Maximum Welfare Allocations under Quantile Valuations

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Abstract

We propose a new model for aggregating preferences over a set of indivisible items based on a quantile value. In this model, each agent is endowed with a specific quantile, and the value of a given bundle is defined by the corresponding quantile of the individual values of the items within it. Our model captures the diverse ways in which agents may perceive a bundle, even when they agree on the values of individual items. It enables richer behavioral modeling that cannot be captured by additive valuation functions. We aim to maximize utilitarian and egalitarian welfare within the quantile-based valuation setting. For each of the welfare functions, we analyze the complexity and provide complementary approximation and exact algorithms. Interestingly, our results show that the complexity of both functions varies significantly, depending on whether the allocation is required to be balanced. We provide near-optimal approximation algorithms for utilitarian welfare, and for egalitarian welfare, we present exact algorithms for many cases.

1 Introduction

The problem of allocating indivisible items in a fair and efficient manner has been well-studied in recent years (Feige 2006; Feige and Vondrák 2010; Budish 2011; Caragiannis et al. 2019; Aziz et al. 2024). The overwhelming majority of this work focuses on settings where agents have monotone valuations for the items being assigned[1] partially because of the underlying structure imposed by them. In practice however, agent preferences may be unreasonably non-monotone, and alternate models of preferences are needed.

Consider a setting where an incoming class of school students need to be divided into sections each with a different teacher. School teachers are some of the most overworked and underpaid professionals. Consequently, it is imperative to try and ensure good allocations for them. Teachers get satisfaction from the students' learning and growth throughout the year. Each teacher may have different levels of satisfaction with a given set of students assigned to them, even when the students even if the students were to perform similarly. One teacher may be upset if even one student performs poorly, whereas a different teacher may be satisfied if at least half their class does well. Alternately, some teachers may be delighted if they have even one exceptional child in their class, even if the others do not do as well. These opinions can be captured by different quantile values for the set of students assigned. The pessimistic or critical teacher bases their satisfaction on the lowest quantile, the teacher whose satisfaction is based on at least half the class doing well can be captured by the median quantile and the optimistic teacher bases their satisfaction on the highest quantile.

We introduce a novel valuation class, termed *quantile valuations*, which encompasses the aforementioned scenarios. In this framework, each agent is endowed with a specific quantile value $\tau \in [0, 1]$, and the value that she assigns for a bundle S is the τ-quantile of the distribution of item values in S. Quantiles are widely used across data analysis and statistics because they provide a robust description of value distributions. Compared to measures like average or total/gross, most quantile based measures are significantly less susceptible to outliers. As a result, quantiles are commonly used in practical settings, in measures like median household income, median age, and median house price in a given neighborhood etc. Quantiles have also been used in decision theory to model agent preferences in settings where agents have preferences over stochastic outcomes. Specifically, quantiles have been considered in settings where an agent faces a choice of actions, each yielding a distribution over outcomes. Here, modeling the agent's choice as a quantile maximizer has been shown to provide a better approximation of human behavior than modeling them as an expected utility maximizer (de Castro and Galvao 2019, 2022).

Other allocation settings where quantile valuations are relevant include allocating repair tasks to workers and assigning submitted papers to reviewers. In these settings, as well typical school settings with in-person classes, there often is an added restriction that each teacher be assigned an equal sized set of students. In contrast, when it comes to online classes such as those for distance learning or preparatory classes for specific exams, there is no underlying restriction on class size. Hence, we consider both the space of *balanced* allocations as well as that of all *unconstrained* allocations.

[1] a monotone valuation function is one where the marginal value of an item is always non-negative or always non-positive.

		Goods		Chores		Identical
Balanced	**USW**	APX-h	(Th. 1)	NP-h	(Th. 10)	in P (Th. 8)
		$\min(\frac{m}{n}+1, n)$-USW†	(Th. 2)	n-USC	(Prop. 6)	
	ESW	in P†	(Th. 3)	in P†	(Th. 12)	in P (Th. 3)
Unconstrained	**USW**	NP-h	(Th. 4)	APX-h	(Th. 11)	?
		$(1+\frac{1}{n-1})$-USW†	(Th. 5)	$\log m$-USC	(Cor. 2)	
	ESW	APX-h	(Th. 7)	APX-h	(Th. 13)	in P (Th. 9)
		in P for $\tau \in \{0, {}^1/_3, \frac{t}{t+1}, 1\}$, $t \in \mathbb{Z}_+$	(Th. 6)	in P for $\tau \in \{0, 1\}$	(Prop. 7)	

Table 1: Complexity of computing USW and ESW optimal allocations given n agents and m items. α-USW refers to an α approximation to the optimal USW. Algorithmic results marked with † hold when agents have heterogeneous quantiles. All other algorithmic results require agents to have the same (homogeneous) quantile value of τ. All complexity results hold for both homogeneous and heterogenous quantiles.

1.1 Our Results

We study the problem of maximizing welfare for agents with quantile valuations. Under quantile valuations, each agent i specifies their value for individual items and a quantile value $\tau_i \in [0, 1]$. Given a bundle B, agent i's value for B is the τ_ith quantile of the values of the items in B. We provide comprehensive results on *utilitarian social welfare* (USW) (see for e.g. Harsanyi (1955)), which captures efficiency, and *egalitarian social welfare* (ESW) (see for e.g. Moulin (2004)), which captures fairness. USW is essentially the sum of the agents' valuations for their assigned bundles, and ESW is the minimum of these values. We study each objective for both balanced and unconstrained allocations. Our results are summarized in Table 1.

Utilitarian Welfare. We first show that the problem of maximizing the Utilitarian Social Welfare is NP-hard for both balanced and all allocations. Over balanced allocations (where each agent receives m/n items), we prove that it is NP-hard to approximate the optimal USW within a factor $O(\frac{m/n}{\log(m/n)})$ for instances with $m \leq n^2$. We then present a $\min(\frac{m}{n}+1, n)$-approximation algorithm, which matches the hardness of approximation bound asymptotically.

In the unconstrained setting, we present a $\left(1+\frac{1}{n-1}\right)$-approximation algorithm to the optimal USW. Our results thus demonstrate that the complexity of both problems differs significantly depending on whether the allocations are required to be balanced.

Egalitarian Welfare. In the setting where allocations are constrained to be balanced, we prove that ESW optimal allocations can be computed in polynomial time, even when agents have arbitrary heterogeneous quantiles. This is in contrast to USW, where we have hardness of approximation.

When not restricted to balanced allocations, we show that the complexity of maximizing ESW is highly dependent on the agents' quantile values. Specifically, we prove that when agents have homogeneous quantiles τ, the problem is solvable in polynomial time for $\tau \in \{0, {}^1/_3, 1\} \cup \{{}^t/_{t+1} \mid t \in \mathbb{Z}_+\}$. In contrast, for $\tau \in (0, {}^1/_4] \cup ({}^3/_8, {}^2/_5] \cup ({}^5/_9, {}^3/_5]$, the problem becomes APX-hard, with no multiplicative approximation is possible unless P=NP.

Near-optimal results for goods. Our results for goods are in fact quite tight. For USW, it is straightforward to see that the hardness bounds are quite close to the guarantees provided by the approximation bounds. Specifically, in the balanced case, it is NP-hard to get an $O(\frac{m/n}{\log m/n})$ approximation algorithm, and our approximation algorithm matches this by giving a $\min(\frac{m}{n}+1, n)$. For the unconstrained case, it is straightforward to see that an $O(1-\frac{1}{n})$ approximation is near-optimal for an NP-hard problem.

For egalitarian welfare, we show that even with binary values for the items, the problem of maximizing ESW is NP-hard. As this holds for binary valuations, any $\alpha > 0$ approximation on ESW would imply exact ESW. As a result, the problem of maximizing ESW is APX-hard with no non-trivial approximation possible. We then identify a subset of quantiles where the problem of maximizing ESW can be completed in polynomial time, while showing NP-hardness for a large sub-class of the remaining quantiles. The intractability holds when all agents have the same quantile, thus easily extends to heterogeneous quantiles.

Identical Valuations. When all agents have the same valuation function, the strong intractability results for maximum USW under balanced allocations and maximum ESW over all allocations can be overcome. The problem of maximizing USW in the unconstrained setting with identical valuations remains open.

Chores. For chores, the problem of maximizing welfare is equivalent to that of minimizing social cost. We find that the problem of maximizing *utilitarian social cost* (USC) in the unconstrained setting is equivalent to the weighted set cover problem. For balanced allocations, maximizing USC is NP-hard and we get an $O(n)$-approximation. For *egalitarian social cost* (ESC) in the unconstrained setting, the problem becomes APX-hard, even for quantiles that were tractable for goods. For the balanced case, the algorithm for goods extends to chores as well. Due to space constraints, we defer this entire discussion to Section E.

1.2 Related Work

We defer an extended literature review to Section A.

Quantile based preferences. Quantile based preferences are well-established in mathematical economics and social choice theory. Our proposed valuations are a generalization of *preference set extensions* that lift preferences over individual items to a set of items. The study of preference set extensions has a long-standing history in social choice theory (Barberà, Bossert, and Pattanaik 2004) and has been applied to hedonic coalition formation games (Cechlárová and Hajduková 2003, 2004), committee selection (Aziz and Monnot 2020) and multidimensional matchings (Hosseini, Narang, and Roy 2025). Recently, Caragiannis and Roy (2025) introduced quantile based utilities to the context of randomized social choice and matchings. We discuss these and other generalizations of set extensions in Section A.

Allocating Indivisible Items. The problem of allocating indivisible items fairly and/or efficiently is very well studied (see Amanatidis et al. (2023) for a survey). Existing literature almost exclusively assumes that aggregated preferences are monotone, very often, additive (Caragiannis et al. 2019; Aziz et al. 2022). Some also consider arbitrary valuations (Bérczi et al. 2024; Barman et al. 2024a). Our proposed valuations are non-monotone for most quantiles. Restricted cardinality allocations have been explored for additive valuations (Shoshan, Hazon, and Segal-Halevi 2023; Biswas and Barman 2018; Caragiannis and Narang 2024).

2 Model

We shall use $[t] = \{1, \cdots, t\}$ for any $t \in \mathbb{Z}_+$.

We consider a setting with a set of agents N s.t. $|N| = n$ and a set of items M, s.t. $|M| = m$. Each agent $i \in N$ has a valuation function v_i over M. Informally, a valuation function is a τ quantile valuation, for $\tau \in [0, 1]$, if the value assigned to a bundle $S \subseteq M$[2] is determined by the τ quantile of the distribution of item values in S.

Definition 1 (Quantile Valuations)**.** *Given a set of indivisible items M, we say that $v_i : 2^M \to \mathbb{R}$ is a τ_i quantile for $\tau_i \in [0, 1]$, if for any subset $S \subseteq M$, we have that*

$$v_i(S) = \min_{g \in S} \left\{ v_i(g) : \frac{|\{g' \in S : v_i(g') \leq v_i(g)\}|}{|S|} \geq \tau_i \right\}.$$

An equivalent way of defining quantile valuations is to say that v_i is a τ_i quantile for $\tau_i \in [0, 1]$ if for any subset $S \subseteq M$ where $g_{i_1}, \cdots, g_{i_{|S|}}$ are the items in S s.t. $v_i(g_{i_1}) \leq \cdots \leq v_i(g_{i_{|S|}})$ and $v_i(S) = v_i(g_{i_{\lceil \tau_i |S| \rceil}})$ if $\tau_i > 0$, otherwise, $v_i(S) = v_i(g_{i_1})$. In particular, if τ_i is 0, the agent values the given set as much as their least favorite item and if τ_i is 1, they value it as much as their most favorite item.

We shall use τ_i to denote the quantile of agent i. When all agents have the same quantile, we shall simply use τ. Unless otherwise specified, assume that all agents have the same quantile $\tau \in [0, 1]$. Consequently, an instance of our problem can be expressed by the tuple $I = \langle N, M, v, \tau \rangle$.

Each item must be allocated to some agent. Formally, an allocation $A = (A_1, \cdots, A_n)$ is an n-partition of M, with A_i being the set of items assigned to agent $i \in N$. We shall use $\Pi(n, M)$ to denote the set of all allocations that divide the items in M among n agents. Our aim is to allocations with maximum welfare.

[2]we refer to a subset of items as a *bundle*

Definition 2 (Utilitarian Social Welfare (USW))**.** *Given an instance $I = \langle N, M, v, \tau \rangle$ and an allocation $A = (A_1, \cdots, A_n)$, the utilitarian social welfare is the sum of the values received by the agents* $\text{USW}(A) = \sum_{i \in N} v_i(A_i)$.

Given an instance $I = \langle N, M, v, \tau \rangle$, let A^* be a maximum USW allocation. We shall say that allocation A is α-USW for $\alpha \in [0, 1]$, if $\text{USW}(A) \geq \alpha \text{USW}(A^*)$.

Definition 3 (Egalitarian Social Welfare (ESW))**.** *Given an instance $I = \langle N, M, v, \tau \rangle$ and an allocation $A = (A_1, \cdots, A_n)$, the egalitarian social welfare is the minimum of the values incurred by the agents* $\text{ESW}(A) = \min_{i \in N} v_i(A_i)$.

Balanced Allocations Quantile valuations are very intuitive for settings where we insist on each agent getting an equal number of items, as in the case of assigning papers to reviewers in conferences or assigning students to teachers. We shall consider both Utilitarian and Egalitarian Welfare with and without this requirement.

When considering balanced allocations, we shall assume that the number of agents divides the number of items. That is, $m = kn$ for some $k \in \mathbb{Z}_+$. Thus, we shall look for allocations $A = (A_1, \cdots, A_n)$ where $|A_i| = k$. We shall use $\overline{\Pi}(n, M)$ to denote the set of all balanced allocations for instance I. It is important to note that when we consider maximizing USW or ESW over balanced allocations, we are in fact finding a maximum welfare allocation from $\overline{\Pi}(I)$ alone. That is, we are not holding the allocations to the standard of maximum welfare under unconstrained allocations. When not explicitly specified, assume unconstrained.

Goods and Chores. We shall say that an item $g \in M$ is a *good*, if for all agents $v_i(g) \geq 0$. Analogously, we shall say that an item $g \in M$ is a *chore*, if for all agents, $v_i(g) \leq 0$. Unless specifically mentioned otherwise, the items we refer to will be goods. When referring to chores, we shall often use the term disutilities with d_i denoting agent i's disutility where $d_i = -v_i$. Consequently, an instance of our problem can be denoted equivalently by $\langle N, M, d, \tau \rangle$ when considering an instance with chores.

When we consider instances with only chores, the social welfare notions become *Utilitarian Social Cost (*USC*)* and *Egalitarian Social Cost (*ESC*)*, respectively. Here, $\text{USC}(A) = \sum_{i \in N} d_i(A_i)$ and $\text{ESC}(A) = \max_{i \in N} d_i(A_i)$. We shall say that allocation A is α-USC for $\alpha \geq 1$, if $\text{USC}(A) \leq \alpha \text{USC}(A^*)$ where A^* has minimum USC.

3 Balanced Allocations

We first explore quantile valuations with the requirement that the allocations be balanced. Our results for USW and ESW lie in stark contrast with each other here. We defer any omitted proofs to Section B.

3.1 Utilitarian Social Welfare

We first show, when allocations are balanced, that maximizing USW is NP-hard to approximate to better than a factor of $O(\frac{m/n}{\log(m/n)})$. We then proceed to give a polynomial-time algorithm that matches hardness of approximation bound.

Hardness of Approximation In order to show hardness of approximation, we give an approximation preserving reduction from the k-DIMENSIONALMATCHING(kDM) problem. The kDM problem requires finding a maximum collection of disjoint edges in a k-partite hypergraph where each hyper-edge has size k. Hazan, Safra, and Schwartz (2003) showed that this problem is NP-hard to approximate to a factor better than $O(\frac{k}{\log k})$.

Theorem 1. *Given instance $I = \langle N, M, v, \tau\rangle$ where $m < n^2$ and $m = kn$, it is NP-hard to find an $O\left(\frac{m/n}{\log(m/n)}\right)$-USW balanced allocation.*

Proof. Given an instance of kDM, $\langle G = (X, H), \ell\rangle$, we create an instance of our problem as follows: For each edge $H_i \in H$, we create agent i. For each vertex $x \in X$, we create item g_x. We can assume, without loss of generality, that each vertex is contained in at least one hyper-edge. Thus, we have that $|X| \le k|H|$. To balance the item count, we introduce $k|H| - |X|$ dummy items $g'_1, \cdots, g'_{kn-|X|}$. Thus, we have $n = |H|$ agents and the number of items is $m = k|H|$. As a result, we have that $m = kn$. Recall that balanced allocations require $k = \frac{m}{n}$ items to be allocated to each agent.

For each agent $i \in N$, we set $\tau_i = 0$ for all $i \in N$. Now for i and each g_x, if $x \in H_i$, we set $v_i(g_x) = 1$ else, we set $v_i(g_x) = 0$. Finally, for each $t \in [k|H| - |X|]$, set $v_i(g'_t) = 0$. We now show that a matching of size ℓ in the kDM problem can be transformed into a balanced allocation whose USW is at least ℓ in the reduced instance of our problem, and vice versa. Consider a matching μ of size ℓ in kDM. For each $H_i \in \mu$, allocate the items vertices in H_i. That is, $A_i = \{g_x | x \in H_i\}$. Arbitrarily allocate the remaining items, ensuring $|A_i| = k$. It is easy to see that $\text{USW}(A) \ge \ell$.

Now consider a balanced allocation A in the reduced instance with a USW of ℓ. As the maximum value for any agent is 1, this implies that ℓ agents receive a value of 1 from A. By construction, $v_i(A_i) = 1$ only if A_i contains *all* the items corresponding to the vertices in H_i. From here, it is easy to see that $\mu = \{H_i | v_i(A_i) = 1\}$ is a matching of size ℓ. Hazan, Safra, and Schwartz (2003) proved that there exists a class of instances with $k < n$ such that kDM is NP-hard to approximate within a factor of $O(k/\log(k))$. Thus, we have hardness of approximation for instances where $m < n^2$. □

Observe that this reduction can be extended to all $\tau \in [0, 1)$ by adding enough dummy items s.t. an agent gets a value of 1 only if they get three items of value 1. Consequently, the APX-hardness holds for all quantiles $\tau \in [0, 1)$ where $(m - n\lceil\tau(m/n)\rceil) < n^2$.

Near-Optimal Algorithm We now provide an approximation algorithm that almost matches the lower bound placed by Theorem 1. The greedy algorithm (Algorithm 1) proceeds by iteratively allowing unassigned agents to "demand" their best possible set from the unassigned items. We then choose the agent whose value for their demanded set is highest. We repeat this until all items are assigned.

ALGORITHM 1: $\min(\frac{m}{n} + 1, n)$-USW Greedy Algorithm

Input: Instance with goods and heterogeneous quantiles $\langle N, M, v, \tau\rangle$ where $m = kn$
Output: A balanced allocation A
Initialize set of unallocated goods $P \leftarrow M$;
Initialize set of unassigned agents $N' \leftarrow N$;
Let $k_i \leftarrow \min(k, k - \lceil\tau_i k\rceil + 1)$;
while $N' \neq \emptyset$ **do**
  For each $i \in N'$, let $S_i \leftarrow \underset{S \subseteq P, |S| = k_i}{\arg\max} \sum_{g \in S} v_i(g)$;
  Let $i^* = \underset{i \in N'}{\arg\max}\, (\min_{g \in S_i} v_i(g))$;
  $A_{i^*} \leftarrow S_{i^*}$;
  $P \leftarrow P \setminus A_{i^*}$;
  $N' \leftarrow N' \setminus \{i^*\}$;
Allocate items in P arbitrarily s.t. $|A_i| = k$ for all $i \in N$;
Return A

Theorem 2. *Given an instance $I = \langle N, M, v, \tau\rangle$ with $m = kn$ and heterogeneous quantiles, Algorithm 1 returns a balanced allocation which is* $\min(\frac{m}{n} + 1, n)$-USW.

3.2 Egalitarian Social Welfare

We now move to maximizing egalitarian welfare. We begin with a very useful reduction, which facilitates all our algorithms for ESW. We show that whenever there is an algorithm to find an allocation with maximum ESW under binary valuations, we can use it to find a maximum ESW allocation under general non-negative valuations.

Lemma 1. *The problem of finding an allocation with* ESW *at least $\nu \ge 0$ over allocations in $\Pi' \subseteq \Pi(n, M)$ under heterogeneous quantiles reduces to maximizing* ESW *over Π' under binary goods with heterogeneous quantiles.*

The proof of Lemma 1 shows that given an arbitrary instance I and a value ν, we can construct an alternate instance I' with binary valuations such that I has an allocation with ESW of ν if and only if the maximum ESW under I' is 1. As a result, given I and an algorithm ALG which finds a maximum ESW allocation over Π' for binary goods, we can make at most mn calls to ALG to find a maximum ESW allocation over Π' for I.

This enables us to maximize ESW over balanced allocations, even if the quantile values are heterogeneous. We only consider a setting where $v_i(g) \in \{0, 1\}$ for all $i \in N$ and all $g \in M$. Here, we shall try to see if an allocation with ESW 1 can exist. That is, all agents must get a value of 1. In order to achieve this, we first make the following observation:

Observation 1. *For an agent i with quantile τ_i and binary valuations, given a bundle $B \subseteq M$, the value of i for B satisfies $v_i(B) = 1$ if and only if there are at most $\lceil\tau_i|B|\rceil - 1$ items in B for which i has value 0.*

This follows from the definition of quantile valuations. Thus, to have ESW of 1, each $i \in N$ must receive at least

ALGORITHM 2: Max balanced ESW for binary goods.

Input: Instance with binary values and heterogeneous quantiles $\langle N, M, v, \tau \rangle$ s.t. $m = kn$
Output: Balanced Allocation A

Let $k_i = \min(k, k - \lceil \tau_i k \rceil - 1)$;
Create bipartite graph $G = (X, Y, E)$ where X contains x_g for each $g \in M$, Y contains $y_1^i, \cdots y_{k_i}^i$ for each $i \in N$ and
$(x_g, y_t^i) \in E$ only if $v_i(g) = 1$, for each $i \in N$, $g \in M, t \in [k_i]$;
Let μ be a maximum cardinality matching in G;
if $|\mu| = |Y|$ **then**
  Initialize $A = (A_1, \cdots A_n)$ where $A_i \leftarrow \{g | (x_g, y_t^i) \in \mu$ for some $t \in [k_i]\}$ for each $i \in N$;
  Allocate remaining items arbitrarily but ensuring $|A_i| = k$ for all $i \in N$;
else
  Let A be an arbitrary balanced allocation;
Return A;

$k_i = \min(k, k - \lceil \tau_i k \rceil + 1)$ items of value 1, where $m = kn$. Note that the min argument only comes in when $\tau_i = 0$.

We can check if this is possible using a simple maximum cardinality bipartite matching algorithm. This is shown in Algorithm 2. Here, we create a bipartite graph where for each $i \in N$, we create k_i vertices, and for each $g \in M$ we create one vertex. We add an edge between i and g if $v_i(g) = 1$. A matching of size $\sum_{i \in N} k_i$ exists if and only if the given instance has an allocation with ESW of 1.

Proposition 1. *Given $I = \langle N, M, v, \tau \rangle$ where $m = kn$ with binary goods and heterogeneous quantiles, Algorithm 2 finds a max* ESW *balanced allocation in polynomial time.*

As a consequence of Lemma 1 and Proposition 1, we get the following theorem.

Theorem 3. *Given $I = \langle N, M, v, \tau \rangle$ with heterogeneous quantiles where $m = kn$, a balanced allocation with maximum* ESW *can be found in polynomial time.*

4 Unconstrained Allocations

Typical work on allocating indivisible items does not require allocations to be balanced. Further, there are many practical settings, such as online classrooms in distance education, where allocations need not be balanced. Thus, for completeness, we again explore welfare maximization, now for unconstrained allocations, beginning with utilitarian welfare. We find that it is possible to give significantly better guarantees on USW, compared to the balanced setting. In contrast, maximum ESW now becomes intractable for a large subset of the quantiles. Omitted proofs are deferred to Section C.

4.1 Utilitarian Social Welfare

We find that while maximizing USW still remains intractable, we are able to circumvent hardness of approximation and achieve a near exact approximation algorithm that even works for heterogeneous quantiles.

ALGORITHM 3: Scapegoat Algorithm for $1 + \frac{1}{n-1}$-USW

Input: Instance with heterogeneous quantiles $\langle N, M, v, \tau \rangle$
Output: Allocation A

for *each* $i \in N$ **do**
  Create weighted bipartite graph $G^i = (X, Y, E, w)$ where
  X contains x_g for each $g \in M$,
  Y contains y_j for each $j \in N \setminus i$ and
  $w(x_g, y_j) = v_j(g)$;
  Let μ be a maximum weight matching in G^i;
  Set $A_j^i = \{g | x_g = \mu(y_j)\}$ for all $j \neq i$;
  Set $A_i^i = M \setminus \cup_{j \neq i} A_j^i$;
Let $A \leftarrow \arg\max\{\text{USW}(A^i) | i \in N\}$;
Return A;

Intractability We first show that for non-identical agents with quantile $\tau = 0$ for all agents, the problem of maximizing social welfare proves to be NP-hard for goods. We give a reduction from the EXACT3COVER problem, which is known to be NP-hard (Garey and Johnson 1979).

Theorem 4. *Given instance $I = \langle N, M, v, \tau \rangle$ with goods finding a maximum* USW *allocation is NP-complete.*

A similar reduction can be carried out for all other quantiles in $\tau \in [0, 1)$ by adding a sufficient number of items that would give value 0 to all agents. The only change needed would be add enough "padding" items of value 0 for everyone, so that we can get an analogous mapping of instances. The number of padding items needed depends on the quantile, but can easily be computed for each $\tau \in [0, 1)$.

Near Exact Algorithm. In contrast to the balanced case, we find a near-optimal approximation for USW. We call this the scapegoat algorithm (Algorithm 3). It proceeds by considering allocations where one agent is the "scapegoat" and receives $m - n + 1$ items, while the remaining agents get one item each of high value. Exactly n such allocations are considered, one for each agent as the scapegoat. For a fixed scapegoat, the corresponding allocation is built by a maximum weight one-one matching between the other agents and the items. The algorithm chooses the allocation with the highest USW.

Theorem 5. *Given instance $I = \langle N, M, v, \tau \rangle$ with heterogeneous quantiles, scapegoat algorithm (Algorithm 3) returns an $(1 + \frac{1}{n-1})$-*USW *allocation in polynomial time.*

Building on this approach, we now show that when even one agent has $\tau_i = 1$, we can now maximize USW in poly time. Essentially this agent can be treated as the scapegoat, and we can simply use a maximum weight one-one matching as in Algorithm 3 and allocate all remaining items to the scapegoat.

Proposition 2. *Given instance $I = \langle N, M, v, \tau \rangle$ with heterogeneous quantiles and an agent i^* such that $\tau_{i^*} = 1$, a maximum* USW *allocation can be found in polynomial time.*

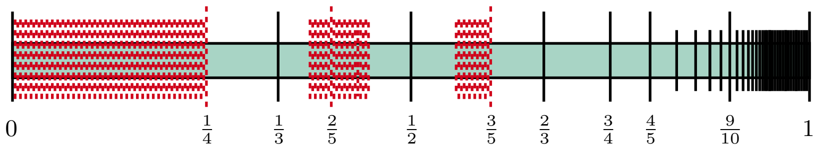

Figure 1: Quantile-wise tractability or intractability of max ESW. Red dashed lines show values of τ for which maximizing ESW is NP-hard, black solid lines show a value for τ for which we have polytime algorithms.

4.2 Egalitarian Social Welfare

Rather surprisingly, we find that maximizing ESW over all allocations is intractable for some quantiles and tractable for others. Here, we assume all agents have the same quantiles. Clearly, the intractability results would extend to settings with arbitrary heterogeneous valuations. We illustrate the spectrum of quantiles for which the problem is tractable vs intractable in Fig. 1. When presenting algorithms, we shall again assume binary valuations. From Lemma 1, a polynomial time algorithm for the binary case is sufficient to get a general algorithm.

Exact Algorithms. We are able to find polynomial time algorithms for maximizing ESW under a class of quantiles which includes many natural quantiles like $\tau = 0, {}^1\!/_2, {}^2\!/_3, {}^3\!/_4, {}^9\!/_{10}$. We begin with an observation which is true for all quantiles: for maximum ESW to be 1, each agent must get at least one item of value 1 simultaneously.

Observation 2. *Under an instance with binary goods, for allocation A,* $\text{ESW}(A) = 1$ *if and only if for each $i \in N$, there exists $g \in A_i$, s.t. $v_i(g) = 1$.*

This gives a necessary condition for an allocation with ESW of 1 to exist. We now specifically consider quantiles of the form $\tau = \frac{t}{t+1}$ for $k \in \mathbb{Z}_+$. For this setting, we have the following simple result.

Lemma 2. *For an agent $i \in N$ with $\tau_i = \frac{t}{t+1}$, where $t \in \mathbb{Z}_+$ is fixed, a bundle $B \subseteq M$ with exactly ℓ goods of value 1 for i, we have that $v_i(B) = 1$ if and only if the number of 0 valued items in B for i is at most $\ell t - 1$.*

Based on Observation 2 and Lemma 2 we develop an algorithm for maximizing ESW over unconstrained allocations when there is a $t \in \mathbb{Z}_+$ s.t. $\tau_i = \frac{t}{t+1}$ for each $i \in N$. We divide the items into two set M_0 and M_1. Items in M_0 are objective 0s, that is, all agents have value 0 for each item in M_0. The items in M_1 are subjective 1s, that is, these are the items for which at least one agent has value 1.

The algorithm checks for two conditions: are there enough items so that each agent can receive a good of value 1 and are there enough items in M_1 to offset the items in M_0. If so, it first assigns each agent an item they have value 1 for. Next, out of the unassigned items in M_1, it arbitrarily selects one such item and allocates it to an agent who has value 1 for it, along with t items from M_0. Finally, if no items remain in one of M_0 or M_1, it allocates the remaining items while ensuring the condition in Lemma 2.

Proposition 3. *Given instance $I = \langle N, M, v, \tau = \frac{t}{t+1} \rangle$ where $t \in \mathbb{Z}_+$, Algorithm 4 returns a maximum* ESW *allocation in polynomial time.*

ALGORITHM 4: Max ESW for binary goods and $\tau = {}^t\!/_{t+1}$

Input: $I = \langle N, M, v, \tau \rangle$ with binary goods and $\tau = {}^t\!/_{t+1}$
Output: An allocation A

Create bipartite graph $G = (X, Y, E)$ where X contains x_g for each $g \in M$, Y contains y_i for each $i \in N$ and $(x_g, y_i) \in E$ only if $v_i(g) = 1$, for each $i \in N$, $g \in M$;
Let μ be a maximum cardinality matching in G;
Let $M_0 = \{g \in M | v_i(g) = 0 \text{ for all } i \in N\}$;
Let $M_1 = M \setminus M_0$;
if $|M_0| > t|M_1| - n$ *OR* $|\mu| < n$ **then**
    Let A be an arbitrary allocation;
else
    Let $A = (A_1, \cdots, A_n)$ be s.t. $A_i \leftarrow \{g | (x_g, y_i) \in \mu\}$;
    $M_1 \leftarrow M_1 \setminus \cup_i A_i$;
    while $M_1 \neq \emptyset$ *AND* $M_0 \neq \emptyset$ **do**
        Arbitrarily pick $g \in M_1$ and $i \in N$ s.t. $v_i(g) = 1$;
        if $|M_0| \geq t$ **then**
            Pick an arbitrary subset $S \subseteq M_0$ s.t. $|S| = t$;
        else
            Let $S \leftarrow M_0$;
        $A_i \leftarrow A_i \cup \{g\} \cup S$;
        $M_1 \leftarrow M_1 \setminus \{g\}$ and $M_0 \leftarrow M_0 \setminus S$;
    if $M_0 \neq \emptyset$ **then**
        Let $B_1 \cdots B_n$ be an arbitrary partition of M_0 s.t. $|B_i| \leq t - 1$ for all $i \in N$;
    if $M_1 \neq \emptyset$ **then**
        Let $B_1 \cdots B_n$ be an arbitrary partition of M_1 s.t. $g \in |B_i|$ only if $v_i(g) = 1$;
    For each $i \in N$, set $A_i \leftarrow A_i \cup B_i$;
Return A;

We can extend this idea to the setting of $\tau_i = {}^1\!/_3$ for all agents. We defer the discussion to Section C.

Proposition 4. *Given $I = \langle N, M, v, \tau = {}^1\!/_3 \rangle$, Algorithm 5 returns a maximum* ESW *allocation in polynomial time.*

We can now summarize our tractability results for maximum ESW over all allocations as follows.

Theorem 6. *A maximum* ESW *allocation can be found in polynomial time for $\tau = \{0, {}^1\!/_3, 1\} \cup \{\frac{t}{t+1} | t \in \mathbb{Z}_+\}$.*

Intractability. We now show that there are several quantile values for which maximizing ESW is APX-hard. Intriguingly, these values interweave between quantile values for which maximizing ESW can be done in polynomial time. We find three ranges of intractability. Namely, for $\tau \in (0, {}^1\!/_4] \cup ({}^3\!/_8, {}^2\!/_5] \cup ({}^5\!/_9, {}^3\!/_5]$. For $\tau \in ({}^3\!/_8, {}^2\!/_5]$ or $\tau \in ({}^5\!/_9, {}^3\!/_5)$ the ratio of additional items of value 1 for a new item of value 0 can vary. We find that deciding between these cases proves to be intractable. We show intractability for binary valuations for each range. Under binary valuations, any $\alpha > 0$ approximation on ESW would be an exact algorithm. Consequently, we get that it is NP-hard to have any α-ESW algorithm for all $\alpha > 0$.

Theorem 7. *Given $I = \langle N, M, v, \tau\rangle$, maximizing* ESW *is APX-hard for $\tau \in (0, \frac{1}{4}] \cup (3/8, 2/5] \cup (5/9, 3/5]$.*

5 Identical Valuations

We now consider identical valuations, that is all agents have the same quantile τ and the same valuation function v. We defer all omitted proofs to Section D.

5.1 Utilitarian Welfare

Maximizing USW remains open for the unconstrained case. A maximum USW balanced allocation can be found in polynomial time for any $\tau \in [0, 1]$. In fact, we have that the same greedy algorithm (Algorithm 1) that was $\min(\frac{m}{n} + 1, n)$-USW that proves to be an exact algorithm in this case.

Theorem 8. *Given an instance with identical valuations $I = \langle N, M, v, \tau\rangle$, Algorithm 1 returns a balanced allocation with maximum* USW.

5.2 Egalitarian Welfare

We again focus our attention to binary goods, as a consequence of Lemma 1. The balanced case is already covered by Theorem 3. We find that a maximum ESW allocation in the unconstrained case can be found in polynomial time. To this end, we first observe a simple fact.

Observation 3. *Given $\tau \in (0, 1]$ and a τ-quantile binary valuation function v and bundle $B \subset M$, let $B_0 = \{g \in B : v(B) = 0\}$. We have that $v(B) = 1 \Leftrightarrow |B| > \frac{|B_0|}{\tau}$.*

We use this to show in order to maximize ESW under identical valuations, it suffices to consider allocations where the number of items of value 0 is balanced across agents.

Lemma 3. *Given instance $I = \langle N, M, v, \tau\rangle$ with identical valuations over binary goods, let r be the number of goods with value* 1. *An allocation with* ESW *of* 1 *exists only if there exists an allocation A in which each A_i has at most $\lceil (m - r)/n \rceil$ goods of value* 0 *and $v(A_i) = 1$ for each $i \in N$.*

Proof. Let there exist an allocation $A = (A_1, \cdots, A_n)$ s.t. $v(A_i) = 1$ for all $i \in N$. Let t_i denote the number of goods of value 0 in A_i. If $\tau = 0$, then if $r \neq m$, no allocation can exist where all agents get value 1. It follows that any allocation with ESW of 1, satisfies the required property.

Now consider the case where $\tau > 0$. Let there exist an agent j s.t $t_j > \lceil \frac{m-r}{n} \rceil$. Then there must exist an agent j' s.t. $t_j - t_{j'} \geq 2$. Thus, $t_{j'} < \lfloor \frac{m-r}{n} \rfloor$. We shall show that there exists an allocation A' where j gets $t_j - 1$ items of value 0, and j' gets $t_{j'} + 1$ items of value 0 and $\text{ESW}(A) = 1$.

Consider A' where $A'_i = A_i$ for all $i \neq j, j'$. We shall now transfer one item of value 0 and just enough items of value 1 from j to j' to get the required allocation. From Observation 3, for $t_{j'} + 1$ items of value 0, in order for $v(A'_{j'}) = 1$, it must be the case that $|A'_{j'}| > \frac{t_{j'}+1}{\tau}$. If $|A_{j'}| + 1 > \frac{t_{j'}+1}{\tau}$, we let $A'_{j'} = A_{j'} \cup g_0$ such that $v(g_0) = 0$ and $g_0 \in A_j$. Finally, let $A'_j = A_j \setminus g_0$, observe that $v(A'_j) = v(A'_{j'}) = 1$.

Suppose $|A_{j'}| + 1 \leq \frac{t_{j'}+1}{\tau}$. Now as $v(A_{j'}) = 1$, by Observation 3, it must be that $|A_{j'}| > \frac{t_{j'}}{\tau}$. We shall show that there are enough goods of value 1 in A_j that can be transferred while maintaining the values of both bundles.

Choose $\ell = \lceil \frac{1}{\tau} \rceil - 1$ goods $\{g_1, ..., g_\ell\}$ of value 1 and g_0 of $v(g_0) = 0$ from A_j. Let $A'_{j'} = A_{j'} \cup \{g_0, g_1, ..., g_\ell\}$, and $A'_j = A_j \setminus \{g_0, g_1, ..., g_\ell\}$. Observe that as $v(A_{j'}) = 1$, we have that $|A_{j'}| > \frac{t_{j'}}{\tau}$, it follows that

$$|A'_{j'}| = |A_{j'}| + 1 + \ell \geq |A_{j'}| + \frac{1}{\tau} > \frac{t'_j + 1}{\tau}.$$

Thus, we have $v(A'_{j'}) = 1$. We now show that $v(A'_j) = 1$. By assumption, $v(A_j) = 1$. Consequently, we have that $\lceil \tau |A_j| \rceil \geq t_j + 1$. Now consider A'_j, we need $\lceil \tau(|A_j| - (1 + \ell)) \rceil \geq t_j$. Consider

$$\begin{aligned}
\lceil \tau |A'_j| \rceil &= \lceil \tau(|A_j| - (1 + \ell)) \rceil \\
&\geq \lceil \tau(|A_j| - \frac{1}{\tau}) \rceil = \lceil \tau |A_j| - 1 \rceil = \lceil \tau |A_j| \rceil - 1 \\
&\geq t_j + 1 - 1 = t_j.
\end{aligned}$$

Hence, we have that $\text{ESW}(A') = 1$. Consequently, we can repeat this procedure till each agent has at most $\lceil \frac{m-r}{n} \rceil$ goods of value 0. As a result, whenever an allocation exists s.t. $\text{ESW}(A) = 1$, there must exist an allocation A', s.t. $\text{ESW}(A') = 1$ and each agent receives either $\lfloor (m - r)/n \rfloor$ or $\lceil (m - n)/r \rceil$ items of value 0. $\square$

This result proves useful in maximizing ESW in the unconstrained setting and even leads to an algorithm for maximum USW for binary goods.

Theorem 9. *Given instance $I = \langle N, M, v, \tau\rangle$ with identical valuations, an allocation with maximum* ESW *can be found in polynomial time.*

Proposition 5. *Given instance $I = \langle N, M, v, \tau\rangle$ with identical valuations over binary goods, an allocation with maximum* USW *can be found in polynomial time.*

6 Conclusions

In this work, we proposed a novel quantile-based preference model in the context of indivisible item allocation. We studied *Utilitarian* and *Egalitarian Welfare*, both with and without the balanced allocation requirement, and provided comprehensive algorithmic and complexity-theoretic results.

Interestingly, our results reveal that the complexity of the problems changes significantly depending on whether the balancedness requirement is imposed. For instance, for balanced allocations there is a strong hardness of approximation bound for maximizing USW, whereas for unconstrained allocations, a near-exact approximation algorithm exists. A similar phenomenon occurs with ESW but in reverse: for balanced allocations, maximizing ESW can be solved efficiently, while for unconstrained allocations, maximizing ESW is APX-hard for many quantile values.

Our work opens up several promising directions for future research. Firstly, while we focused on the two extremes of the p-means (Utilitarian and Egalitarian welfare), exploring other welfare functions, such as Nash welfare, presents an intriguing avenue for study. Secondly, investigating the compatibility between fairness notions, such as EF1 or EFx, and Pareto efficiency within the framework of our valuation class is another interesting direction of further research.

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A Additional Related Work

Allocating Indivisible Items. The problem of allocating indivisible items fairly and/or efficiently is very well studied (See (Amanatidis et al. 2023) for a survey). Existing literature almost exclusively assumes that preferences are aggregated in a monotone manner, often assuming additive valuations (Caragiannis et al. 2019; Aziz et al. 2022), but also at times subadditive (Barman et al. 2024b; Benabbou et al. 2020) or superadditive valuations (Barman, Narayan, and Verma 2023; Viswanathan and Zick 2023). A couple of papers also consider arbitrary valuations, with no underlying structure guaranteed, in addition to monotone valuations (Bérczi et al. 2024; Barman et al. 2024a). Our paper considers quantile preferences which may be monotone for the two extreme quantiles, but non-monotone for all others.

Constrained Allocations. While typical work on allocating indivisible items does not explicitly restrict the type of allocations studied there has been some work restricting the number of items that can be allocated. Shoshan, Hazon, and Segal-Halevi (2023) and (Biswas and Barman 2018) consider a setting where items are partitioned into categories and there is a uniform constraint on how many items of each category can be allocated to a single agent. For this space, Biswas and Barman (2018) focus on envy based fairness, while Shoshan, Hazon, and Segal-Halevi (2023) consider efficiency via pareto optimality and envy-based fairness.

Caragiannis and Narang (2024) study a repeated matching setting where there are T rounds and n agents and n items. In each round, each agent must receive exactly one item. Here, value for an item depends on how many times this agent has received the item in the past. For this space, (Caragiannis and Narang 2024) pursue utilitarian social welfare and envy-based fairness.

Quantile based preferences. Quantile based preferences are well-established in mathematical economics and social choice theory. (de Castro and Galvao 2019, 2022) show that quantile preferences are a more accurate model of real-life behavior of agents in random settings over expected utility. Recently, quantile valuations have been introduced to the setting of randomized social choice as well as one and two-sided matchings (Caragiannis and Roy 2025).

These preferences are a generalization of *preference set extensions* that lift preferences over individual items to a set of items. The study of preference set extensions has a long-standing history in social choice theory (Barberà, Bossert, and Pattanaik 2004) and has been applied to hedonic coalition formation games (Cechlárová and Hajduková 2003, 2004; Cechlárová 2008), committee selection (Aziz and Monnot 2020) and multidimensional matchings (Hosseini, Narang, and Roy 2025). Among them, one is called the *best set extension* in which the sets are compared based on the best item in each set. One is called the *worst set extension*, in which the sets are compared based on the best item in each set. The best and worst extension correspond to the $\tau = 1$ and $\tau = 0$ in our model.

Quantile based set extensions have been explored in prior work through the lens of specific quantiles. The downward lexicographic (DL) and the upward lexicographic (UL) set extension are both natural refinements of the best and worst set extensions, respectively. Both lexicographic extensions are also special cases of set extensions based on additive valuations (Barberà, Bossert, and Pattanaik 2004). Lexicographic preferences have been well studied within fair division (Aziz et al. 2015; Hosseini, Mammadov, and Was 2023; Hosseini et al. 2021, 2023; Ebadian, Peters, and Shah 2022). Other quantiles have also been considered previously. Nitzan and Pattanaik (1984) characterize median quantile preferences that are a special case of $\tau = 1/2$ in our model.

Recently, the idea of quantiles has been introduced on top of the standard additive valuation setting within fair division. (Babichenko et al. 2024) consider settings where an agent assesses the fairness of a bundle by comparing it to her valuation in a random allocation. In this framework, a bundle is considered q-quantile fair, if it is at least as good as a bundle obtained in a uniformly random allocation with probability at least q. In a similar vein, Bhawalkar et al. (2024) introduce the average value problem where the valuations are additive but they require that the average value of the bundles received by the agents meets a certain threshold.

B Omitted Proofs from Section 3

We now present the omitted proofs regarding balanced allocations, beginning with utilitarian welfare.

B.1 Utilitarian Welfare

Recall that we find that the problem of maximizing utilitarian welfare is APX-hard. Specifically, that is NP-hard to get an $O(\frac{m/n}{\log m/n})$ approximation. We now prove the correctness and running time for the greedy algorithm (Algorithm 1) giving a matching approximation guarantee.

Theorem 2. *Given an instance* $I = \langle N, M, v, \tau \rangle$ *with* $m = kn$ *and heterogeneous quantiles, Algorithm 1 returns a balanced allocation which is* $\min(\frac{m}{n} + 1, n)$-USW.

Proof. Given I, let $A^* = (A_1^*, ..., A_n^*)$ be a maximum USW balanced allocation. Without loss of generality, we assume that $v_1(A_1^*) \geq v_2(A_2^*) \geq \cdots \geq v_n(A_n^*)$. Let i_t denote the agent who is allocated a bundle in the t-th iteration of the while loop, and let A_{i_t} denote the corresponding bundle allocated to her under Algorithm 1.

Let $k_i = \min(k, k - \lceil \tau_i k \rceil + 1)$. That is, k_i is minimum number of items in any k sized bundle B s.t. $v_i(g) \geq v_i(B)$. Recall that under the greedy algorithm, agent i "demands" k_i items. Let $k' = \max_i k_i$. Observe that $1 \leq k' \leq k$. We shall now show that the first $\lceil \frac{n}{k'+1} \rceil$ agents to receive a bundle will have value comparable to the value under specific bundles under A^*.

Claim: For each $t = 1, \cdots, \lceil \frac{n}{k'+1} \rceil$, we have that the value of agent i_t, $v_{i_t}(A_{i_t}) \geq v_{(t-1)(k'+1)+1}(A^*_{(t-1)(k'+1)+1})$.

Proof of Claim. We shall prove this by induction. For i_1, as none of the items have been allocated, the best possible bundle A_{i_1} must be such that $v_{i_1}(A_{i_1}) \geq v_1(A_1^*)$.

Suppose we have that for all $t \leq \bar{t}-1$, the claim holds. Let $L = \bigcup_{\ell \in [\bar{t}-1]} A_{i_\ell}$ be the set of items that are allocated up to the $(\bar{t}-1)$th iteration of the while loop. In each iteration at most k' items are allocated. Consequently, we have that $|L| \leq k'(\bar{t}-1)$. It follows that in the worst case, the number of bundles under A^* for which some item has already be allocated in L is $|\{j \in [(\bar{t}-1)(k'+1)+1] : A_j^* \cap L \neq \emptyset\}| \leq |L| = k'(\bar{t}-1)$.

Consequently, we get that among the top $(\bar{t}-1)(k'+1)+1$ bundles under A^*, at least $\bar{t}$ bundles do not intersect with L. Thus far, $\bar{t}-1$ bundles have been allocated. Consequently, at least one bundle and agent pair among these $\bar{t}$ unallocated bundles must remain available for selection . Hence, we must have that $v_{i_{\bar{t}}}(A_{i_{\bar{t}}}) \geq v_{(t-1)(k'+1)+1}(A^*_{(t-1)(k'+1)+1})$. ☐

We can now prove the approximation guarantee. Let $\alpha = \min(k'+1, n)$. Observe that $\lceil \frac{n}{k'+1} \rceil = \lceil \frac{n}{\alpha} \rceil$. The USW of A is lower bounded by $\sum_{t=1}^{\lceil \frac{n}{k'+1} \rceil} v_{i_t}(A_{i_t})$. From the proof of the claim, we know that

$$\sum_{t=1}^{\lceil \frac{n}{\alpha} \rceil} v_{i_t}(A_{i_t}) \geq \sum_{t=1}^{\lceil \frac{n}{\alpha} \rceil} v_{(t-1)(k'+1)+1}(A^*_{(t-1)(k'+1)+1})$$

Recall that agents are ordered according to A^*, that is, $v_1(A_1^*) \geq \cdots \geq v_n(A^*)$. As a result, we get that

$$\begin{aligned} \mathrm{USW}(A^*) &= \sum_{t \in [n]} v_t(A_t^*) \\ &\geq \alpha \sum_{t=1}^{\lceil \frac{n}{k'+1} \rceil} v_{(t-1)(k'+1)+1}(A^*_{(t-1)(k'+1)+1}). \end{aligned}$$

From the claim, we know that this is greater than or equal to α times the value obtained by the first $\lceil \frac{n}{k'+1} \rceil$ agents under Algorithm 1. Hence, $\mathrm{USW}(A) \geq \frac{\mathrm{USW}(A^*)}{\alpha}$.

Observe that when each agent demands $k_i < n-1$ items, we are guaranteed $(k'+1)$-USW which may be even better than $(k+1)$-USW. However, when $k' \geq n-1$, the greedy algorithm can only guarantee n-USW. Consequently, for an arbitrary I, Algorithm 1 is $\min(\frac{m}{n}+1, n)$-USW. ☐

B.2 Egalitarian Welfare

In contrast to the previous case, we find that for ESW, we are able to provide an exact algorithm, that works even for heterogeneous quantiles. To this end, we first provide the proof of a very useful reduction.

Lemma 1. *The problem of finding an allocation with* ESW *at least $\nu \geq 0$ over allocations in $\Pi' \subseteq \Pi(n, M)$ under heterogeneous quantiles reduces to maximizing* ESW *over Π' under binary goods with heterogeneous quantiles.*

Proof. Consider an arbitrary instance $I = \langle N, M, v, \tau \rangle$ and $\Pi' \subseteq \Pi(n, M)$. Each agent can receive at most m distinct values, consequently, the ESW of an allocation can take at most mn different value. As a result, we can check for at most mn distinct threshold values for ν s.t. we wish to find an allocation in $A \in \Pi'$ where $\mathrm{ESW}(A) \geq \nu$.

Fix a value for the threshold ν. We can construct an alternate instance $I' = \langle N, M, v', \tau \rangle$ with binary valuations as follows: $v_i'(g) = 1$ if and only if $v_i(g) \geq \nu$. We can now show that an allocation $A \in \Pi'$ has $\mathrm{ESW}(A) \geq \nu$ under v if and only if $\mathrm{ESW}(A) = 1$ under v'.

Suppose we have an allocation $A \in \Pi'$ s.t. A has $\mathrm{ESW}(A) \geq \nu$ under v. Thus, for each $i \in N$, A_i must contain enough goods each with value at least ν so that $v_i(A_i) \geq \nu$. Thus, for the same quantile τ_i, it must be that $v_i'(A_i) \geq 1$. Consequently, $\mathrm{ESW}(A) \geq 1$ under v'.

Now, suppose we have an allocation $A \in \Pi'$ s.t. A has $\mathrm{ESW}(A) \geq 1$ under v'. We can analogously see that $v_i'(A_i) \geq 1$ if and only if $v_i(A_i) \geq \nu$. As a result, it must be that $v_i(A_i) \geq \nu$ for each $i \in N$, and thus, $\mathrm{ESW}(A) \geq \nu$ under v.

Hence, given an algorithm that finds a maximum ESW allocation for Π' under an instance with binary goods, we can make at most mn calls to it to find a maximum ESW allocation for Π' under arbitrary goods. ☐

Based on this previous result, we henceforth only consider instances with binary goods when maximizing ESW. For the space of balanced allocations, Algorithm 2 finds a maximum ESW allocation for agents with heterogeneous quantiles. We now prove the corrctness and running time of this algorithm.

Proposition 1. *Given $I = \langle N, M, v, \tau \rangle$ where $m = kn$ with binary goods and heterogeneous quantiles, Algorithm 2 finds a max* ESW *balanced allocation in polynomial time.*

Proof. For an instance with binary goods, the maximum ESW possible is 1. For a balanced allocation to have ESW of 1, from Observation 1 each $i \in N$ must receive at least $k_i = \min(k, k - \lceil \tau_i k \rceil + 1)$ items which give i value 1. Algorithm 2 checks if this is simultaneously possible for all $i \in N$. Let A be the allocation returned by Algorithm 2 on I. We shall now show that $\mathrm{ESW}(A) = 1$ whenever a balanced allocation with ESW 1 exists for I.

Suppose a matching of size $\sum_{i \in N} k_i$ exists in the graph G constructed in Algorithm 2. Consequently, A_i contains at least k_i items for which i has value 1. Thus, $\mathrm{ESW}(A) = 1$.

Conversely, let an allocation A' with $\mathrm{ESW}(A') = 1$ exist for the given instance. Then for each $i \in N$, A_i' must contain at least k_i items of value 1 for i. Let $g_i^1 \cdots g_i^{k_i} \in A_i'$ be distinct items s.t. $v_i(g_i^t) = 1$ for all $t \in [k_i]$. Consider $\mu' = \{(x_{g_i^t}, y_t^i) | t \in [k_i]\}$. Clearly, $|\mu'| = \sum_i k_i$. That is, a maximum cardinality matching of the required size must exist. As a result, the allocation returned by A is s.t. $\mathrm{ESW}(A) = 1$.

Running Time. The bipartite graph has $\sum_i k_i \leq kn = m$ vertices in X and m vertices in Y and thus, at most m^2 edges. A maximum cardinality matching on a bipartite graph can be found in polynomial time using the Ford-Fulkerson Algorithm. As a result, Algorithm 2 runs in polynomial time. ☐

C Omitted Proofs from Section 4

We now move to the case of unconstrained allocations, again beginning with utilitarian welfare.

C.1 Utilitarian Welfare

In contrast to the balanced case, for unconstrained allocation, while maximizing USW remains intractable, it is nolonger inapproximable.

Intractability

Theorem 4. *Given instance $I = \langle N, M, v, \tau\rangle$ with goods finding a maximum* USW *allocation is NP-complete.*

Proof. It is straightforward to see that this problem is in NP. Given instance I and value α, such that there exists an allocation A with USW at least α, it can be checked in polynomial time that A has USW at least α.

We give a reduction from the well known EXACT3COVER problem, which is known to be NP-hard (Garey and Johnson 1979). Under the EXACT3COVER problem we are given a universe of $3t$ elements $\mathcal{U}$ and a family of sets $\mathcal{S} = \{S_1, \cdots, S_\ell\}$ s.t. for each $j \in [t]$, $S_j \subset \mathcal{U}$ and $|S_j| = 3$. The aim is to find t mutually disjoint sets $S_{j_1}, \cdots, S_{j_t}$ that cover the given set of elements, i.e., $\bigcup_{p\in[\ell]} S_{j_p} = \mathcal{U}$.

Given an instance of EXACT3COVER $\langle \mathcal{U}, \mathcal{S}\rangle$, we construct an instance of our problem with ℓ agents and $2t + \ell$ items as follows: For each set S_j, create an agent i_j. For each element $e \in \mathcal{U}$, create an item g_e. Create an additional set of prized $\ell - t$ items $g'_1, \cdots, g'_{\ell-t}$.

For each agent i, set the quantile value $\tau_i = 0$. Set the agent values as follows: for any $i \in N$, we set i's value for a prized item g' as $v_i(g') = 2$ and for each $j \in [t]$ we set $v_{i_j}(g_e) = 1$ if $e \in S_j$ and $v_{i_j}(g_e) = 0$ otherwise.

We can now show that $\langle \mathcal{U}, \mathcal{S}\rangle$ has an EXACT3COVER if and only if there exists an allocation $A = (A_1, \cdots, A_n)$ under the constructed instance $\langle N, M, v\rangle$ with USW at least $2\ell - t$.

First assume that an Exact 3-Cover does exist, and that it is, without loss of generality, $S_1, \cdots, S_k$. Consider the allocation A where each agent corresponding a set in the exact 3 cover receives the items corresponding to its constituent elements and the remaining agents get one prized item each. That is, for $j \in [t]$, $A_{i_j} = \{g_e | e \in S_j\}$ and for $j \in [\ell] \setminus [t]$, $A_{i_j} = \{g'_{j-t}\}$. Observe that, here agents $i_1, \cdots, i_t$ each get a value of 1 while the remaining $\ell - t$ agents receive value 2, making the USW of A to be $2\ell - t$.

Now let there not exist an exact 3-cover. Then, even if we assign each item corresponding to an element g_e to a set containing it, we will need to give these elements to at least $t+1$ distinct agents, in which case at least $t+1$ agents receive value 1 and at most $\ell - t - 1$ agents receive value 2.

If we were to assign an element item to an agent who does not contain it, they would get value 0. Thus, in either case the optimal USW cannot be more than $2\ell - t - 1$. □

Near Exact Algorithm We complement the intractability result, by providing a near exact approximation algorithm which we call the scapegoat algorithm (Algorithm 3). We now prove its correctness and running time.

Theorem 5. *Given instance $I = \langle N, M, v, \tau\rangle$ with heterogeneous quantiles, scapegoat algorithm (Algorithm 3) returns an $(1 + \frac{1}{n-1})$-*USW *allocation in polynomial time.*

Proof. Given I, let A^i and A be as in Algorithm 3 when run on I. Let A^* be a maximum USW allocation for I. Further, let $i^* \in N$ be such that $A = A^i$.

By definition of quantile valuations, for each $j \in N$, there is some $g_j \in A^*_j$ s.t. $v_j(A^*_j) = v_j(g_j)$. Without loss of generality we can assume that $v_1(A^*_1) \geq v_2(A^*_2) \geq \cdots \geq v_n(A^*_n)$. As a result, $\frac{n-1}{n}\text{USW}(A^*) \leq \sum_{j=1}^{n-1} v_j(A^*_j)$.

Now, as A^i has maximum USW over all the allocations constructed, it is straightforward to see that its USW must be at least the weight of the best max weight matching constructed under Algorithm 3. This matching in turn must have weight at least $\sum_{j=1}^{n-1} v_j(g_j) = \sum_{j=1}^{n-1} v_j(A^*_j)$. Thus, we get that

$$\text{USW}(A) \geq \sum_{j=1}^{n-1} v_j(g_j) \geq \frac{n-1}{n}\text{USW}(A^*).$$

Hence, A is $(1 + \frac{1}{n-1})$-USW.

Note that since the maximum weight matching can be computed in $O(nm)$ time, and such a matching is computed n times, our algorithm terminates in $O(mn^2)$ time. □

Building on this approach, whenever even one agent is optimistic, that is, for at least one agent $\tau_i = 1$, we can make this agent a scapegoat and exactly maximize utilitarian social welfare.

Proposition 2. *Given instance $I = \langle N, M, v, \tau\rangle$ with heterogeneous quantiles and an agent i^* such that $\tau_{i^*} = 1$, a maximum* USW *allocation can be found in polynomial time.*

Proof. Given I, let A^* be a maximum USW allocation. For each $j \in N$, let $g_j \in A^*_j$ be such that $v_j(A^*_j) = v_j(g_j)$.

Consider a maximum weight matching μ on the bipartite graph with *all* agents and all items. Observe that the weight of μ is at least $\sum_j v_j(A^*_j) = \text{USW}(A)$. Now define allocation A to be such that for all $j \neq i^*$, they are allocated only their matched item under μ. Agent i^* is allocated the matched item under μ along with all remaining items.

Clearly $\text{USW}(A)$ is the weight of μ. Hence, A has maximum USW. □

C.2 Egalitarian Welfare

For the unconstrained case, we find that the quantiles for which maximizing ESW is tractable and those for which it is intractable, interlace. We first present the correctness of our exact algorithms.

Exact Algorithms We first wish to prove the correctness and running time of our algorithm for $\tau = \frac{t}{t+1}$ (Algorithm 4). To this end, we first prove a useful lemma for this class of quantiles.

Lemma 2. *For an agent $i \in N$ with $\tau_i = \frac{t}{t+1}$, where $t \in \mathbb{Z}_+$ is fixed, a bundle $B \subseteq M$ with exactly ℓ goods of value 1 for i, we have that $v_i(B) = 1$ if and only if the number of 0 valued items in B for i is at most $\ell t - 1$.*

Proof. Given agent i and bundle B with exactly ℓ goods of value 1 for i. Observe that it is sufficient to compare the case when there are either exactly $\ell t - 1$ items of value 0 or ℓt items of value 0.

Suppose the number of items of value 0 is ℓt. The value of agent i for B would be from the $\lceil(\ell t + \ell)\frac{t}{t+1}\rceil = \ell t$'th lowest valued item, which would have value 0. Consequently $v_i(B) = 0$.

On the other hand, if B contained $\ell t - 1$ items of value 0, then i's value would come from the item which has the pth lowest value where

$$\begin{aligned} p &= \left\lceil(\ell t - 1 + \ell)\frac{t}{t+1}\right\rceil \\ &= \left\lceil\frac{\ell t^2 + \ell t - t}{t+1}\right\rceil \\ &= \left\lceil(\ell t - \frac{t}{t+1})\right\rceil \\ &= \ell t && (\text{As } \tfrac{t}{t+1} < 1.) \end{aligned}$$

As a result, when there are at most $\ell t - 1$ items of value 0, $v_i(B) = 1$. ☐

We can now prove the running time and correctness of Algorithm 4. This algorithm first matches each agent to one item they like, and then chooses up to t items all agents have value 0 for, and matches them to an agent along with an item they like, if possible. We show that these checks are necessary and sufficient whenever $\tau = \frac{t}{t+1}$.

Proposition 3. *Given instance $I = \langle N, M, v, \tau = \frac{t}{t+1}\rangle$ where $t \in \mathbb{Z}_+$, Algorithm 4 returns a maximum* ESW *allocation in polynomial time.*

Proof. We now show that given an instance with binary goods $I = \langle N, M, v, \tau = \frac{t}{t+1}\rangle$, Algorithm 4 finds an allocation with ESW of 1 whenever it exists.

Let μ, M_0 and M_1 be as defined in Algorithm 4 by step 4. We shall now show that whenever an allocation of ESW 1 exists, Algorithm 4 will return an allocation with ESW 1. We first show that when $|\mu| = n$ and $|M_0| \leq t|M_1| - n$, Algorithm 4 creates an allocation where if agent $i \in N$ receives $\ell_i > 1$ items of value 1 then they receive at most $t\ell_i - 1$ items of value 0. We have that $\ell_i > 1$ as μ matches each agent to an item of value 1.

Further, in the while loop, whenever i receives at most t 0 valued items from M_0, they are accompanied with one item of value 1. After the while loop, an additional $t - 1$ items of value 0 may be allocated to i. As a result, $v_i(A_i) = 1$ in this case.

Consequently, when $|\mu| = n$ and $|M_0| \leq t|M_1| - n$, we have that Algorithm 4 finds an allocation with $\text{ESW}(A) = 1$.

Conversely, assume that an allocation A^* exists s.t. $\text{ESW}(A^*) = 1$. Now, clearly each agent i must receive at least one good of value 1, thus we have that $|\mu| = n$.

Now let $A^*_{i,0}$ and $A^*_{i,1}$ respectively denote the 0 and 1 valued items i is allocated under A^*. We have that $M_0 \subseteq \cup_{i\in N} A^*_{i,0}$ and $\cup_{i\in N} A^*_{i,1} \leq M_1$.

ALGORITHM 5: Max ESW binary goods for $\tau = 1/3$

Input: $I = \langle N, M, v, \tau\rangle$ with binary goods and $\tau = 1/3$
Output: An allocation A

Let $M_0 = \{g \in M | v_i(g) = 0 \text{ for all } i \in N\}$;
Let $M_1 = M \setminus M_0$;
Create graph $G = (X \cup Y, E)$ where X contains x_g for each $g \in M_1$, Y contains y_i for each $i \in N$ and $(x_g, y_i) \in E$ only if $v_i(g) = 1$ and $(x_g, x_{g'}) \in E$ only if there exists $i \in N$ s.t. $v_i(g) = v_i(g') = 1$;
Define edge weight function w where $w(x, y) = |X \cup Y| + 1$ and $w(y, y') = 1$;
Let μ be a maximum weight matching in G_2;
if $w(\mu) < |M_0| + n(|X \cup Y| + 1)$ **then**
    Let A be an arbitrary allocation;
else
    Let $A = (A_1, \cdots, A_n)$ be such that $A_i = \{g | (x_g, y_i) \in \mu\}$;
    $M_1 \leftarrow M_1 \setminus \cup_i A_i$;
    while $M_0 \neq \emptyset$ **do**
        Arbitrarily pick $g_0 \in M_0$ and g, g' s.t. $(x_g, x_{g'}) \in \mu$;
        Pick $i \in N$ s.t. $v_i(g) = v_i(g') = 1$;
        $A_i \leftarrow A_i \cup \{g_0, g, g'\}$;
        $M_1 \leftarrow M_1 \setminus \{g, g'\}$ and $M_0 \leftarrow M_0 \setminus \{g_0\}$;
    if $M_1 \neq \emptyset$ **then**
        Let $B_1 \cdots B_n$ be an arbitrary partition of M_1 s.t. $g \in |B_i|$ only if $v_i(g) = 1$;
    For each $i \in N$, set $A_i \leftarrow A_i \cup B_i$;
Return A;

As $v_i(A^*_i) = 1$, from Lemma 2, we have that $|A^*_{i,0}| \leq t|A^*_{i,1}| - 1$. Consequently, we have that

$$|M_0| \leq \sum_i |A^*_{i,0}| \leq \sum_i t|A^*_{i,1}| - 1 \leq t|M_1| - n$$

Hence, we have that the necessary conditions will be satisfied and Algorithm 4 will return an allocation of ESW 1. As a result, Algorithm 4 will return an allocation with ESW 1 if and only if one exists. ☐

$1/3$ quantile. We now consider the case where $\tau = 1/3$. To this end, we begin with the following simple observation, analogous to Lemma 2.

Observation 4. *For an agent $i \in N$ with $\tau_i = 1/3$, a bundle $B \subseteq M$ with exactly ℓ items of value 0 for i, we have that $v_i(B) = 1$ if and only if the number of 1 valued items in B is at least $2\ell + 1$.*

Thus, when M_0 and M_1 are as in Algorithm 4, we need two items from M_1 to offset one from M_0. We can now build Algorithm 5 where we need to check if we can satisfy both Observations 2 and 4.

Proposition 4. *Given $I = \langle N, M, v, \tau = 1/3\rangle$, Algorithm 5 returns a maximum* ESW *allocation in polynomial time.*

Proof. We now show that given an instance with binary goods $I = \langle N, M, v, \tau = 1/3\rangle$, Algorithm 5 finds an allocation with ESW of 1 whenever it exists.

Let μ, M_0 and M_1 be as initially defined in Algorithm 5. We shall now show that whenever an allocation of ESW 1 exists, Algorithm 4 will return an allocation with ESW 1. We first show that when $w(\mu) \geq |M_0| + n(|X \cup Y| + 1)$, Algorithm 5 creates an allocation where if agent $i \in N$ receives ℓ_i items of value 0, then they receive at least $2\ell_i + 1$ items of value 1.

First, μ matches each agent to one item of value 1, so for agents with $\ell_i = 0$, the requirement is satisfied. Further, in the while loop, whenever i receives two items of value 1 for every item from M_0. After the while loop, only items of value 1 may be allocated to i. As a result, $v_i(A_i) = 1$ in this case.

Consequently, when $w(\mu) \geq |M_0| + n(|X \cup Y| + 1)$, we have that Algorithm 5 finds an allocation with $\text{ESW}(A) = 1$.

Conversely, assume that an allocation A^* exists s.t. $\text{ESW}(A^*) = 1$. Now let $A^*_{i,0}$ and $A^*_{i,1}$ respectively denote the 0 and 1 valued items i is allocated under A^*. We have that $M_0 \subseteq \cup_{i \in N} A^*_{i,0}$ and $\cup_{i \in N} A^*_{i,1} \leq M_1$.

As $v_i(A^*_i) = 1$, from Lemma 2, we have that $|A^*_{i,1}| \geq 2|A^*_{i,0}| + 1$. Consequently, we have can build a matching μ' in in G_2 matching $|A^*_{i,0}|$ pairs of items from $A^*_{i,1}$ to each other and one additional item to i. Now as μ_2 is a maximum weight matching in G_2, it must have weight at least

$$\begin{aligned} w(\mu_2) &\geq w(\mu') \\ &\geq \sum_i (|A^*_{i,0}| + |X \cup Y| + 1) \\ &= n(|X \cup Y| + 1) + \sum_i |A^*_{i,0}| \\ &\geq n(|X \cup Y| + 1) + |M_0|. \end{aligned}$$

Hence, we have that the necessary condition will be satisfied and Algorithm 4 will return an allocation of ESW 1. As a result, Algorithm 4 will return an allocation with ESW 1 if and only if one exists. □

Intractability

Lemma 4. *Given an instance with binary goods $I = \langle N, M, v, \tau \in (1/5, 1/4] \rangle$, maximizing* ESW *is NP-hard.*

Proof. We shall give a reduction from EXACT3COVER (X3C) [3]. Given an instance of X3C $\langle \mathcal{U}, \mathcal{S} \rangle$ where $|\mathcal{U}| = 3t$ and $|\mathcal{S}| = \ell$, we shall create an instance of our problem as follows:

For each $S_j \in \mathcal{S}$, create a set agent i_j and a set item g_j.

For each element $u \in \mathcal{U}$, we create an element item g_u.

Create t dummy items $g'_1, \cdots, g'_t$.

As a result, we have created ℓ agents and $\ell + 4t$ items. We define agent preferences as follows. For any $i_j \in N$, $v_{i_j}(g') = 0$ for any dummy item g'. Further, for any $u \in \mathcal{U}$, if $u \in S_j$, set $v_{i_j}(g_u) = 1$, otherwise set $v_{i_j}(g_u) = 0$. Finally, for any $j' \in [\ell]$, set $v_{i_j}(g_{j'}) = 1$ if $j = j'$ otherwise, set $v_{i_j}(g_{j'}) = 0$. Lastly, choose τ_i arbitrarily from the range $\in (1/5, 1/4]$, for all $i \in N$.

[3] See the proof of Theorem 4 for a definition of the exact 3 cover problem

We shall now show that an allocation with ESW of 1 exists if and only if the given X3C instance has an exact 3-cover.

First, assume that $S_{j_1}, \cdots, S_{j_t}$ form an exact 3 cover of $\langle \mathcal{U}, \mathcal{S} \rangle$. Consider the following allocation A

$$A_{i_j} = \begin{cases} \{g_j\} & \text{if } j \notin \{j_1 \cdots, j_t\} \\ \{g_j, g'_p\} \cup \{g_u | u \in S_j\} & \text{if } j = j_p \text{ for some } p \in [t] \end{cases}$$

That is, agents corresponding to sets in the exact 3 cover receive their set item, one dummy item and constituent items. Agents corresponding to sets not in the exact 3 cover only receive their corresponding set item. Firstly, observe that as $S_{j_1}, \cdots, S_{j_t}$ form an exact 3 cover of $\langle \mathcal{U}, \mathcal{S} \rangle$, A must be a valid allocation, where all items are allocated, and the bundles of agents are disjoint. Now, for any agent i_j, where $j \notin \{j_1 \cdots, j_t\}$, we have that $A_{i_j} = \{g_j\}$, thus, $v_{i_j}(A_{i_j}) = 1$.

Further, for $j \in \{j_1 \cdots, j_t\}$, A_{i_j} contains one item of value 0, the dummy item g' and 4 items of value 1, the set item and element items. As, $\tau \in (1/5, 1/4]$, we have that $v_{i_j}(A_{i_j}) = 1$. Consequently, $\text{ESW}(A) = 1$.

Now, conversely, assume that an allocation A^* exists s.t. $\text{ESW}(A^*) = 1$. That is, for each $i \in N$, $v_i(A^*_i) = 1$.

Observe that for each agent there are exactly four items of value 1: the corresponding set item and constituent element items. As $\tau_i \in (1/5, 1/4]$, A^*_i can contain at most 1 item of value 0 for i. If it does contain one item of value 0, all four of the value 1 items must also be contained in order to ensure $v_i(A^*_i) = 1$.

In particular, as there are t dummy items for which each agent has value 0, each agent can be allocated at most one dummy item under A^*. Let the set of agents who receive one dummy item be $i_{j_1}, \cdots, i_{j_t}$. Further, each for $j \in \{j_1, \cdots, j_t\}$, we have that A_{i_j} must contain g_j and all three items in $\{g_u | u \in S_j\}$. As a result, $S_{j_1}, \cdots, S_{j_t}$ must be mutually disjoint. Consequently, $S_{j_1}, \cdots, S_{j_t}$ form an exact 3 cover of $\langle \mathcal{U}, \mathcal{S} \rangle$.

Hence, the problem of finding a maximum ESW allocation is NP-hard for binary goods and $\tau \in (1/5, 1/4]$. □

We can do an analogous reduction from the t-dimensional matching problem for $t \geq 3$, to an instance with binary goods and $\tau \in (\frac{1}{t+2}, \frac{1}{t+1})$ where 1 item of value 0 needs to be offset by t items of value 1 to ensure that the bundle has value 1 for the corresponding agent.

Corollary 1. *Given $I = \langle N, M, v, \tau \in (0, 1/4] \rangle$ with binary goods, maximizing* ESW *is NP-hard.*

Intractability with $\tau \in (3/8, 2/5]$. The main source of intractability in this range of quantiles comes from differing number of value 1 items that can are needed to offset an additional item of value 0. Considering only bundles that give value 1 to an agent, with four items of value 1, there can be at most two items of value 0. However with five items of value 1 there can be at most three items of value 0. Thus when the number items which give value 0 is strictly greater than the number of items that give value 1 to at least one agent, deciding if an ESW 1 allocation may not be possible with polynomially many greedy decisions.

Lemma 5. *Finding a maximum* ESW *allocation is NP-hard for* $\tau \in (3/8, 2/5]$.

Proof. We shall give a reduction from EXACT3COVER (X3C). Given an instance of X3C $\langle \mathcal{U}, \mathcal{S} \rangle$ where $|\mathcal{U}| = 3t$ and $|\mathcal{S}| = \ell$, we shall create an instance of our problem as follows:

- For each $S_j \in \mathcal{S}$, we create a set agent i_j and a set items g_j^1 and g_j^2.
- For each element $u \in \mathcal{U}$, we create an element item g_u.
- Create $\ell + 2t$ dummy items $g'_1, \cdots, g'_{\ell+2t}$.

As a result, we have created $n = \ell$ agents and $m = 3\ell + 5t$ items. We define agent preferences as follows. For any $i_j \in N$, $v_{i_j}(g') = 0$ for any dummy item g'. Further, for any $u \in \mathcal{U}$, if $u \in S_j$, set $v_{i_j}(g_u) = 1$, otherwise set $v_{i_j}(g_u) = 0$. Finally, for any $j' \in [\ell]$, set $v_{i_j}(g^1_{j'}) = v_{i_j}(g^2_{j'}) = 1$ if $j = j'$ otherwise, set $v_{i_j}(g^1_{j'}) = v_{i_j}(g^2_{j'}) = 0$. Lastly, set $\tau_i = 2/5$, for all $i \in N$.

We shall now show that an allocation with ESW of 1 exists if and only if the given X3C instance has an exact 3-cover.

First, assume that $S_{j_1}, \cdots, S_{j_t}$ form an exact 3 cover of $\langle \mathcal{U}, \mathcal{S} \rangle$. Consider the following allocation A where $A_{i_j} = \{g_j^1, g_j^2, g'_j\}$ if $j \notin \{j_1 \cdots, j_t\}$ and $A_{i_j} = \{g_j^1, g_j^2, g'_j, g'_{\ell+2p-1}, g'_{\ell+2p}\} \cup \{g_u | u \in S_j\}$ if $j = j_p$ for some $p \in [t]$

That is, agents corresponding to sets in the exact 3 cover receive their set items, three dummy items and their constituent items. Agents corresponding to sets not in the exact 3 cover only receive their corresponding set items and one dummy item. Firstly, observe that as $S_{j_1}, \cdots, S_{j_t}$ form an exact 3 cover of $\langle \mathcal{U}, \mathcal{S} \rangle$, A must be a valid allocation, where all items are allocated, and the bundles of agents are disjoint. Now, for any agent i_j, where $j \notin \{j_1 \cdots, j_t\}$, we have that $A_{i_j} = \{g_j^1, g_j^2, g'_j\}$. As $\tau_i > 1/3$, we have that, $v_{i_j}(A_{i_j}) = 1$.

Further, for $j \in \{j_1 \cdots, j_t\}$, A_{i_j} contains three items of value 0, the dummy items $g'_j, g'_{\ell+2p-1}, g'_{\ell+2p}$ and five items of value 1, the set items and element items. As, $\tau_i > 3/8$, we have that $v_{i_j}(A_{i_j}) = 1$. Consequently, ESW$(A) = 1$.

Now, conversely, assume that an Exact 3 Cover does not exist.

Observe that for each agent there are exactly five items of value 1: the corresponding set items and constituent element items. This along with the fact that $\tau_i \leq 2/5$ implies that any bundle of value 1 for i can contain at most three items of value 0 for i. If it does contain three items of value 0, all five of the value 1 items must also be contained to ensure the bundle has value 1.

Recall that there are $\ell + 2t$ dummy items for which each agent has value 0. These items can only be offset by 2ℓ set items and $3t$ element items. Further, as for all agents $\tau_i \in (3/8, 2/5]$, we have that if an agent received $p < 5$ items of value 1, they must have at most $\lfloor p/2 \rfloor$ items of value 0.

Consequently, in order to offset all the dummy items, we must have at least t agents who each receive three dummy items and their corresponding set item and constituent items. Now as no exact 3 cover exists, at most $t-1$ (set) agents can receive all three constituent element items. Thus, no allocation exists with an ESW of 1. □

Intractability with $\tau \in (5/9, 3/5]$**.** The main source of intractability in this range of quantiles comes from differing number of value 0 items that can be added with an additional item of value 1. Considering only bundles that give value 1 to an agent, with three items of value 1, there can be at most three items of value 0. However with four items of value 1 there can be at most five items of value 0. Thus when the number items which give value 0 is strictly greater than the number of items that give value 1 to at least one agent, deciding if an ESW 1 allocation may not be possible with polynomially many greedy decisions.

Lemma 6. *Finding a maximum* ESW *allocation is NP-hard for* $\tau \in (5/9, 3/5]$.

Proof. We shall give a reduction from EXACT3COVER (X3C). Given an instance of X3C $\langle \mathcal{U}, \mathcal{S} \rangle$ where $|\mathcal{U}| = 3t$ and $|\mathcal{S}| = \ell$, we shall create an instance of our problem as follows:

For each $S_j \in \mathcal{S}$, we create a set agent i_j and a set item g_j.

For each element $u \in \mathcal{U}$, we create an element item g_u.

Create $\ell + 4t$ dummy items $g'_1, \cdots, g'_{\ell+4t}$.

Thus, we have created ℓ agents and $2\ell + 7t$ items. We define agent preferences as follows. For any $i_j \in N$, $v_{i_j}(g') = 0$ for any dummy item g'. Further, for any element $u \in \mathcal{U}$, if $u \in S_j$, set $v_{i_j}(g_u) = 1$, otherwise set $v_{i_j}(g_u) = 0$. Finally, for any $j' \in [\ell]$, set value for set item $g_{j'}$ $v_{i_j}(g_{j'}) = 1$ if $j = j'$ otherwise, set $v_{i_j}(g_{j'}) = 0$. Lastly, arbitrarily set $\tau_i \in (5/9, 3/5]$, for all $i \in N$.

We shall now show that an allocation with ESW of 1 exists if and only if the given X3C instance has an exact 3-cover.

First, assume that $S_{j_1}, \cdots, S_{j_t}$ form an exact 3 cover of $\langle \mathcal{U}, \mathcal{S} \rangle$. Consider the following allocation A where $A_{i_j} = \{g_j, g'_j\}$ if $j \notin \{j_1 \cdots, j_t\}$ and $A_{i_j} = \{g_j, g'_j, g'_{\ell+4p-3}, g'_{\ell+4p-2}, g'_{\ell+4p-1}, g'_{\ell+4p}\} \cup \{g_u | u \in S_j\}$ if $j = j_p$ for some $p \in [t]$

That is, agents corresponding to sets in the exact 3 cover receive their set item, five dummy items and their constituent items. Agents corresponding to sets not in the exact 3 cover only receive their corresponding set item and one dummy item. Firstly, observe that as $S_{j_1}, \cdots, S_{j_t}$ form an exact 3 cover of $\langle \mathcal{U}, \mathcal{S} \rangle$, A must be a valid allocation, where all items are allocated, and the bundles of agents are disjoint. Now, for any agent i_j, where $j \notin \{j_1 \cdots, j_t\}$, we have that $A_{i_j} = \{g_j, g'_j\}$. As $\tau_i > 0.5$, we have that, $v_{i_j}(A_{i_j}) = 1$.

Further, for $j \in \{j_1 \cdots, j_t\}$, A_{i_j} contains five items of value 0, the dummy items g'_j, $g'_{\ell+4p-3}$, $g'_{\ell+4p-2}$, $g'_{\ell+4p-1}, g'_{\ell+4p}$ and 4 items of value 1, the set item and element items. As, $\tau_i > 5/9$, we have that $v_{i_j}(A_{i_j}) = 1$. Consequently, ESW$(A) = 1$.

Now, conversely, assume that an Exact 3 Cover does not exist.

Observe that for each agent there are exactly four items of value 1: the corresponding set item and constituent element items. This along with the fact that $\tau_i \leq 3/5$ implies that any bundle of value 1 for i can contain at most 5 items of value 0 for i. If it does contain five items of value 0, all four of the value 1 items must also be contained to ensure the bundle has value 1.

Recall that there are $\ell + 4t$ dummy items for which each agent has value 0. These items can only be offset by d set items and $3t$ element items. Clearly there are fewer items that can give an agent value 1 than the number of items that give all agents value 0. Further, as for all agents $\tau_i \in (5/9, 3/5]$, we have that if an agent received $p < 4$ items of value 1, they must have at most p items of value 0.

Consequently, in order to offset all the dummy items, we must have at least t agents who each receive 5 dummy items and their corresponding set item and constituent items. Now as no exact 3 cover exists, at most $t-1$ (set) agents can receive all three constituent element items. Thus, no allocation exists with an ESW of 1. □

D Omitted Proofs from Section 5

We now present the omitted proofs on identical valuations. Here we find exact algorithms for all cases studied.

D.1 Utilitarian Welfare

The problem of maximizing USW over unconstrained allocations with identical valuations remains open. However, for balanced allocations we find that the greedy algorithm which gave an approximation to the optimal USW for the non-identical case, proves to be an exact algorithm for the identical case.

Theorem 8. *Given an instance with identical valuations $I = \langle N, M, v, \tau\rangle$, Algorithm 1 returns a balanced allocation with maximum* USW.

Proof. Given I with identical valuation v, let the items be such that $v(g_1) \geq v(g_2) \geq \cdots \geq v(g_m)$. Let A be the allocation returned by Algorithm 1. Furthermore, let k_i be as defined in Algorithm 1. Since the quantiles are identical, we have that $k_j = \min(k, k - \lceil \tau k \rceil + 1)$ for all $j \in N$. Let $k' = \min(k, k - \lceil \tau k\rceil + 1)$.

Let the order in which agents are first assigned their demanded set under Algorithm 1 be $i_1, i_2, \cdots i_n$. Without loss of generality, we may assume that the set demanded by agent i_t is $S_{i_t} = \{g_{(t-1)k'+1}, \cdots, g_{tk'}\}$. Thus, $v(A_{i_t}) = v(g_{tk'})$ for each $t \in [n]$.

Let A^* be a balanced allocation with maximum USW. As agents have identical valuations, we assume without loss of generality that $v(A^*_{i_1}) \geq v(A^*_{i_2}) \geq \cdots \geq v(A^*_{i_n})$.

We now show that $v(A_{i_t}) \geq v(A^*_{i_t})$ for all $t \in [n]$. Suppose, for the sake of contradiction, that there exists $\ell \in [n]$ such that $v(A_{i_\ell}) < v(A^*_{i_\ell})$. It follows that, we have $v(A^*_{i_\ell}) > v(g_{\ell k'})$. Since $v(A^*_{i_1}) \geq \cdots \geq v(A^*_{i_{\ell-1}}) \geq v(A^*_{i_\ell})$, we see that in A^* the number of agents who get value strictly higher than $v(g_{\ell k'})$ is at least ℓ. However, as $|A^*_j| = k$ for all $j \in N$, this implies that the number of items value strictly more that $v(g_{\ell k'})$ is at least $k'\ell$, contradicting the fact that number of items that are valued strictly more than $v(g_{\ell k'})$ is at most $\ell k' - 1$ by the item ordering. Consequently, for each $t \in [n]$, we must have that $v(A_{i_t}) \geq v(A^*_{i_t})$. Hence, $\text{USW}(A) \geq \text{USW}(A^*)$. □

D.2 Egalitarian Welfare

The balanced case is already resolved for all valuations, identical or otherwise by Algorithm 2. We now provide an algorithm for the unconstrained case based on Lemma 3 which shows that it is sufficient to consider allocations where the number of items with value 0 are allocated as equally as possible across the agents.

Theorem 9. *Given instance $I = \langle N, M, v, \tau\rangle$ with identical valuations, an allocation with maximum* ESW *can be found in polynomial time.*

Proof. Given I, as a consequence of Lemma 1, we assume I is an instance with binary goods.

First, consider the set of items of value 0, that is, $M_0 = \{g \in M | v(g) = 0\}$. From Lemma 3, we know that it is sufficient to consider only allocations where items in M_0 are distributed uniformly. Let $t = |M_0| - n\lfloor \frac{|M_0|}{n} \rfloor$. That is, t is the number of agents who need to receive more than $\lfloor \frac{|M_0|}{n} \rfloor$ items from M_0.

Now, consider the set of items of items of value 1, that is $M_1 = M \setminus M_0$. If an agent receives ℓ items from M_0, by Observation 3, we can easily calculate the minimum bundle size k s.t. $\lceil \tau k \rceil > \ell$. In particular, $k = \min\{k' \in \mathbb{Z}_+ | k > \frac{\ell}{\tau}\}$. Consequently, let $k_1 = \min\{k' \in \mathbb{Z}_+ | k > \frac{\lfloor |M_0|/n \rfloor}{\tau}\} - \lfloor |M_0|/n \rfloor$ and $k_2 = \min\{k' \in \mathbb{Z}_+ | k > \frac{\lceil |M_0|/n \rceil}{\tau}\} - \lceil |M_0|/n \rceil$.

As a result, an agent receiving $\lfloor |M_0|/n \rfloor$ items from M_0 requires at least k_1 items from M_1 to have value 1. Analogously, an agent receiving $\lceil |M_0|/n \rceil$ items from M_0 requires at least k_2 items from M_1 to have value 1. Thus, an allocation of ESW 1 exists, if and only if $|M_1| \geq tk_2 + (n-t)k_1$.

This can easily be checked and an appropriate allocation can accordingly be built. Thus, a maximum ESW allocation can be found in polynomial time. □

Maximum USW for binary goods via ESW. We now show that our algorithm for maximum ESW implies an algorithm for maximum USW under binary goods.

Proposition 5. *Given instance $I = \langle N, M, v, \tau\rangle$ with identical valuations over binary goods, an allocation with maximum* USW *can be found in polynomial time.*

Proof. Let M_0 and M_1 be the sets of items of value 0 and 1, respectively.

Observe that with binary goods, an allocation with USW n exists, if and only if an allocation with ESW 1 exists. Thus, we first find a maximum ESW allocation. If it has ESW 1, it must have USW n.

If an ESW 1 allocation does not exist, we can check if $|M_1| > n-1$. If so we can give $n-1$ agents exactly one item each from M_1 and give all remaining items in M_1 and M_0 to the remaining agent.

Finally, if $|M_1| \leq n-1$, we can give $|M_1|$ agents one item each from M_1 and the items in M_0 are distributed arbitrarily among the remaining agents. In this case, no agent can get an item from both M_1 and M_0.

It is easy to see that this can be done in polynomial time. $\square$

E Chores

We now turn our attention to the case of *chores*. Here, all items give all agents non-positive values. Recall that for chores, we capture agent preferences via *disutilities* and aim to minimize utilitarian and egalitarian social *costs*.

E.1 Utilitarian Social Cost

Balanced Allocations. Under balanced allocations, we had seen in Section 3.1 that maximum USW is hard to approximate. While the same reduction does not extend well to the case, of chores, it can be seen from a simple reduction from 3DIMENSIONALMATCHING (3DM) that the problem is NP-hard.

Theorem 10. *Given instance $I = \langle N, M, d, \tau\rangle$ with chores, it is NP-hard to obtain a minimum* USC *balanced allocation.*

Proof. Given an instance of 3DM, $\langle G = (X, H), \ell\rangle$, we create an instance of our problem as follows: For each edge $H_i \in H$, we create agent i. For each vertex $x \in X$, we create item g_x. We can assume, without loss of generality, that each vertex is contained in at least one hyper-edge. Thus, we have that $|X| \leq 3|H|$. To balance the item count, we introduce $3|H| - |X|$ dummy items $g'_1, \cdots, g'_{3n-|X|}$. Thus, we have $n = |H|$ agents and the number of items is $m = 3|H|$. As a result, we have that $m = 3n$. Recall that balanced allocations require $k = \frac{m}{n}$ items to be allocated to each agent.

For each agent $i \in N$, we set $\tau_i = 0$ for all $i \in N$. Now for i and each g_x, if $x \in H_i$, we set $d_i(g_x) = 0$ else, we set $d_i(g_x) = 1$. Finally, for each $t \in [3|H| - |X|]$, set $d_i(g'_t) = 1$. We now show that a matching of size ℓ in the 3DM problem can be transformed into a balanced allocation whose USC is at most $n - \ell$ in the reduced instance of our problem, and vice versa. Consider a matching μ of size ℓ in 3DM. For each $H_i \in \mu$, allocate the items vertices in H_i. That is, $A_i = \{g_x | x \in H_i\}$. Arbitrarily allocate the remaining items, ensuring $|A_i| = 3$. Here, for each $H_i \in \mu$, $d_i(A_i) = 0$ otherwise $d_i(A_i) = 1$.

Now consider a balanced allocation A in the reduced instance with a USC of $n - \ell$. As the maximum disutility for any agent is 1, this implies that $n - \ell$ agents receive a disutility of 1 from A. By construction, $d_i(A_i) = 0$ only if A_i contains *all* the items corresponding to the vertices in H_i. From here, it is easy to see that $\mu = \{H_i | d_i(A_i) = 0\}$ is a matching of size ℓ.

Observe that this proof can be extended to any $\tau \in [0, 1)$, by simply padding with enough items of disutility 1 to match the quantile value. That is, for some fixed $\tau \in [0, 1)$, we simply need to add nk' extra dummy items of disutility 1 to all agents, where k' is such that, $\lceil \tau(k' + 3)\rceil = k' + 1$. $\square$

While the greedy algorithm (Algorithm 1), that worked for the case of goods, can be arbitrarily bad for chores, a minimum ESC allocation (which we shall discuss in Section E.2 and can be found in polynomial time) can be shown to be at most n-USC.

Proposition 6. *Given instance $I = \langle N, M, d, \tau\rangle$ with chores, a balanced allocation which is n-*USC *can be found in polynomial time.*

Proof. Observe that given an instance with chores, the minimum possible egalitarian cost must always be less than or equal to the minimum possible utilitarian cost. This follows from the fact that under any allocation A,

$$\mathrm{ESC}(A) = \max_{i \in N} d_i(A_i) \leq \sum_i d_i(A_i) = \mathrm{USC}(A).$$

Further, under any allocation A,

$$\mathrm{USC}(A) = \sum_{i \in N} d_i(A_i) \leq n \max_i d_i(A_i) = n\mathrm{ESC}(A).$$

Consequently, an allocation with minimum ESC will be n-USW. We have from Theorem 12 that a minimum ESC allocation can always be found in polynomial time. $\square$

Unconstrained Allocations. While goods allow for a near exact algorithm, for chores, one is not possible. We now show that minimizing USC when all agents quantiles is 0, is equivalent to the WEIGHTEDSETCOVER problem. This enables us to show that with chores and pessimistic agents (quantile $\tau = 0$) no polytime algorithm can give an $(1 - o(1)) \log m$ approximation to minimum USC, if $P \neq NP$.

Theorem 11. *Given instance $I = \langle N, M, d, \tau = 0\rangle$ with chores, it is NP-hard to get $(1 - o(1)) \log m$-*USC*.*

Proof. We give an approximation preserving reduction from the SETCOVER problem. Given an instance of SETCOVER with element set $E = [\ell]$ and family of subsets $\mathcal{S} = \{S_1, \cdots, S_t\}$, we construct an instance of our problem with $n = t$ agents and $m = \ell$ items as follows:

Create an agent i_j for each set $S_j \in \mathcal{S}$. Create an item (chore) g_e for each element $e \in E$. Set the quantile value $\tau_i = 0$ for each agent i.

We set the disutilities as follows: for $j \in [t]$, the disutility of agent i_j for item g_e is:

$$d_{i_j}(g_e) = \begin{cases} 1 & \text{if } e \in S_j \\ t + 1 & \text{if } e \notin S_j \end{cases}$$

It is straightforward to see that an allocation with USC at most $\alpha \leq t$ exists if and only if a set cover of size at most $\alpha \leq t$ exists. Let there exist a set cover of size α. Without loss of generality, let it be $S_1, \cdots, S_\alpha$. Then construct allocation A where

$$A_{i_j} = \begin{cases} \{g_e : e \in S_j \setminus (\cup_{j' \leq j-1} S_{j'})\} & \text{if } j \leq \alpha \\ \emptyset & \text{if } j > \alpha \end{cases}$$

Clearly, agents $i_1, \cdots, i_\alpha$ receive disutility 1 and all remaining agents receive disutility 0. As a result, the USC of A is α.

We can similarly argue the other case. Let an allocation A have USC$(A) = \alpha \leq t$. Thus, each agent i_j must only be assigned items belonging to its corresponding set S_j and exactly α agents are assigned any items. Consequently, sets corresponding to these agents must form set cover of size α.

As a result, we have an approximation preserving reduction from SETCOVER. It is known that, for SETCOVER problem is hard to approximate to within factor $(1 - o(1)) \log \ell$, unless P=NP (Feige 1998). Consequently, as our constructed instance has $m = \ell$ items, the problem of minimizing USC is NP-hard to approximate to factor within $(1 - o(1)) \log m$. $\square$

Given an instance with chores, $I = \langle N, M, d, \tau = 0 \rangle$, we can get an analogous reduction to weighted set cover. Here we construct an element for each item and a set for each agent and each disutility level. That is, for each agent, we create a set with top ℓ best chores, for each $\ell \in [m]$. The corresponding weight of the set will be the agent's disutility for the set. We can now see that there is a mapping between set covers of weight at most ν and to allocations of USC at most ν. Thus, the $\log m$-approximation algorithm for weighted set cover provides a $\log m$-USC algorithm for pessimistic agents.

Corollary 2. *Given $I = \langle N, M, d, \tau \rangle$, an $\log m$-USC allocation can be found in polynomial time.*

E.2 Egalitarian Social Cost

We had begun our exploration of egalitarian welfare in Lemma 1, by showing that for goods, it is sufficient to find algorithms for binary goods. We can get an analogous result for the case of egalitarian cost.

Lemma 7. *The problem of finding an allocation with* ESC *at most $\nu \geq 0$ over allocations in $\Pi' \subseteq \Pi(n, M)$ under heterogeneous quantiles reduces to the problem of minimizing* ESC *over Π' under binary chores with heterogeneous quantiles.*

We can now focus our discussion on binary chores where $d_i(g) \in \{0, 1\}$ for all $i \in N$ and all $g \in M$. For balanced allocations, it is straightforward to see that the ideas for binary goods for *balanced allocations* extend to binary chores. Recall Algorithm 2. Instead of adding edges for items of value 1, we add an edge for chores for disutility 0. The result will then analogously follow.

Theorem 12. *Given an instance with chores and heterogeneous quantiles $I = \langle N, M, d, \tau \rangle$ where $m = kn$, a minimum* ESC *balanced allocation can be found in polynomial time.*

Unconstrained Allocations. Recall that for goods, the tractability of egalitarian welfare was highly quantile dependent. Algorithm 4 found a maximum ESW allocation in polynomial time for quantiles of the form $\tau = {}^{t}/_{t+1}$ for $t \in \mathbb{Z}^+$. We now show that for chores, not only does Algorithm 4 not extend, but the corresponding quantiles prove to be NP-hard.

Theorem 13. *Given an instance $I = \langle N, M, d, \tau = {}^{1}/_{2} \rangle$, finding a minimum* ESC *allocation is NP-hard.*

Proof. We give a reduction from VERTEXCOVER (Johnson and Garey 1979) where given a connected undirected graph $G = (X, E)$ and a positive integer $\alpha > 0$, we wish to find a set of at most α vertices, such that each edge has at least one endpoint in this set.

Given an instance of vertex cover $\langle G = (X, E), \alpha \rangle$, we construct an instance with binary chores as follows. For each vertex $x \in X$, we create an agent i_x. For each edge e, we create an item g_e. We create additionally, $|E| - \alpha$ universal bads $g'_1, \cdots, g'_{|E|-\alpha}$. For each $x \in X$ and $e \in E$, $d_{i_x}(g_e) = 0$ if e is incident on x, otherwise, $d_{i_x}(g_e) = 1$. Further for all agents and all $t \in [|E| - \alpha]$, we have that $d_i(g'_t) = 1$. Finally, for each i, we set $\tau_i = {}^{1}/_{2}$.

Now, observe that if an agent is allocated an empty set, clearly their disutility will be 0. For non-empty sets, as $\tau_i = {}^{1}/_{2}$ for all i, i needs to have strictly more items of disutility 0 than 1 to have a disutility of 0 for the allocated set.

We now show that the given instance has a vertex cover of size at most α if and only if the constructed instance has an allocation with ESC of 0.

First, assume that there is a minimum vertex cover $S \subseteq X$ s.t. $|S| \leq \alpha$. Thus, we can assume without loss of generality, there a must be a partition of the edges in $E \cup_{x \in S} P_x = E$ s.t. $|P_x| \geq 2$. If not, as the graph is connected, a smaller vertex cover must exist. Based on this, we create a partition of the universal bads g' s.t. $\cup_{x \in S} M_x = \{g'_1, \cdots, g'_{|E|-\alpha}\}$ and $|M_x| = |P_x| - 1$.

We now construct the following allocation where $A_{i_x} = \emptyset$ if $x \notin S$ and if $x \in S$, $A_{i_x} = \{g_e | e \in P_x\} \cup M_x$. Clearly, for $x \notin S$ $d_{i_x}(A_{i_x}) = 0$. For agents i_x s.t. $x \in S$, i_x receives $|P_x|$ incident edge items and $|M_x| - 1$ universal bads. Consequently, $d_{i_x}(A_{i_x}) = 0$. Thus, ESC$(A) = 0$.

Conversely, let an allocation A exist s.t. ESC$(A) = 0$. By construction, there are $|E|$ items that give disutility 0 to at least some agent, and $|E| - \alpha$ items that give disutility 1 to all.

Now, as ESC$(A) = 0$, at most α agents can receive non-empty sets, as there are only α extra edge items than universal bad. Further, all the edge items must have been allocated to an agents corresponding to an endpoint of the edge. Thus, the agents allocated non-empty sets must correspond to a vertex cover of size at most α.

Hence, a vertex cover of size at most α exists in G if and only if the constructed instance has an allocation of ESC 0. $\square$

Exact Algorithms. In this manner, most of the quantiles for which we had exact algorithms for goods become intractable. For two specific quantiles however, we continue to have simple exact algorithms: $\tau = 0$ and $\tau = 1$. If $\tau = 0$, it is sufficient to check for the given binary chores instance, if there is an item that is a universal bad, in which case all allocations will have ESC of 1. If not, each item has at least one agent who has no disutility for it, and can be arbitrarily assigned to such an agent to get an allocation with ESC of 0. For $\tau = 1$, if any one agent has even item with no disutility, they can be given all the items in M to get an allocation with ESC of 0. Otherwise, each allocation must have ESC of 1.

Proposition 7. *Given instance $I = \langle N, M, d, \tau \in \{0, 1\}\rangle$, a minimum* ESC *allocation can be found in polynomial time.*

Proof. Note that, here we assume homogeneous quantiles. That is, either for all $i \in N$, $\tau_i = 0$ or for all $i \in N$, $\tau_i = 1$. We again assume binary chores, that is, $d_i(g) \in 0, 1$.

Pessimistic quantile $\tau = 0$. In this case, it is sufficient to check if for each g, there exists at least one agent i such that $d_i(g) = 0$. In this case, allocating each chore to any one such agent ensures that each agent receives only chores that they have no disutility for. Thus the resultant allocation will have ESC of 0. Suppose that there exists at least one $g \in M$ s.t. for all $i \in N$, $d_i(g) = 1$. In this case, no allocation can have an ESC of 0, as whoever receives this item will have strictly positive disutility. In fact, in this case, all allocations will have an ESC of 1, so an arbitrary allocation can be selected. Observe that this condition can be checked in time $O(mn)$, and thus a minimum ESC allocation can be found in polynomial time.

Optimistic quantile $\tau = 1$. In this case, it is sufficient to check if for at least one agent $i \in N$ and one item $g \in M$, $d_i(g) = 0$. In this case, the allocation where $A_i = M$ and $A_j = \emptyset$ for all $j \neq i$ will clearly have an ESC of 0. Suppose no such agent exists. Thus, for all $i \in N$ and all $g \in M$, we have that $d_i(g) = 1$. Then, all allocations will have an ESC of 1 and thus an arbitrary allocation can be selected. This condition can be checked in time $O(mn)$, thus a minimum ESC allocation can be found in polynomial time. $\square$