Contact-Aware Covariance Control of Stochastic Contact-Rich Systems

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***Abstract*— Planning and control for uncertain contact systems is challenging as it is not clear how to propagate uncertainty for planning. We use a particle filter-based approach to propagate moments for stochastic complementarity system. To circumvent the issues of open-loop chance constrained planning, we propose a contact-aware controller for covariance steering of the complementarity system. Our optimization problem is formulated as Non-Linear Programming (NLP) using bilevel optimization. We verify that our contact-aware controller is able to steer the covariance of the states for stochastic contact-rich tasks.**

I. Introduction

Contacts lead to discontinuous dynamics and thus, planning through contacts requires careful treatment of constraints arising due to these discontinuities. Complementarity constraints offer an efficient way of modeling contact systems. However, uncertainty in contact systems could lead to stochastic complementarity systems [1], [2]. Even though complementarity systems are well studied, stochastic complementarity systems are not well understood. The state and complementarity variables are implicitly related via the complementarity constraints – uncertainty in one leads to stochastic evolution of other. This makes uncertainty propagation challenging. Furthermore, multiplicity of solutions to the complementarity variables also makes it difficult to characterize the stochastic evolution.

We present a particle-based technique to perform feedback control of stochastic complementarity systems. We present covariance control for the stochastic complementarity systems by solving for a trajectory-centric feedback controller to enable efficient control along long-horizon trajectories. See [3] for more details.

II. Related Work

Our proposed stochastic optimization problem is related to three major areas of work. The first area is optimization with complementarity constraints. This topic has been well studied in optimization and robotics literature [4], [5], [6], [7], [8], [9]. This approach has been shown to work well for generating trajectories for manipulation and locomotion problems. However, it cannot be trivially extended to stochastic complementarity systems to introduce robustness.

Using stochastic complementarity constraints for planning robust manipulation is not so well understood in literature. Some of the recent work could be found in [10], [1]. However, the problem with these approaches is that the uncertainty needs to be very small otherwise the optimization might be infeasible. Consequently, these approaches could fail to provide robust plans for uncertain contact systems. Furthermore, uncertainty propagation for stochastic complementarity systems is not properly modeled in these approaches. One of the reasons is the implicit relationship between contact and state variables in complementarity constraints.

Open-loop Chance-Constrained Optimization (CCO) could lead to quite conservative solutions to satisfy chance constraints, covariance steering methods have gained attention[11], [12], [13]. Covariance steering methods are able to design feedforward and feedback gains simultaneously to satisfy chance constraints. However, these cannot be directly applied to contact-rich systems.

III. Problem Formulation

A. Stochastic Discrete-time Linear Complementarity Systems

In this work, we consider the Stochastic Discrete-time Linear Complementarity Systems (SDLCS):

$$\begin{aligned} x_{k+1} =& A_k(\xi)x_k + B_k u_k + C_k(\xi)\lambda_{k+1} + g_k(\xi) \\ &+ w_k(\xi) \end{aligned} \tag{1a}$$

$$\begin{aligned} 0 \leq \lambda_{k+1} \perp & D_k(\xi)x_k + E_k u_k + F_k(\xi)\lambda_{k+1} + h_k(\xi) \\ &+ l_k(\xi) \geq 0 \end{aligned} \tag{1b}$$

where k is the time-step index, $x_k \in \mathbb{R}^{n_x}$ is the state, $u_k \in \mathbb{R}^{n_u}$ is the control input, and $\lambda_k \in \mathbb{R}^{n_c}$ is the algebraic variable (e.g., contact forces). We define $x = [x_1, \dots, x_T], u = [u_0, \dots, u_{T-1}], \lambda = [\lambda_1, \dots, \lambda_T]$. The parameter $\xi \sim \Xi$ is the uncertain parameter with distribution Ξ. In addition, $A_k(\xi) \in \mathbb{R}^{n_x \times n_x}$, $B_k \in \mathbb{R}^{n_x \times n_u}$, $C_k(\xi) \in \mathbb{R}^{n_x \times n_c}$, $g_k(\xi) \in \mathbb{R}^{n_x}$, $D_k(\xi) \in \mathbb{R}^{n_c \times n_x}$, $E_k \in \mathbb{R}^{n_c \times n_u}$, $F_k(\xi) \in \mathbb{R}^{n_c \times n_c}$, and $h_k(\xi) \in \mathbb{R}^{n_c}$ are all dependent on the uncertain parameter ξ. For simplicity, we abbreviate ξ from these matrices for the discussion in the following sections. The notation $0 \leq a \perp b \geq 0$ denotes the complementarity constraints $a \geq 0, b \geq 0, ab = 0$. The initial state of the system $x_0(\xi)$ is also assumed to be uncertain.

In the following, we make the assumption that $F_k(\xi)$ is a P-matrix [14] for all k and ξ. Under this assumption, there is an unique solution λ_{k+1} to (1b) for each ξ and any u_k, x_k. From this it is easy to infer that there exists an unique trajectory x and λ for any realization of uncertainty $\xi \sim \Xi$ and controls u from every initial condition $x_0(\xi)$.

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B. Stochastic Control for Contact-Rich Systems

To design a robust controller satisfying chance constraints over SDLCS, the following optimization problem can be formulated:

$$\min_{u} \sum_{k=1}^{T} \|\mathbb{E}_{\xi\sim\Xi}\left[\mathbf{x}_k(\xi,u)\right] - x_d\|_Q^2 + \sum_{k=0}^{T-1} \|u_k\|_R^2 \tag{2a}$$

$$\text{s.t. } u_k \in \mathcal{U} \tag{2b}$$

$$\mathrm{Pr}_{\xi\sim\Xi}\left(\mathbf{x}(\xi,u) \in \mathcal{X}\right) \geq \Delta \tag{2c}$$

where $Q = Q^\top$ is positive semidefinite, $R = R^\top$ is positive definite, $\mathcal{U}$ is a convex polytope consisting of a finite number of linear inequality constraints. x_d is the target state at $t = T$. The set $\mathcal{X}$ represents a convex safe region where the entire state trajectory has to lie in. We assume that $\mathcal{X} = \{x \in \mathbb{R}^{n_x T} \,|\, g_i(x) \leq 0 \,\forall\, i = 1,\ldots,n_g\}$. Pr denotes the probability of an event and Δ is the user-defined minimum safety probability, where the probability of satisfying constraints is at least greater than Δ.

IV. Covariance Steering

A. Particle-based Control for Contact-Rich Systems

We propose to solve (2) approximately using the Sample Average Approximation (SAA). In particular, we obtain N realizations of the uncertainty $\Xi^N = \{\xi^1,\ldots,\xi^N\}$ by sampling the distribution Ξ. In other words, we approximate the distribution Ξ using a finite-dimensional distribution Ξ^N which follows an uniform distribution on the samples. Accordingly, the SAA for (2) is given as

$$\min_{u} \sum_{k=1}^{T} \left\|\mathbb{E}_{\xi\sim\Xi^N}\left[\mathbf{x}_k(\xi,u)\right] - x_d\right\|_Q^2 + \sum_{k=0}^{T-1} \|u_k\|_R^2 \tag{3a}$$

$$\text{s.t. } u_k \in \mathcal{U} \tag{3b}$$

$$\mathrm{Pr}_{\xi\sim\Xi^N}\left(\mathbf{x}(\xi,u) \in \mathcal{X}\right) \geq \Delta. \tag{3c}$$

There remains the implicit function $\mathbf{x}(\xi,u)$ which requires us to simulate the SDLCS for every realization of $\xi \in \Xi^N$. We opt to remove this difficulty by replacing the implicit functions with the corresponding trajectories x^i, λ^i for each $\xi^i \in \Xi^N$. Our formulation using N particles is given by:

$$\min_{x^i,u,\lambda^i} \sum_{k=1}^{T} \left\|\frac{1}{N}\sum_{i=1}^{N} x_k^i - x_d\right\|_Q^2 + \sum_{k=0}^{T-1} \|u_k\|_R^2 \tag{4a}$$

$$\text{s.t. } x_{k+1}^i = A_k^i x_k^i + B_k u_k + C_k^i \lambda_{k+1}^i + g_k^i + w_k^i \tag{4b}$$

$$0 \leq \lambda_{k+1}^i \perp D_k^i x_k^i + E_k u_k + F_k^i \lambda_{k+1}^i + h_k^i + l_k^i \geq 0 \tag{4c}$$

$$x_0^i = x_0(\xi^i) \tag{4d}$$

$$u_k \in \mathcal{U} \tag{4e}$$

$$\frac{1}{N}\sum_{i=1}^{N} \mathbb{I}\left(x^i \in \mathcal{X}\right) \geq \Delta \tag{4f}$$

where $\mathbb{I}(\cdot)$ is an indicator function returning 1 when the conditions in the operand are satisfied and 0 otherwise. Note that x^i, λ^i represent the state and algebraic variable trajectory, respectively, propagated from a particular set of particles x_0^i, θ_k^i where $\theta_k^i = [A_k^i, C_k^i, g_k^i, D_k^i, F_k^i, h_k^i, w_k^i, v_k^i]$. Using N trajectories obtained from N particles, we approximate mean of random variables as $\mathbb{E}_{\xi\sim\Xi}[\mathbf{x}_k(\xi,u)] \approx \frac{1}{N}\sum_{i=1}^{N} x_k^i, \mathbb{E}_{\xi\in\Xi}[\boldsymbol{\lambda}_k(\xi,u)] \approx \frac{1}{N}\sum_{i=1}^{N}\lambda_k^i$. In (4), we approximate (2a) using the mean variable as shown in (4a). Chance constraints (2c) can be also approximated as (4f) using N realization trajectories, which can be formulated as integer constraints (see [15]).

In this work, we consider the following controllers:

$$\textbf{feedforward}: u_k = v_k \tag{5a}$$

$$\textbf{feedback}: u_k = v_k + K_k(x_k - \bar{x}_k) + L_k(\lambda_k - \bar{\lambda}_k) \tag{5b}$$

where K_k, L_k are feedback gains to control covariance. For brevity, we use $\bar{x}_k = \frac{1}{N}\sum_{i=1}^{N} x_k^i, \bar{\lambda}_k = \frac{1}{N}\sum_{i=1}^{N}\lambda_k^i$. We emphasize that controlling both states and contact variables is critical for contact-rich systems and thus we also introduce $L_k(\lambda_k - \bar{\lambda}_k)$ to (5b) to stabilize the system. Here, we focus on discussing feedback controller (5b) for (4). Our optimization formulation for covariance steering of SDLCS would be:

$$\min_{x^i,v,K,L,\lambda^i} \sum_{k=1}^{T} ||\bar{x}_k - x_d||_Q^2 + \sum_{k=0}^{T-1} \|u_k\|_R^2 \tag{6a}$$

$$\begin{aligned}\text{s. t. } x_{k+1}^i &= (A_k^i + B_k K_k)x_k^i + B_k v_k \\ &+ (C_k^i + B_k L_k)\lambda_{k+1}^i + \bar{g}_k^i \\ &- B_k K_k \bar{x}_k - B_k L_k \bar{\lambda}_{k+1} + w_k^i\end{aligned} \tag{6b}$$

$$\begin{aligned}0 \leq \lambda_{k+1}^i &\perp (D_k^i + E_k K_k)x_k^i \\ &+ E_k v_k + (F_k^i + E_k L_k)\lambda_{k+1}^i \\ &+ h_k^i - E_k K_k \bar{x}_k - E_k L_k \bar{\lambda}_{k+1} + l_k^i \geq 0\end{aligned} \tag{6c}$$

$$(4d), (4e), (4f) \tag{6d}$$

To solve (6), we need to take care of, (6b), (6c) and (4f). One method is mixed-integer programming. It is possible that binary variables can be used to deal with integer constraints (4f) using Big-M formulation. Also, bilinear terms in (6b) and (6c) can be approximated using McCormick envelopes, leading to additional binary variables. As a result, a number of binary variables are introduced and we observed that it is almost impossible to obtain a single feasible solution. Instead, we use NLP which can solve (6b) as nonlinear constraints and (6c) as complementarity constraints.

B. Bilevel Optimization for Particle-based Control

To solve (6) using NLP, we need to solve integer constraints (4f) in NLP fashion. To achieve this, we propose the following bilevel optimization problem.

$$\min_{x^i,v,K,L,\lambda^i,t^i,z^*} \sum_{k=1}^{T} \|\bar{x}_k - x_d\|_Q^2 + \sum_{k=0}^{T-1} \|u_k\|_R^2 \tag{7a}$$

$$\text{s. t. } (6b), (6c), (4e) \tag{7b}$$

$$\forall j = 1,\ldots,n_g,\ g_j(x) \leq t^i, \tag{7c}$$

$$\frac{1}{N}\sum_{i=1}^{N} z^{i,*} \geq \Delta \tag{7d}$$

$$\forall i = 1, \ldots, N, \; z^{i,*} = \arg\min_{z^i} t^i z^i | 0 \leq z^i \leq 1 \qquad (7e)$$

We introduce time-invariant parameter $t^i \in \mathbb{R}^1$ for each set of trajectory realization i. If $x^i \in \mathcal{X}$, $t^i \geq -\epsilon$ with $\epsilon \geq 0$. In contrast, if $x \notin \mathcal{X}$, $t^i \geq 0$. This condition is encoded in (7c). We have in total N lower-level optimization problems (7e), where each optimization is formulated as linear programming. $z^i \in \mathbb{R}^1$ is the decision variable used in i -th lower-level optimization problem.

The purpose of (7e) is to count the number of trajectory realizations that are inside $\mathcal{X}$. The optimal solution of (7e) can be as follows:

$$z^i = \begin{cases} 1, & t^i < 0 \\ [0,1], & t^i = 0 \\ 0, & t^i > 0 \end{cases} \qquad (8)$$

If $t^i < 0$, (7c) argues that $x^i \in \mathcal{X}$ and thus we count this i-th trajectory propagated from i-th particles as one. If $t^i = 0$, (7c) argues that x^i lies on the boundary of $\mathcal{X}$ and thus we count this i-th trajectory propagated from i-th particles as one. If $t^i > 0$, then x^i is not within $\mathcal{X}$, and thus we count it as zero. Then (7d) considers the approximated chance constraints.

Since the upper-level optimization decision variable t^i can be influenced by other upper-level decision variables, we need to solve these two problems simultaneously, leading to a bilevel optimization problem. Since the lower-level optimization problems are formulated as N linear programming problems, we can efficiently solve the entire bilevel optimization problem using the Karush-Kuhn-Tucker (KKT) condition as follows:

$$\min_{x^i, v, K, L, \lambda^i, t^i, z^{i,*}, w^i_+, w^i_-} (7a) \qquad (9a)$$

$$\text{s. t. } (7b), (7c), (7d) \qquad (9b)$$

$$\forall i = 1, \ldots, N, \; 0 \leq z^{i,*} \leq 1, w^i_+, w^i_- \geq 0 \qquad (9c)$$

$$w^i_+(z^{i,*} - 1) = 0, w^i_-(z^{i,*}) = 0, \qquad (9d)$$

$$t^i + w^i_+ - w^i_- = 0 \qquad (9e)$$

where w^i_+, w^i_- are Lagrange multipliers associated with $z^i - 1 \leq 0$, $-z^i \leq 0$, respectively.

V. Results

We implement our method using IPOPT [16] with PY-ROBOCOP [5]. When we run (9), we use 1000 samples to calculate the empirical probability of failure to evaluate the satisfaction of chance constraints. Readers are encouraged to read [3] to see all results.

Here we explain how we simulate trajectories (i.e., perform MC simulation for SDLCS, see [1] for more details). We propagate the dynamics by finding the roots of the complementarity system with sampled parameters given the control sequence obtained from optimization. We run each case for 1000 trials with different sampled parameters to estimate the probability of failure. Note that, unlike the continuous-domain dynamics, we cannot rollout the dynamics for SDLCS with the given control sequences since we do not have the access to λ_{k+1}.

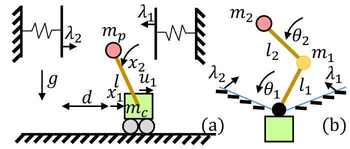

Fig. 1: (a): cartpole with softwalls. (b): acrobot with soft joints.

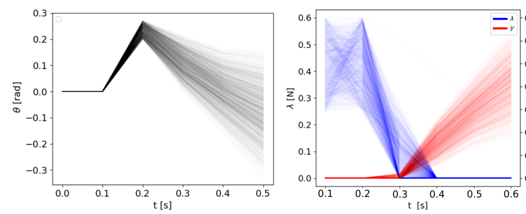

Fig. 2: Uncertainty propagation for cartpole system.

A. Uncertainty Propagation for SDLCS

We demonstrate uncertainty propagation for a cartpole system with softwalls (see [9] for more details). Here we consider both k_1 and k_2 follows uniform distributions where upper bound of uniform distribution for k_1 and k_2 is 14, 12, respectively, and the lower bound is 5 for both k_1 and k_2. In this experiment, we do not apply any controller: we simply propagate SDLCS given uncertain parameters in order to show how the SDLCS behaves.

Fig. 2 shows the evolution of uncertainty for the aforementioned system. At $t = 0$ s, there is no uncertainty for state $\theta_{t=0}$. However, because we provide uncertainty with k_1 and k_2, $\lambda_{t=0.1}$ has uncertainty. This is again because given realization of uncertain parameters, complementarity constraints give a realization of λ and y, resulting in uncertainty in λ and y. This stochastic $\lambda_{t=0.1}$ brings uncertainty in $\theta_{t=0.1}$ based on (1). As shown in Fig. 2, both state and complementarity variables are stochastic. This can not be captured in approximations like Expected Residual Minimization (ERM) [10].

B. Cartpole with Softwalls

We demonstrate our open- and closed-loop controllers for cartpole with softwalls system. x is the cart position and θ is the pole angle. u_1 is the control and λ_1, λ_2 are the reaction forces at from the wall 1, 2, respectively. For the actual value of the physical parameters, see [3]. We assume that the uncertainty arises from the k_1, k_2 and use the same distribution in Sec V-A. We set $dt = 0.1$ for the explicit Euler integration and $T = 6$.

The results using ERM and our controller for the open-loop trajectory are shown in Fig. 3, Fig. 4. We observed that the proposed open-loop controller shows the better satisfaction of chance constraints compared to the ERM-based method. Also, we observe that the gap between the commanded Δ used in our optimization and Δ_{test} obtained from MC simulation over testing dataset is smaller the gap between the commanded Δ used in ERM method and Δ_{test} obtained from MC simulation over testing dataset. This is because our method explicitly considers propagation of uncertainty for SDLCS while the ERM-based method is

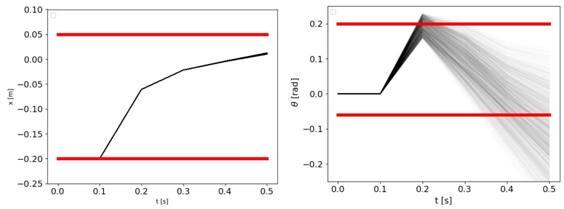

Fig. 3: Simulated trajectories for cartpole system using ERM-based controller. $\Delta = 0.2$ and $\Delta_{\text{test}} = 0.083$. Red lines show boundaries specified in chance constraints.

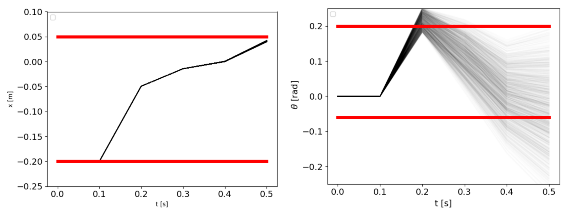

Fig. 4: Simulated trajectories for cartpole system using our open-loop controller. $\Delta = 0.2$ and $\Delta_{\text{test}} = 0.190$ where Δ is input of optimization and Δ_{test} is the empirically obtained success rate from MC simulation. Red lines show boundaries specified in chance constraints.

unable to consider.

We observe in Table I, (9) for open-loop controller with high Δ was unable to find feasible solutions but (9) for closed-loop controller could find feasible solutions. Since the closed-loop controller can change feedback gains to satisfy chance constraints, it could find feasible solutions with high Δ. Also, Table I shows that the contact-aware closed-loop controller could find the feasible solution with high $\Delta = 0.8, 0.7$ but the non-contact-aware controller (i.e., $L_k = 0, \forall k$ in (5b)) could not. For SDLCS, introducing feedback to both states and forces is important to realize the robust motion. The MC simulation results using our contact-aware closed-loop controller are shown in Fig. 5. In contrast to Fig. 3 and Fig. 4, the closed-loop controller could bound the distribution of the states because it controls covariance.

C. Acrobot with Soft Joints

We demonstrate our controller for acrobot with soft joints system (see [9] for more details). θ_1 is the first joint angle and θ_2 is the second joint angle. u_1 is the control at the second joint and λ_1, λ_2 are the reaction forces at from the wall 1, 2, respectively. For the actual value of the physical parameters, see [3]. We consider the stochastic physical parameters k and l_2 where k is the stiffness of the walls and l_2 is the length of the second rod. We assume that k follows uniform distribution where the upper bound and the lower bound of the distribution is 1.6 and 0.6, respectively. We assume that l_2 follows a truncated Gaussian distribution where we set the mean to 1.0, variance to 0.01, the upper bound of the interval is 1.3, and the lower bound of the interval is 0.7. We set $dt = 0.04$ for the explicit Euler integration and $T = 15$.

The open- and closed-loop trajectories are shown in Fig. 6. We observed that both controller could satisfy chance constraints over the testing dataset and the closed-loop controller shows the better performance.

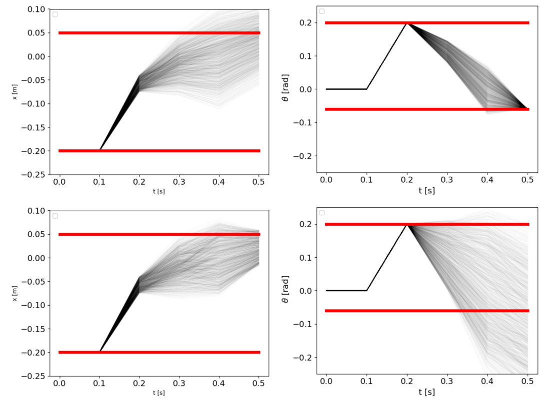

Fig. 5: Simulated trajectories for cartpole system using our closed-loop controller. Top: $\Delta = 0.6$ and $\Delta_{\text{test}} = 0.510$, bottom: $\Delta = 0.2$ and $\Delta_{\text{test}} = 0.188$, where Δ is input of optimization and Δ_{test} is the empirically obtained success rate from MC simulation. Red lines show boundaries specified in chance constraints.

TABLE I: Comparison of feasibility for cartpole system among open-, non-contact-aware closed, and contact-aware-closed controllers with different Δ. $\circ$ and $\times$ show if optimization finds a feasible solution or not, respectively.

Δ	0.8	0.7	0.6	0.4	0.2
Open-loop	$\times$	$\times$	$\times$	$\circ$	$\circ$
Non-contact-aware closed-loop	$\times$	$\times$	$\circ$	$\circ$	$\circ$
Contact-aware closed-loop	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$

VI. Conclusion

This paper presents a study of SDLCS to perform covariance steering using particles. It is shown that our work could design open- and closed-loop controllers with chance constraints by appropriately considering the evolution of uncertainty for SDLCS.

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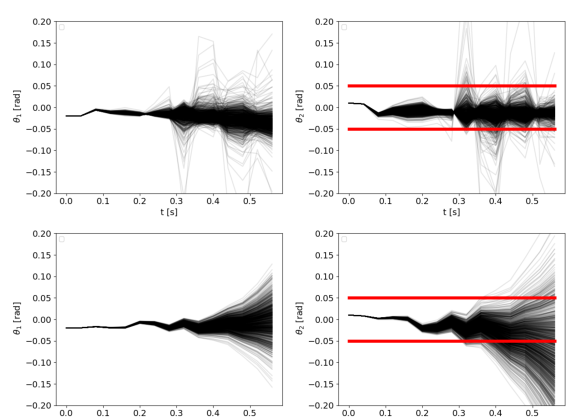

Fig. 6: Simulated trajectories for acrobot using our open- and closed-loop controllers. Top: closed-loop controller with $\Delta = 0.8$ and $\Delta_{\text{test}} = 0.771$, bottom: open-loop controller with $\Delta = 0.4$ and $\Delta_{\text{test}} = 0.366$. Red lines show boundaries specified in chance constraints.

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