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Cubature Quadrature Filter for One-step Randomly Delayed Measurements

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Abstract— **The paper proposes an extension of the conventional cubature quadrature Kalman filter (CQKF), in order to enable it for dealing with randomly delayed measurement problems for nonlinear systems. The proposed extension is bounded with an assumption that the maximum delay does not exceed unit sampling time which is alternatively called as one step delayed measurements in this paper. The proposed method which uses third-degree spherical cubature rule and arbitrary order Gauss-Laguerre quadrature rule to solve the intractable integrals is more accurate compared to its existing cubature Kalman filter (CKF) based counterpart. The efficacy of the proposed algorithm has been demonstrated for a maneuvering target tracking problem.**

I. Introduction

Filters estimate the states of a dynamic system recursively from given noisy measurements. The conventional and widely accepted Kalman filter [1] gives optimal estimate for the linear systems corrupted with additive Gaussian noise. However, no optimal solution exists for the estimation of nonlinear systems. Researchers developed several approximated estimators namely the extended Kalman filter (EKF) [1], [2], unscented Kalman filter (UKF) [3] and its variant [4], central difference filter (CDF) [5], particle filter [6], Gauss-Hermite filter (GHF) [7], [8] *etc.* to estimate the states of a nonlinear system.

The conventional nonlinear filters come with a critical assumption that the present measurement is available without any delay. However, the practical scenario mostly differs and the measurements may be randomly delayed in time, and so, it may contain the information about some previous time instant. Some of the real-life filtering problems with randomly delayed measurements could be noticed in the varying domains of communication [9], control applications [10], aerospace and underwater target tracking [11] *etc.*.

The work on the state estimation for a randomly delayed system was initiated by Ray *et al.* [12], which modified the general linear filtering algorithm for the randomly delayed measurements. The later literature refers this problem with varying names, as such, filtering with random sample delay [13], filtering with random time delayed measurements [14], out of sequence measurement (OOSM) [15] *etc.*

The literature for randomly delayed measurements is not significantly large. This could be concluded from limited publications appeared in linear[12], [14], [16], [17] and nonlinear [18], [19], [20] domains. Focusing over the literature for nonlinear systems, it began with the works of Hermoso-Carazo *et al.* [18], [19] which extended the conventional EKF for dealing with delayed measurement problems. It is to be noted that these publications appear with a cap over the extent of delay as one and two times of the sampling interval. The same work was later corrected to deal with the problems if the noises are correlated. [21]. Further, in a significant contribution [20], an efficient nonlinear filter, named as cubature Kalman filter (CKF) [22], was extended for dealing with the problems of randomly delayed measurements.

Recently an algorithm, named as cubature quadrature Kalman filter (CQKF) [23], [24] has been proposed in literature. In this paper, an extension of this filter is proposed for dealing with the nonlinear filtering problems with possible delayed measurements. The proposed delay filter has been validate by implementing it for tracking of a maneuvering target. The simulation results reflect an enhanced efficacy for the proposed algorithm compared to its CKF and UKF counterparts.

II. Problem Formulation

The study in this paper considers a discrete nonlinear system, represented by the following state space model,

$$\mathbf{x}_k = \phi_{k-1}(\mathbf{x}_{k-1}) + q_{k-1}, \tag{1}$$

$$z_k = \gamma_k(\mathbf{x}_k) + w_k, \tag{2}$$

where $\mathbf{x}_k \in \Re^n$ and $z_k \in \Re^p$ are state and measurement variables at an instant k *i.e.* $k = \{0, 1, 2, 3, \cdots, N\}$. $\phi_k(\mathbf{x}_k)$ and $\gamma_k(\mathbf{x}_k)$ are known nonlinear functions of $\mathbf{x}_k$ and k. The process noise $q_k \in \Re^n$ and measurement noise $w_k \in \Re^p$ are bounded with several assumptions, such as, uncorrelated, white and zero mean Gaussian. The covariances for the noises were assumed to be Q_k and R_k respectively.

As the proposition falls under the cases where the measurements may be delayed by unit sampling interval, the actual measurement, y_k, could be modeled as

$$y_k = (1 - \beta_k)z_k + \beta_k z_{k-1}; \quad y_1 = z_1 \tag{3}$$

where β_k is a sequence of Bernoulli random variables which can be either 0 or 1, following certain probabilistic bounds, as

$$\begin{aligned} P(\beta_k = 1) &= p_k = E[\beta_k] \\ P(\beta_k = 0) &= 1 - p_k \\ \text{and,} \quad E[(\beta_k - p_k)^2] &= p_k(1 - p_k). \end{aligned} \tag{4}$$

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In equation (3), $\beta_k = 0$ leads to a non-delayed condition, whereas, $\beta_k = 1$ reflects that the measurement is delayed by unit sampling interval. The proposed algorithm is motivated to construct the posterior pdf $P(\mathbf{x}_k|Y_k)$ where $Y_k = \{y_i\}$ with $i = \{1, 2, \cdots, k\}$ denotes the set of delayed measurements.

III. Filtering Under Bayesian Framework for Single Step Delayed (Random) Measurements

Recalling equation (3), a mathematical model of unit step delayed measurements could be formed as

$$\begin{aligned} y_k &= (1-\beta_k)z_k + \beta_k z_{k-1} \\ &= (1-\beta_k)(\gamma_k(\mathbf{x}_k) + w_k) + \beta_k(\gamma_{k-1}(\mathbf{x}_{k-1}) + w_{k-1}). \end{aligned} \tag{5}$$

The case of delayed measurement needs to construct $P(w_k|Y_k)$, prior to constructing the desired pdf, $P(\mathbf{x}_k|Y_k)$. So, the state vector has been augmented with measurement noise, *i.e.*

$$\mathbf{x}^a_{k+1} = [\mathbf{x}^T_{k+1} \quad w^T_{k+1}]^T, \tag{6}$$

hence the posterior pdf $P(x^a_k|Y_k)$ needs to be estimated.

A. Assumptions

The proposed algorithm is an extension of the conventional CQKF which is bounded with the assumption that the pdfs appeared during the filtering are Gaussian. The same assumption propagates in the proposed algorithm as well, and hence, we assume the following

- $P(\mathbf{x}_{k+1}|Y_{k+1})$ is Gaussian with mean $\hat{\mathbf{x}}_{k+1|k+1}$, and covariance $\mathbf{P}_{k+1|k+1}$.
- $P(\mathbf{w}_{k+1}|Y_k+1)$ is Gaussian with mean $\hat{w}_{k+1|k+1}$ and covariance $\mathbf{P}^{ww}_{k+1|k+1}$.
- The pdf of augmented states is distributed normally

$$\mathbf{P}(\mathbf{x}^a_{k+1}|Y_{k+1}) = \aleph(\mathbf{x}^a_{k+1}; \hat{\mathbf{x}}^a_{k+1|k+1}, \mathbf{P}^a_{k+1|k+1}), \tag{7}$$

where

$$\mathbf{x}^a_{k+1|k+1} = \begin{bmatrix} \hat{\mathbf{x}}_{k+1|k+1} \\ \hat{w}_{k-1|k-1} \end{bmatrix}, \tag{8}$$

and

$$\mathbf{P}^a_{k+1|k+1} = \begin{bmatrix} \mathbf{P}_{k+1|k+1} & \mathbf{P}^{xw}_{k+1|k+1} \\ (\mathbf{P}^{xw}_{k+1|k+1})^T & \mathbf{P}^{ww}_{k+1|k+1} \end{bmatrix} \tag{9}$$

- The prior density function for $\mathbf{x}_{k+1}$ is Gaussian

$$P(\mathbf{x}_{k+1}|Y_k) = \aleph(\mathbf{x}_{k+1}; \hat{\mathbf{x}}_{k+1|k}, \mathbf{P}_{k+1|k}). \tag{10}$$

- The prior density function for delayed measurement $\mathbf{y}_{k+1}$ is Gaussian, *i.e.*

$$P(y_{k+1}|Y_k) = \aleph(y_{k+1}; \hat{y}_{k+1|k}, \mathbf{P}^{yy}_{k+1|k}). \tag{11}$$

- The density function for the non-delayed measurement, $P(z_k|Y_k)$ and $P(z_{k+1}|Y_k)$, are Gaussian with mean $\hat{z}_{k|k}$, $\hat{z}_{k+1|k}$ and covariance $\mathbf{P}^{zz}_{k|k}$, and $\mathbf{P}^{zz}_{k+1|k}$ respectively.

The states, as a result of augmentation, could be estimated in two steps: (i) state estimation and (ii) measurement noise estimation.

B. Estimation of measurement noise

The prior and posterior density function for non-delayed measurement could be given as

$$\hat{z}_{k+1|k} = \int \gamma_{k+1}(\mathbf{x}_{k+1}) \aleph(\mathbf{x}_{k+1}; \hat{\mathbf{x}}_{k+1|k}, \mathbf{P}_{k+1|k}) d\mathbf{x}_{k+1}, \tag{12}$$

$$\begin{aligned} \mathbf{P}^{zz}_{k+1|k} = \int \gamma_{k+1}(\mathbf{x}_{k+1}) \gamma^T_{k+1}(\mathbf{x}_{k+1}) \aleph(\mathbf{x}_{k+1}; \hat{\mathbf{x}}_{k+1|k}, \\ \mathbf{P}_{k+1|k}) d\mathbf{x}_{k+1} - \hat{z}_{k+1|k}\hat{z}^T_{k+1|k} + R_{k+1}, \end{aligned} \tag{13}$$

$$\hat{z}_{k|k} = \int [\gamma_k(\mathbf{x}_k) + w_k] \aleph(\mathbf{x}^a_k; \hat{\mathbf{x}}^a_{k|k}, \mathbf{P}^a_{k|k}) d\mathbf{x}^a_k, \tag{14}$$

$$\begin{aligned} \mathbf{P}^{zz}_{k|k} = \int [\gamma_k(\mathbf{x}_k) + w_k][\gamma_k(\mathbf{x}_k) + w_k]^T \aleph(\mathbf{x}^a_k; \hat{\mathbf{x}}^a_{k|k} \\ , \mathbf{P}^a_{k|k}) d\mathbf{x}^a_k - \hat{z}_{k|k}\hat{z}^T_{k|k}. \end{aligned} \tag{15}$$

The prior density function for the delayed measurements could be given as

$$\hat{y}_{k+1|k} = (1-p_{k+1})\hat{z}_{k+1|k} + p_{k+1}\hat{z}_{k|k}, \tag{16}$$

$$\begin{aligned} \mathbf{P}^{yy}_{k+1|k} = (1-p_{k+1})\mathbf{P}^{zz}_{k+1|k} + p_{k+1}\mathbf{P}^{zz}_{k|k} + p_{k+1} \times \\ (1-p_{k+1})(\hat{z}_{k+1|k} - \hat{z}_{k|k})(\hat{z}_{k+1|k} - \hat{z}_{k|k})^T. \end{aligned} \tag{17}$$

The expression for Kalman gain during noise estimation is

$$K^w_k = \mathbf{P}^{wy}_{k+1|k}(\mathbf{P}^{yy}_{k+1|k})^{-1}, \tag{18}$$

where

$$\mathbf{P}^{wy}_{k+1|k} = (1-p_{k+1})R_{k+1}. \tag{19}$$

The posterior estimate of measurement noise, as well as the error covariance are

$$\hat{w}_{k+1|k+1} = K^w_k(y_{k+1} - \hat{y}_{k+1|k}), \tag{20}$$

$$\mathbf{P}^{ww}_{k+1|k+1} = R_{k+1} - K^w_k \mathbf{P}^{yy}_{k+1|k}(K^w_k)^T. \tag{21}$$

C. State estimation

The prior mean and covariance are

$$\hat{\mathbf{x}}_{k+1|k} = \int \phi_k(\mathbf{x}_k) \aleph(\mathbf{x}_k; \hat{\mathbf{x}}_{k|k}, \mathbf{P}_{k|k}) d\mathbf{x}_k, \tag{22}$$

$$\begin{aligned} \mathbf{P}_{k+1|k} = \int \phi_k(\mathbf{x}_k)\phi^T_k(\mathbf{x}_k) \aleph(\mathbf{x}_k; \hat{\mathbf{x}}_{k|k}, \mathbf{P}_{k|k}) d\mathbf{x}_k \\ - \hat{\mathbf{x}}_{k+1|k}\hat{\mathbf{x}}^T_{k+1|k} + Q_k. \end{aligned} \tag{23}$$

Cross covariances are

$$\begin{aligned} \mathbf{P}^{xz}_{k+1,k|k} = E[\tilde{\mathbf{x}}_{k+1|k}\tilde{z}^T_{k|k}|Y_k] = \int \phi_k(\mathbf{x}_k)[\gamma_k(\mathbf{x}_k) \\ + w_k]^T \aleph(\mathbf{x}^a_k; \hat{\mathbf{x}}^a_{k|k}, \mathbf{P}^a_{k|k}) d\mathbf{x}^a_k - \hat{\mathbf{x}}_{k+1|k}\hat{z}^T_{k|k}, \end{aligned} \tag{24}$$

$$\begin{aligned} \mathbf{P}^{xz}_{k+1|k} = E[\tilde{\mathbf{x}}_{k+1|k}\tilde{z}^T_{k+1|k}|Y_k] = \int \mathbf{x}_{k+1}\gamma_{k+1}(\mathbf{x}_{k+1})^T \\ \aleph(\mathbf{x}_{k+1}; \hat{\mathbf{x}}_{k+1|k}, \mathbf{P}_{k+1|k}) d\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k}\hat{z}^T_{k+1|k}. \end{aligned} \tag{25}$$

The expression of Kalman gain for state estimation is

$$K_k^x = \mathbf{P}_{k+1|k}^{xy}(\mathbf{P}_{k+1|k}^{yy})^{-1}, \tag{26}$$

where

$$\mathbf{P}_{k+1|k}^{xy} = (1-p_{k+1})\mathbf{P}_{k+1|k}^{xz} + p_{k+1}\mathbf{P}_{k+1,k|k}^{xz}. \tag{27}$$

Finally the posterior mean and error covariance are

$$\hat{\mathbf{x}}_{k+1|k+1} = \hat{\mathbf{x}}_{k+1|k} + K_k^x(y_{k+1} - \hat{y}_{k+1|k}), \tag{28}$$

$$\mathbf{P}_{k+1|k+1} = \mathbf{P}_{k+1|k} - K_k^x\mathbf{P}_{k+1|k}^{yy}(K_k^x)^T. \tag{29}$$

The cross covariance of measurement noise and state is

$$\mathbf{P}_{k+1|k+1}^{xw} = -K_k^x\mathbf{P}_{k+1|k}^{yy}(K_k^w)^T. \tag{30}$$

All the expressions summarized in (12) - (30) are proved in [20].

Note-1 For $\beta_k = 0$, recalling equation (5), the case of delayed measurements comes down to non-delayed measurements, and hence the proposed algorithm reduces down to a filtering algorithm with non-delayed measurements.

Note-2 The integrals appeared in equations (12)-(15) and (22)-(25) are generally intractable. The computation of these integrals are based on numerical approximation and hence the filter accuracy is subject to the accuracy of the numerical approximation.

This paper utilizes the cubature quadrature rule [24], described in next section, for approximating the integrals.

IV. Cubature Quadrature Evaluation of Multidimensional Integral

A. Approach

The concerned multidimensional intractable integrals appeared in (12)-(15) and (22)-(25) are broken in two parts which gives a surface integral along with a line integral. The two integrals are dealt separately. In a first, the third-degree spherical cubature rule is implemented for dealing with the surface integral. Later, a simple arbitrary order Gauss-Laguerre quadrature rule could do the task for the line integral. In the earlier literature [24], the resultant rule has been regarded as cubature quadrature rule. The sample points coming out of this rule is popularly known as cubature quadrature points.

For a random variable, $X \in \Re^n$, the integral of interest is

$$I(f) = \frac{1}{\sqrt{|\Sigma|(2\pi)^n}}\int_{\Re^n} f(X)e^{-(1/2)(X-\mu)^T\Sigma^{-1}(X-\mu)}dX \tag{31}$$

where μ and Σ are mean and covariance of X. The above integral could be decomposed as

$$I(f) = \frac{1}{\sqrt{(2\pi)^n}}\int_{r=0}^{\infty}\int_{U_n}[f(CrZ+\mu)d\sigma(Z)]\, r^{n-1}e^{-r^2/2}dr \tag{32}$$

with $X = CrZ + \mu$ if C is Cholesky decomposition Σ and $\parallel Z \parallel = 1$. It is to be noted that U_n is the surface obtained as a result of n-dimensional unit hyper-sphere.

If a zero mean and unity covariance system is assumed, the integral

$$\int_{U_n} f(CrZ+\mu)d\sigma(Z) \tag{33}$$

reduces to

$$\int_{U_n} f(rZ)d\sigma(Z). \tag{34}$$

As a result of third-degree spherical cubature rule, the integral (34) reduces to [22]

$$\int_{U_n} f(rZ)d\sigma(Z) \approx \frac{2\sqrt{\pi^n}}{2n\Gamma(n/2)}\sum_{i=1}^{2n} f(r[u]_i), \tag{35}$$

where $[u]_i$ $(i = 1, 2, \cdots, 2n)$ are cubature points. The cubature points are axial locations at which a unit-hyper sphere intersects the axis of the multidimensional space. For a Gaussian distribution with arbitrary mean and covariance, the cubature points are further transformed as $(C[u]_i + \mu)$.

B. Gauss-Laguerre quadrature rule

The Gauss-Laguerre quadrature rule could be stated as a numerical approximation method with following approximation property

$$\int_{\lambda=0}^{\infty} f(\lambda)\lambda^\alpha e^{-\lambda}d\lambda \approx \sum_{i'=1}^{n'} A_{i'}f(\lambda_{i'}), \tag{36}$$

where $\lambda_{i'}$ being the quadrature points and $A_{i'}$ being the corresponding weights. The quadrature points could be given as roots of n' order Chebyshev-Laguerre polynomial equation

$$L_{n'}^\alpha(\lambda) = (-1)^{n'}\lambda^{-\alpha}e^\lambda\frac{d^{n'}}{d\lambda^{n'}}\lambda^{\alpha+n'}e^{-\lambda} = 0. \tag{37}$$

On the other hand, the weights could be computed as [25]

$$A_{i'} = \frac{n'!\Gamma(\alpha+n'+1)}{\lambda_{i'}[\dot{L}_{n'}^\alpha(\lambda_{i'})]^2}. \tag{38}$$

C. Cubature quadrature rule

The cubature quadrature rule is a result of the combination of the cubature and the quadrature rules. Hence, the mathematical aspects of the cubature quadrature rule could be obtained by substituting (35) and (36) in (32). Hence, the integral of interest could be approximated as

$$I(f) \approx \frac{1}{2n\Gamma(n/2)}\left[\sum_{i=1}^{2n}\sum_{i'=1}^{n'} A_{i'}f(\sqrt{2\lambda_{i'}}[u]_i)\right]. \tag{39}$$

The above equation concludes that the implementation of the cubature quadrature rule requires $2nn'$ number of sample points for approximating an n-dimensional integral with n'-order of Gauss-Laguerre approximation. Further, the accuracy improves with increasing order of Gauss-Laguerre approximation, but at the same time, it comes at the cost of increased computational burden.

D. Calculation of Cubature Quadrature (CQ) points

Here, a short review is provided over the steps required for computation of the cubature quadrature points and associated weights:

- The very first step is to construct the cubature points $[u]_{i(i=1,2\cdots,n)}$ under the statement made in this regard earlier.
- Replace $\alpha = n/2 - 1$ in equation (37) and solve it for obtaining $(\lambda_{i'})$ which is nothing but the quadrature points to be used for quadrature rule.
- The cubature quadrature (CQ) points could be obtained as $\xi_j = \sqrt{2\lambda_{i'}}[u]_i$. At the same time, the corresponding weights could be computed as

$$w_j = \frac{1}{2n\Gamma(n/2)}(A_{i'}) = \frac{1}{2n\Gamma(n/2)} \frac{n'!\Gamma(\alpha+n'+1)}{\lambda_{i'}[\dot{L}^{\alpha}_{n'}(\lambda_{i'})]^2},$$

for $i = 1, 2, \cdots, 2n$, $i' = 1, 2, \cdots, n'$ and $j = 1, 2, \cdots, 2nn'$.

V. Cubature Quadrature Kalman Filter for Single Step Randomly Delayed Measurements

The algorithm of the proposed CQKF with unit step randomly delayed measurements is summarized as follows.

Step i Initialization of filters

- The filter is initialized with $\hat{\mathbf{x}}^a_{0|0}$ and $\mathbf{P}^a_{0|0}$ *i.e.*

$$\hat{\mathbf{x}}^a_{0|0} = \begin{bmatrix} \hat{\mathbf{x}}_0 \\ 0 \end{bmatrix}, \qquad \mathbf{P}^a_{0|0} = \begin{bmatrix} \mathbf{P}_{0|0} & 0 \\ 0 & R \end{bmatrix} \tag{40}$$

- Calculate the $2n'(n+p)$ number of CQ points, ξ_j, and their corresponding weights w_j.
- Calculate another set of CQ points, ξ_i, and their corresponding weights w_i, where $i = 1, 2, \cdots, 2n'n$.

Step ii Propagation of CQ points predictor step

- By performing the Cholesky decomposition of the error covariance

$$\mathbf{P}^a_{k|k} = S^a_{k|k}(S^a_{k|k})^T. \tag{41}$$

- Generate CQ points of augmented system

$$\chi_{j,k|k} = [(\chi^x_{j,k|k})^T \quad (\chi^w_{j,k|k})^T]^T = \hat{\mathbf{x}}^a_{k|k} + S^a_{k|k}\xi_j, \tag{42}$$

where $j = 1, 2, \cdots, 2n'(n+p)$

- Compute the time updated mean and covariance

$$\hat{\mathbf{x}}_{k+1|k} = \sum_{j=1}^{2n'(n+p)} w_j \phi_k(\chi^x_{j,k|k}), \tag{43}$$

$$\mathbf{P}_{k+1|k} = \sum_{j=1}^{2n'(n+p)} w_j \phi_k(\chi^x_{j,k|k})\phi_k(\chi^x_{j,k|k})^T - \hat{\mathbf{x}}_{k+1|k}\hat{\mathbf{x}}^T_{k+1|k} + Q_k. \tag{44}$$

Step iii Estimation of measurement noise

- By performing the Cholesky decomposition over the error covariance

$$\mathbf{P}_{k+1|k} = S_{k+1|k}(S_{k+1|k})^T. \tag{45}$$

- Generate CQ points around prior estimate of state

$$\chi_{i,k+1|k} = \hat{\mathbf{x}}_{k+1|k} + S_{k+1|k}\xi_i. \tag{46}$$

- Calculate the statistics of non-delayed measurements

$$\hat{z}_{k+1|k} = \sum_{i=1}^{2nn'} w_i \gamma_k(\chi_{i,k+1|k}), \tag{47}$$

$$\mathbf{P}^{zz}_{k+1|k} = \sum_{i=1}^{2nn'} w_i \gamma_k(\chi_{i,k+1|k})\gamma_k(\chi_{i,k+1|k})^T - \hat{z}_{k+1|k}\hat{z}^T_{k+1|k} + R_{k+1}, \tag{48}$$

$$\hat{z}_{k|k} = \sum_{j=1}^{2n'(n+p)} w_j[\gamma_k(\chi^x_{j,k|k}) + \chi^w_{j,k|k}], \tag{49}$$

$$\mathbf{P}^{zz}_{k|k} = \sum_{j=1}^{2n'(n+p)} w_j[\gamma_k(\chi^x_{j,k|k}) + \chi^w_{j,k|k}][\gamma_k(\chi^x_{j,k|k}) + \chi^w_{i,k|k}]^T - \hat{z}_{k|k}\hat{z}^T_{k|k}. \tag{50}$$

- Mean and covariance of delayed measurement

$$\hat{y}_{k+1|k} = (1-p_{k+1})\hat{z}_{k+1|k} + p_{k+1}\hat{z}_{k|k}, \tag{51}$$

$$\mathbf{P}^{yy}_{k+1|k} = (1-p_{k+1})\mathbf{P}^{zz}_{k+1|k} + p_{k+1}\mathbf{P}^{zz}_{k|k} + P_{k+1}\times (1-p_{k+1})(\hat{z}_{k+1|k} - \hat{z}_{k|k})(\hat{z}_{k+1|k} - \hat{z}_{k|k})^T. \tag{52}$$

- Calculate Kalman gain for noise estimation

$$(K^w_k) = \mathbf{P}^{wy}_{k+1|k}(\mathbf{P}^{yy}_{k+1|k})^{-1}, \tag{53}$$

where

$$\mathbf{P}^{wy}_{k+1|k} = (1-p_{k+1})R_{k+1}. \tag{54}$$

- Posterior mean and error covariance for the measurement noise could be computed as

$$\hat{w}_{k+1|k+1} = K^w_k(y_{k+1} - \hat{y}_{k+1|k}), \tag{55}$$

$$\mathbf{P}^{ww}_{k+1|k+1} = R_{k+1} - K^w_k\mathbf{P}^{yy}_{k+1|k}(K^w_k)^T. \tag{56}$$

Step iv State estimation

- Calculate the cross covariances, as

$$\mathbf{P}^{xz}_{k+1,k|k} = \sum_{j=1}^{2n'(n+p)} w_j\phi_k(\chi^x_{j,k|k})[\gamma_k(\chi^x_{j,k|k}) + \chi^w_{j,k|k}]^T - \hat{\mathbf{x}}_{k+1|k}\hat{z}^T_{k|k}, \tag{57}$$

$$\mathbf{P}^{xz}_{k+1|k} = \sum_{i=1}^{2nn'} w_i\chi^x_{i,k+1|k}\gamma_k(\chi^x_{i,k+1|k})^T - \hat{\mathbf{x}}_{k+1|k}\hat{z}^T_{k+1|k}, \tag{58}$$

$$\mathbf{P}^{xy}_{k+1|k} = (1-p_{k+1})\mathbf{P}^{xz}_{k+1|k} + p_{k+1}\mathbf{P}^{xz}_{k+1,k|k}. \tag{59}$$

- Evaluate Kalman gain for state estimation

$$K_k^x = \mathbf{P}^{xy}_{k+1|k}(\mathbf{P}^{yy}_{k+1|k})^{-1}. \tag{60}$$

- The posterior state estimate and covariance are given as

$$\hat{\mathbf{x}}_{k+1|k+1} = \hat{\mathbf{x}}_{k+1|k} + K_k^x(y_{k+1} - \hat{y}_{k+1|k}), \tag{61}$$

$$\mathbf{P}_{k+1|k+1} = \mathbf{P}_{k+1|k} - K_k^x \mathbf{P}^{yy}_{k+1|k}(K_k^x)^T. \tag{62}$$

- Cross covariance of state and measurement noise error covariance is calculated as

$$\mathbf{P}^{wx}_{k+1|k+1} = -K_k^x \mathbf{P}^{yy}_{k+1|k}(K_k^w)^T. \tag{63}$$

Note-3 The computational cost of delayed CQKF is little more compared to the delayed UKF and CKF. However all the filters are free from the *curse of dimensionality* problem.

VI. Simulation

In this section, the proposed extension for delayed filtering is applied to track a nonlinear maneuvering target. To this regard, the measurements were assumed to be randomly delayed in time with maximum delay not exceeding unit time step.

A. Tracking of maneuvering target

For the simulation and efficacy verification of the extended delayed filtering algorithm, a maneuvering target tracking problem has been taken into consideration. The turn rate of the maneuvering target has been assumed to be constant. The major challenge lies in the second assumption where the turn rate is considered to be unknown to the practitioner. Upto a limited extent, this model can effectively be assumed under varying turn rate scenario, looking at the fact that the noise has been incorporated to deal with the variability. The model of maneuvering target with constant turn rate has been specifically polpular with the name coordinated turn model [26], [27].

1) Process model: The formulation of the problem in this section considers a maneuvering aerial vehicle. It is assumed that the vehicle motion is in a plane parallel to the ground, and hence it could be said that the height of the plane remains constant with time. Under further assumption, the vehicle maneuver is restricted to follow a constant turn model. At this point, it is to be noted that the system becomes linear if the constant turn rate is known to the practitioners. However, an unknown turn rate results in a nonlinear mathematical model, and hence it requires to be dealt using nonlinear filtering techniques. Let us assume the motion of the aerial vehicle is in a plane (x, y), then the equation of motion could be given as

$$\ddot{\mathrm{x}} = -\Omega\dot{\mathrm{y}}, \quad \ddot{\mathrm{y}} = \Omega\dot{\mathrm{x}}, \quad \text{and,} \ \dot{\Omega} = 0 \tag{64}$$

where x and y is representation of position in x and y directions respectively and Ω represents a constant turn rate. Under the state space form, the above model could be represented as

$$\dot{X} = AX + w, \tag{65}$$

where $X = [\mathrm{x}\ \dot{\mathrm{x}}\ \mathrm{y}\ \dot{\mathrm{y}}\ \Omega]^T$. The model encounters several uncertainties which mainly arises due to variation in turn rate, wind speed, velocity change *etc*. To counter these uncertainties, the process noise w has been incorporated in the above equation.

Further, the discrete process model could be given as

$$X_{k+1} = F_k X_k + w_k, \tag{66}$$

where

$$F_k = \begin{bmatrix} 1 & \frac{\sin(\Omega_{k-1}T)}{\Omega_{k-1}} & 0 & -\frac{1-\cos(\Omega_{k-1}T)}{\Omega_{k-1}} & 0 \\ 0 & \cos(\Omega_{k-1}T) & 0 & -\sin(\Omega_{k-1}T) & 0 \\ 0 & \frac{1-\cos(\Omega_{k-1}T)}{\Omega_{k-1}} & 1 & \frac{\sin(\Omega_{k-1}T)}{\Omega_{k-1}} & 0 \\ 0 & \sin(\Omega_{k-1}T) & 0 & \cos(\Omega_{k-1}T) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

2) Measurement model: It is assumed that the sensor senses the range and bearing angle. Hence, the mathematical representation of the measurement could follow the following structure

$$\gamma(X_k) = \begin{bmatrix} \sqrt{\mathrm{x}_k^2 + \mathrm{y}_k^2} \\ \text{atan2}(\mathrm{y}_k, \mathrm{x}_k) \end{bmatrix} + v_k, \tag{67}$$

with atan2 being four quadrant inverse tangent function and T being the sampling time. The process noise covariance (Q) could be calculated as

$$Q = q\begin{bmatrix} \frac{T^3}{3} & \frac{T^2}{2} & 0 & 0 & 0 \\ \frac{T^2}{2} & T & 0 & 0 & 0 \\ 0 & 0 & \frac{T^3}{3} & \frac{T^2}{2} & 0 \\ 0 & 0 & \frac{T^2}{2} & T & 0 \\ 0 & 0 & 0 & 0 & 1.75 \times 10^{-3}T \end{bmatrix}, \tag{68}$$

In the above equation, T represents the sampling time. It is assumed to be 0.5 second for the implementation purpose. Further, $q = 0.1$ is constant. The measurement noise covariance is assumed as $R = \text{diag}([\sigma_r^2\ \sigma_t^2])$ where $\sigma_r = 120$m and $\sigma_t = \sqrt{70}$mrad.

3) Simulation results: The initial estimate $\hat{X}_0$ is generated from the Gaussian distribution of mean X_0 and covariance P_0, where $X_0 = [1000\text{m}\ \ 30\text{m/s}\ \ 1000\text{m}\ \ 0\text{m/s}\ \ -3°/s]^T$ is true initial value of states and $P_0 = \text{diag}([200\text{m}^2\ \ 20\text{m}^2/\text{s}^2\ \ 200\text{m}^2\ \ 20\text{m}^2/\text{s}^2\ \ 100\text{mrad}^2/\text{s}^2])$ is initial error covariance.

The effects of probability of delay (p_k) have been studied. The RMSE averaged over the time horizon is plotted for position velocity and turn rate for different p_k in Fig. 1 - Fig. 3. It has been observed from the figures that for all the values of p_k, delayed CQKF performs better compared to its UKF and CKF counterparts. Moreover, the number of sample points for CQKF is n'-times higher than that for CKF. Looking at the fact that the computational burden remains almost similar for UKF and CKF, the proposition leads to increase the computational time by n'-times.

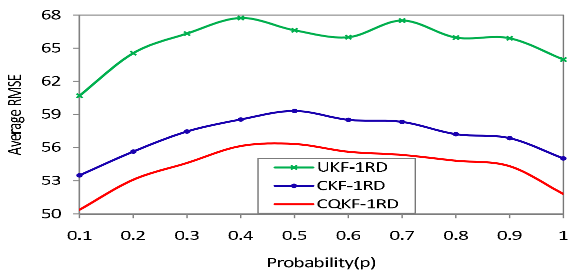

Fig. 1. RMSE vs probability plot for position

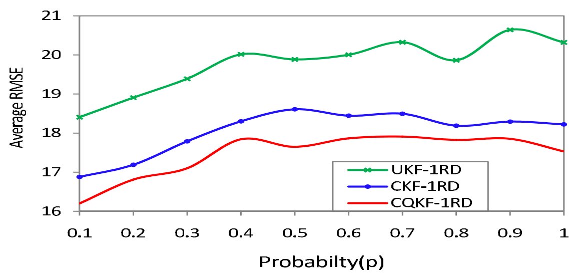

Fig. 2. RMSE vs probability plot for velocity

VII. Discussions and Conclusions

In this paper an algorithm which is a more generalized form of one step randomly delayed CKF, is proposed for solving the nonlinear filtering problems. The superiority of the proposed method has been compared to its CKF and UKF counterparts by implementing them for a nonlinear filtering problem. In this problem, the states of a maneuvering target is tracked. Owing to the enhanced accuracy, the proposed filter could be recommended for the implementation to the real-life filtering problems if the delayed measurements are expected.

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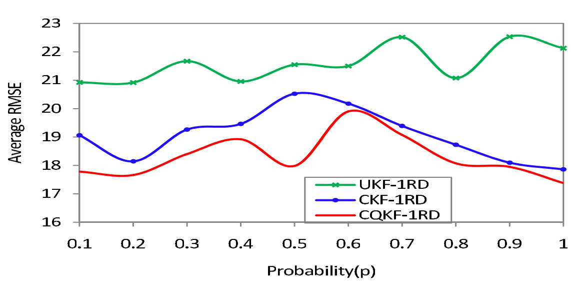

Fig. 3. RMSE vs probability plot for turn rate in rad/sec

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