Time-Optimal Control of Quantum Lambda Systems in the KP Configuration

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Abstract

We consider the problem of time-optimal control for a three-level quantum system where one level is coupled by the control field to the lowest two, which are not coupled to each other. A bound is assumed on the norm of the control at every time. Such a problem belongs to the class of KP sub-Riemannian problems for which we can perform a symmetry reduction and reduce to a Riemannian problem on the associated quotient space. We prove several properties of such a quotient space in our case, including the fact that it is an example of an almost-Kähler manifold which is not Kähler. We provide the explicit form of the optimal controls for any unitary transformation on the lowest two levels and discuss the geometric and practical implications of this result.

Keywords: Time optimal control of quantum systems, sub-Riemannian geometry, Lambda systems, symmetry reduction

1 Introduction

Optimal control of quantum systems has a long and successful history [14], [20], [27] but explicit, analytic, solutions of optimal control problems are rare. The optimal control of two-level quantum systems has been treated in detail in several papers (see, e.g., [3], [10], [11], [19]) and there are some results on the three level case [5], [8], [9]. These studies are motivated by the implementation of quantum information processing. In particular, two-level quantum systems represent quantum bits in the circuit model of quantum computation [25]. On the other hand, three-level quantum systems may also represent quantum bits when only two of the levels are used as carriers of information. A common scenario is the one of *Lambda systems* where the energy level diagram takes the form reported in Figure 1 with only the lowest two levels coupled to the third level via an electromagnetic field. The lowest two levels may be used to implement a quantum bit. Besides quantum computation, the Lambda configuration is very common in several additional applications of quantum mechanics [16], [29], including, for example, electromagnetic induced transparency [7], [18]. From a control perspective, Lambda systems have been investigated mostly in the context of *adiabatic* techniques (see, e.g., [6]).The time-optimal control of three-level quantum systems in the Lambda configuration is the subject of this paper.

The model we consider is a *Schrödinger operator equation* of the form

$$\dot{U}(t) = \hat{A}U(t) + \sum_{j=1}^{m} B_j U(t)\hat{u}_j(t), \qquad U(0) = \mathbf{1}, \tag{1}$$

where U is the evolution operator varying in the Lie group $SU(3)$ of 3×3 unitary matrices with determinant equal to 1, and $\mathbf{1}$ is the 3×3 identity. We choose units in which the Planck constant $\hbar$ is equal to 1 and $\hat{A}$ is diagonal

$$\hat{A} := \begin{pmatrix} i\lambda_0 & 0 & 0 \\ 0 & i\lambda_1 & 0 \\ 0 & 0 & i\lambda_2 \end{pmatrix}, \tag{2}$$

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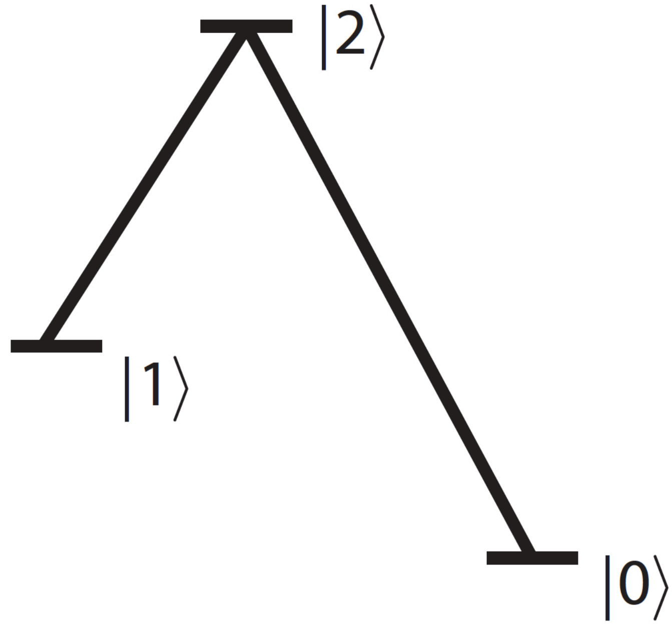

Figure 1: Energy level diagram for a Lambda system

with $\lambda_0, \lambda_1, \lambda_2$ the energy levels corresponding to eigenstates $|0\rangle$, $|1\rangle$, $|2\rangle$, respectively (with $\lambda_0 + \lambda_1 + \lambda_2 = 0$). In (1), we will have $m = 4$ and the *controls* $\hat{u}_{1,2,3,4}$ represent electromagnetic fields coupling the levels $|0\rangle$ and $|2\rangle$ and the levels $|1\rangle$ and $|2\rangle$. The matrices B_j are the orthonormal (under the inner product $\langle B, C\rangle := Tr(BC^\dagger)$) matrices in $su(3)$,

$$B_1 := \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_2 := \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_3 := \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad B_4 := \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}. \tag{3}$$

The general optimal control problem for system (1) we shall consider is to find the control functions $\hat{u} = \hat{u}_{1,2,3,4}$ driving the state U from the identity $\mathbf{1}$ to a desired final condition in $U_f \in SU(3)$ subject to the constraint that $\|\hat{u}\| \leq M$, for a given bound M. In particular, we shall completely solve such a problem for the case where U_f has the form

$$U_f := \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & \hat{U}_f \end{pmatrix}, \tag{4}$$

with $\hat{U}_f \in U(2)$ and $\det(\hat{U}_f) = e^{i\phi}$, i.e., the desired final condition U_f in (4) is an arbitrary transformation on the lowest two levels with an arbitrary phase shift with the level $|2\rangle$.

The problem for system (1) has a KP structure as it was introduced in [22] [8]. This is because the Lie algebra of $SU(3)$, $su(3)$, has a Cartan decomposition [21], $su(3) = \mathcal{K} \oplus \mathcal{P}$ with $\mathcal{K}$ and $\mathcal{P}$ satisfying the commutation relations

$$[\mathcal{K}, \mathcal{K}] \subseteq \mathcal{K}, \qquad [\mathcal{K}, \mathcal{P}] \subseteq \mathcal{P}, \qquad [\mathcal{P}, \mathcal{P}] \subseteq \mathcal{K}, \tag{5}$$

and the matrix $\hat{A}$ in (1) belongs to $\mathcal{K}$ while the matrices $B_{1,2,3,4}$ form an orthonormal basis of $\mathcal{P}$. In fact, the relevant Cartan decomposition for system (1) is the one where $\mathcal{K}$ is spanned by block diagonal matrices in $su(3)$ with blocks of dimension 1 and 2, i.e., matrices of the form

$$K := \begin{pmatrix} if & 0 \\ 0 & Q \end{pmatrix} \tag{6}$$

with $Q \in u(2)$ and f in $\mathbb{R}$. The matrices in $\mathcal{P}$ are the corresponding block anti-diagonal matrices, i.e., matrices of the form $P := \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha^* & 0 & 0 \\ -\beta^* & 0 & 0 \end{pmatrix}$ with arbitrary complex numbers α and β, that is, linear combinations of the matrices in (3). This KP structure of the problem allows us to use several properties which we shall describe in the next section. Among these, we have that we can effectively transform the system into a *driftless* one, i.e., eliminate the term $\hat{A}U$ in (1), displaying only the part that contains the control and that belongs to the $\mathcal{P}$ portion of the Lie algebra $su(3)$. By defining $X := e^{-\hat{A}t}U$, we obtain for X the Schrödinger operator equation in the interaction picture

$$\dot{X} = \sum_{j=1}^{m} B_j u_j X, \quad X(0) = \mathbf{1}, \tag{7}$$

where the new controls u_j are related to the controls $\hat{u}_j$ in (1) by a (time varying) transformation (cf. (8) in the next section), which is orthogonal and therefore does not modify the norm. We shall focus on the time-optimal control problem to drive X to a desired final condition for system (7) and then explain in Section 5 how to obtain from this the result for the original problem of equation (1).

The contribution of this paper is organized as follows. In Section 2, we recall the main facts on KP problems, including the fact that the optimal trajectories have an explicit form. Because of the KP property, the problem admits a symmetry reduction and can be treated on a quotient space which has the structure of a stratified space. The regular part of such a stratified space (see Section 2 for definitions) is a Riemannian manifold with a metric related to the sub-Riemannian metric of the original problem. We describe this correspondence. In Section 3 we specialize to the problem on $SU(3)$ which is the main topic of this paper. We describe the structure of the quotient space in this case and prove several properties of the corresponding regular part and in particular the fact that it is an example of an almost-Kähler mainfold which is not Kähler. From a practical perspective, the main results are presented in Section 4 where we give a method to determine the parameters of the optimal control for any final condition of the form (4). We obtain therefore the complete optimal synthesis for this type of final condition. The connection between the complete optimal synthesis and the geometry of the reachable sets allows us to extend the method to find the optimal control for the system with drift (1). This is done in Section 5 where we also present some numerical examples and simulations.

This paper can be seen as a continuation of the research presented in [5] and in fact we shall use several results from that paper. With respect to [5], we present a new characterization of the sub-Riemannian problem as a Riemannian problem on the quotient space, we prove several properties of this Riemannian manifold, and, in the solution of the optimal control problem, we remove the restrictions on the eigenvalues of the final condition used in [5] so as to give the general solution. This also allows us to generalize the results to the system with drift (1).

2 Generalities on KP problems

KP problems are a class of control problems with dynamics taking place on a given real, finite-dimensional semisimple[4] Lie group G which we shall assume to be compact. We denote the corresponding Lie algebra by $\mathfrak{g}$. These control problems are defined in terms of a Cartan decomposition of the Lie algebra $\mathfrak{g}$, that is a decomposition $\mathfrak{g} = \mathcal{K} \oplus \mathcal{P}$ with $\mathcal{K}, \mathcal{P}$ vector subspaces satisfying (5). The subspaces $\mathcal{K}$ and $\mathcal{P}$ are orthogonal with respect to the *Killing form*[5] on $\mathfrak{g}$. For $\mathfrak{g} = su(n)$, the Killing form coincides (up to a proportionality constant) with $\langle B, C \rangle := Tr(BC^\dagger)$ and therefore we shall denote it with $\langle \cdot, \cdot \rangle$ in the following for a general semisimple Lie algebra $\mathfrak{g}$. We shall also assume that $e^{\mathcal{K}}$, the Lie group associated with the Lie algebra $\mathcal{K}$, is compact.

A KP problem is the problem to drive the state U of a system of the form (1) from the identity to a desired final condition U_f with Euclidean norm of the control, $\|\hat{u}\|$, bounded at ever time t, in minimum time. The elements $\{B_j\}$ are assumed to form an orthonormal basis of $\mathcal{P}$ in the Cartan decomposition of $\mathfrak{g}$, and $\hat{A}$ is assumed to belong to the corresponding $\mathcal{K}$. Often one can consider the special case with $\hat{A} = 0$ and the system takes the form (7) with the state in G now denoted by X and the control denoted by u. The transformation $X := e^{-\hat{A}t}U$ allows

[4]A Lie group is said to be semisimple if its Lie algebra may be decomposed into a direct sum of simple Lie subalgebras, none of which is abelian.

[5]Recall that the Killing form on a Lie algebra $\mathfrak{g}$ is defined as follows: Fix $X \in \mathfrak{g}$ and let $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ be the linear map $\mathrm{ad}_X(Y) = [X, Y]$ for any $Y \in \mathfrak{g}$. Then the Killing form $K(X, Y) := Tr(\mathrm{ad}_X \circ \mathrm{ad}_Y)$. This quadratic form has the property from Cartan's semisimplicity criterion [21] that it is non-degenerate if and only if $\mathfrak{g}$ is semisimple.

us to transform system (1) to system (7). With this transformation, we get $\dot{X}(t) = \sum_{j=1}^m e^{-\hat{A}t}B_j e^{\hat{A}t}u_j(t)X(t)$, $X(0) = \mathbf{1}$. Since $\{B_j\}$ form an orthonormal basis of $\mathcal{P}$ and $e^{\hat{A}t}Pe^{-\hat{A}t} \in \mathcal{P}$ for any $P \in \mathcal{P}$ and $\hat{A} \in \mathcal{K}$, from (5), we may write:

$$e^{-\hat{A}t}B_j e^{\hat{A}t} := \sum_{k=1}^m a_{jk}(t)B_k, \tag{8}$$

so that we obtain (7) with

$$u_k(t) := \sum_{j=1}^m a_{jk}(t)\hat{u}_j(t). \tag{9}$$

Moreover, since the matrix $a(t)$ with entries $a_{jk}(t)$ is orthogonal for every t, we have $||\hat{u}(t)|| = ||u(t)||$. Therefore the constraint on the norm of $\hat{u}$ translates to the same constraint on the norm of u. Also, given the fact that $\{B_j\}$ is an orthonormal basis of $\mathcal{P}$, the constraint $\|u\| \leq M$ is equivalent to $\left\|\sum_j B_j u_j\right\| \leq M$ where the norm is now the one induced by the (Killing) inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$.

2.1 Sub-Riemannian geometry

KP problems can be treated in the setting of *sub-Riemannian geometry* [1] [24]. A *sub-Riemannian manifold* (G, Δ, g) is a smooth manifold, G, together with a sub-bundle Δ of the tangent bundle TG and a positive-definite, symmetric, bilinear form g defined on Δ. When $\Delta = TG$, this is the usual definition of a Riemannian manifold [17], and often, for a sub-Riemannian manifold, the metric g is taken as the restriction to $\Delta \subseteq TG$ of a Riemannian metric defined on all of TG. For KP problems, we may define a sub-Riemannian structure by taking $\Delta = \bigsqcup_{x \in G} R_{x*}\mathcal{P}$ where R_{x*} is the push-forward of the right-multiplication map on G, i.e. $R_x(y) = yx$ for $x, y \in G$, where we have used the fact that the tangent bundle of any Lie group is parallelizable, i.e. $TG \simeq \bigsqcup_{x \in G} R_{x*}\mathfrak{g}$;[6] We define the sub-Riemannian metric as $g(R_{x*}P, R_{x*}Q) := \langle P, Q \rangle$ for any $P, Q \in \mathcal{P}$ and $x \in G$. A trajectory $\gamma : [0, T] \to G$ for a sub-Riemannian manifold is called *horizontal* if $\dot{\gamma}(t) \in \Delta_{\gamma(t)}$ for every $t \in [0, T]$. The length of a horizontal curve γ from p to q is defined using the sub-Riemannian metric g as

$$l(\gamma) := \int_0^T \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))}dt. \tag{10}$$

The length of the curve does not change with a reparametrization of the time t. Thus we can assume that $T = 1$ in (10). Furthermore, we can reparametrize the time so that $\sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))}$ is constant in (10) [1], [4]. The sub-Riemannian distance between two points p and q in G is defined as

$$d(p, q) := \inf_\gamma \int_0^1 \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))}dt, \tag{11}$$

where the infimum is taken over all horizontal trajectories γ with $\gamma(0) = p$, $\gamma(1) = q$. Under the assumption that 1) G is connected 2) Δ is *bracket generating*, i.e., (repeated) Lie brackets of vector fields with values in Δ span all of the tangent space T_xG for every $x \in G$, the Chow-Raschevskii theorem guarantees that there exists a horizontal trajectory attaining the infimum in (11) (c.f. [1] [24]). Such a curve is called a *sub-Riemannian* geodesic.

The relation between the time-optimal control for a control system (7) and the corresponding sub-Riemannian optimal distance is as follows (c.f., e.g., [1] and [4](Theorem 1)): *The horizontal curve γ is a sub-Riemannian geodesic with constant speed $\sqrt{g(\dot{\gamma}, \dot{\gamma})}$ equal to M joining the points p and q in G in time T if and only if γ is the time optimal trajectory joining p and q with optimal control u, bounded by $\|u\| \leq M$ and T is the minimum time.*

The time-optimal trajectories for KP problems (7) were explicitly described in [22] by utilizing the Pontryagin Maximum Principle [2] of optimal control. In particular, the optimal control satisfies

$$\sum_{k=1}^m B_k u_k(t) = e^{At}Pe^{-At}, \tag{12}$$

[6] Here, as it is customary, we identify the Lie algebra $\mathfrak{g}$ with the tangent space of G at the identity.

for some fixed $A \in \mathcal{K}$ and $P \in \mathcal{P}$ with $||P|| = M$, the bound on the norm of the control. Furthermore, plugging this into equation (7) and integrating yields that the optimal trajectories for KP systems with initial condition at the identity are given by:

$$\gamma(t) = e^{At}e^{(-A+P)t}. \tag{13}$$

So, given a fixed final condition $X_f \in G$, in order to drive the system (7) to X_f in optimal time, subject to the bound on the controls, one must find a matrix $A \in \mathcal{K}$ (so $\dim(\mathcal{K})$ parameters); a matrix $P \in \mathcal{P}$ with $||P|| = M$ (so $\dim(\mathcal{P}) - 1$ parameters); and a *minimum time* $T \geq 0$ (so one further parameter dimension) so that $\gamma(T) = e^{AT}e^{(-A+P)T} = X_f$. That is, in order to find the optimal trajectory joining $\mathbf{1}$ to X_f, one must search through a parameter space with dimension: $\dim(\mathcal{K}) + (\dim(\mathcal{P}) - 1) + 1 = \dim(\mathfrak{g})$. Symmetry reduction can be used to reduce the number of parameters.

2.2 Symmetry reduction for KP systems

Let us examine the action of $e^{\mathcal{K}}$ on G via conjugation. That is, for $K \in e^{\mathcal{K}}$ and $X \in G$, $K \cdot X := KXK^{-1}$; this action is a *proper action* [12] because $e^{\mathcal{K}}$ is compact. Furthermore, it is such that it fixes the identity $\mathbf{1} \in G$; the induced conjugation action of $e^{\mathcal{K}}$ on $\mathfrak{g}$ has as invariant subspaces $\mathcal{K}$ and $\mathcal{P}$; moreover, this action preserves the length of a horizontal curve. To see this, write $\dot{\gamma}$ in terms of a curve $P = P(t)$ in $\mathcal{P}$, i.e., $\dot{\gamma} = P(t)\gamma$. For $K \in e^{\mathcal{K}}$, we have

$$l(K\gamma K^{-1}) = \int_0^1 \sqrt{\langle KP(t)K^{-1}, KP(t)K^{-1}\rangle}dt = \int_0^1 \sqrt{\langle P(t), P(t)\rangle}dt = l(\gamma). \tag{14}$$

In particular, this length preservation property implies that $\gamma(t)$ is an optimal sub-Riemannian geodesic joining $\mathbf{1}$ to X_f if and only if $K\gamma(t)K^{-1}$ is an optimal sub-Riemannian geodesic joining $\mathbf{1}$ to KX_fK^{-1} for each $K \in e^{\mathcal{K}}$. Using the form of the optimal trajectories (13), we have,

$$K\gamma(t)K^{-1} = Ke^{At}e^{(-A+P)t}K^{-1} = e^{KAK^{-1}t}e^{(-KAK^{-1}+KPK^{-1})t}. \tag{15}$$

It is therefore reasonable to consider the minimum length problem in the lower-dimensional quotient space $G/e^{\mathcal{K}}$, the space of equivalence classes under the conjugation action of $e^{\mathcal{K}}$ on G.[7] We can first find the minimal length trajectory joining $\mathbf{1} \in G$ to any element in the same class as X_f, say $\hat{X}_f$, that is, find the matrices A and P and minimum time t in (13), for $\hat{X}_f$. After that, we can find $K \in e^{\mathcal{K}}$ so that $X_f = K\hat{X}_fK^{-1}$, so that the corresponding optimal matrices for X_f, will be KAK^{-1}, KPK^{-1} and the minimum time t will be the same. Since the matrices A and P to be found in the first step of this procedure can be 'rotated' via a matrix $K \in e^{\mathcal{K}}$, they are defined up to a shared conjugation by K and therefore can be assumed to be in a special form. This reduces the number of the unknown parameters of the problem in A and P.

Since the optimal sub-Riemannian problem on G is reduced to a problem on $G/e^{\mathcal{K}}$, it is natural to investigate this space. The conjugation action of $e^{\mathcal{K}}$ is not a free action. Therefore $G/e^{\mathcal{K}}$ is not guaranteed to be a manifold but it has, in general, the structure of a *stratified space* [26]. Denote by π the natural projection $\pi : G \to G/e^{\mathcal{K}}$. The theory of transformation groups (see, e.g., [12]) says that the strata S making up such a space can be classified according to the isotropy group of the elements in $\pi^{-1}(S)$. This is the so-called *stratification by isotropy type*. Also the theory says that exists a 'minimal' subgroup $K_{min} \subseteq e^{\mathcal{K}}$ such that 1) Every element $X \in G$ has isotropy group containing a subgroup of $e^{\mathcal{K}}$ of the form $KK_{min}K^{-1}$ for some $K \in e^{\mathcal{K}}$ 2) The set elements of G having as isotropy group exactly $KK_{min}K^{-1}$ for some $K \in e^{\mathcal{K}}$, called G_{reg}, is such that $G_{reg}/e^{\mathcal{K}}$ is an open and dense connected smooth manifold in $G/e^{\mathcal{K}}$. The set G_{reg} ($G_{reg}/e^{\mathcal{K}}$) is called the *regular part* of G ($G/e^{\mathcal{K}}$) while $G - G_{reg}$ ($(G - G_{reg})/e^{\mathcal{K}}$) is called the *singular part* of G ($G/e^{\mathcal{K}}$) .

2.3 Riemannian geometry on $G_{reg}/e^{\mathcal{K}}$

We have seen above that conjugated curves from the identity to conjugated final conditions have the same length and can be projected to a single curve on $G/e^{\mathcal{K}}$. Moreover, an open and dense subset of $G/e^{\mathcal{K}}$, $G_{reg}/e^{\mathcal{K}}$, has the structure of a connected manifold. It is therefore natural to investigate whether we can put a Riemannian metric on $G_{reg}/e^{\mathcal{K}}$, so that the length of curves is preserved by the projection π, and, in particular,the portion in G_{reg} of optimal sub-Riemannian geodesics in G is mapped by π to Riemannian geodesics in $G_{reg}/e^{\mathcal{K}}$, with the same length. This can be done [15] [28] under the assumption that the minimal isotropy group K_{min} is

[7] Notice this is **not** the usual left or right coset space considered in the thoery of Riemannian symmetric spaces [21].

discrete. In particular, consider the map $\pi : G_{reg} \to G_{reg}/e^{\mathcal{K}}$ and, for a point $x \in G_{reg}$, the induced push-forward $\pi_* : T_x G_{reg} \to T_{\pi(x)} G_{reg}/e^{\mathcal{K}}$ between the tangent spaces. Then, if we restrict π_* to $R_{x*}\mathcal{P}$, π_* can be proven to be an isomorphism, and for a tangent vector $V \in T_{\pi(x)} G_{reg}/e^{\mathcal{K}}$ we denote by $\pi_*^{-1} V$ its preimage in $R_{x*}\mathcal{P}$. Then we define the metric g_Q on $G_{reg}/e^{\mathcal{K}}$ by

$$g_{Q\pi(x)}(V, W) := g_x(\pi_*^{-1} V, \pi_*^{-1} W) = \langle R_{x^{-1}*}\pi_*^{-1} V, R_{x^{-1}*}\pi_*^{-1} W \rangle, \tag{16}$$

where g is the sub-Riemannian metric on G which we have defined in terms of the (Killing) inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{P}$. This can be seen to be independent of the representative x taken for $\pi(x)$ [15] [28]. With this definition, sub-Riemannian geodesics γ in G which are defined in $[0, T]$ and such that $\gamma(0) = \mathbf{1}$ and $\gamma(t) \in G_{reg}$ for every $t \in (0, T)$ are such that $\pi \circ \gamma$ defined for $t \in (0, T)$ is a Riemannian geodesic in $G_{reg}/e^{\mathcal{K}}$. Therefore, for KP problems, sub-Riemannian geodesics in G are mapped to Riemannian geodesics in $G_{reg}/e^{\mathcal{K}}$ and viceversa they can be obtained as the 'lift' of certain geodesics in $G_{reg}/e^{\mathcal{K}}$. We refer to [15], [28] for details.

3 The KP problem $SU(3)/S(U(2) \times U(1))$

The problem we consider in this paper, which was described in the introduction, is a KP problem where the Lie algebra $\mathfrak{g}$ is $su(3)$ and $\mathcal{K}$ is the Lie subalgebra of matrices in $su(3)$ of the form (6) and $\mathcal{P}$ is spanned by the orthonormal basis in (3). The Lie group G is $SU(3)$ and the Lie subgroup $e^{\mathcal{K}}$ is $S(U(2) \times U(1))$, which is the Lie group of block diagonal matrices of the form (4) with determinant equal to 1. As we have mentioned in the previous section, the problem of finding the matrices A and P in (13) can be simplified by assuming a canonical form for these matrices which can be defined up to a conjugation by an element $K \in e^{\mathcal{K}}$. Furthermore, we can, without loss of generality, assume a fixed value for the norm of P. This is the same value as the norm of the control from (12) and once one has found the optimal control for a value of the bound M in $\|u\| \leq M$, one can simply scale the control and time accordingly for any other bound [4]. Using [5] (Proposition II.I) , we take the unknown A and P as

$$A := i \begin{pmatrix} a+b & 0 & 0 \\ 0 & -a & -c \\ 0 & -c & -b \end{pmatrix}, \qquad P := i \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{17}$$

for unknown real parameters a, b, and c. In the next section, we shall solve the problem to find the parameters in A and the optimal time t to drive the state X of (7) to a final condition of the form (4) in minimum time. This will solve the problem of optimal control for a quantum lambda system on the lowest two energy levels.

The rest of this section is devoted to describing the Riemannian geometry of $SU(3)_{reg}/S(U(2) \times U(1))$. While the results here are of mostly mathematical interest, the properties we shall describe might be useful in the case one wants to find optimal sub-Riemannian geodesics leading to a final condition different from the form (4). This is due to the correspondence, described in the previous section, between the Riemannian geodesics of $G_{reg}/e^{\mathcal{K}}$ and the sub-Riemannian geodesics of the KP problem.

3.1 Riemannian geometry of $SU(3)_{reg}/S(U(2) \times U(1))$

The stratified space $SU(3)/S(U(2) \times U(1))$ was described in [5] (Section III) . Such a description is based on the fact that, via conjugation by an element $K \in S(U(2) \times U(1))$, any matrix in $SU(3)$ can be placed in the *canonical form*

$$U = \begin{pmatrix} x & \sqrt{1-|x|^2} & 0 \\ -\sqrt{1-|x|^2} & x^* & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & \sqrt{1-|z|^2} \\ 0 & -\sqrt{1-|z|^2} & z^* \end{pmatrix}, \tag{18}$$

for complex parameters x and z, with $|x| \leq 1$ and $|z| \leq 1$. If $|x| < 1$ and $|z| < 1$, then U, and every matrix conjugate to U, belong to the regular part of $SU(3)$ and x and z can be taken as complex coordinates in $SU(3)_{reg}/S(U(2) \times U(1))$. The isotropy subgroup of $S(U(2) \times U(1))$ for matrices U in (18) with $|x| < 1$, $|z| < 1$, is the *finite* (and therefore discrete) group of scalar 3×3 matrices $\{e^{i\frac{2\pi k}{3}}\mathbf{1}\}$, for integer k.[8] Therefore we can apply the reduction

[8]In order to see this, write a general matrix $F \in S(U(2) \times U(1))$ in the form of U_f in (4). Comparing the first column of $FU = UF$ (and using the fact that $|x| < 1$) we get that $F = \texttt{diag}(e^{i\phi}, e^{i\phi}, e^{-2i\phi})$. From this, comparing the second columns and using the fact that $|z| < 1$, we obtain that F must be a scalar matrix.

to a Riemannian manifold $SU(3)_{reg}/S(U(2)\times U(1))$ described in the previous section. To extract the coordinates x and z from a matrix U which belongs to the regular part $SU(3)_{reg}$ but which is not necessarily in the canonical form (18), we notice that by denoting by $u_{j,k}$ the entries of U, $T_1 := u_{1,1}$ and $T_2 := u_{2,2}+u_{3,3}$ are *invariant* under conjugation by an element of $S(U(2)\times U(1))$ and they are related to x and z by $T_1 = x$ and $T_2 = x^*z + z^*$ which can be inverted as $z = \frac{T_2^* - xT_2}{1-|x|^2}$.

3.1.1 Expression of the metric on $SU(3)_{reg}/S(U(2)\times U(1))$

We calculate now the induced metric on $SU(3)_{reg}/S(U(2)\times U(1))$. We shall see that it takes a simple block-diagonal form in the coordinates (x_R, x_I, z_R, z_I), real and imaginary parts of x and z, respectively.[9] It is convenient to do the calculations in the coordinates $\{T_{1R}, T_{1I}, T_{2R}, T_{2I}\}$ first and then perform a change of coordinates to obtain the expression in the coordinates (x_R, x_I, z_R, z_I). Therefore a general tangent vector in $T_{\pi(U)}SU(3)_{reg}/S(U(2)\times U(1))$ has the form

$$V = V_1\frac{\partial}{\partial T_{1R}} + V_2\frac{\partial}{\partial T_{1I}} + V_3\frac{\partial}{\partial T_{2R}} + V_4\frac{\partial}{\partial T_{2I}}, \tag{19}$$

while a general tangent vector in $R_{U*}\mathcal{P}$ can be written in terms of the orthonormal basis (3) of $\mathcal{P}$ as $R_{U*}P$, with $P = \sum_{j=1}^4 b_j B_j$. Since π_* is an isomorphism from $R_{U*}\mathcal{P}$ onto $T_{\pi(U)}SU(3)_{reg}/S(U(2)\times U(1))$, we seek to compute the corresponding matrix which transforms the vector $[b_1, b_2, b_3, b_4]^T$ to the vector $[V_1, V_2, V_3, V_4]^T$ in the above definition. Denoting by $\Pi_{l,k}$, the l,k entry of such a matrix, we have,

$$\Pi_{l,k} = \pi_*(R_{U*}B_k)\hat{T}_l = \frac{d}{dt}|_{t=0}(\hat{T}_l(\pi(e^{B_k t}U))), \tag{20}$$

where the functions $\hat{T}_l$ are defined as $(\hat{T}_1, \hat{T}_2, \hat{T}_3, \hat{T}_4) := (T_{1R}, T_{1I}, T_{2R}, T_{2I})$. This gives the following matrix, which we also denote, with some abuse of notation, by π_*:

$$\pi_* = \frac{1}{\sqrt{2}}\begin{pmatrix} u_{2,1R} & -u_{2,1I} & u_{3,1R} & -u_{3,1I} \\ u_{2,1I} & u_{2,1R} & u_{3,1I} & u_{3,1R} \\ -u_{1,2R} & -u_{1,2I} & -u_{1,3R} & -u_{1,3I} \\ -u_{1,2I} & u_{1,2R} & -u_{1,3I} & u_{1,3R} \end{pmatrix}. \tag{21}$$

From the definition (16), since the matrix associated with g is the identity in the basis (3), the matrix $\tilde{G}$ giving the metric is $\tilde{G} = \pi_*^{-T}\pi_*^{-1}$. It is easier to compute $\tilde{G}^{-1} = \pi_*\pi_*^T$ first and then find the inverse. If we do that, using in (21) the form of the matrix U in (18) and the fact that the metric does not change within equivalence classes, that is, it is well-defined, we obtain

$$\tilde{G}^{-1} = \frac{1-|x|^2}{2}\begin{pmatrix} 1 & 0 & z_R & z_I \\ 0 & 1 & z_I & -z_R \\ z_R & z_I & 1 & 0 \\ z_I & -z_R & 0 & 1 \end{pmatrix} \qquad \tilde{G} = \frac{2}{(1-|x|^2)(1-|z|^2)}\begin{pmatrix} 1 & 0 & -z_R & -z_I \\ 0 & 1 & -z_I & z_R \\ -z_R & -z_I & 1 & 0 \\ -z_I & z_R & 0 & 1 \end{pmatrix}. \tag{22}$$

Using $\tilde{G}$ given by (22) we can write the expression of the metric tensor in the coordinates T_{1R}, T_{1I}, T_{2R}, T_{2I}. Moreover, using the relations, $T_1 = x$ and $T_2 = x^*z + z^*$, we have $dT_{1R} = dx_R$, $dT_{1I} = dx_I$, $dT_{2R} = z_R dx_R + x_R dz_R + x_I dz_I + z_I dx_I + dz_R$, $dT_{2I} = z_I dx_R + x_R dz_I - z_R dx_I - x_I dz_R - dz_I$. Using this in the expression of the metric tensor we obtain the metric in the x and z coordinates

$$G_Q = \frac{2}{1-|x|^2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1+|x|^2+2x_R}{1-|z|^2} & \frac{2x_I}{1-|z|^2} \\ 0 & 0 & \frac{2x_I}{1-|z|^2} & \frac{1+|x|^2-2x_R}{1-|z|^2}. \end{pmatrix}. \tag{23}$$

The metric G_Q is singular when $|x|$ or $|z|$ approaches 1. It has a block-diagonal structure where the first block is a scalar matrix. The second block can be written in the form $\frac{2}{1-|z|^2}\begin{pmatrix} \hat{a} & \hat{b} \\ \hat{b} & \hat{d} \end{pmatrix}$, with $\hat{a} := \frac{1+|x|^2+2x_R}{1-|x|^2}$, $\hat{b} := \frac{2x_I}{1-|x|^2}$

[9] We always follow the convention of indicating by y_R and y_I the real and imaginary part, respectively, of a variable y.

$\hat{d} := \frac{1+|x|^2-2x_R}{1-|x|^2}$, and we have $\hat{a}\hat{d} - \hat{b}^2 = 1$. Using this expression, one can write down the geodesic equations or, equivalently, the Euler-Lagrange equations for the geodesics. Optimal geodesics in the regular part have to satisfy these equations and this, together with the correspondence between sub-Riemannian and Riemannian geodesics, is an alternative description of the sub-Riemannian optimal trajectories as compared to (13).

3.2 $SU(3)_{reg}/S(U(2) \times U(1))$ is an almost-Kähler manifold

We can define on $SU(3)_{reg}/S(U(2) \times U(1))$ the structure of an almost-Kähler manifold [23] as follows: Working in the coordinates (x_R, x_I, z_R, z_I), we define a linear operator, J, on each point of the tangent space at (x, z), by $J := \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}$ with

$$J_1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad J_2 := \begin{pmatrix} \hat{b} & \hat{d} \\ -\hat{a} & -\hat{b} \end{pmatrix}. \tag{24}$$

From $\hat{a}\hat{d} - \hat{b}^2 = 1$, we have $J^2 = -\mathbf{1}$. Therefore, J gives an almost complex structure. Furthermore, since $J_1^T \mathbf{1} J_1 = \mathbf{1}$, and $J_2^T \begin{pmatrix} a & b \\ b & d \end{pmatrix} J_2 = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$, for any $V, W \in T_{(x,z)}G_{reg}/K$, we have

$$g_Q(V, W) = g_Q(JV, JW),$$

i.e. compatibility with the metric is verified. Lastly, the 2−form $\omega(X, Y) := g(JX, Y)$, in the $\{x_R, x_I, z_R, z_I\}$ coordinates, is given by

$$\omega = -\frac{2}{1-|x|^2} dx_R \wedge dx_I - \frac{2}{1-|z|^2} dz_R \wedge dz_I$$

and is closed, since $d\omega = 0$. Therefore we have an almost complex structure and compatible Riemannian and symplectic structure on $SU(3)_{reg}/S(U(2) \times U(1))$. However, this is not a Kähler structure as it may be checked using one of the equivalent conditions for an almost-Kähler manifold to be Kähler [23]. In particular if ∇ is the Levi-Civita connection, then one can compute $\nabla J \neq 0$.

3.3 About the cut locus and the curvature

In Riemannian and sub-Riemannian geometry, the *cut locus* is defined as the set of points in the manifold where geodesics, starting from a given point of interest, for example the identity of the group in KP problems, stop being optimal. Describing the cut locus is of both theoretical and practical importance. From a mathematical point of view, the cut locus gives important information about the geometry of the manifold under consideration (see, e.g., [17] (Chapter 13)). From a more practical point of view, if one wants to obtain all the optimal geodesics one can obtain only the geodesics leading to the points in the cut locus (*cut points*). This is sufficient to obtain *all* the optimal geodesics, since such geodesics are optimal before reaching their final point in the cut locus, and by existence theorems, the optimal geodesics exist for every point. Therefore, one way to approach the problem to obtain the complete optimal synthesis is to first describe the cut locus and then obtain the optimal trajectory for any point in it.

For KP problems, properties of the cut locus were shown in [4]:[10] The cut locus is the inverse image under the natural projection π of a set in $G/e^{\mathcal{K}}$. Furthermore, every trajectory which crosses the regular part G_{reg} and reaches the singular part G_{sing}, has to lose optimality at the final singular point. Therefore if there is no cut point in G_{reg}, then the whole cut locus belongs to the singular part of the space. Given the correspondence between sub-Riemannian geodesics in G and Riemannian geodesics in $G_{reg}/e^{\mathcal{K}}$, such a property would be proven if we proved that there are no cut points in $G_{reg}/e^{\mathcal{K}}$ for geodesics $\gamma = \gamma(t)$ with $\lim_{t\to 0^+} \gamma(t) = \pi(\mathbf{1})$. This was proven in [15] for the case where $G = SU(2)$ and $e^{\mathcal{K}}$ is the one dimensional subgroup of diagonal matrices in $SU(2)$. The proof was based on the fact that the sectional curvature of $G_{reg}/e^{\mathcal{K}}$ is negative. We believe that the property that the cut locus is a subset of G_{sing} is more general, and it applies, for example, to the $SU(3)/S(U(2) \times U(1))$ treated in this paper. However the proof of [15] does not extend directly to this case because a calculation of the sectional curvature for $SU(3)_{reg}/S(U(2) \times U(1))$ shows that it changes sign at different points.

[10] The cut locus was called critical locus in [4] reserving the name cut locus for points where two different geodesics meet.

4 Complete optimal synthesis for Lambda systems

A desired final condition of the form U_f in (4) is characterized up to similarity transformations in $e^{\mathcal{K}}$ by two parameters: the eigenvalue $e^{-i\phi}$ and one eigenvalue of the matrix $\hat{U}_f$ (the other eigenvalue being determined by the condition on the determinant). In this section we shall scale all the angles by a factor 2π and the times t also by a factor 2π in order to avoid carrying the 2π factor in all the calculations. Therefore the problem is characterized by two real numbers x and ϕ both in $[0,1)$ such that the eigenvalues of U_f in (4) are $e^{-i\phi}$, e^{ix} and $e^{i(\phi-x)}$. The parameters ϕ and x determine the equivalence class for the desired final condition. Given the simplifications due to the symmetry reduction described in the previous sections, the problem is to find three real parameters a, b, c in (17) and a smallest time $t>0$ such that $e^{At}e^{(-A+P)t} \in e^{\mathcal{K}}$ with eigenvalues $e^{-i\phi}$, e^{ix} and $e^{i(\phi-x)}$.

We shall assume in (17) that $c \neq 0$. If $c=0$, the desired final condition must be necessarily in the same class as $\texttt{diag}(e^{-i\phi}, e^{i\phi}, 1)$, that is, $x=0$. We shall show in Remark 4.2 that this is done without loss of generality. Under the condition $c \neq 0$, it follows from Proposition II.3 in [5] that $e^{(-A+P)t}$ must be a scalar matrix $e^{i\frac{2\pi k}{3}}\mathbf{1}$, for some integer k. We now quote one of the main results of [5] (cf. section IV in that paper) which transforms the optimal problem into an *integer optimization problem.* We then proceed to the solution of this integer optimization problem and show how this gives the minimum time and the optimal parameters a, b and c in (17). Our solution is simpler than the one in [5] and more general since it avoids assumptions on the eigenvalues of U_f which were made in [5].

4.1 Relation to an integer optimization problem

Consider the following submatrix of $-iA$ in (17): $\tilde{C} = \begin{pmatrix} -a & -c \\ -c & -b \end{pmatrix}$, and denote by $\hat{M}$ and $\hat{m}$ its largest and smallest eigenvalues (which are real since the matrix is symmetric), respectively. These depend on a, b, and c. Since $e^{(-A+P)t} = e^{i\frac{2\pi k}{3}}\mathbf{1}$, for the optimal time t, we must have (recall we scale t and angles by 2π) $\hat{M}t = \hat{\alpha} - \frac{k}{3} + \hat{l}$, and $\hat{m}t = \hat{\beta} - \frac{k}{3} + \hat{r}$, where $\hat{\alpha}$ is equal to $\phi - x$ and $\hat{\beta} = x$ or viceversa and $\hat{l}$ and $\hat{r}$ are integer numbers. We define α and β, and l and r so that

$$\phi_l := \hat{M}t = \hat{\alpha} - \frac{k}{3} + \hat{l} = \alpha - \frac{k}{3} + l, \tag{25}$$

$$\psi_r := \hat{m}t = \hat{\beta} - \frac{k}{3} + \hat{r} = \beta - \frac{k}{3} + r \tag{26}$$

with α and β in $[0,1)$ and l and r integers.[11] We summarize one of the main results of [5] (cf. formulas (40) and (45) in that paper) in the following theorem:

Theorem 1. *Given $\alpha, \beta \in [0,1)$ consider the minimum of*

$$t^2 = \frac{k^2}{12} + s^2 - (\phi_l^2 + \psi_r^2 + \phi_l\psi_r), \tag{27}$$

for k, l, r integer and s integer if k is even and half integer if k is odd subject to

$$\frac{k}{6} - s < \psi_r < -\frac{k}{3} < \phi_l < \frac{k}{6} + s. \tag{28}$$

Then the minimum time of the time optimal control problem is the minimum $t>0$ in (27) for values of α and β consistent with the values of x and ϕ (as from definition (25) (26)).

Therefore the solution of the optimal control problem is found once we know the solution of the problem to minimize (27) with the constraint (28), for any value of α and β in $[0,1)$. Once we know such a solution we replace for α and β the values deriving from the prescribed eigenvalues for U_f in (4) according to (25) and (26). We choose the values of α and β consistent with the values of x and ϕ, which give the smallest t (see subsection 4.3 below). The values of a, b, and c, once we have the minimum t and ϕ_l and ψ_r for the optimal l and r and the optimal k and s, are obtained as follows. One obtains b from

$$bt^3 = \frac{k}{3}\left(s^2 - \frac{k^2}{36}\right) + (\phi_l + \psi_r)(\phi_l\psi_r), \tag{29}$$

[11] It is not necessarily true that $\hat{\alpha} = \alpha$, $\hat{l} = l$, $\hat{\beta} = \beta$, $\hat{r} = r$, because $\phi - x$ may be negative. Thus, if, for instance $\hat{\alpha} = \phi - x < 0$ we choose $\alpha = \phi - x + 1$ and $l = \hat{l} - 1$.

then a from

$$(a+b)t = -(\phi_l + \psi_r), \tag{30}$$

and then c (c can be chosen positive or negative) from

$$c^2t^2 = abt^2 - \phi_l\psi_r. \tag{31}$$

Remark 4.1. In [5], formulas (27) and (29) come from the requirement that $e^{(-A+P)t} = e^{i\frac{2\pi k}{3}}\mathbf{1}$, using the relation between the coefficients of the characteristic polynomial and the eigenvalues of the 3×3 matrix $(-A+P)t$ [13] and using (31) which also follows from such a relation but for the matrix $\tilde{C}$. Formulas (30) and (31) also come from the relation between the eigenvalues and the coefficients of the characteristic polynomial for the matrix $\tilde{C}$. The inequality (28) follows from $c^2 > 0$, the fact that by definition $\phi_l \geq \psi_r$ (in fact it can be shown in [5] that strict inequality holds) and the fact which is proven in [5] (cf. Lemmas V.1, V.2 and V.3) that, in the search of the values (k,s,l,r) we can always reduce ourselves to the region in the (k,s) plane where $s > \frac{|k|}{2}$.

4.2 Solution of the integer optimization problem

We now show how to solve the optimization problem of Theorem 1. Here our derivation differs from the one in [5]. It is simpler and it generalizes it. As in [5], we shall make use of the functions **SI** and **LI**. The number $\mathbf{SI}(y)$ is the smallest integer which is strictly greater than y, i.e., the integer part of y+1, while $\mathbf{LI}(y)$ is the largest integer which is strictly smaller than y, i.e., the integer part of y-1. If p is an integer we have $\mathbf{SI}(y+p) = \mathbf{SI}(y) + p$ and $\mathbf{LI}(y+p) = \mathbf{LI}(y) + p$.

Using the definitions (25) and (26) in (28) and subtracting $-\frac{k}{3}$ in all terms we obtain

$$\frac{k}{2} - s < \beta + r < 0 < \alpha + l < \frac{k}{2} + s,$$

which gives the two inequalities

$$\frac{k}{2} - s - \beta < r < -\beta, \qquad -\alpha < l < \frac{k}{2} + s - \alpha. \tag{32}$$

This gives, using the definition of **SI** and **LI**,

$$\mathbf{SI}\left(\frac{k}{2} - s - \beta\right) \leq r \leq \mathbf{LI}(-\beta), \qquad \mathbf{SI}(-\alpha) \leq l \leq \mathbf{LI}\left(\frac{k}{2} + s - \alpha\right). \tag{33}$$

We are going to assume that $\alpha \in (0,1)$, $\beta \in (0,1)$, i.e., a range for these parameters with *open* interval on the left. We will discuss in Remark 4.2 why this is not a restriction.

With this assumption, and using the properties of the functions **SI** and **LI** with the fact that $\frac{k}{2} \pm s$ is always an integer number, we arrive at the inequalities

$$\frac{k}{2} - s \leq r \leq -1, \qquad 0 \leq l \leq \frac{k}{2} + s - 1. \tag{34}$$

The function t^2 to be minimized subject to the constraint (34) is the function (27) where we substitute the expressions of ϕ_l and ψ_r in (25) (26). This gives

$$t^2 = s^2 - \frac{k^2}{4} - \alpha^2 - \beta^2 - \alpha\beta + (\alpha+\beta)k - F_k(l,r), \tag{35}$$

where $F_k(l,r) := l^2 + r^2 + lr + l(2\alpha + \beta - k) + r(2\beta + \alpha - k)$ and we have separated the part that depends on (l,r) from the part that depends only on k and s. The minimum is found as

$$\min_{k,s} s^2 - \frac{k^2}{4} - \alpha^2 - \beta^2 - \alpha\beta + (\alpha+\beta)k - \max_{l,r} F_k(l,r),$$

where, for every k and s, the $\max_{l,r} F_k(l,r)$ is taken over the box defined in the inequalities (34). Because of the (quadratic) form of the function F_k such a maximum is attained at one of the corners. In particular, if we define

$p_1 := \left(0, \frac{k}{2} - s\right)$, $p_2 := (0, -1)$, $p_3 := \left(\frac{k}{2} + s - 1, \frac{k}{2} - s\right)$, and $p_4 = \left(\frac{k}{2} + s - 1, -1\right)$, this is the $\max_{p=p_1,p_2,p_3,p_4} F_k(p)$. Therefore the minimum time t^2 is calculated as the

$$\min_{p=p_1,p_2,p_3,p_4} \min_{k,s} Y_p(k,s),$$

for the function $Y_p(k,s) := s^2 - \frac{k^2}{4} - \alpha^2 - \beta^2 - \alpha\beta + (\alpha+\beta)k - F_k(p)$. Let us calculate $Y_p(k,s)$ for the values of $p = p_1, p_2, p_3, p_4$. We have $Y_{p_1}(k,s) = \alpha\left(s + \frac{k}{2}\right) + 2\beta s - \alpha^2 - \beta^2 - \alpha\beta$, $Y_{p_2}(k,s) = s^2 - \frac{k^2}{4} - \alpha^2 - \beta^2 - \alpha\beta + (\alpha+\beta-1)k + 2\beta + \alpha - 1$, $Y_{p_3}(k,s) = -\alpha^2 - \beta^2 - \alpha\beta + (s-1) + \frac{k}{2}(1-\alpha-\beta) + (\beta-\alpha)s + 2\alpha + \beta$, $Y_{p_4}(k,s) = -\alpha^2 - \beta^2 - \alpha\beta + 3(s-1) + \frac{k}{2}(\beta-1) - s(2\alpha+\beta) + 3\alpha + 3\beta$. To minimize these functions over k and s, it is convenient to write $s = \frac{|k|}{2} + j$ with $j \geq 1$ an integer (recall that $s > \frac{|k|}{2}$ and is an integer if k is even and a half integer if k is odd (cf. Remark 4.1)). We can then minimize over j and k. This process leads to the fact that, for all the values p_1, p_2, p_3, p_4, the minimum is achieved when $k = 0$ and $j = s = 1$. Furthermore the minima coincide, and we have

$$\min_{p=p_1,p_2,p_3,p_4} \min_{k,s} Y_p(k,s) = Y_{p_1}(0,1) = Y_{p_2}(0,1) = Y_{p_3}(0,1) = Y_{p_4}(0,1) = 2\beta + \alpha - \alpha^2 - \beta^2 - \alpha\beta. \tag{36}$$

Using the values $k = 0$ and $s = 1$ in (34), we find that the optimal l and r are $l = 0$ and $r = -1$. We arrive therefore at the solution of the integer optimization problem of Theorem 1. We have the following

Theorem 2. *For every α and β in $(0,1)$ with the definition of ϕ_l and ψ_r in (25) (26) the minimum over (l,r,k,s) of the function (27) subject to (28) is attained for $l = 0$, $r = -1$, $k = 0$, $s = 1$ and it is equal to*

$$t^2_{min} = -\alpha^2 - \beta^2 - \alpha\beta + 2\beta + \alpha. \tag{37}$$

From the theorem we also obtain the optimal values of ϕ_l and ψ_r. They are $\phi_l = \alpha$ and $\psi_r = \beta - 1$.

4.3 Optimal Synthesis

Our goal is now to translate the above solution of the optimization problem of Theorem 1 to the solution of the minimum time optimal control problem where we are given the two angles ϕ and x in $[0,1)$. This involves finding the correct corresponding values of α and β and then finding the optimal time and the corresponding ϕ_l and ψ_r, from which we can find the parameters a, b and c in (17) for the optimal synthesis.

We shall assume $x \neq 0$ and $\phi - x \neq 0$ and refer to Remark 4.2 below to remove these assumptions. There are a few cases to consider. Recall that x is always assumed ≥ 0. Assume first $\phi - x > 0$. There are two possibilities to choose α and β in (25) (26) both of them in principle possible: $\alpha = x$, $\beta = \phi - x$ or $\alpha = \phi - x$ and $\beta = x$. In both cases $\alpha, \beta \in (0,1)$ and therefore we can use (37). In the first case (37) gives

$$t^2_{min} = V_3(x,\phi) := 2\phi - x - \phi^2 + \phi x - x^2. \tag{38}$$

In the second case, it gives

$$t^2_{min} = V_4(x,\phi) := x + \phi - \phi^2 + x\phi - x^2. \tag{39}$$

We make the choice which gives the smallest value for t^2 which depends on the value of x and ϕ. In particular if $\phi > 2x$ we choose V_4 otherwise we choose V_3. The two values coincide when $\phi = 2x$. Assume now that $\phi - x < 0$. In order to define α or β we need to add $+1$ to make this positive. Therefore, in this case, we can choose $\alpha = 1 + \phi - x$ and $\beta = x$ or, viceversa, $\alpha = x$ and $\beta = 1 + \phi - x$. In both cases $\alpha, \beta \in (0,1)$ and therefore we can use (37). In the first case (37) gives

$$t^2_{min} = V_1(x,\phi) := -\phi^2 - x^2 - \phi + x\phi + 2x. \tag{40}$$

In the second case, it gives

$$t^2_{min} = V_2(x,\phi) := 1 - x^2 + \phi x - \phi^2.$$

We choose the minimum of the two which depends on the point (x,ϕ). In particular if $\phi < 2x - 1$ the minimum is given by V_2, if $\phi > 2x - 1$ it is given by V_1. The two values coincide when $\phi = 2x - 1$. The situation is summarized in the Figure 2 which represents the box $\{(x,\phi) | 0 \leq x \leq 1, 0 \leq \phi \leq 1\}$. According to the region where the desired final condition (x,ϕ) is in the box, we choose the function V_1, V_2, V_3 or V_4 to calculate the minimum time. Also the corresponding values for ϕ_l and ψ_r are found by the corresponding values of α and β as $\phi_l = \alpha$, $\psi_r = \beta - 1$.

Remark 4.2. In the above discussion to find the minimum for the optimal control problem we have assumed $x \neq 0$ and $\phi - x \neq 0$. This was done for us to be able to use formula (37) which was derived under the assumption of $\alpha, \beta \in (0,1)$, i.e., $\alpha, \beta \in [0,1)$ which is the case in Theorem 1, and $\alpha, \beta \neq 0$. However, from standard results in sub-Riemannian geometry (see, e.g., Chapter 3 in [1]), we know that the sub-Riemannian distance and therefore the minimum time function is continuous and therefore the values of the minimum time extend by continuity to the set $\phi = x$ as well as to the set $x = 0$. Furthermore the box $\{(x,\phi) \,|\, 0 \leq x < 1, 0 \leq \phi < 1\}$ should be seen as a torus, i.e., with the segments corresponding to $\phi = 0$ and $\phi = 1$ identified, as well as the segments corresponding to $x = 0$ and $x = 1$ identified. A direct verification shows that the minimum time function is indeed continuous on this segments as well as on the line $\phi = x$, as expected.

The condition $x \neq 0$ was also used to rule out $c = 0$ in (17). If $x = 0$, we have to consider the case $c = 0$ as well, since Theorem 1 assumes that $c \neq 0$. In fact if $x = 0$ the optimal is achieved with $c = 0$. To prove this, assume by contradiction that the optimal time is obtained with $c \neq 0$. Therefore using Theorem 1 and the formula for t^2 in (39), $t^2_{min} = \phi - \phi^2$. The parameters ϕ_l and ψ_r are chosen according to Figure 2 (with $x = 0$) to be $\phi_l = \phi$ and $\psi_r = -1$. With these values, we can find b, a, and c in sequence according to (29) (30) and (31). In particular, from (29) (recall $k = 0$), we get $b = \frac{1}{t_{min}}$. Using this in (30), we get $a = -\frac{\phi}{t_{min}}$, and using the expressions of a and b in (31) we obtain $c^2 t^2_{min} = 0$, which is a contradiction with the assumption that $c \neq 0$. Therefore for $x = 0$, we have to assume $c = 0$. We now show how to find the optimal control and time in this case, when the final condition is in the same class as $\texttt{diag}(e^{-i\phi}, e^{i\phi}, 1)$. With $c = 0$ we can write A in (17) as $A = B + S$ where $B = \texttt{diag}(i\frac{b}{2}, i\frac{b}{2}, 0)$ $S = \texttt{diag}(ia + i\frac{b}{2}, -ia - i\frac{b}{2}, ib)$. We have (see (13)) that we have to find the matrix S and the minimum t such that $e^{At}e^{(-A+P)t} = e^{St}e^{(-S+P)t}$, with P given in (17), is in the same class as $\texttt{diag}(e^{-i\phi}, e^{i\phi}, 1)$, which, given the form of $e^{St}e^{(-S+P)t}$ is equivalent to $e^{St}e^{(-S+P)t} = \texttt{diag}(e^{-i\phi}, e^{i\phi}, 1)$. Defining $\eta := a + \frac{b}{2}$, this means that we have to find the minimum t and a real parameter η such that

$$e^{\hat{S}t}e^{(-\hat{S}+\hat{P})t} = \texttt{diag}(e^{-i\phi}, e^{i\phi}), \qquad \hat{S} := \begin{pmatrix} i\eta & 0 \\ 0 & -i\eta \end{pmatrix}, \qquad \hat{P} := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{41}$$

This is the same as the KP problem on $SU(2)$ treated in [3] and the solution can be found in that paper. In particular, this is a special case of Theorem 1 in [3] as the final condition is in the singular part of the quotient space associated with the KP problem in the $SU(2)$ case. Using the results of that paper, we obtain $t^2_{min} = \phi - \phi^2$ and $\eta = \sqrt{\frac{1}{\phi(1-\phi)} - 4} = a + \frac{b}{2}$.[12] We remark that the value of the minimum time coincides with the limit $x \to 0$ of the minimum time obtained when $c \neq 0$ as expected from the continuity of the sub-Riemannian distance (and therefore the minimum time function).

We give a simple example of application of the above technique and postpone to the next section more complicated examples, further discussion and generalizations.

Example 4.3. Assume we want to reach a final condition of the form $X_f = \begin{pmatrix} 1 & 0 \\ 0 & X_f \end{pmatrix}$, that is, $\phi = 0$. Choose $0 < x \leq \frac{1}{2}$. We are in the region of V_3 and the minimum time is $t^2_{min} = 2x - x^2$. We have $\alpha = 1 - x$ and $\beta = x$ so that $\phi_l = 1 - x$, $\psi_r = x - 1$. Application of the formulas (29), (30) and (31) give $a = b = 0$ and $c^2 = \frac{1-x}{2x}$. As expected if we repeat the process for replacing x with $1 - x$, which represents the same equivalence class of final conditions, we obtain the same values for the minimum time and parameters, a, b and c.

5 Extensions and examples

We start with an example of optimal synthesis for a system without drift such as (7).

[12] To translate the result in [3] the notation η in this paper corresponds to $\frac{\omega}{2}$ in that paper; $\frac{T}{2\pi}$ in [3] is t for us; and γ in [3] has to be taken equal to 2. Then one refers to formulas (18) and (19) in [3] calculated for $|\omega| = \sqrt{\frac{4\pi^2}{T^2} - \gamma^2}$, i.e., at the endpoints of the curve $\mathcal{F}_T$ defined in [3] which simplifies the formulas. Imposing $x + iy = e^{-i\phi}$ one obtains the condition on T which gives the minimum time, and, from that, one derives the value of ω.

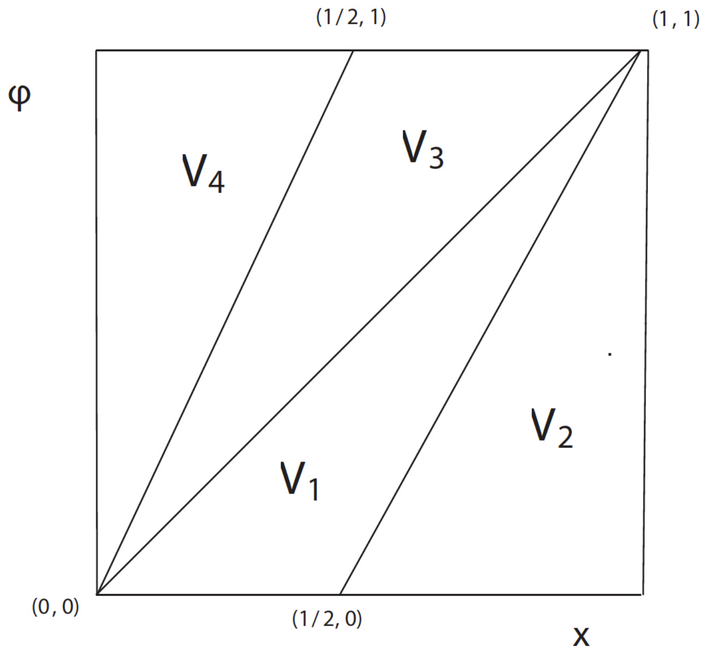

Figure 2: Functions to be used according to the point (x, ϕ) to calculate square of the minimum time (normalized by 2π). In the region of V_1 the choice of (α, β) is $\alpha = 1 + \phi - x$, $\beta = x$, so that $\phi_l = 1 + \phi - x$, $\psi_r = x - 1$. In the region of V_2 the choice of (α, β) is $\alpha = x$, $\beta = 1 + \phi - x$, so that $\phi_l = x$, $\psi_r = \phi - x$. In the region of V_3 the choice of (α, β) is $\alpha = x$, $\beta = \phi - x$, so that $\phi_l = x$, $\psi_r = \phi - x - 1$. In the region of V_4 the choice of (α, β) is $\alpha = \phi - x$, $\beta = x$, so that $\phi_l = \phi - x$, $\psi_r = x - 1$. These values along with $k = 0$ are used in (29), (30) and (31) in order to obtain the parameters a, b, and c of the optimal control.

5.1 Example 1

Let us assume that we would like to reach a final condition given by the following Hadamard-like gate:

$$U_f = \begin{pmatrix} -i & 0 & 0 \\ 0 & -\frac{1}{2} + \frac{i}{2} & -\frac{1}{2} + \frac{i}{2} \\ 0 & -\frac{1}{2} + \frac{i}{2} & \frac{1}{2} - \frac{i}{2} \end{pmatrix}. \tag{42}$$

The eigenvalues of U_f are $e^{-i\frac{\pi}{2}}$, $e^{i\frac{3\pi}{4}}$ and $e^{-i\frac{\pi}{4}}$. Therefore, using the notation in Section 4, after scaling by 2π, we choose

$$\phi = \frac{1}{4}, \qquad x = \frac{3}{8}, \qquad \phi - x = -\frac{1}{8},$$

and we are in the region V_1 of Figure 2. According to the caption of Figure 2, we have

$$\alpha = \frac{7}{8}, \qquad \beta = \frac{3}{8}, \qquad \phi_l = \frac{7}{8}, \qquad \psi_r = -\frac{5}{8}.$$

Using (40), we obtain the value of the minimum time

$$t_{min} = \frac{5}{8}.$$

With the values of t_{min}, ϕ_l and ψ_r, by equation (29), we obtain $b = -\frac{14}{25}$ and using (30), (31), we obtain $a = \frac{4}{25}$, $c = \frac{3\sqrt{91}}{25}$. Therefore, the matrix A in (17) is

$$A = i \begin{pmatrix} -\frac{2}{5} & 0 & 0 \\ 0 & -\frac{4}{25} & -\frac{3\sqrt{91}}{25} \\ 0 & -\frac{3\sqrt{91}}{25} & \frac{14}{25} \end{pmatrix}.$$

Going back to the original time scale, let $t = 2\pi t_{min} = \frac{5\pi}{4}$, one then computes the final condition to be

$$e^{At}e^{(-A+P)t} = e^{At} = \begin{pmatrix} -i & 0 & 0 \\ 0 & -\frac{3}{10\sqrt{2}} + \frac{3i}{10\sqrt{2}} & \left(-\frac{1}{10} + \frac{i}{10}\right)\sqrt{\frac{91}{2}} \\ 0 & \left(-\frac{1}{10} + \frac{i}{10}\right)\sqrt{\frac{91}{2}} & \frac{3}{10\sqrt{2}} - \frac{3i}{10\sqrt{2}} \end{pmatrix} := \hat{U}_f,$$

since $e^{(-A+P)t} = \mathbf{1}$. This is not the same as the desired final condition U_f in (42) but, as expected, it is in the same equivalence class. The matrix

$$K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}\sqrt{2 + \frac{3+\sqrt{91}}{5\sqrt{2}}} & -\frac{1}{2}\sqrt{2 - \frac{3+\sqrt{91}}{5\sqrt{2}}} \\ 0 & \frac{1}{2}\sqrt{2 - \frac{3+\sqrt{91}}{5\sqrt{2}}} & \frac{1}{2}\sqrt{2 + \frac{3+\sqrt{91}}{5\sqrt{2}}} \end{pmatrix} \in e^{\mathcal{K}}, \tag{43}$$

is such that $K\hat{U}_f K^{-1} = U_f$. Therefore the matrices A and P for the optimal synthesis (12) have to be changed to KAK^{-1} and KPK^{-1}, with K in (43).

5.2 Extension of the procedure to systems with drift

The treatment for the case of a system without drift (7) also gives the solution for the optimal control problem for the original system with drift (1). This is due to the fact that in Section 4 we have constructed the function t_{min} which gives the minimum time for *any* final condition in $e^{\mathcal{K}}$. Recall also, from Sections 1 and 2, that there is an explicit relation between the trajectory $X = X(t)$ of the driftless system (7) and the corresponding trajectory of the system with drift (1), $U = U(t)$, given by $X(t) = e^{-\hat{A}t}U(t)$, and a one-to-one correspondence between the controls given by (8) (9). In view of these facts, we have the following observation.

Proposition 5.1. The minimum time to reach the final condition U_f for system (1) is the smallest t such that

$$t = t_{min}(e^{-\hat{A}t}U_f). \tag{44}$$

Proof. Consider the continuous function $s(t) := t - t_{min}(e^{-At}U_f)$ which is negative when $t = 0$ (assuming $U_f \neq \mathbf{1}$) and positive for large t since t_{min} is bounded. Therefore there is at least one point t such that $s(t) = 0$, i.e., (44) is verified. Consider the first point $t = \hat{t}$ where $s(t) = 0$. We have $s(t) < 0$ for $t < \hat{t}$. The value $t = \hat{t}$ is such that there exists an admissible control driving the system (1) to U_f, in time $\hat{t}$. This is the (modified) control used to drive the driftless system to $e^{-\hat{A}\hat{t}}U_f$. Therefore the minimum time for the system with drift t_d has to be such that $t_d \leq \hat{t}$. However if $t_d < \hat{t}$ we have $s(t_d) < 0$, that is $t_d - t_{min}(e^{-\hat{A}t_d}U_f) < 0$. The control driving the system with drift in time t_d will drive (after modification) the system without drift to $e^{-\hat{A}t_d}U_f$ in time less than $t_{min}(e^{-\hat{A}t_d}U_f)$ which contradicts the minimality of t_{min}. Thus, equality must hold. □

The proof of this proposition also gives us information about the optimal control synthesis for the system with drift. Once we know the minimum time, the control for the system without drift gives us the control for the system with drift, using the modification described in (8) (9). We illustrate the procedure with an example.

5.3 Example of application to a system with drift (1)

Suppose that matrix $\hat{A}$ in (1) is given by

$$\hat{A} = \begin{pmatrix} 2\pi i & 0 & 0 \\ 0 & \pi i & 0 \\ 0 & 0 & -3\pi i \end{pmatrix}, \tag{45}$$

and assume that the desired final condition is a NOT gate on the lowest two energy levels, with a phase shift between these two levels and the highest level:

$$U_f = \begin{pmatrix} e^{-i\frac{\pi}{4}} & 0 & 0 \\ 0 & 0 & e^{i\frac{\pi}{4}} \\ 0 & -1 & 0 \end{pmatrix}. \tag{46}$$

We have

$$e^{-\hat{A}t}U_f = \begin{pmatrix} e^{-i\frac{\pi}{4}-i2\pi t} & 0 & 0 \\ 0 & 0 & e^{i\frac{\pi}{4}-i\pi t} \\ 0 & -e^{i3\pi t} & 0 \end{pmatrix}. \tag{47}$$

The eigenvalues of $e^{-\hat{A}t}U_f$ are $e^{-i\frac{\pi}{4}-i2\pi t}$, $ie^{i\frac{\pi}{8}+i\pi t}$ and $-ie^{i\frac{\pi}{8}+i\pi t}$. We scale the phases by 2π. Using the notation in Section 4, we denote these eigenvalues by $e^{-i\phi}$, e^{ix} and $e^{i(\phi-x)}$. We have

$$\phi = \phi(t) = \frac{1}{8} + t \ (\texttt{mod}\, 1),$$

$$x = \frac{5}{16} + \frac{t}{2} \ (\texttt{mod}\, 1).$$

The initial point, $x = \frac{5}{16}$, $\phi = \frac{1}{8}$ is in the region V_1 in Figure 2. As t increases the trajectory $x = x(t)$, $\phi = \phi(t)$ crosses into the region V_3 when $t = \frac{3}{8}$ and $x = \phi = \frac{1}{2}$. Then the points stay in the region V_3 until $t = \frac{7}{8}$, crossing into region V_2 when $x = \frac{3}{4}$, $\phi = 1$. Let us look for a $t < \frac{7}{8}$ first. Replacing $x = \frac{5}{16} + \frac{t}{2}$, $\phi = \frac{1}{8} + t$ in the expression of t_{min} in (40), we obtain, for $0 \le t \le \frac{3}{8}$,

$$t_{min} = \sqrt{\frac{109}{256} - \frac{3}{4}t^2 - \frac{3}{16}t}.$$

However there is no $t \in (0, \frac{3}{8}]$ such that $t = t_{min}$ with the above expression of t_{min}. We consider then the next interval $(\frac{3}{8}, \frac{7}{8})$, where we use the expression (38) for t_{min}, with $x = \frac{5}{16} + \frac{t}{2}$, $\phi = \frac{1}{8} + t$. This gives $t_{min} = \sqrt{\frac{21}{16}t - \frac{3}{4}t^2 - \frac{35}{256}}$. With this expression of t_{min}, the condition (44) gives

$$t = \sqrt{\frac{21}{16}t - \frac{3}{4}t^2 - \frac{35}{256}},$$

which is solved for $t \in (\frac{3}{8}, \frac{7}{8})$ by $t = \frac{5}{8}$. This is the minimum time we were looking for.

As a result,

$$\phi = \frac{1}{8} + t = \frac{3}{4}, \qquad x = \frac{5}{16} + \frac{t}{2} = \frac{5}{8}, \qquad \phi - x = \frac{1}{8},$$

and using the prescription for region V_3 from figure 2 we have

$$\phi_l = \frac{5}{8}, \qquad \psi_r = -\frac{7}{8}.$$

With these values, by solving (29), (30) and (31) (with $k = 0$), we obtain that

$$a = -\frac{4}{25}, \qquad b = \frac{14}{25}, \qquad c = \frac{3\sqrt{91}}{25}$$

The matrix A in (17) is

$$A = i \begin{pmatrix} \frac{2}{5} & 0 & 0 \\ 0 & \frac{4}{25} & -\frac{3\sqrt{91}}{25} \\ 0 & -\frac{3\sqrt{91}}{25} & -\frac{14}{25} \end{pmatrix}.$$

Therefore, we obtain that, with $t = \frac{5}{8}$,

$$X(t) = e^{2\pi A t} e^{2\pi(-A+P)t} = \begin{pmatrix} i & 0 & 0 \\ 0 & -\frac{3}{10\sqrt{2}} - \frac{3}{10\sqrt{2}}i & \left(\frac{1}{10} + \frac{i}{10}\right)\sqrt{\frac{91}{2}} \\ 0 & \left(\frac{1}{10} + \frac{i}{10}\right)\sqrt{\frac{91}{2}} & \frac{3}{10\sqrt{2}} + \frac{3}{10\sqrt{2}}i \end{pmatrix}, \tag{48}$$

with $e^{2\pi(-A+P)t} = \mathbf{1}$.

This is in the same class as (47) with $t = \frac{5}{8}$. In fact, the two matrices differ by by conjugation with

$$K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \left(-\frac{1}{20} + \frac{i}{20}\right)\left(\sqrt{35} + \sqrt{65}\right) e^{\frac{15\pi i}{8}} & \frac{\sqrt{35}-\sqrt{65}}{10\sqrt{2}} \\ 0 & \left(\frac{1}{20} - \frac{i}{20}\right)\left(\sqrt{35} - \sqrt{65}\right) e^{\frac{15\pi i}{8}} & \frac{\sqrt{35}+\sqrt{65}}{10\sqrt{2}} \end{pmatrix} \in e^{\mathcal{K}}.$$

The optimal synthesis for the system without drift is obtained using (12) with A replaced by $KAK^\dagger$ and P replaced by $KPK^\dagger$ with the above K. Formulas (8) (9) give the corresponding optimal control for the system with drift.

5.4 Some more generalizations and remarks

As we have mentioned in the Introduction, Lambda systems have been mostly treated in the context of adiabatic control. In particular the STIRAP protocol [6] achieves population transfer between the lowest two levels by minimizing the average population in the highest level. In its basic formulation, it consists of a sequence of two pulses with slowly varying amplitudes and achieves its goal in a large (theoretically infinite) amount of time.

Minimizing the population in the highest level is required because this is the population mostly subject to decaying to lower (un-modeled) levels, thus causing de-coherence. In this paper, we have taken a different approach to counteract de-coherence, namely we have looked for the fastest possible transfer. In this sense, our protocol is the opposite of adiabatic protocols, such as STIRAP, which take a large amount of time. However, our scheme does not consider any constraint in the population in the highest level. In fact a direct calculation shows that such a population must be nonzero in the control interval.[13] Given the explicit nature of our solution however we can calculate the actual cost in terms of the average population in the highest level for any state transfer. We also know that the optimal time is inversely proportional to the bound on the control and we can study how to find the best compromise between population in the highest level and the time of transfer.

Finally we remark that our protocols can be used not only for Lambda systems but for any three-level quantum system where we want to transfer population between two states that are connected only through a third one. In fact, many of our results, in particular the ones of Sections 2 and 3 are valid for general three level quantum systems in the KP configuration and can be potentially used in the future to find optimal protocols for state transfers different from the ones considered in Sections 4 and 5.

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