GENERATION OF A RANDOM SELF-SIMILARITY CURVE

Ilya Pershin
Innopolis University
i.pershin@innopolis.ru

Dmitrii Tumakov
Kazan Federal University
dtumakov@kpfu.ru

ABSTRACT

This work presents a novelty method for generating a random self-similarity curve by fractal dimension. Fractal dimension is the main feature of self-similarity objects, which can be used for for efficient representation of complex structures. Experiments show that the proposed method can be used to design a wire dipole antenna.

1 INTRODUCTION

The present work proposes a deterministic algorithm to generate a random self-similarity curve, which was inspired by the box counting method (Falconer, 2003). A fractal is an object whose fractal dimension is greater than its topological dimension (Mandelbrot, 1983). Fractal dimension in a broad sense characterizes the complexity of a fractal. An algorithm can be considered deterministic if it is not required to generate a random curve for each dimension value. Related works (Brown et al., 2010; Ringl & Urbassek, 2013) for generating objects by fractal dimension are not intended for generating curves. The generation of continuous curves is necessary in some applications, such as those related to miniature antenna modeling (Tumakov et al., 2020), calculating molecular complexity for drug discovery (von Korff & Sander, 2019), etc.

2 METHOD

The calculation of fractal dimension (Minkowski dimension) d of set F is based on the coverage by boxes with side ε

$$d = \lim_{\varepsilon \to 0} \frac{\log |N_\varepsilon(F)|}{-\log \varepsilon}, \tag{1}$$

where $N_\varepsilon(F)$ is set and contains minimum number $|N_\varepsilon(F)|$ of boxes, which cover the set F.

Box-counting algorithm is the numerical approach for estimation of fractal dimension. The algorithm is based on covering a set with boxes of different sizes and calculating the dimension as the slope of a straight line, which is constructed using the values of ε and N_ε. The calculated dimension is called the box-counting dimension. To do this, equation 1 is represented as

$$d \log \varepsilon + \log |N_\varepsilon(F)| = 0. \tag{2}$$

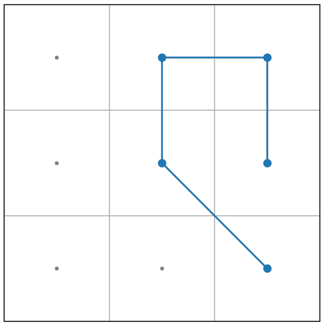
Let us impose a constraint on the connectivity of neighboring elements on the set F. The sequence $\mathcal{E}$ of side of boxes is defined as follows:

$$\mathcal{E} = \{3^{-i}\}_{k=1}^{n}, \tag{3}$$

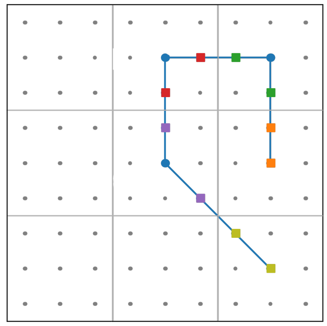
which is that all boxes from the previous step are divided into 9 smaller boxes.

Let a set $F^{(i)}$ be constructed that satisfies the connectivity condition on the subset $\mathcal{E}^{(i)} = \{\varepsilon_1, \varepsilon_2, ..., \varepsilon_i\} \subset \mathcal{E}$. At the next $(i+1)$-th step, the curve with a step 3 times smaller (according to equation 3), which was at i-th step is resampled. As a result, we get a resampled set $\widetilde{F}^{(i)}$ (figure 1b).

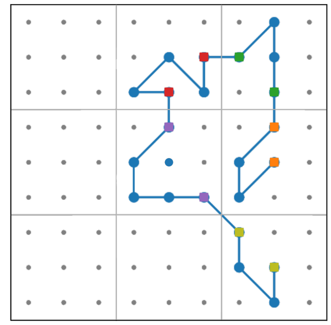
From equation 1 follows the number of boxes $N(\varepsilon, d) = \varepsilon^{-d}$. At the next step each filled box from the previous step must be broken into 9 parts and filled $L = \lceil N(\varepsilon_{i+1}, d)/|N_{\varepsilon_i}(F^{(i)})| \rceil$ boxes. To do this, for each box $\tilde{b} \in N_{\varepsilon_i}(\widetilde{F}^{(i)})$ the corresponding subset of the constructed curve $\widetilde{M}^{(i)} = \widetilde{F}^{(i)} \cap \tilde{b}$



(a) Generated curve $F^{(i)}$ on step i.



(b) Resampled curve $\widetilde{F}^{(i)}$.



(c) Generated curve $F^{(i+1)}$ on step $i+1$.

Figure 1: Algorithm steps for generating a random curve on the value of the fractal dimension. The colored squares mark the starting and ending points of the generated local curves.

Table 1: Root mean squared error (RMSE) and relative mean absolute error (rMAE) for regressional models of the base frequency f dependent on the length L and fractal dimension d of the dipole arm.

Error	Model		
	linear, by L	nonlinear, by L	linear, by d
RMSE	65.6	40.6	29.52
rMAE	12.25%	7%	5.14%

is obtained. Take the first $m^{(i)}_{first} \in \widetilde{M}^{(i)}$ and the last $m^{(i)}_{last} \in \widetilde{M}^{(i)}$ elements of subset $\widetilde{M}^{(i)}$. An example of the first and last curve points inside each box is shown in figure 1b. Generate a random local curve $C = \{c_1, c_2, ..., c_L\}$ inside each box (using the recursive Breadth First Search algorithm), where $c_1 = m^{(i)}_{first}$ and $c_L = m^{(i)}_{last}$. The set $F^{(i+1)}$ is constructed by combining all local curves. The resulting curve is shown in figure 1c. The process is repeated until the resulting set $F^{(n)}$ is constructed. To construct the set $F^{(1)}$ the first and last elements are chosen randomly, without resampling.

3 Wire dipole modeling experiments

A symmetric wire dipoles with a linear arm length of 7.5 cm made of a wire with a diameter of 1 mm are considered. We randomly generate the geometry of the arms, as described in the 2 section, with the fractal dimension d from 1.1 to 1.7 with a step of 0.003. For each fractal dimension value we generate two curves: with $\varepsilon = 3^{-2}$ (second iteration of generation) and $\varepsilon = 3^{-3}$ (third iteration of generation). We did not use more complex curves because such antennas are difficult to produce in practice. For each generated antenna geometry, we numerically simulate the electrodynamic characteristics (base frequency f and bandwidth BW). The simulation was performed using professional computational electromagnetics software.

We fit regression models of electrodynamic characteristics on the length L and fractal dimension d of the dipole arm. Table 1 shows values of the root mean squared error (RMSE) and relative mean absolute error (rMAE). The smallest values of RMSE and rMAE corresponds to the linear models on the fractal dimension.

4 Conclusion

The new method for generating a random self-similarity curve by fractal dimension on a set of length scales is proposed. This approach is used to generate the geometry of the arms for the symmetric wire dipole antenna. Empirical evaluation demonstrated significant outperformance of linear models on fractal dimension compared to the nonlinear models on the length of dipole arm.

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URM STATEMENT

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A TECHNICAL BACKGROUND ON FRACTAL DIMENSION

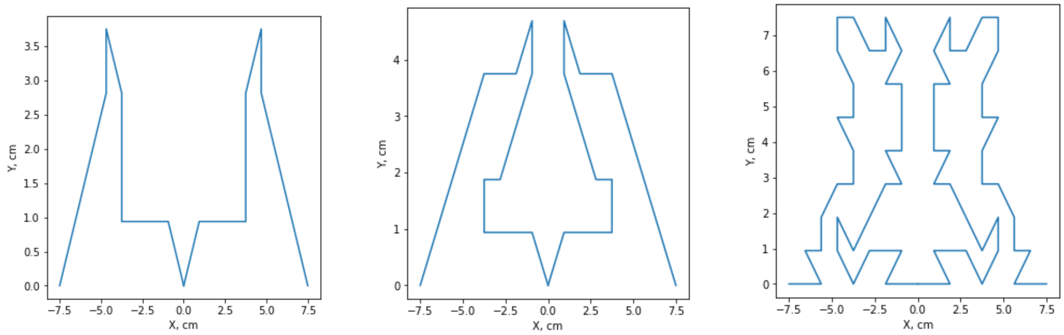
Fractal dimension is the most important characteristic of self-similarity objects. This paper considers the widely used Minkowski dimension, which is also known as the box-counting dimension. Its idea is based on the analysis of the size of the set, ignoring irregularities less than $\varepsilon \to 0$. For example, if the set F is a plane curve, then F can be covered with a collection of segments of $N_\varepsilon(F)$ length ε. Such a coating is used in the coastline paradox, which was first systematized by Richardson (Mandelbrot, 1983). Evaluating the length of a border (or coastline) by covering the segments of the length ε he found that the total length leads to infinity when tends to $\varepsilon \to 0$ and the equation is fulfilled

$$N(\varepsilon, \gamma) \sim c\varepsilon^{-\gamma} \tag{4}$$

where $N(\varepsilon, \gamma)$ is the minimum number of length segments ε, which cover the object, and c, γ are constants.

From equation 4 can be expressed

$$\gamma = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon, \gamma)}{-\log \varepsilon}. \tag{5}$$



(a) Fractal dimension $d = 1.1$. (b) Fractal dimension $d = 1.28$. (c) Fractal dimension $d = 1.64$.

Figure 2: Example of a symmetric dipole geometry based on a random second order prefractal.

Box-counting dimension. An arbitrary non-empty bounded set $F \subset \mathbb{R}^d$ can be covered by a set $N_\varepsilon(F)$ of boxes of a side $\varepsilon > 0$. Then box-counting dimension of the set F is defined as

$$d = \lim_{\varepsilon \to 0} \frac{\log |N_\varepsilon(F)|}{-\log \varepsilon}, \tag{6}$$

where $N_\varepsilon(F)$ contains minimum number $|N_\varepsilon(F)|$ of boxes, which cover the set F. The equation 6 can be represented as

$$d \log \varepsilon + \log |N_\varepsilon(F)| = 0. \tag{7}$$

Equation 7 is the equation of a straight line with slope coefficient d. It is necessary to calculate $\log \varepsilon$ and $|\log N_\varepsilon(F)|$ to estimate it for different values $\varepsilon \in E$. At the same time, it is inconvenient to operate with the size value of the ε box side, therefore, one often operates with the number of boxes $s = \varepsilon^{-1}$, that are contained in the side of the grid (for example, when calculating the dimension of objects in an image). Then equation 7 can be rewritten as

$$-d \log s + \log |N_s(F)| = 0$$

and write a ratio based in equation 4 and equation 6

$$N(s, d) = s^d, \tag{8}$$

where d is the value of the fractal dimension.

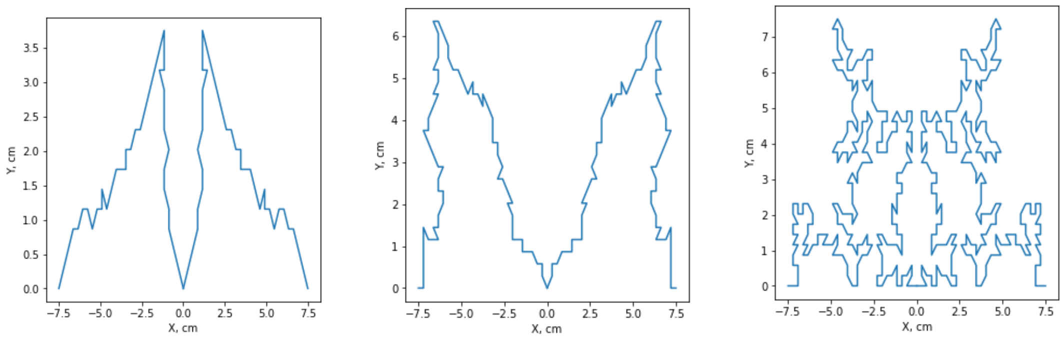
The set S is often formed as power values. As a result, the value in the set $S = \{\log s_i\}_{i=1}^n$ will be distributed evenly.

B Visualization of generated curves and relationship between electrodynamics and geometric characteristics

Figure 1 shows the steps of the algorithm for self-similarity curve generation by fractal dimension value. Figure 1a shows the resulting curve from iteration i. On the next iteration $i + 1$, the curve is resampled with less discretization step (figure 1b) and a curve is generated from the first to last point with length L in each box(figure 1c).

Figure 2 and 3 shows a visualization of generated symmetrical wire dipole geometries for second and third iteration, retrospectively.

Figure 4 shows the relationship between the base frequency and the geometric features of the dipole arm. For the relationship with the length of the dipole arm, a linear (figure 4a and nonlinear (figure 4b) regression model was fitted. For the relationship with the fractal dimension of the dipole arm, only a linear model (figure 4c) was fitted. Figure 5 shows a similar visualization for the relationship with base frequency.

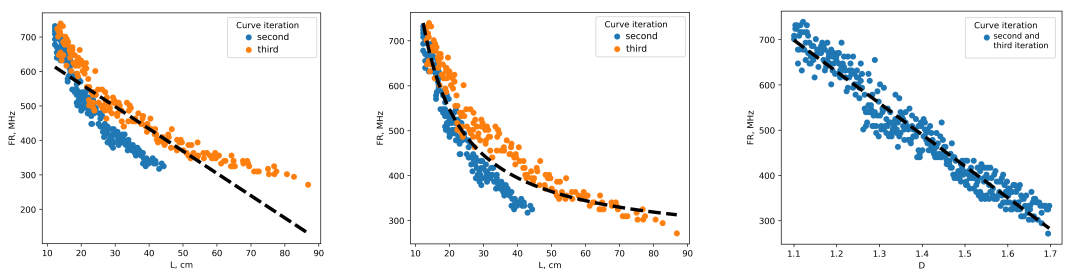


(a) Fractal dimension $d = 1.106$.

(b) Fractal dimension $d = 1.292$.

(c) Fractal dimension $d = 1.628$.

Figure 3: Example of a symmetric dipole geometry based on a random third-order prefractal.

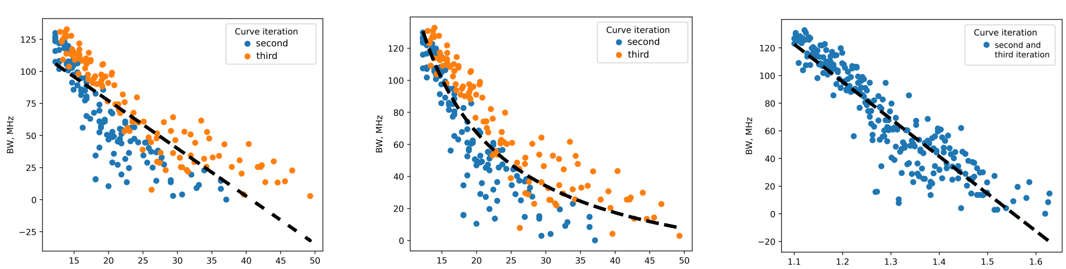


(a) Linear model dependent on length L (cm) of dipole arm.

(b) Nonlinear model dependent on length L (cm) of dipole arm.

(c) Linear model dependent on fractal dimension d of dipole arm geometry.

Figure 4: Regression models of base frequnecy f (MHz).



(a) Linear model dependent on length L (cm) of dipole arm.

(b) Nonlinear model dependent on length L (cm) of dipole arm.

(c) Linear model dependent on fractal dimension d dipole arm geometry.

Figure 5: Regression models of bandwidth BW (MHz).