lin, Stephanie Lin, Mateusz Litwin, Theresa Lopez, Ryan Lowe, Patricia Lue, Anna Makanju, Kim Malfacini, Sam Manning, Todor Markov, Yaniv Markovski, Bianca Martin, Katie Mayer, Andrew Mayne, Bob McGrew, Scott Mayer McKinney, Christine McLeavey, Paul McMillan, Jake McNeil, David Medina, Aalok Mehta, Jacob Menick, Luke Metz, Andrey Mishchenko, Pamela Mishkin, Vinnie Monaco, Evan Morikawa, Daniel Mossing, Tong Mu, Mira Murati, Oleg Murk, David Mély, Ashvin Nair, Reiichiro Nakano, Rajeev Nayak, Arvind Neelakantan, Richard Ngo, Hyeonwoo Noh, Long Ouyang, Cullen O'Keefe, Jakub Pachocki, Alex Paino, Joe Palermo, Ashley Pantuliano, Giambattista Parascandolo, Joel Parish, Emy Parparita, Alex Passos, Mikhail Pavlov, Andrew Peng, Adam Perelman, Filipe de Avila Belbute Peres, Michael Petrov, Henrique Ponde de Oliveira Pinto, Michael, Pokorny, Michelle Pokrass, Vitchyr H. Pong, Tolly Powell, Alethea Power, Boris Power, Elizabeth Proehl, Raul Puri, Alec Radford, Jack Rae, Aditya Ramesh, Cameron Raymond, Francis Real, Kendra Rimbach, Carl Ross, Bob Rotsted, Henri Roussez, Nick Ryder, Mario Saltarelli, Ted Sanders, Shibani Santurkar, Girish Sastry, Heather Schmidt, David Schnurr, John Schulman, Daniel Selsam, Kyla Sheppard, Toki Sherbakov, Jessica Shieh, Sarah Shoker, Pranav Shyam, Szymon Sidor, Eric Sigler, Maddie Simens, Jordan Sitkin, Katarina Slama, Ian Sohl, Benjamin Sokolowsky, Yang Song, Natalie Staudacher, Felipe Petroski Such, Natalie Summers, Ilya Sutskever, Jie Tang, Nikolas Tezak, Madeleine B. Thompson, Phil Tillet, Amin Tootoonchian, Elizabeth Tseng, Preston Tuggle, Nick Turley, Jerry Tworek, Juan Felipe Cerón Uribe, Andrea Vallone, Arun Vijayvergiya, Chelsea Voss, Carroll Wainwright, Justin Jay Wang, Alvin Wang, Ben Wang, Jonathan Ward, Jason Wei, CJ Weinmann, Akila Welihinda, Peter Welinder, Jiayi Weng, Lilian Weng, Matt Wiethoff, Dave Willner, Clemens Winter, Samuel Wolrich, Hannah Wong, Lauren Workman, Sherwin Wu, Jeff Wu, Michael Wu, Kai Xiao, Tao Xu, Sarah Yoo, Kevin Yu, Qiming Yuan, Wojciech Zaremba, Rowan Zellers, Chong Zhang, Marvin Zhang, Shengjia Zhao, Tianhao Zheng, Juntang Zhuang, William Zhuk, and Barret Zoph. Gpt-4 technical report, 2024. URL `https://arxiv.org/abs/2303.08774`.

Francesco Orabona. A modern introduction to online learning, 2025. URL `https://arxiv.org/abs/1912.13213`.

Aldo Pacchiano, Mohammad Ghavamzadeh, Peter Bartlett, and Heinrich Jiang. Stochastic bandits with linear constraints. In *International conference on artificial intelligence and statistics*, pp. 2827–2835. PMLR, 2021.

Martin L Puterman. *Markov decision processes: discrete stochastic dynamic programming.* John Wiley & Sons, 2014.

Shuang Qiu, Xiaohan Wei, Zhuoran Yang, Jieping Ye, and Zhaoran Wang. Upper confidence primal-dual reinforcement learning for cmdp with adversarial loss. In *Proceedings of the 34th International Conference on Neural Information Processing Systems*, NIPS '20, Red Hook, NY, USA, 2020. Curran Associates Inc. ISBN 9781713829546.

Aviv Rosenberg and Yishay Mansour. Online stochastic shortest path with bandit feedback and unknown transition function. In H. Wallach, H. Larochelle, A. Beygelzimer, F. d'Alché-Buc, E. Fox, and R. Garnett (eds.), *Advances in Neural Information Processing Systems*, volume 32. Curran Associates, Inc., 2019a. URL `https://proceedings.neurips.cc/paper_files/paper/2019/file/a0872cc5b5ca4cc25076f3d868e1bdf8-Paper.pdf`.

Aviv Rosenberg and Yishay Mansour. Online convex optimization in adversarial markov decision processes. In *International Conference on Machine Learning*, pp. 5478–5486. PMLR, 2019b.

David Silver, Julian Schrittwieser, Karen Simonyan, Ioannis Antonoglou, Aja Huang, Arthur Guez, Thomas Hubert, Lucas Baker, Matthew Lai, Adrian Bolton, Yutian Chen, Timothy Lillicrap, Fan Hui, Laurent Sifre, George van den Driessche, Thore Graepel, and Demis Hassabis. Mastering the game of go without human knowledge. *Nature*, 550:354–, October 2017. URL `http://dx.doi.org/10.1038/nature24270`.

Abhishek Sinha and Rahul Vaze. Optimal algorithms for online convex optimization with adversarial constraints. *Advances in Neural Information Processing Systems*, 37:41274–41302, 2024.

Laura Smith, Yunhao Cao, and Sergey Levine. Grow your limits: Continuous improvement with real-world rl for robotic locomotion. In *2024 IEEE International Conference on Robotics and Automation (ICRA)*, pp. 10829–10836. IEEE, 2024.

Francesco Emanuele Stradi, Matteo Castiglioni, Alberto Marchesi, and Nicola Gatti. Online learning in CMDPs with adversarial losses and stochastic hard constraints. In *Seventeenth European Workshop on Reinforcement Learning*, 2024a. URL `https://openreview.net/forum?id=1Jl0PErLDe`.

Francesco Emanuele Stradi, Jacopo Germano, Gianmarco Genalti, Matteo Castiglioni, Alberto Marchesi, and Nicola Gatti. Online learning in cmdps: handling stochastic and adversarial constraints. In *Proceedings of the 41st International Conference on Machine Learning*, ICML'24. JMLR.org, 2024b.

Francesco Emanuele Stradi, Anna Lunghi, Matteo Castiglioni, Alberto Marchesi, and Nicola Gatti. Learning constrained markov decision processes with non-stationary rewards and constraints. *CoRR*, 2024c.

Francesco Emanuele Stradi, Matteo Castiglioni, Alberto Marchesi, and Nicola Gatti. Optimal strong regret and violation in constrained mdps via policy optimization. In *International Conference on Learning Representations (ICLR)*, 2025a.

Francesco Emanuele Stradi, Matteo Castiglioni, Alberto Marchesi, Nicola Gatti, et al. Learning adversarial mdps with stochastic hard constraints. In *Forty-Second International Conference on Machine Learning*, pp. 1–8, 2025b.

Richard S. Sutton and Andrew G. Barto. *Reinforcement Learning: An Introduction.* The MIT Press, second edition, 2018.

Honghao Wei, Arnob Ghosh, Ness Shroff, Lei Ying, and Xingyu Zhou. Provably efficient model-free algorithms for non-stationary cmdps. In *International Conference on Artificial Intelligence and Statistics*, pp. 6527–6570. PMLR, 2023.

Xiaohan Wei, Hao Yu, and Michael J. Neely. Online learning in weakly coupled markov decision processes: A convergence time study. *Proc. ACM Meas. Anal. Comput. Syst.*, 2(1), April 2018. doi: 10.1145/3179415. URL `https://doi.org/10.1145/3179415`.

Lu Wen, Jingliang Duan, Shengbo Eben Li, Shaobing Xu, and Huei Peng. Safe reinforcement learning for autonomous vehicles through parallel constrained policy optimization. In *2020 IEEE 23rd International Conference on Intelligent Transportation Systems (ITSC)*, pp. 1–7. IEEE, 2020.

Xinlei Yi, Xiuxian Li, Tao Yang, Lihua Xie, Tianyou Chai, and Karl Johansson. Regret and cumulative constraint violation analysis for online convex optimization with long term constraints. In *International conference on machine learning*, pp. 11998–12008. PMLR, 2021.

Xinlei Yi, Xiuxian Li, Tao Yang, Lihua Xie, Yiguang Hong, Tianyou Chai, and Karl Henrik Johansson. Distributed online convex optimization with adversarial constraints: Reduced cumulative constraint violation bounds under slater's condition. *ArXiv*, abs/2306.00149, 2023. URL `https://api.semanticscholar.org/CorpusID:258999517`.

Liyuan Zheng and Lillian Ratliff. Constrained upper confidence reinforcement learning. In *Learning for Dynamics and Control*, pp. 620–629. PMLR, 2020.

Jiahui Zhu, Kihyun Yu, Dabeen Lee, Xin Liu, and Honghao Wei. An optimistic algorithm for online cmdps with anytime adversarial constraints. In *Forty-second International Conference on Machine Learning*, 2025.

Martin Zinkevich. Online convex programming and generalized infinitesimal gradient ascent. In *Proceedings of the Twentieth International Conference on International Conference on Machine Learning*, ICML'03, pp. 928–935. AAAI Press, 2003. ISBN 1577351894.

A Appendix

Here, we present all the omitted proofs, calculations, and algorithmic descriptions in a sequential order of appearance in the main paper. We frequently make use of some algebraic inequalities throughout the section: (1) $(a+b)^2 \leq 2(a^2+b^2)$, $\forall a,b \in \mathbb{R}$; (2) $\sqrt{a+b} \leq \sqrt{a}+\sqrt{b}$, $\forall a,b \geq 0$, (3) $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, $\forall a,b,c \in \mathbb{R}$; and (4) $\sqrt{a+b+c} \leq \sqrt{a}+\sqrt{b}+\sqrt{c}$, $\forall a,b,c \geq 0$.

From the definition of $\varphi(\cdot)$, we observe that $\varphi'(\cdot)$ is non-decreasing. Additionally, we have $\tilde{\nu}_t \geq 0$ due to the clipping and scaling of the constraints, which implies $\zeta_1 \leq \zeta_2 \leq \cdots \leq \zeta_t$, for any $t \geq 1$. Therefore, we obtain two relations, which we also use throughout this section: 1) $\sum_{\tau=1}^{t} \varphi'(\zeta_\tau) \leq t \cdot \varphi'(\zeta_t)$; and 2) $\sum_{\tau=1}^{t} \varphi'(\zeta_\tau)^2 \leq t \cdot \varphi'(\zeta_t)^2$.

A.1 Upper bound of the surrogate regret in Section 4.1

We make use of Eqn. 26 and also of Eqn. 29 from Theorem 1.

$$
\begin{aligned}
\text{Regret}'_t(\rho^\star) &\leq \sqrt{2}D\sqrt{\sum_{\tau=1}^{t} \|\nabla_\tau\|^2} \\
&\leq \sqrt{2}D\sqrt{\sum_{\tau=1}^{t} (\omega L)^2 \left(1+\varphi'(\zeta_\tau)\right)^2} \\
&= \sqrt{2}D\omega L\sqrt{\sum_{\tau=1}^{t}\left(1+\varphi'(\zeta_\tau)\right)^2} \\
&\leq \sqrt{2}D\omega L\sqrt{\sum_{\tau=1}^{t} 2\left(1+\varphi'(\zeta_\tau)^2\right)} \\
&\leq 2D\omega L\sqrt{t} + 2D\omega L\sqrt{\sum_{\tau=1}^{t}\varphi'(\zeta_\tau)^2} \\
&\leq 2D\omega L\sqrt{t}\left(1+\varphi'(\zeta_t)\right).
\end{aligned}
$$

A.2 Proof of Lemma 1 in Section 4.2

The random variable $\mathbf{1}_t(s,a)$ is Bernoulli with success probability $\rho_t(s,a)$. We show via direct calculations,

$$\begin{aligned}\mathbb{E}_t[\hat{\ell}_{t,h}(s,a)] &= \mathbb{E}_t\left[\frac{\ell_{t,h}(s,a)}{\rho_t(s,a)+\Lambda_t}\mathbf{1}_t(s,a)\right] \\ &= \frac{\ell_{t,h}(s,a)}{\rho_t(s,a)+\Lambda_t}\mathbb{E}_t\left[\mathbf{1}_t(s,a)\right] \\ &= \frac{\ell_{t,h}(s,a)}{\rho_t(s,a)+\Lambda_t}\rho_t(s,a).\end{aligned}$$

Also, we can show that

$$\begin{aligned}\mathbb{E}_t[\hat{\ell}_{t,h}(s,a)^2] &= \mathbb{E}_t\left[\frac{\ell_{t,h}(s,a)^2}{(\rho_t(s,a)+\Lambda_t)^2}\mathbf{1}_t(s,a)\right] \\ &= \frac{\rho_t(s,a)}{(\rho_t(s,a)+\Lambda_t)^2} \\ &\le \frac{\rho_t(s,a)+\Lambda_t}{(\rho_t(s,a)+\Lambda_t)^2} \\ &\le \frac{1}{\rho_t(s,a)+\Lambda_t}.\end{aligned}$$

Similarly, we can easily prove that $\mathbb{E}_t[\hat{c}_{t,h}(s,a)] = \frac{c_{t,h}(s,a)}{\rho_t(s,a)+\Lambda_t}\rho_t(s,a)$ and $\mathbb{E}_t[\hat{c}_{t,h}(s,a)^2] \le \frac{1}{\rho_t(s,a)+\Lambda_t}$.

A.3 Proof of Lemma 2 in Section 4.2

By direct calculations, we have:

$$\begin{aligned}&\ell_{t,h}(s,a) - \mathbb{E}_t[\hat{\ell}_{t,h}(s,a)] \\ &= \ell_{t,h}(s,a) - \frac{\ell_{t,h}(s,a)}{\rho_t(s,a)+\Lambda}\rho_t(s,a) \\ &= \ell_{t,h}(s,a)\left(1 - \frac{\rho_t(s,a)}{\rho_t(s,a)+\Lambda}\right) \\ &= \frac{\Lambda\ell_{t,h}(s,a)}{\rho_t(s,a)+\Lambda},\end{aligned}$$

which is always non-negative and $\ell_{t,h}(s,a) - \mathbb{E}_t[\hat{\ell}_{t,h}(s,a)] \le \frac{\Lambda\ell_{t,h}(s,a)}{\rho_t(s,a)}$. Proceeding similarly, we also have: $0 \le c_{t,h}(s,a) - \mathbb{E}_t[\hat{c}_{t,h}(s,a)] \le \frac{\Lambda c_{t,h}(s,a)}{\rho_t(s,a)}$.

A.4 Bounding the norm of the bias of the gradient estimate in Section 4.2

Recall that $\hat{\boldsymbol{\ell}}_t$ and $\hat{\boldsymbol{c}}_t$ are biased estimators of $\boldsymbol{\ell}_t$ and $\boldsymbol{c}_t$. It is clear from Eqn. 25 and Eqn. 38 that the bias vector $\boldsymbol{b}_t$ should be given by:

$$\begin{aligned}\boldsymbol{b}_t &= \mathbb{E}_t[\widehat{\nabla}_t] - \nabla_t \\ &= \omega\mathbb{E}_t[\hat{\boldsymbol{\ell}}_t] + \varphi'(\zeta_t)\omega\mathbb{E}_t[\hat{\boldsymbol{c}}_t\cdot\mathbf{1}_{\{\mathcal{C}_t>0\}}] - \omega\boldsymbol{\ell}_t - \varphi'(\zeta_t)\omega\boldsymbol{c}_t\cdot\mathbf{1}_{\{\langle\rho_t,\omega\boldsymbol{c}_t\rangle>0\}} \\ &= \omega\big(\mathbb{E}_t[\hat{\boldsymbol{\ell}}_t] - \boldsymbol{\ell}_t\big) + \omega\varphi'(\zeta_t)\big(\mathbb{E}_t[\hat{\boldsymbol{c}}_t\cdot\mathbf{1}_{\{\mathcal{C}_t>0\}}] - \boldsymbol{c}_t\cdot\mathbf{1}_{\{\langle\rho_t,\omega\boldsymbol{c}_t\rangle>0\}}\big).\end{aligned}$$

where $\mathbf{1}_{\{\mathcal{C}_t>0\}}$ and $\mathbf{1}_{\{\langle\rho_t,\omega\boldsymbol{c}_t\rangle>0\}}$ are equal to 1 if $\mathcal{C}_t>0$ and $\langle\rho_t,\omega\boldsymbol{c}_t\rangle>0$ respectively (0 otherwise). By the triangle inequality for norms, we have:

$$\|\boldsymbol{b}_t\| \leq \overbrace{\left\|\omega\big(\mathbb{E}_t[\hat{\boldsymbol{\ell}}_t]-\boldsymbol{\ell}_t\big)\right\|}^{\|\boldsymbol{b}_{t,\boldsymbol{\ell}}\|} + \underbrace{\left\|\omega\varphi'(\zeta_t)\big(\mathbb{E}_t[\hat{\boldsymbol{c}}_t\cdot\mathbf{1}_{\{\mathcal{C}_t>0\}}]-\boldsymbol{c}_t\cdot\mathbf{1}_{\{\langle\rho_t,\omega\boldsymbol{c}_t\rangle>0\}}\big)\right\|}_{\|\boldsymbol{b}_{t,\boldsymbol{c}}\|}.$$

$$\implies \|\boldsymbol{b}_t\| \leq \|\boldsymbol{b}_{t,\boldsymbol{\ell}}\| + \|\boldsymbol{b}_{t,\boldsymbol{c}}\|. \tag{69}$$

Observe that $\|\boldsymbol{b}_{t,\boldsymbol{\ell}}\|^2 = \left\|\omega\big(\mathbb{E}_t[\hat{\boldsymbol{\ell}}_t]-\boldsymbol{\ell}_t\big)\right\|^2 = \omega^2\left\|\mathbb{E}_t[\hat{\boldsymbol{\ell}}_t]-\boldsymbol{\ell}_t\right\|^2$. Since the squared norm is the sum of the squared differences over all the (s,a,h) components, we get from Lemma 2:

$$\left\|\mathbb{E}_t[\hat{\boldsymbol{\ell}}_t]-\boldsymbol{\ell}_t\right\|^2 = \sum_{(s,a,h)}\big(\mathbb{E}_t[\hat{\ell}_{t,h}(s,a)]-\ell_{t,h}(s,a)\big)^2 = \sum_{(s,a,h)}\frac{\Lambda_t^2\cdot\ell_{t,h}(s,a)^2}{(\rho_t(s,a)+\Lambda_t)^2}.$$

Note that the losses are bounded, i.e., $\ell_{t,h}(s,a)\in[0,1]$, for all $t\in[T]$ and for all $h\in[H]^{-1}$. Also, in the earlier expression, the denominator is at least Λ_t^2, since $\rho_t(s,a)\geq 0$. Therefore, we have:

$$\left\|\mathbb{E}_t[\hat{\boldsymbol{\ell}}_t]-\boldsymbol{\ell}_t\right\|^2 \leq \sum_{(s,a,h)}\frac{\Lambda_t^2\cdot 1^2}{\Lambda_t^2} = \sum_{(s,a,h)} 1 \leq SHA.$$

$$\implies \|\boldsymbol{b}_{t,\boldsymbol{\ell}}\| \leq \omega\sqrt{SHA}. \tag{70}$$

We will now upper bound the term $\|\boldsymbol{b}_{t,\boldsymbol{c}}\|$ in Eqn. 69. Decomposing $\boldsymbol{b}_{t,\boldsymbol{c}}$ without the norm as follows:

$$\begin{aligned}
\boldsymbol{b}_{t,\boldsymbol{c}} &= \omega\varphi'(\zeta_t)\big(\mathbb{E}_t[\hat{\boldsymbol{c}}_t\cdot\mathbf{1}_{\{\mathcal{C}_t>0\}}]-\boldsymbol{c}_t\cdot\mathbf{1}_{\{\langle\rho_t,\omega\boldsymbol{c}_t\rangle>0\}}\big)\\
&= \omega\varphi'(\zeta_t)\big(\mathbb{E}_t[\hat{\boldsymbol{c}}_t\cdot\mathbf{1}_{\{\mathcal{C}_t>0\}}]-\boldsymbol{c}_t\cdot\mathbf{1}_{\{\langle\rho_t,\omega\boldsymbol{c}_t\rangle>0\}}+\mathbb{E}_t[\hat{\boldsymbol{c}}_t\cdot\mathbf{1}_{\{\langle\rho_t,\omega\boldsymbol{c}_t\rangle>0\}}]-\mathbb{E}_t[\hat{\boldsymbol{c}}_t\cdot\mathbf{1}_{\{\langle\rho_t,\omega\boldsymbol{c}_t\rangle>0\}}]\big)\\
&= \omega\varphi'(\zeta_t)\Bigg(\big(\mathbb{E}_t[\hat{\boldsymbol{c}}_t]-\boldsymbol{c}_t\big)\cdot\mathbf{1}_{\{\langle\rho_t,\omega\boldsymbol{c}_t\rangle>0\}}+\mathbb{E}_t[\hat{\boldsymbol{c}}_t\cdot(\mathbf{1}_{\{\mathcal{C}_t>0\}}-\mathbf{1}_{\{\langle\rho_t,\omega\boldsymbol{c}_t\rangle>0\}})]\Bigg).
\end{aligned}$$

Applying the triangle inequality on the norm of $\boldsymbol{b}_{t,\boldsymbol{c}}$,

$$\|\boldsymbol{b}_{t,\boldsymbol{c}}\| \leq \omega\varphi'(\zeta_t)\Bigg(\left\|\big(\mathbb{E}_t[\hat{\boldsymbol{c}}_t]-\boldsymbol{c}_t\big)\cdot\mathbf{1}_{\{\langle\rho_t,\omega\boldsymbol{c}_t\rangle>0\}}\right\|+\left\|\mathbb{E}_t[\hat{\boldsymbol{c}}_t\cdot(\mathbf{1}_{\{\mathcal{C}_t>0\}}-\mathbf{1}_{\{\langle\rho_t,\omega\boldsymbol{c}_t\rangle>0\}})]\right\|\Bigg).$$

We separately bound each term inside the parentheses. For the first term, we have

$$\left\|\big(\mathbb{E}_t[\hat{\boldsymbol{c}}_t]-\boldsymbol{c}_t\big)\cdot\mathbf{1}_{\{\langle\rho_t,\omega\boldsymbol{c}_t\rangle>0\}}\right\| \leq \|\mathbb{E}_t[\hat{\boldsymbol{c}}_t]-\boldsymbol{c}_t\|\cdot\mathbf{1}_{\{\langle\rho_t,\omega\boldsymbol{c}_t\rangle>0\}} \leq \|\mathbb{E}_t[\hat{\boldsymbol{c}}_t]-\boldsymbol{c}_t\|.$$

Again from Lemma 2 and using the fact that $c_{t,h}(s,a)\in[-1,1]$, for all $t\in[T]$ and for all $h\in[H]^{-1}$,

$$\begin{aligned}
&\|\mathbb{E}_t[\hat{\boldsymbol{c}}_t]-\boldsymbol{c}_t\|^2\\
&= \sum_{(s,a,h)}\big(\mathbb{E}_t[\hat{c}_{t,h}(s,a)]-c_{t,h}(s,a)\big)^2\\
&= \sum_{(s,a,h)}\frac{\Lambda_t^2\cdot c_{t,h}(s,a)^2}{(\rho_t(s,a)+\Lambda_t)^2}\\
&\leq \sum_{(s,a,h)}\frac{\Lambda_t^2\cdot 1^2}{\Lambda_t^2} = \sum_{(s,a,h)} 1 \leq SHA\\
&\implies \|\mathbb{E}_t[\hat{\boldsymbol{c}}_t]-\boldsymbol{c}_t\| \leq \sqrt{SHA}.
\end{aligned} \tag{71}$$

On applying Jensen's inequality to the second term, we obtain:

$$\begin{aligned}
\left\|\mathbb{E}_t[\hat{\boldsymbol{c}}_t\cdot(\mathbf{1}_{\{\mathcal{C}_t>0\}}-\mathbf{1}_{\{\langle\rho_t,\omega\boldsymbol{c}_t\rangle>0\}})]\right\| &\leq \mathbb{E}_t\Big[\left\|\hat{\boldsymbol{c}}_t\cdot(\mathbf{1}_{\{\mathcal{C}_t>0\}}-\mathbf{1}_{\{\langle\rho_t,\omega\boldsymbol{c}_t\rangle>0\}})\right\|\Big]\\
&\leq \mathbb{E}_t\Big[\|\hat{\boldsymbol{c}}_t\|\cdot(\mathbf{1}_{\{\mathcal{C}_t>0\}}-\mathbf{1}_{\{\langle\rho_t,\omega\boldsymbol{c}_t\rangle>0\}})\Big] \leq \mathbb{E}_t\Big[\|\hat{\boldsymbol{c}}_t\|\Big].
\end{aligned}$$

Bounding the square L^2-norm of the sparse vector $\hat{\boldsymbol{c}}_t$ (i.e., having only H non-zero entries),

$$
\begin{aligned}
\|\hat{\boldsymbol{c}}_t\|^2 &= \sum_{h=0}^{H-1} \left(\frac{c_{t,h}(s,a)}{\rho_t(s,a) + \Lambda_t} \right)^2 \cdot \mathbf{1}_t(s,a) \\
&= \sum_{h=0}^{H-1} \frac{c_{t,h}(s,a)^2}{(\rho_t(s,a) + \Lambda_t)^2} \\
&\leq \sum_{h=0}^{H-1} \frac{1}{\Lambda_t^2} = \frac{H}{\Lambda_t^2}. \implies \|\hat{\boldsymbol{c}}_t\| = \frac{\sqrt{H}}{\Lambda_t}.
\end{aligned}
$$

The final bound on the second term of the parentheses is

$$
\left\| \mathbb{E}_t[\hat{\boldsymbol{c}}_t \cdot (\mathbf{1}_{\{\mathcal{C}_t > 0\}} - \mathbf{1}_{\{\langle \rho_t, \omega \boldsymbol{c}_t \rangle > 0\}})] \right\| \leq \mathbb{E}_t \Big[\|\hat{\boldsymbol{c}}_t\| \Big] \leq \frac{\sqrt{H}}{\Lambda_t}. \tag{72}
$$

Using Eqn. 71 and Eqn. 72 we arrive at

$$
\|\boldsymbol{b}_{t,\boldsymbol{c}}\| \leq \omega \varphi'(\zeta_t) \left(\sqrt{SHA} + \frac{\sqrt{H}}{\Lambda_t} \right). \tag{73}
$$

Putting Eqn. 70 and Eqn. 73 in Eqn. 69, we have the final upper bound on the L^2-norm of the bias as

$$
\|\boldsymbol{b}_t\| \leq \omega \sqrt{SHA} + \omega \varphi'(\zeta_t) \left(\sqrt{SHA} + \frac{\sqrt{H}}{\Lambda_t} \right) \leq \omega L + \omega \varphi'(\zeta_t) \left(L + \frac{\sqrt{H}}{\Lambda_t} \right). \tag{74}
$$

A.5 Upper bounding the component terms in Eqn. 40 of Section 4.2

We will use the Cauchy-Schwarz inequality, which is stated as: for all vectors $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}$, $|\langle \boldsymbol{p}, \boldsymbol{q} \rangle| \leq \|\boldsymbol{p}\| \, \|\boldsymbol{q}\|$.

$$
\begin{aligned}
T_1 &= \sum_{\tau=1}^{t} \langle \rho_\tau - \rho^\star, \mathbb{E}_\tau[\widehat{\nabla}_\tau] \rangle \\
&\leq \sqrt{2} D \sqrt{\sum_{\tau=1}^{t} \left\| \mathbb{E}_\tau[\widehat{\nabla}_\tau] \right\|^2}
\end{aligned}
$$

Setting $\Lambda_\tau = \omega\sqrt{H}$, for all $\tau \geq 1$, and from Eqn. 39, we have:

$$
\begin{aligned}
T_1 &\leq \sqrt{2}D\sqrt{\sum_{\tau=1}^{t}\left\|\mathbb{E}_\tau[\widehat{\nabla}_\tau]\right\|^2} \\
&\leq \sqrt{2}D\sqrt{\sum_{\tau=1}^{t}\left(\omega L + \omega\varphi'(\zeta_\tau)\left(L + \frac{\sqrt{H}}{\Lambda_\tau}\right) + \omega L\big(1+\varphi'(\zeta_\tau)\big)\right)^2} \\
&\leq \sqrt{2}D\sqrt{\sum_{\tau=1}^{t}3\left(\omega^2L^2 + \left(\omega\varphi'(\zeta_\tau)\left(L + \frac{\sqrt{H}}{\Lambda_\tau}\right)\right)^2 + \omega^2L^2\big(1+\varphi'(\zeta_\tau)\big)^2\right)} \\
&= \sqrt{6}D\sqrt{\sum_{\tau=1}^{t}\omega^2L^2 + \left(\omega\varphi'(\zeta_\tau)\left(L + \frac{\sqrt{H}}{\Lambda_\tau}\right)\right)^2 + \omega^2L^2\big(1+\varphi'(\zeta_\tau)\big)^2} \\
&\leq D\omega L\sqrt{6t} + \sqrt{6}D\sqrt{\sum_{\tau=1}^{t}\left(\omega L\varphi'(\zeta_\tau) + \varphi'(\zeta_\tau)\right)^2} + D\omega L\sqrt{6}\sqrt{\sum_{\tau=1}^{t}\big(1+\varphi'(\zeta_\tau)\big)^2} \\
&\leq D\omega L\sqrt{6t} + \sqrt{6}D\sqrt{\sum_{\tau=1}^{t}\left(\omega L\varphi'(\zeta_\tau) + \varphi'(\zeta_\tau)\right)^2} + D\omega L\sqrt{6}\sqrt{\sum_{\tau=1}^{t}\big(1+\varphi'(\zeta_\tau)\big)^2} \\
&\leq D\omega L\sqrt{6t} + \sqrt{12}D\sqrt{\sum_{\tau=1}^{t}\omega^2L^2\varphi'(\zeta_\tau)^2 + \varphi'(\zeta_\tau)^2} + D\omega L\sqrt{12}\sqrt{\sum_{\tau=1}^{t}1 + \varphi'(\zeta_\tau)^2} \\
&\leq D\omega L\sqrt{6t} + D\omega L\sqrt{12}\sqrt{\sum_{\tau=1}^{t}\varphi'(\zeta_\tau)^2} + D\sqrt{12}\sqrt{\sum_{\tau=1}^{t}\varphi'(\zeta_\tau)^2} + D\omega L\sqrt{12t} + D\omega L\sqrt{12}\sqrt{\sum_{\tau=1}^{t}\varphi'(\zeta_\tau)^2}
\end{aligned}
$$

On putting $\omega = \frac{1}{2LD}$ and employing the non-decreasing property of $\varphi'(\cdot)$,

$$
\begin{aligned}
T_1 &\leq \frac{\sqrt{6t}}{2} + \frac{\sqrt{12t}}{2}\varphi'(\zeta_t) + D\sqrt{12t}\cdot\varphi'(\zeta_t) + \frac{\sqrt{12t}}{2} + \frac{\sqrt{12t}}{2}\varphi'(\zeta_t) \\
&= \sqrt{12t}\cdot\varphi'(\zeta_t) + \frac{\sqrt{6t}}{2} + \frac{\sqrt{12t}}{2} + D\sqrt{12t}\cdot\varphi'(\zeta_t).
\end{aligned} \tag{75}
$$

By the Cauchy-Schwarz inequality, $|\langle\rho_\tau - \rho^\star, \boldsymbol{b}_\tau\rangle| \leq \|\rho_\tau - \rho^\star\| \cdot \|\boldsymbol{b}_\tau\| \leq D\,\|\boldsymbol{b}_\tau\|$. Therefore, we have the following upper bound on T_2:

$$
\begin{aligned}
T_2 &= \sum_{\tau=1}^{t}\langle\rho_\tau - \rho^\star, \boldsymbol{b}_\tau\rangle \\
&\leq D\sum_{\tau=1}^{t}\|\boldsymbol{b}_\tau\| \\
&\leq D\sum_{\tau=1}^{t}\omega L + \omega\varphi'(\zeta_\tau)\left(L + \frac{\sqrt{H}}{\Lambda_\tau}\right) \\
&\leq D\omega Lt + D\omega L\sum_{\tau=1}^{t}\varphi'(\zeta_\tau) + D\omega\sqrt{H}\sum_{\tau=1}^{t}\frac{\varphi'(\zeta_\tau)}{\Lambda_\tau} \\
&\leq D\omega Lt + D\omega L\sum_{\tau=1}^{t}\varphi'(\zeta_\tau) + D\sum_{\tau=1}^{t}\varphi'(\zeta_\tau) \\
&\leq \frac{t}{2} + \frac{t}{2}\cdot\varphi'(\zeta_t) + Dt\cdot\varphi'(\zeta_t).
\end{aligned} \tag{76}
$$

A.6 Bounding the term "Error" in Eqn. 50 of Section 5.1

From Eqn. 50, we have:

$$\text{Error} = \sum_{\tau=1}^{t} \langle \rho_\tau - \hat{\rho}_\tau, \nabla_\tau \rangle.$$

Since $\|\nabla_t\| \leq \omega L\big(1+\varphi'(\zeta_t)\big)$, and by the Cauchy-Schwarz inequality,

$$\begin{aligned}
\text{Error} &\leq \sum_{\tau=1}^{t} \|\rho_\tau - \hat{\rho}_\tau\| \cdot \|\nabla_\tau\| \\
&\leq \omega L \sum_{\tau=1}^{t} \|\rho_\tau - \hat{\rho}_\tau\| \cdot \big(1+\varphi'(\zeta_\tau)\big).
\end{aligned}$$

Since $\hat{\rho}_\tau$ is obtained from a transition function in the confidence set $\mathcal{P}_{i_\tau}$ (where i_τ is the epoch index for episode τ), Lemma 3 implies that with probability at least $1-6\delta$:

$$\sum_{\tau=1}^{t} \|\rho_\tau - \hat{\rho}_\tau\|_1 \leq HS\sqrt{At\ln\left(\frac{SAt}{\delta}\right)},$$

where $\|\cdot\|_1$ is the L^1-norm. Owing to the fact that $\|\rho_\tau - \hat{\rho}_\tau\| \leq \|\rho_\tau - \hat{\rho}_\tau\|_1$, we get:

$$\sum_{\tau=1}^{t} \|\rho_\tau - \hat{\rho}_\tau\| \leq \sum_{\tau=1}^{t} \|\rho_\tau - \hat{\rho}_\tau\|_1 \leq HS\sqrt{At\ln\left(\frac{SAt}{\delta}\right)}.$$

Combining all the above results, we have the final bound on "Error" as:

$$\text{Error} \leq \omega LHS\sqrt{At\ln\left(\frac{SAt}{\delta}\right)} \cdot \big(1+\varphi'(\zeta_t)\big). \tag{77}$$

A.7 Upper bound of the surrogate regret in Section 5.1

From Eqn. 50 and Eqn. 77, we see:

$$\begin{aligned}
\text{Regret}'_t(\rho^\star) &\leq \overbrace{\sum_{\tau=1}^{t} \langle \hat{\rho}_\tau - \rho^\star, \nabla_\tau \rangle}^{\text{Reg}} + \overbrace{\sum_{\tau=1}^{t} \langle \rho_\tau - \hat{\rho}_\tau, \nabla_\tau \rangle}^{\text{Error}} \\
&\leq \sqrt{2}D\sqrt{\sum_{\tau=1}^{t} \|\nabla_\tau\|^2} + \omega LHS\sqrt{At\ln\left(\frac{SAt}{\delta}\right)} \cdot \big(1+\varphi'(\zeta_t)\big) \\
&\leq \sqrt{2}D\omega L\sqrt{\sum_{\tau=1}^{t} \big(1+\varphi'(\zeta_\tau)\big)^2} + \omega LHS\sqrt{At\ln\left(\frac{SAt}{\delta}\right)} \cdot \big(1+\varphi'(\zeta_t)\big) \\
&= 2D\omega L\sqrt{t} + 2D\omega L\sqrt{\sum_{\tau=1}^{t} \varphi'(\zeta_\tau)^2} + \omega LHS\sqrt{At\ln\left(\frac{SAt}{\delta}\right)} \cdot \big(1+\varphi'(\zeta_t)\big) \\
&\leq 2D\omega L\sqrt{t} + 2D\omega L\sqrt{t} \cdot \varphi'(\zeta_t) + \omega LHS\sqrt{At\ln\left(\frac{SAt}{\delta}\right)} \cdot \big(1+\varphi'(\zeta_t)\big) \\
&= \big(1+\varphi'(\zeta_t)\big)\left(2D\omega L\sqrt{t} + \omega LHS\sqrt{At\ln\left(\frac{SAt}{\delta}\right)}\right).
\end{aligned} \tag{78}$$

A.8 Bounding the components of Eqn. 60 in Section 5.2

We set $\Lambda_t = \omega\sqrt{H}$ for all $t \in [T]$, to bound each component of Eqn. 60. First, we bound the term "Reg":

$$
\begin{aligned}
\text{Reg} = \sum_{\tau=1}^{t} \langle \hat{\rho}_\tau - \rho^\star, \widehat{\nabla}_\tau \rangle &\leq \sqrt{2} D \sqrt{\sum_{\tau=1}^{t} \left\| \widehat{\nabla}_\tau \right\|^2} \\
&\leq \sqrt{2} D \sqrt{\sum_{\tau=1}^{t} \frac{\omega^2 H}{\Lambda_\tau^2} \left(1 + \varphi'(\zeta_\tau)\right)^2} \\
&= \sqrt{2H} D \omega \sqrt{\sum_{\tau=1}^{t} \frac{\left(1 + \varphi'(\zeta_\tau)\right)^2}{\Lambda_\tau^2}} \\
&= \frac{\sqrt{2H} D \omega}{\omega\sqrt{H}} \sqrt{\sum_{\tau=1}^{t} \left(1 + \varphi'(\zeta_\tau)\right)^2} \\
&= D\sqrt{2} \sqrt{\sum_{\tau=1}^{t} \left(1 + \varphi'(\zeta_\tau)\right)^2} \\
&\leq D\sqrt{2} \sqrt{\sum_{\tau=1}^{t} 2\left(1 + \varphi'(\zeta_\tau)^2\right)} \\
&\leq 2D\sqrt{t} + 2D \sqrt{\sum_{\tau=1}^{t} \varphi'(\zeta_\tau)^2} \\
&\leq 2D\sqrt{t} + 2D\sqrt{t} \cdot \varphi'(\zeta_t) \\
&= 2D\sqrt{t}\left(1 + \varphi'(\zeta_t)\right).
\end{aligned} \tag{79}
$$

We have the bound on "Error" from Eqn. 77 as,

$$
\text{Error} = \sum_{\tau=1}^{t} \langle \rho_\tau - \hat{\rho}_\tau, \nabla_\tau \rangle \leq \omega LHS \sqrt{At \ln\left(\frac{SAt}{\delta}\right)} \cdot \big(1 + \varphi'(\zeta_t)\big). \tag{80}
$$

From Section 5.2, we know that $\|\boldsymbol{b}_t\| = \left\| \mathbb{E}_t[\widehat{\nabla}_t] - \nabla_t \right\| \leq \omega L + \omega \varphi'(\zeta_t)\big(L + \sqrt{H}/\Lambda_t\big)$. Now, we upper bound the term "Bias1",

$$
\begin{aligned}
\text{Bias1} &= \sum_{\tau=1}^{t} \langle \hat{\rho}_\tau, \nabla_\tau - \widehat{\nabla}_\tau \rangle \\
&= \sum_{\tau=1}^{t} \langle \hat{\rho}_\tau, \nabla_\tau - \mathbb{E}_\tau[\widehat{\nabla}_\tau] \rangle + \sum_{\tau=1}^{t} \langle \hat{\rho}_\tau, \mathbb{E}_\tau[\widehat{\nabla}_\tau] - \widehat{\nabla}_\tau \rangle \\
&= \underbrace{\sum_{\tau=1}^{t} \langle \hat{\rho}_\tau, \mathbb{E}_\tau[\widehat{\nabla}_\tau] - \widehat{\nabla}_\tau \rangle}_{T_1} - \underbrace{\sum_{\tau=1}^{t} \langle \hat{\rho}_\tau, \mathbb{E}_\tau[\widehat{\nabla}_\tau] - \nabla_\tau \rangle}_{T_2}.
\end{aligned} \tag{81}
$$

It is easily seen that $T_2 = \sum_{\tau=1}^{t} \langle \hat{\rho}_\tau, \mathbb{E}_\tau[\widehat{\nabla}_\tau] - \nabla_\tau \rangle = \sum_{\tau=1}^{t} \langle \hat{\rho}_\tau, \boldsymbol{b}_\tau \rangle$. By the Cauchy-Schwarz inequality:

$$\begin{aligned} T_2 &\leq \sum_{\tau=1}^{t} \|\hat{\rho}_\tau\| \cdot \|\boldsymbol{b}_\tau\| \\ &\leq \sum_{\tau=1}^{t} \|\boldsymbol{b}_\tau\| \\ &\leq \omega L + \omega\big(L + \sqrt{H}/\Lambda_\tau\big) \sum_{\tau=1}^{t} \varphi'(\zeta_\tau) \\ &\leq \omega L + \omega t\big(L + \sqrt{H}/\Lambda_t\big)\varphi'(\zeta_t). \end{aligned}$$

Finally, we have on putting $\Lambda_t = \omega\sqrt{H}$ for all $t \in [T]$ that:

$$T_2 \leq \omega L + \omega L t \cdot \varphi'(\zeta_t) + t \cdot \varphi'(\zeta_t).$$

We define a random variable $X_\tau = \langle \hat{\rho}_\tau, \mathbb{E}_\tau[\widehat{\nabla}_\tau] - \widehat{\nabla}_\tau \rangle$ for all $\tau \in [t]$. Here, $\hat{\rho}_\tau$ is $\mathcal{F}_{\tau-1}$-measurable and $\mathbb{E}_\tau[\cdot] = \mathbb{E}[\cdot \mid \mathcal{F}_{\tau-1}]$ is the conditional expectation. By construction, $\mathbb{E}_\tau[X_\tau] = 0$, so $\{X_\tau\}_{\tau=1}^{t}$ is a martingale difference sequence adapted to the filtration $\{\mathcal{F}_\tau\}$. For each τ, we have $|X_\tau| \leq n_\tau$, where $n_\tau = \frac{2\omega}{\Lambda_\tau}\big(1+\varphi'(\zeta_\tau)\big)$. Considering $\epsilon = \sqrt{2\ln(2/\delta) \cdot \sum_{\tau=1}^{t} n_\tau^2}$, where $\delta \in (0,1)$, and applying the Azuma-Hoeffding inequality, we get: $T_1 \leq \frac{2\omega\left(1+\varphi'(\zeta_t)\right)}{\Lambda_t}\sqrt{2t\ln(2/\delta)}$. Therefore, we have the following upper bound on "Bias1":

$$\text{Bias1} \leq \frac{2\big(1+\varphi'(\zeta_t)\big)}{\sqrt{H}}\sqrt{2t\ln(2/\delta)} - \omega L - \omega L t \cdot \varphi'(\zeta_t) - t \cdot \varphi'(\zeta_t). \tag{82}$$

Before proceeding to bound the term "Bias2", we state and prove the following lemma, which is a slightly different form of Lemma 1 from Neu (2015). The proof draws inspiration from the techniques given in the proof of Lemma 1 of Neu (2015).

Lemma 4. *For all $t \in [T]$ and for all $h \in [H]^{-1}$, let $\{\alpha_{t,h}\}$ be a sequence such that each $\alpha_{t,h} \in [0, 2\Lambda_t]^{\mathcal{S}\times\mathcal{A}}$ is $\mathcal{F}_t$-measurable. Then, with probability at least $1-\delta$, we get:*

$$\begin{aligned} &\sum_{t=1}^{T} \sum_{(s,a,h)} \alpha_{t,h}(s,a) \left(\hat{c}_{t,h}(s,a) - \frac{\rho_t(s,a)}{u_t(s,a)} c_{t,h}(s,a) \right) \leq H \ln \frac{H}{\delta}, \text{ and} \\ &\sum_{t=1}^{T} \sum_{(s,a,h)} \alpha_{t,h}(s,a) \left(\hat{\ell}_{t,h}(s,a) - \frac{\rho_t(s,a)}{u_t(s,a)} \ell_{t,h}(s,a) \right) \leq H \ln \frac{H}{\delta}. \end{aligned}$$

Proof. Recall $\frac{x}{1+\frac{x}{2}} \leq \ln(1+x)$ for all $x \geq 0$. For any pair (s,a) and let $\Delta = 2\Lambda_t$, we get:

$$\begin{aligned} \hat{c}_{t,h}(s,a) &= \frac{c_{t,h}(s,a)}{u_t(s,a) + \Lambda_t} \mathbf{1}_t(s,a) \\ &\leq \frac{c_{t,h}(s,a)}{u_t(s,a) + \Lambda_t c_{t,h}(s,a)} \mathbf{1}_t(s,a) \\ &= \frac{\mathbf{1}_t(s,a)}{\Delta} \times \frac{2\Lambda \frac{c_{t,h}(s,a)}{u_t(s,a)}}{1 + \Lambda_t \frac{c_{t,h}(s,a)}{u_t(s,a)}} \\ &\leq \frac{1}{\Delta} \ln\left(1 + \frac{\Delta c_{t,h}(s,a)\mathbf{1}_t(s,a)}{u_t(s,a)}\right). \end{aligned} \tag{83}$$

For all $h \in [H]^{-1}$, let us have

$$\widehat{J}_{t,h} = \sum_{(s,a,h)} \alpha_{t,h}(s,a)\hat{c}_{t,h}(s,a), \text{ and}$$
$$J_{t,h} = \sum_{(s,a,h)} \alpha_{t,h}(s,a)\frac{\rho_t(s,a)}{u_t(s,a)}c_{t,h}(s,a).$$

By Eqn. 83, we have:

$$\begin{aligned}
\mathbb{E}_t\left[\exp(\widehat{J}_{t,h})\right] &\leq \mathbb{E}_t\left[\exp\left(\sum_{(s,a,h)} \frac{\alpha_{t,h}(s,a)}{\Delta}\ln\left(1+\frac{\Delta c_{t,h}(s,a)\mathbf{1}_t(s,a)}{u_t(s,a)}\right)\right)\right] \\
&\leq \mathbb{E}_t\left[\prod_{(s,a,h)}\left(1+\frac{\alpha_{t,h}(s,a)c_{t,h}(s,a)\mathbf{1}_t(s,a)}{u_t(s,a)}\right)\right] \\
&= \mathbb{E}_t\left[1+\sum_{(s,a,h)}\frac{\alpha_{t,h}(s,a)c_{t,h}(s,a)\mathbf{1}_t(s,a)}{u_{t,h}(s,a)}\right] \\
&= 1 + J_{t,h} \leq \exp(J_{t,h}).
\end{aligned}$$

The second inequality is because $a\ln(1+b) \leq \ln(1+ab)$ for all $b \geq -1$ and $a \in [0,1]$, and we apply it with $a = \frac{\alpha_{t,h}(s,a)}{\Delta}$ which is in $[0,1]$ by the condition $\alpha_{t,h}(s,a) \in [0, 2\Lambda_t]$. The first arises since $\mathbf{1}_t(s,a)\mathbf{1}_t(s',a') = 0$ for any $s \neq s'$ or $a \neq a'$. On using Markov's inequality, we get:

$$\begin{aligned}
\mathbb{P}\left[\sum_{t=1}^{T}(\widehat{J}_{t,h} - J_{t,h}) > \ln\left(\frac{H}{\delta}\right)\right] &\leq \frac{\delta}{H}\cdot\mathbb{E}\left[\exp\left(\sum_{t=1}^{T}(\widehat{J}_{t,h} - J_{t,h})\right)\right] \\
&= \frac{\delta}{H}\cdot\mathbb{E}\left[\exp\left(\sum_{t=1}^{T-1}(\widehat{J}_{t,h} - J_{t,h})\right)\mathbb{E}_T\left[\exp\left(\widehat{J}_{T,h} - J_{T,h}\right)\right]\right] \\
&\leq \frac{\delta}{H}\cdot\mathbb{E}\left[\exp\left(\sum_{t=1}^{T-1}(\widehat{J}_{t,h} - J_{t,h})\right)\right] \\
&\leq \cdots \leq \frac{\delta}{H}.
\end{aligned} \tag{84}$$

On applying the union bound over all $h \in [H]^{-1}$, we have the following holds with probability at least $1-\delta$,

$$\sum_{t=1}^{T}\sum_{(s,a,h)}\alpha_{t,h}(s,a)\left(\hat{c}_t(s,a) - \frac{\rho_t(s,a)}{u_t(s,a)}c_{t,h}(s,a)\right) = \sum_{h=0}^{H-1}\sum_{t=1}^{T}(\widehat{J}_{t,h} - J_{t,h}) \leq H\ln\frac{H}{\delta}.$$

Similarly, we can also show that $\sum_{t=1}^{T}\sum_{(s,a,h)}\alpha_{t,h}(s,a)\left(\hat{\ell}_{t,h}(s,a) - \frac{\rho_t(s,a)}{u_t(s,a)}\ell_{t,h}(s,a)\right) \leq H\ln\frac{H}{\delta}$. □

Recall the definitions of $\widehat{\nabla}_t$ and ∇_t from Section 5.2 and Section 5.1,

$$\widehat{\nabla}_t = \begin{cases}\omega\hat{\boldsymbol{\ell}}_t + \varphi'(\zeta_t)\omega\hat{\boldsymbol{c}}_t, & \text{if } \mathcal{C}_t > 0,\\ \omega\hat{\boldsymbol{\ell}}_t, & \text{if } \mathcal{C}_t \leq 0,\end{cases} \quad \text{and} \quad \nabla_t = \begin{cases}\omega\boldsymbol{\ell}_t + \varphi'(\zeta_t)\omega\boldsymbol{c}_t, & \text{if } \langle\hat{\rho}_t, \omega\boldsymbol{c}_t\rangle > 0,\\ \omega\boldsymbol{\ell}_t, & \text{if } \langle\hat{\rho}_t, \omega\boldsymbol{c}_t\rangle \leq 0.\end{cases}$$

We perform the decomposition below for "Bias2":

$$
\begin{aligned}
\text{Bias2} &= \sum_{\tau=1}^{t} \langle \rho^\star, \widehat{\nabla}_\tau - \nabla_\tau \rangle \\
&= \sum_{\tau=1}^{t} \langle \rho^\star, \omega \hat{\boldsymbol{\ell}}_\tau + \varphi'(\zeta_\tau)\omega \hat{\boldsymbol{c}}_\tau \cdot \mathbf{1}_{\{\mathcal{C}_t > 0\}} - \omega \boldsymbol{\ell}_\tau - \varphi'(\zeta_\tau)\omega \boldsymbol{c}_\tau \cdot \mathbf{1}_{\{\langle \hat{\rho}_t, \omega \boldsymbol{c}_t \rangle > 0\}} \rangle \\
&= \overbrace{\omega \sum_{\tau=1}^{t} \langle \rho^\star, \hat{\boldsymbol{\ell}}_\tau - \boldsymbol{\ell}_\tau \rangle}^{L_1} + \underbrace{\omega \sum_{\tau=1}^{t} \varphi'(\zeta_\tau) \langle \rho^\star, \hat{\boldsymbol{c}}_\tau \cdot \mathbf{1}_{\{\mathcal{C}_t > 0\}} - \boldsymbol{c}_\tau \cdot \mathbf{1}_{\{\langle \hat{\rho}_t, \omega \boldsymbol{c}_t \rangle > 0\}} \rangle}_{L_2}.
\end{aligned} \tag{85}
$$

Note that $\rho^\star(s,a) \in [0,1] \subseteq [0, 2\Lambda_\tau]$ for $\Lambda_\tau \geq 1/2$. Since $\frac{\rho_\tau(s,a)}{u_\tau(s,a)} \leq 1$ and $\ell_{\tau,h}(s,a) \in [0,1]$, the term $\sum_{(s,a,h)} \rho^\star(s,a) \left(\frac{\rho_\tau(s,a)}{u_\tau(s,a)} - 1 \right) \ell_{\tau,h}(s,a) \leq 0$. Thus,

$$
\langle \rho^\star, \hat{\boldsymbol{\ell}}_\tau - \boldsymbol{\ell}_\tau \rangle \leq \sum_{(s,a,h)} \rho^\star(s,a) \left(\hat{\ell}_{\tau,h}(s,a) - \frac{\rho_\tau(s,a)}{u_\tau(s,a)} \ell_{\tau,h}(s,a) \right).
$$

Using Lemma 4, with $\alpha_{\tau,h}(s,a) = \rho^\star(s,a)$, we have with probability at least $1 - \delta$:

$$
L_1 = \omega \sum_{\tau=1}^{t} \langle \rho^\star, \hat{\boldsymbol{\ell}}_\tau - \boldsymbol{\ell}_\tau \rangle \leq \omega \sum_{\tau=1}^{t} \sum_{(s,a,h)} \rho^\star(s,a) \left(\hat{\ell}_{\tau,h}(s,a) - \frac{\rho_\tau(s,a)}{u_\tau(s,a)} \ell_{\tau,h}(s,a) \right) \leq \omega H \ln \frac{H}{\delta}. \tag{86}
$$

We split $L_2 = \omega \sum_{\tau=1}^{t} \varphi'(\zeta_\tau) \langle \rho^\star, \hat{\boldsymbol{c}}_\tau \cdot \mathbf{1}_{\{\mathcal{C}_t > 0\}} - \boldsymbol{c}_\tau \cdot \mathbf{1}_{\{\langle \hat{\rho}_t, \omega \boldsymbol{c}_t \rangle > 0\}} \rangle$ into two components as:

$$
L_2 = \omega \left(\sum_{\tau=1}^{t} \varphi'(\zeta_\tau) \langle \rho^\star, (\hat{\boldsymbol{c}}_\tau - \boldsymbol{c}_\tau) \cdot \mathbf{1}_{\{\mathcal{C}_t > 0\}} \rangle - \sum_{\tau=1}^{t} \varphi'(\zeta_\tau) \langle \rho^\star, \boldsymbol{c}_\tau \cdot \left(\mathbf{1}_{\{\langle \hat{\rho}_t, \omega \boldsymbol{c}_t \rangle > 0\}} - \mathbf{1}_{\{\mathcal{C}_t > 0\}} \right) \rangle \right). \tag{87}
$$

<u>First Term of L_2:</u> Again, as $\frac{\rho_\tau(s,a)}{u_\tau(s,a)} \leq 1$, so:

$$
\langle \rho^\star, \hat{\boldsymbol{c}}_\tau - \boldsymbol{c}_\tau \rangle \leq \sum_{(s,a)} \rho^\star(s,a) \left(\hat{c}_\tau(s,a) - \frac{\rho_\tau(s,a)}{u_\tau(s,a)} c_\tau(s,a) \right).
$$

With probability at least $1 - \delta$, on using Lemma 4, we have:

$$
\sum_{\tau=1}^{t} \varphi'(\zeta_\tau) \langle \rho^\star, \hat{\boldsymbol{c}}_\tau - \boldsymbol{c}_\tau \rangle \leq \sum_{\tau=1}^{t} \varphi'(\zeta_\tau) \sum_{(s,a,h)} \rho^\star(s,a) \left(\hat{c}_{\tau,h}(s,a) - \frac{\rho_\tau(s,a)}{u_\tau(s,a)} c_{\tau,h}(s,a) \right) \leq \varphi'(\zeta_t) \cdot H \ln \frac{H}{\delta}. \tag{88}
$$

<u>Second Term of L_2:</u> Let $F_\tau = \mathbf{1}_{\{\langle \hat{\rho}_t, \omega \boldsymbol{c}_t \rangle > 0\}} - \mathbf{1}_{\{\mathcal{C}_t > 0\}}$. Note that $|F_\tau| \leq 1$ for all τ. Additionally, since $\rho^\star$ is a probability distribution and each component of $\boldsymbol{c}_\tau$ lies in $[-1,1]$, we have $|\langle \rho^\star, \boldsymbol{c}_\tau \rangle| \leq 1$. Therefore, $\left| \sum_{\tau=1}^{t} \varphi'(\zeta_\tau) \langle \rho^\star, \boldsymbol{c}_\tau \cdot \left(\mathbf{1}_{\{\langle \hat{\rho}_t, \omega \boldsymbol{c}_t \rangle > 0\}} - \mathbf{1}_{\{\mathcal{C}_t > 0\}} \right) \rangle \right| \leq \sum_{\tau=1}^{t} \varphi'(\zeta_\tau) \cdot |\langle \rho^\star, \boldsymbol{c}_\tau \rangle| \cdot |F_\tau| \leq \sum_{\tau=1}^{t} \varphi'(\zeta_\tau) \leq t \cdot \varphi'(\zeta_t)$.

Hence, we have an upper bound on L_2 as

$$
L_2 \leq \omega \varphi'(\zeta_t) \cdot H \ln \frac{H}{\delta} - \omega t \cdot \varphi'(\zeta_t). \tag{89}
$$

Combining Eqn. 86 and Eqn. 89 we obtain an upper bound on "Bias2":

$$
\text{Bias2} \leq \omega H \ln \frac{H}{\delta} + \omega \varphi'(\zeta_t) \cdot H \ln \frac{H}{\delta} - \omega t \cdot \varphi'(\zeta_t). \tag{90}
$$