Deinterleaving Periodic Processes

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***Abstract*—The analysis of periodic processes is an important area of signal analysis. We introduce an algorithm (EQUIMEA), which analyses multi-period data sets and can be applied to the analysis of periodic processes with a single generator and the deinterleaving and analysis of processes with multiple generators. We address both theory and computation by finishing with a computational study, demonstrating how the continuants of different periods affect the efficiency of the algorithm.**

I. Introduction

A fundamental problem in Harmonic Analysis and Signal Processing is the analysis of point processes. Is it possible to identify one (or several) periodic generator(s)? If there are several, can we deinterleave the signals? We present an algorithm (with theoretical justification) that answers these questions. This works on all periodic point processes, but in particular, when the standard tools, e.g., Wiener's Periodogram, break down. The procedure is computationally straightforward, stable with respect to noise, and converges quickly. In the case of a single generator, a preliminary algorithm uses number theory in novel ways to extract the underlying period by modifying the Euclidean algorithm to determine the period from a sparse set of noisy measurements [5], [6], [12]. The elements of the set are the noisy occurrence times of a periodic event with (perhaps very many) missing measurements. Variations on equidistribution theory lead to a procedure that can be applied to the deinterleaving and analysis of processes with multiple generators. The proposed algorithms are computationally straightforward and converge quickly. The preliminary algorithm is justified by a theorem that shows, for a set of randomly chosen positive integers, the probability that they do not all share a common prime factor approaches one quickly as the cardinality of the set increases. The theorem is in essence a probabilistic interpretation of the Riemann Zeta Function.

II. Single Generator Case: The MEA

We first analyze a single periodic pulse train, getting an estimate of the underlying period. This estimate, while not maximum likelihood, is used as initialization in a three-step algorithm that achieves the Cramer-Rao bound for moderate noise levels, as shown by comparing Monte Carlo results with the Cramer-Rao bounds [5], [6], [12]. The data sets arise in radar pulse repetition interval (PRI) analysis, in bit synchronization in communications, in biomedical applications, and other scenarios. We assume our data is a finite set of real numbers $S = \{s_j\}_{j=1}^{n}$, with $s_j = k_j\tau + \phi + \eta_j$, where τ (the period) is a fixed positive real number, the k_j's are non-repeating positive integers, ϕ (the phase) is a real random variable uniformly distributed over the interval $[0, \tau)$, and the η_j's are zero-mean independent identically distributed (iid) error terms. We assume that the η_j's have a symmetric probability density function (pdf), and that $|\eta_j| < \frac{\tau}{2}$ for all j. We develop an algorithm for isolating the period of the process from this set, which we shall assume is (perhaps very) sparse. In the noise-free case our basic algorithm, given below, is equivalent to the Euclidean algorithm and converges with very high probability given only $n = 10$ data samples, independent of the number of missing measurements. We assume that the original data set is in descending order, i.e., $s_j \geq s_{j+1}$. Let $\widehat{\tau}$ denote the value the algorithm gives for τ, and let "$\longleftarrow$" denote *replacement*, e.g., "$a \longleftarrow b$" means that the value of the variable a is to be replaced by the current value of the variable b.

The Modified Euclidean Algorithm (MEA)

Initialize: Set `iter` $= 0$.
1.) [Adjoin 0 after first iteration.] If `iter` > 0, then $S \longleftarrow S \cup \{0\}$.
2.) [Form the new set with elements $(s_j - s_{j+1})$.] Set $s_j \longleftarrow (s_j - s_{j+1})$.
3.) Sort the elements in descending order.
4.) [Eliminate noise.] If $0 \leq s_j \leq \eta_0$, then $S \longleftarrow S \setminus \{s_j\}$.
5.) The algorithm terminates if S has only one element s_1. Declare $\widehat{\tau} = s_1$. If not, then set
`iter` $\longleftarrow$ (`iter` $+ 1$). Go to **(1.)**.

Here, $0 < \eta_0 < \tau$ is a noise threshold. Noise-free simulation examples demonstrate successful estimation of τ for $n = 10$ with 99.99% of the possible measurements missing. The algorithm is based on several theoretical results. First, we can modify the basic Euclidean algorithm, allowing a reformulation using subtraction rather than division. Given a set $\{k_1\tau, \ldots, k_n\tau\}$ with $\gcd \gamma$, where $\gamma = \tau \gcd(k_1, \ldots, k_n)$, we say that the elements in the set are *commensurate* to γ. The number γ serves as the fundamental unit for the set. Every element in the set can be expressed as a multiple of γ. For example, the elements of the set $\{\frac{1}{2}, \frac{5}{4}, 2\}$ are commensurate to $\frac{1}{4}$. If no such fundamental unit exists, we call the elements of the set *incommensurate*. All finite sets of rational numbers are commensurate. Add a single irrational number to the set and this new set is incommensurate. The standard Euclidean algorithm involves repeated division. In our problem, we are

dealing with numbers that are essentially "noisy integers." Remainder terms could be noise and thus could be non-zero numbers arbitrarily close to zero. Subsequent iterations in the procedure may involve dividing by such small values, resulting in arbitrarily large numbers. Thus, the standard algorithm is unstable under perturbation by noise. However, the algorithm may be changed so that the process of subtraction replaces division as follows:
$\gcd(k_1, \ldots, k_n) = \gcd((k_1 - k_2), \ldots, (k_{n-1} - k_n), k_n)$.

We have shown that our procedure almost surely converges to the period by proving the following result. The Riemann Zeta Function is defined on the complex half plane $\{z \in \mathbb{C} : \Re(z) > 1\}$ by $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$. Euler demonstrated the connection of ζ with number theory by showing that $\zeta(z) = \prod_{j=1}^{\infty} \frac{1}{1-(p_j)^{-z}}$, $\Re(z) > 1$, where $\mathbb{P} = \{p_1, p_2, p_3, \ldots\} = \{2, 3, 5, \ldots\}$ is the set of all prime numbers. In the following, we let $P\{\cdot\}$ denote probability, $\mathrm{card}\{\cdot\}$ denote the cardinality of the set $\{\cdot\}$, and let $\{1, \ldots, \ell\}^n$ denote the sublattice of positive integers in $\mathbb{R}^n$ with coordinates c such that $1 \leq c \leq \ell$. Therefore, $N_n(\ell) = \mathrm{card}\{(k_1, \ldots, k_n) \in \{1, \ldots, \ell\}^n : \gcd(k_1, \ldots, k_n) = 1\}$ is the number of relatively prime elements in $\{1, \ldots, \ell\}^n$. Two different proofs of this result may be found in [5], [6].

Theorem II.1. *Let* $N_n(\ell) = \mathrm{card}\{(k_1, \ldots, k_n) \in \{1, \ldots, \ell\}^n : \gcd(k_1, \ldots, k_n) = 1\}$. *For* $n \geq 2$*, we have that*

$$\lim_{\ell \to \infty} \frac{N_n(\ell)}{\ell^n} = [\zeta(n)]^{-1} . \tag{II.1}$$

We have that $[\zeta(n)]^{-1} \longrightarrow 1$ quickly as n increases. In fact, the rate of convergence is exponential – $\lim_{\omega \to \infty} [\zeta(\omega)]^{-1} = 1$, converging to 1 from below faster than $(1 - 2^{1-\omega})$. This estimate shows that the algorithm produces the period in the noise-free case with as few as 10 elements.

III. Multiple Generator Case: The EQUIMEA

The analysis of the multi-periodic case uses the MEA procedure as the "engine" of a larger algorithm. Our data model is the union of M copies of our previous datasets, each with different periods or "generators" $\Gamma = \{\tau_i\}$, k_{ij}'s and phases. Let $\tau = \max_i\{\tau_i\}$. Then our data is $S = \bigcup_{i=1}^{M} \{\phi_i + k_{ij}\tau_i + \eta_{ij}\}_{j=1}^{n_i}$, where n_i is the number of elements from the i^{th} generator, $\{k_{ij}\}$ is a linearly increasing sequence of natural numbers with missing observations, ϕ_i is a random variable uniformly distributed in $[0, \tau_i)$, and the η_{ij}'s are zero-mean iid Gaussian with standard deviation $3\sigma_{ij} < \tau/2$. We think of the data as events from M periodic processes, and represent it, after reindexing, as $S = \{\alpha_l\}_{l=1}^{N}$. We difference as in the MEA, but we compute **all** of the differences. We repeat this m times, and saving the elements from each iteration. We form a union of all of these data elements. The relative primeness of data generated by one generator will "fill in" the missing elements for that generator, whereas the data from two different generators will become "Weyl flat." The number of times m that this process is applied is determined experimentally. Assuming only minimal knowledge of the range of $\{\tau_i\}$, namely bounds T_L, T_U such that $0 < T_L \leq \tau_i \leq T_U$, we phase wrap the data by the mapping $\Phi_\rho(\alpha_l) = \left\langle \frac{\alpha_l}{\rho} \right\rangle = \frac{\alpha_l}{\rho} - \left\lfloor \frac{\alpha_l}{\rho} \right\rfloor$, where $\rho \in [T_L, T_U]$, and $\lfloor \cdot \rfloor$ is the floor function. Thus $\langle \cdot \rangle$ is the fractional part, and so $\Phi_\rho(\alpha_l) \in [0, 1)$.

Definition III.1. A sequence of real random variables $\{x_j\} \subset [0, 1)$ is essentially uniformly distributed in the sense of Weyl if given a, b, $0 \leq a < b < 1$, $\frac{1}{n}\mathrm{card}\{1 \leq j \leq n : x_j \in [a, b]\} \longrightarrow (b - a)$ as $n \longrightarrow \infty$ almost surely.

We recall Weyl's Theorem

Theorem III.2 (Weyl)**.** *Given a fixed irrational number* γ*, then for every* a, b *such that* $0 \leq a < b < 1$*,*

$$\lim_{n \to \infty} \frac{1}{n} \mathrm{card}\{1 \leq k \leq n : a \leq \langle k\gamma \rangle \leq b\} = (b - a) . \tag{III.1}$$

For our variation of Weyl's Theorem, we assume that for each i, $\{k_{ij}\}$ is a linearly increasing infinite sequence of natural numbers with missing observations such that $k_{ij} \longrightarrow \infty$ as $j \longrightarrow \infty$. We must make this assumption because the result is only approximately true for a finite length sequence.

Theorem III.3. *For almost every (a.e.) choice of* ρ *(in the sense of Lebesgue measure)* $\Phi_\rho(\alpha_l)$ *is essentially uniformly distributed in the sense of Weyl.*

Proof. Let $\rho \in [T_L, T_U]$, $\lfloor \cdot \rfloor$ be the floor function, and $\langle \cdot \rangle$ be the fractional part. The mapping

$$\Phi_\rho(s_l) = \left\langle \frac{s_l}{\rho} \right\rangle = \frac{s_l}{\rho} - \left\lfloor \frac{s_l}{\rho} \right\rfloor , \tag{III.2}$$

is a measurable and measure-preserving mapping into $[0, 1)$.

Claim . For a.e. choice of ρ, Φ_ρ is ergodic.
Proof of Claim. Normalize $[T_L, T_U)$ to $[0, 1)$ Let X be a absolutely and square Lebesgue integrable random variable, i.e., $X \in L^1 \cap L^2$. Therefore, we can expand X in a Fourier series

$$X(t) = \sum_{n \in \mathbb{Z}} \widehat{X}[n] \exp(i\pi nt) , \tag{III.3}$$

with the Fourier coefficients $\widehat{X}[n]$ given by

$$\widehat{X}[n] = \int_0^1 X(t) \exp(-i\pi nt)\, dt . \tag{III.4}$$

Then

$$X(\Phi_\rho(t)) = \sum_{n \in \mathbb{Z}} \widehat{X}[n] \exp(i\pi nt/\rho) . \tag{III.5}$$

For X to be invariant,

$$\widehat{X}[n](1 - \exp(i\pi nt/\rho)) = 0 \text{ for all } n \tag{III.6}$$

for $a.e.\, t$. This implies either

$$\widehat{X}[n] = 0 \text{ or } \exp(i\pi nt/\rho) = 1 . \tag{III.7}$$

But since ρ is irrational a.e., by Weyl's Equidistribution Theorem,

$$\exp(i\pi nt/\rho) \neq 1 \tag{III.8}$$

on a set of full measure. □

Moreover, the set of ρ's for which this is not true are rational multiples of $\{\tau_i\}$. Since $\mathbb{Q}$ is a countable set and a finite union of countable sets is countable, the set of rational multiples of $\{\tau_i\}$ is countable, and therefore a set of measure zero. Thus, except for those values, $\Phi_\rho(s_l)$ is essentially uniformly distributed in $[0,1)$. The values at which $\Phi_\rho(s_l) = 0$ almost surely are $\rho \in \{\tau_i/n : n \in \mathbb{N}\}$, which is a set of measure zero. These values of ρ cluster at zero, but spread out for lower values of n. The equidistribution of $\Phi_\rho(s_l)$ for $a.e.$ ρ leads to a "flat" range $[0, \eta_0]$ for S for $\rho \notin \{\tau_i/n : n \in \mathbb{N}\}$. Given that we have to produce an answer in finite time, and therefore, have to terminate, we pre-set a noise floor η_0 and two degree of accuracy parameters, $\mathcal{E}_\beta$ and $\mathcal{E}_\eta$. The first, $\mathcal{E}_\beta$, sets a "Weyl equidistribution floor," while $\mathcal{E}_\eta$ sets an iterative convergence baseline. The convergence rate of the algorithm is related to η_0, $\mathcal{E}_\beta$ and $\mathcal{E}_\eta$. At each step, the data is sorted in descending order $S_{iter} = \{s_l\}_{l=1}^N = \{s_1, \ldots, s_N\}$. We phase wrap the data by computing modulus of the spectrum, i.e., compute $|\text{Spec}_{iter}(\tau)| = |\sum_{l=1}^N e^{(2\pi i s_l/\tau)}|$. The values of $|\text{Spec}_{iter}(\tau)|$ will have peaks at the periods τ_i and their harmonics $(\tau_i)/k$. The "noise-like" behavior of $\Phi_\rho(\alpha_l)$ for $a.e.$ ρ leads to a "flat" range $[0, \mathcal{E}]$ for S for $\rho \notin \{\tau_i/n : n \in \mathbb{N}\}$. In turn, this gives the following. Let i_0 denote the index of the most prolific generator. We then isolate the data generated by τ_{i_0} by convolution with a pulse train of width τ_{i_0}, and subtract it out of the original data. We then repeat the process, terminating when we have the empty set. We refer to this as the EQUIMEA algorithm.

The EQUIMEA Algorithm

Initialize: Sort the elements of S in descending order. Form the new set with elements $(s_l - s_{l+1})$. Set $s_l \longleftarrow (s_l - s_{l+1})$. Set `iter` $= 1$, $i = 1$, η_0, $\mathcal{E}_\beta$, and $\mathcal{E}_\eta$.
1.) [Adjoin 0.] $S_{iter} \longleftarrow S \cup \{0\}$.
2.) [Sort.] Sort the elements of S_{iter} in descending order.
3.) [Compute all differences.] Set $S_{iter} = \bigcup(s_j - s_k)$ with $s_j > s_k$.
4.) [Eliminate noise.] If $0 \leq s_j \leq \eta_0$, then $S \longleftarrow S \setminus \{s_j\}$.
5.) [Adjoin previous iteration.] Form $S_{iter} \longleftarrow S_{iter} \cup S_{iter-1}$, sort and reindex.
6.) [Compute spectrum.] Compute $|\text{Spec}_{iter}(\tau)| = |\sum_{l=1}^N e^{(2\pi i s_l/\tau)}|$.
7.) [Threshold.] Choose the rightmost peak. Label it as τ_{iter}.
8.) [Test.] If $|\text{Spec}_{iter}(\tau_{iter})| > \mathcal{E}_\beta$ and $|\tau_{iter} - \tau_{iter-1}| < \mathcal{E}_\eta$, declare $\widehat{\tau_i} = \tau_{iter}$.
If not, `iter` $\longleftarrow$ (`iter` $+ 1$). Go to **1.)**.
9.) [Remove τ_i and harmonics.] Given τ_i, remove it and its harmonics $|\text{Spec}_{iter}(\tau)|$ for $\widehat{\tau_i}/m$, $m \in \mathbb{N}$. Label as $\text{Notch}_{iter}(\tau)$.
10.) [Recompute frequency notched spectrum.] Compute $|\text{Spec}_{iter}(\tau) - \text{Notch}_{iter}(\tau)|$.
11.) [Threshold.] If $|\text{Spec}_{iter}(\tau) - \text{Notch}_{iter}(\tau)| \leq \mathcal{E}_\beta$ algorithm terminates. Else, let $i \longleftarrow i + 1$. Go to step **7.)**.

We then have to deinterleave the data. We use a standard discrete matched filtering algorithm, correlating a known delayed signal (a "template") with an unknown signal to detect the presence of the template in the unknown signal. Here, our known signal has the form $\sum_{k\in\mathbb{Z}} \delta(t - k\tau_i)$, a pulse train of period τ_i with no missing observations, and our unknown signal consists of those elements of the original data S generated by all of the underlying periods $\Gamma = \{\tau_i\}$.

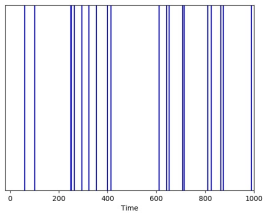

Fig. 1. Two Periods Original Data

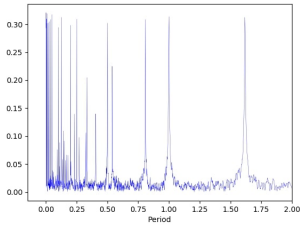

Fig. 2. Two Periods EQUIter2 Spectrum

Given the original data and the set of generating periods $\{\tau_1, \ldots, \tau_n\}$, convolve the data with $\sum_{k\in\mathbb{Z}} \delta(t - k\tau_1)$. This convolution will identify the elements in the original data set S that are generated by the generating period τ_1. Call these elements S_{τ_1}. Let $S_2 = S \setminus S_{\tau_1}$. Convolve S_2 with $\sum_{k\in\mathbb{Z}} \delta(t - k\tau_2)$. This convolution will identify the elements in the data set S_2 that are generated by the generating period τ_2. Call these elements S_{τ_2}. Let $S_3 = S \setminus S_{\tau_2}$. Repeat the process for τ_3 up to τ_n. This process deinterleaves the data into components generated by individual generators τ_i.

We now demonstrate the algorithm. The data in Figure 1 has two underlying periods equaling 1 and $\varphi = (1 + \sqrt{5})/2$, with 90% of the information randomly removed, and 10% jitter noise. Figure 2 shows the $|S_{iter}|$ after two iterations. The two periods are clearly visible as outside nodes. The original data deinterleaved (red – 1, green – φ) is in Figure 3.

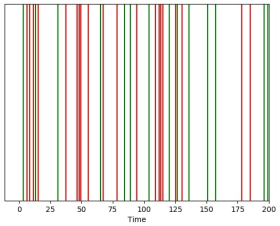

Fig. 3. Two Periods Deinterleaved

IV. Theory vs. Computation

This brings up two important points. First, if two underlying processes have commensurate (or even equal) periods, the EQUIMEA will first capture the largest of the periods (or the period). It is highly probable that the two or more events will not have the same phase. This data can then be separated by the deinterleaving process.

The second point is more subtle, and leads to a discussion of computability. The strengths of the signals $|\text{Spec}_{iter}|$ is reliant on the fact that the generators are incommensurate, and so the differences of data elements $(\varphi_i + k_{ij}\tau_i + \eta_{ij}) - (\varphi_{i'} + k_{i'j'}\tau_{i'} + \eta_{i'j'})$, for $i \neq i'$, are incommensurate to τ_i and $\tau_{i'}$. These data elements in $|\text{Spec}_{iter}|$ become "Weyl flat." But this is a function of the degree of the accuracy of the computation, the ergodicity of the periods relative to each other, and, if applicable, to Diophantine approximations of the periods. We note that very deep work of Katok, Stepin, Margulis, et al. [9], [10] addresses this.

The data in Figure 1 had two underlying periods equaling 1 and $\varphi = (1+\sqrt{5})/2$, with 90% of the information randomly removed and 10% jitter noise. Figure 2 shows $|\text{Spec}_{iter}|$ after two iterations. By proceeding right to left, one can easily see the two underlying periods. Each will reinforce the data elements from the specific generator, allowing the elements to be extracted. The algorithm "backfilled" the multiples of 1 and $\varphi = (1+\sqrt{5})/2$, while the data from two different periods became "Weyl flat." There was not only a sufficient amount of data, but also, the ergodicity of the periods relative to each other, played a role in making the periods stand out against the "flat" data. Ergodicity can be demonstrated with two numbers – $\varphi = \frac{1+\sqrt{5}}{2}$ and $\mathcal{L} = \sum 1/10^{k!}$. Here, φ, the Golden Mean, is poorly approximated by rationals, whereas $\mathcal{L}$ is a *Liouville number*, a transcendental well approximated by rationals.

Definition : A real number β is a *Liouville number*, denoted by $\beta \in \mathbb{L}$, if β is irrational and if for every integer $m \geq 2$, there exists integers p, q with $q \geq 2$ such that $|\beta - p/q| < 1/|q|^m$.

Liouville developed $\mathbb{L}$ as a special class of transcendentals. The algebraic irrationals are not a subset $\mathbb{L}$. The complement of $\mathbb{L}$ includes *poorly approximated by rationals*. The Golden Mean $\varphi = (1+\sqrt{5})/2$ is the least rapidly approximated, and this is seen in its continuant $[1;\overline{1}]$. In contrast, the continuant of $\mathcal{L}$ is $[0;9,11,99,\ldots]$, with gaps between increasingly large sequences of 9's. In terms of their approximation by rationals, φ and $\mathcal{L}$ are quite different. $|\text{Spec}_{iter}|$ will become "Weyl flat" considerably more quickly for φ than $\mathcal{L}$. The ergodicity of the periods relative to each other, plays a role in making the periods stand out against the "flat" data. In this sense, although Weyl's theorem holds for all irrationals, its "numerical erogdicity" is a function of the continuants, and where these continuants are situated between φ and $\mathcal{L}$. This is evident in the following numerical computation. We plot $\exp(2\pi i n \tau_j)$ for $\tau_1 = \varphi = \frac{1+\sqrt{5}}{2}$ and $\tau_2 = \mathcal{L} = \sum 1/10^{k!}$. In Figure 4, we see the ergodicity of φ, essentially filling the circle with only 300 points. In contrast, Figure 5 shows that $\mathcal{L}$ at first seems to follow the distribution of $\tau = 11/100$, and, even after 3000 points, has noticeable gaps. But $\mathcal{L}$ is irrational, and therefore, by Weyl's Theorem, the points will eventually fill in. Numerically, they do so, albeit at a *much* slower rate.

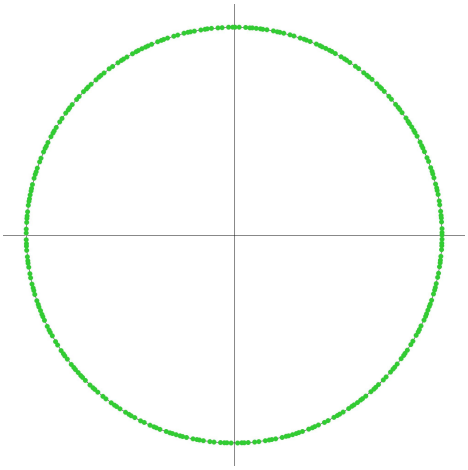

Fig. 4. φ 300 dots

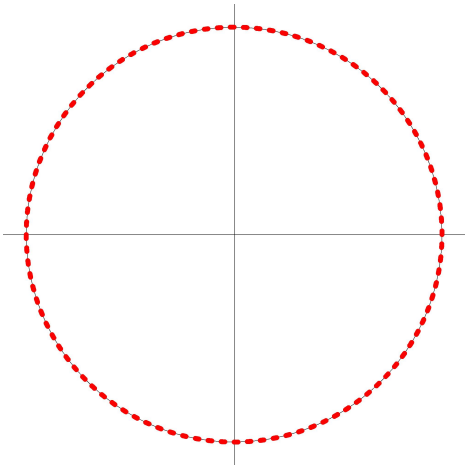

Fig. 5. $\mathcal{L}$ 3000 dots

However, encountering an element of $\mathbb{L}$ is unlikely.

Theorem IV.1. *$\mathbb{L}$ has Lebesgue measure zero.*

In fact, $\mathbb{L}$ has σ-dimensional Hausdorff measure zero for all $\sigma > 0$.

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