Super-resolution via Prony-Type Polynomials

Anna Veselovska
Department of Mathematics & Munich Data Science Institute
Technical University of Munich
Garching (Munich), Germany
anna.veselovska@tum.de

Jürgen Prestin
Institute of Mathematics
University of Lübeck
Lübeck, Germany
juergen.prestin@uni-luebeck.de

***Abstract*—The problem of hidden periodicity in three dimensions is to recover frequency vectors $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_N \in [0, 2\pi)^3$ using finitely many samples of the exponential sum $f(\mathbf{n}) = \sum_{j=1}^{N} a_j \exp\left(-\mathrm{i}\langle \boldsymbol{\omega}_j, \mathbf{n}\rangle\right)$, where $a_1, \ldots, a_N \in \mathbb{C}\backslash\{0\}$ and $\mathbf{n} \in \mathbb{Z}^3$. Inspired by the approaches developed in [11], [30], we consider specifically constructed polynomials, which are called Prony-type polynomials, and show that the frequency vectors $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_N$ can be recovered via a set of common zeros of such polynomials. By employing Cantor tuple functions, we position the method of Prony-type polynomials within the spectrum of sampling requirements between the methods proposed in [21], [22]. While the Prony-type polynomial method demands more samples than the approach in [22], numerical experiments indicate that it exhibits greater stability in the presence of noisy data.**

***Index Terms*—Prony's method, exponential sums, Prony-type polynomials, super-resolution, common zeros**

I. Introduction

The super-resolution problem or the problem of hidden periodicity is a fundamental problem in digital signal processing with many practical applications [7], [14], [17] and is to find the set frequency vectors $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \cdots, \boldsymbol{\omega}_N \in [0, 2\pi)^d$ and parameters $a_1, a_2, \ldots, a_N \in \mathbb{C}\backslash\{0\}$ out of finitely many samples of the exponential sums

$$f(\mathbf{n}) = \sum_{j=1}^{N} a_j \exp\left(-\mathrm{i}\langle \boldsymbol{\omega}_j, \mathbf{n}\rangle\right), \qquad (1)$$

where $\mathbf{n} = (n_1, \ldots n_d) \in \mathbb{Z}^d = \mathbb{Z} \times \ldots \times \mathbb{Z}$ and the product $\langle \boldsymbol{\omega}_j, \mathbf{n}\rangle = \omega_{j,1} n_1 + \ldots + \omega_{j,d} n_d$.

The univariate SR problem has been considered initially by G.R. de Prony in 1795 [1], where it has been shown that the hidden periodicity can be found via analyzing the zeros of the so-called Prony polynomial. This approach has been named *Prony's method*. Recently, the SR problem in general – and Prony's method in particular – has garnered significant attention, leading to the development of various approaches for finding solutions. On the one hand, several techniques have been introduced to enhance the stability of Prony's method (see [3], [4], [11]). Recently, in [25], [32], exponential analysis was reformulated as a Padé approximation problem, allowing parameter recovery even in cases of small separation distances.

On the other hand, efforts have been made to extend Prony's method to the multidimensional case. Among these, the first comprehensive generalization was introduced in [21], which employs kernel basis analysis of the multilevel Toeplitz matrix of moment data. This method requires at least $(2N + 1)^d$ samples, with d representing the dimension.

In contrast, the algorithms in [16], [20], [22] take a different approach, relying on function sampling along multiple lines within a hyperplane. This transformation reduces the problem to a univariate setting, which can then be addressed using classical one-dimensional techniques. A related study in [26] explores the problem on a hyperbolic cross, demonstrating that Prony's problem with N frequency vectors can be solved with at most $(d + 1)N^2 \log^{2d-2} N$ function evaluations.

Besides, less than ten years ago, several authors [13], [19] considered the SR problem from the perspective of an atomic measure recovery problem in light of convex optimization. Hereupon, the generalization to higher dimensions on the torus has been considered in [23], [27], [28], [31].

Nevertheless, the stability of numerical solutions in the case of noisy data still has a lot of open questions, especially when the number of parameters increases. As it has been shown in [11], [30], stability can be achieved by incorporating orthogonal polynomial approaches and autocorrelation sequences.

In this work, we generalize the line of methods proposed in [6], [11], [30] to the three-dimensional case by considering *Prony-type polynomials*. We show that the parameters $\mathbf{z}_1, \ldots, \mathbf{z}_N$ can be recovered as a set of common zeros of a particular set of Prony-type polynomials and illustrate a stable behavior of the method in the noisy data case.

The outline of this paper is as follows. In Section II, we fix the notation and recall basic concepts related to the multivariate polynomials. In Section III we define Prony-type polynomials in three dimensions, introduce the method of Prony-type polynomials, and prove the main result of this paper. Numerical results are provided in Section IV.

II. Notations

A. Problem Formulation

Let $N \in \mathbb{N}$ be an integer, $a_1, a_2, \ldots, a_N \in \mathbb{C}\backslash\{0\}$ and $\boldsymbol{\omega}_j = (\omega_{j,1}, \ldots, \omega_{j,d}) \in [0, 2\pi)^d$ with $\boldsymbol{\omega}_j \neq \boldsymbol{\omega}_k$ for $j \neq k$, $j, k{=}1, \ldots, N$. Let us consider a function $f\colon\ \mathbb{Z}^d \to \mathbb{C}$ of the form

$$f(\mathbf{n}) = \sum_{j=1}^{N} a_j \exp\left(-\mathrm{i}\langle \boldsymbol{\omega}_j, \mathbf{n}\rangle\right), \tag{2}$$

where $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ and $\langle \boldsymbol{\omega}_j, \mathbf{n}\rangle = \omega_{j,1}n_1 + \ldots + \omega_{j,d}n_d$. The function f is called an *N-sparse multivariate exponential sum* with the pairwise distinct frequency vectors $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \ldots, \boldsymbol{\omega}_N$ and coefficients $a_1, a_2, \ldots, a_N$.

It is convenient to use the following notation for the exponent vectors

$$\begin{aligned}\exp\left(-\mathrm{i}\boldsymbol{\omega}_j\right) &= \left(\exp(-\mathrm{i}\omega_{j,1}), \ldots, \exp(-\mathrm{i}\omega_{j,d})\right)\\ &= (z_{j,1}, \ldots, z_{j,d}) = \mathbf{z}_j, \quad j = 1, \ldots, N.\end{aligned}$$

This together with the multi-index notation $\mathbf{z}_j^{\mathbf{n}} = z_{j,1}^{n_1} \ldots, z_{j,d}^{n_d}$ for $\mathbf{n}{=}(n_1, \ldots, n_d) \in \mathbb{N}^d$ allows us to rewrite the exponential sum (2) in a little bit more compact form as follows

$$f(\mathbf{n}) = \sum_{j=1}^{N} a_j \mathbf{z}_j^{\mathbf{n}}. \tag{3}$$

In representation (3), the elements $\mathbf{z}_1, \ldots, \mathbf{z}_N \in \mathbb{T}^d = \mathbb{T} \times \cdots \times \mathbb{T}$, where $\mathbb{T}{=}\{z \in \mathbb{C} : |z|{=}1\}$, are called *parameters* of the exponential sums f. In such a way, instead of dealing with detecting the frequency vectors $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \ldots, \boldsymbol{\omega}_N$ one can consider an analogous problem and search for the parameters $\mathbf{z}_1, \ldots, \mathbf{z}_N$. Once the parameters $\mathbf{z}_j$ are detected, the coefficients of $a_1, \ldots, a_N$ can be computed by simply solving a linear system of equations.

B. Monomials and Cantor functions

In this subsection based on [10], [18], we recall some notations and definitions related to multivariate monomials. For a tuple of non-negative integers $\mathbf{k} = (k_1, k_2, \ldots, k_d) \in \mathbb{Z}_+^d$ and a multivariate complex variable $\mathbf{z} = (z_1, z_2 \ldots, z_d)$ we use multi-index notations

$$\mathbf{z}^{\mathbf{k}} = z_1^{k_1} \ldots z_d^{k_d}, \tag{4}$$

$$\langle \mathbf{z}, \mathbf{k}\rangle = \langle \mathbf{k}, \mathbf{z}\rangle = z_1 k_1 + \cdots + z_d k_d$$

and $\alpha\mathbf{z}{=}(\alpha z_1, \alpha z_2, \cdots, \alpha z_d)$, for any real number $\alpha \in \mathbb{R}$. The product (4) is called a monomial in variables $z_1, \ldots, z_d$ and the sum of exponents $|\mathbf{k}| = k_1 + \ldots, k_d$ is called the *total degree* of the monomial $\mathbf{z}^{\mathbf{k}}$.

Let $\boldsymbol{\alpha}{=}(\alpha_1, \ldots, \alpha_d)$ and $\boldsymbol{\beta}{=}(\beta_1, \ldots, \beta_d)$ be elements of $\mathbb{Z}_+^d$; we say that $\boldsymbol{\alpha}$ is greater than $\boldsymbol{\beta}$ with respect to the Graded Lexicographic Order (Grlex) $\boldsymbol{\alpha} >_{\mathrm{grlex}} \boldsymbol{\beta}$, if $|\boldsymbol{\alpha}| > |\boldsymbol{\beta}|$ or $|\boldsymbol{\alpha}|{=}|\boldsymbol{\beta}|$ and in the vector difference $\boldsymbol{\alpha}{-}\boldsymbol{\beta}$ the leftmost nonzero entry is positive. Accordingly, we say that a monomial $\mathbf{z}^{\boldsymbol{\alpha}}$ is greater than a monomial $\mathbf{z}^{\boldsymbol{\beta}}$ with respect to the Grlex, $\mathbf{z}^{\boldsymbol{\alpha}} >_{\mathrm{grlex}} \mathbf{z}^{\boldsymbol{\beta}}$, if $\boldsymbol{\alpha} >_{\mathrm{grlex}} \boldsymbol{\beta}$. For every $n \in \mathbb{Z}_+$ there is the fixed number of monomials $\mathbf{z}^{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}_+^d$, of the total degree equal to n. Having fixed the Grlex monomial order, one gets also the limited number of monomials of the total degree less than or equal to n [18], namely, $\#\{\mathbf{z}^{\mathbf{k}}\colon\ |\mathbf{k}| = n\} = \binom{n+d-1}{n}$ and $\#\{\mathbf{z}^{\mathbf{k}}\colon\ |\mathbf{k}| \leq n\} = \binom{n+d}{n}$. Besides, due to the Grlex, all monomials can be placed into one row of ordered monomials. Enumerating we get the following

$$\begin{aligned}&\underbrace{1}_{0}, \underbrace{z_1,}_{1} \ldots, \underbrace{z_d,}_{d} \ldots \quad \underbrace{z_1^n,}_{\binom{n+d}{n},} \underbrace{z_1^{n-1}z_2,}_{\binom{n+d}{n}+1,} \quad \ldots, \\ &\underbrace{z_1^{n-i}z_2^{i}}_{\binom{n+d}{n}+i,} \ldots, \quad \underbrace{z_d^n,}_{\binom{n+d}{n}+\binom{n+d-1}{n}-1,} \quad \underbrace{z_1^{n+1},}_{\binom{n+1+d}{n+1}} \ldots\end{aligned} \tag{5}$$

The one-to-one correspondence between the set of all monomials $\mathbf{z}^{\mathbf{k}}$ and between the set of nonnegative integers, i.e. numbers of positions that these monomials take in the row of ordered monomials (5) is provided by the Cantor tuple function and its inverse. From now on we focus on the three-dimensional case, that is $d = 3$, and consider the Cantor tuple function

$$c(k_1, k_2, k_3) = \frac{(k_1+k_2+k_3)^3+3(k_1+k_2+k_3)^2+3(k_2+k_3)^2+2k_1+5k_2+11k_3}{6}$$

that maps the integer grid, $\mathbb{Z}_+^3$, onto the set of nonnegative integers $\mathbb{Z}_+$, by assigning to each vector $\mathbf{k} = (k_1, k_2, k_3) \in \mathbb{Z}_+^3$ the nonnegative integer $c(k_1, k_2, k_3) \in \mathbb{Z}_+$, see, e.g., [2]. Herewith, there exist the inverting Cantor functions

$$l_1, l_2, l_3 : \mathbb{Z}_+ \to \mathbb{Z}_+,$$

such that the Cantor map is one-to-one [12], that is for all $N, k_1, k_2, k_3 \in \mathbb{Z}_+$ it holds that

$$c\big(l_1(N), l_2(N), l_3(N)\big) \equiv N, \qquad \text{and } l_j(c(k_1, k_2, k_3)) = k_j,$$

for $j = 1, 2, 3$. For convenience, we will denote the vector of inverse Cantor functions by

$$\mathbf{l}(N) = \big(l_1(N), l_2(N), l_3(N)\big) \in \mathbb{Z}_+^3.$$

III. Prony-type polynomials

Let $f\colon\ \mathbb{Z}_+^3 \to \mathbb{C}$ be an N-sparse exponential sum in three dimensions with parameters $\mathbf{z}_1, \ldots, \mathbf{z}_N$ and coefficients $a_1, a_2, \ldots, a_N \in \mathbb{C}\backslash\{0\}$

$$f(\mathbf{n}) = \sum_{j=1}^{N} a_j \mathbf{z}_j^{\mathbf{n}}. \tag{6}$$

Since f depends on $\mathbf{n}$, we consider f as a three-dimensional sequence $f(\mathbf{n})$. Let us remark, that further in the paper the number $N \in \mathbb{Z}_+$ will always denote the number of parameters in (6). In the two-dimensional case using f we build an analog of the Toeplitz matrix mentioned in the Prony algorithm. Namely, let us consider the matrix

$$\mathcal{T}_N \quad = \quad \left(f_{\mathbf{l}(k)-\mathbf{l}(j)}\right)_{k,j=0}^{N-1}$$

which we call the *multivariate Toeplitz matrix* of f-samples. The index set of the elements of $\mathcal{T}_N$ we denote by

$$I_N = \left\{\mathbf{i} \in \mathbb{Z}_+^3 \colon \mathbf{i} = \mathbf{l}(k) - \mathbf{l}(j),\ k, j = 0, \ldots, N-1\right\}.$$

For the same $N \in \mathbb{Z}_+$, we denote by

$$\widehat{\mathbf{z}}_N = \left(\mathbf{z}^{\mathbf{l}(j)}\right)_{j=0}^{N-1}$$

the *row vector of monomials* that obviously consists of the first N monomials from the row of ordered monomials (5).
The next object we consider is some set of elements from the integer grid $D_N \subset \mathbb{Z}_+^3$ defined in the following way:

$$D_N = \left\{\mathbf{l}(j) \colon j = N, \ldots, N + l_3(N) + \binom{|\mathbf{l}(N)| + 2}{|\mathbf{l}(N)|}\right\}. \tag{7}$$

The set D_N is called the *degree set* of f, and it consists of exponents we will use further for constructing Prony-type polynomials. For all vectors $\mathbf{m} = (m_1, m_2, m_3) \in D_N$, let us denote by

$$\mathbf{f}_{N,\mathbf{m}} = \left(f_{\mathbf{m}-\mathbf{l}(j)}\right)_{j=0}^{N-1}$$

the column vectors called the *column vectors of additional samples*. The set of indices of the vectors $\mathbf{f}_{N,\mathbf{m}}$ for all $\mathbf{m} \in D_N$ we denote by

$$\begin{aligned} I_N^+ = \{\mathbf{i} \in \mathbb{Z}_+^3 :\ & \mathbf{i} = \mathbf{m} - \mathbf{l}(j), \\ & \mathbf{m} \in D_N,\ j = 0, \ldots, N-1\}, \end{aligned} \tag{8}$$

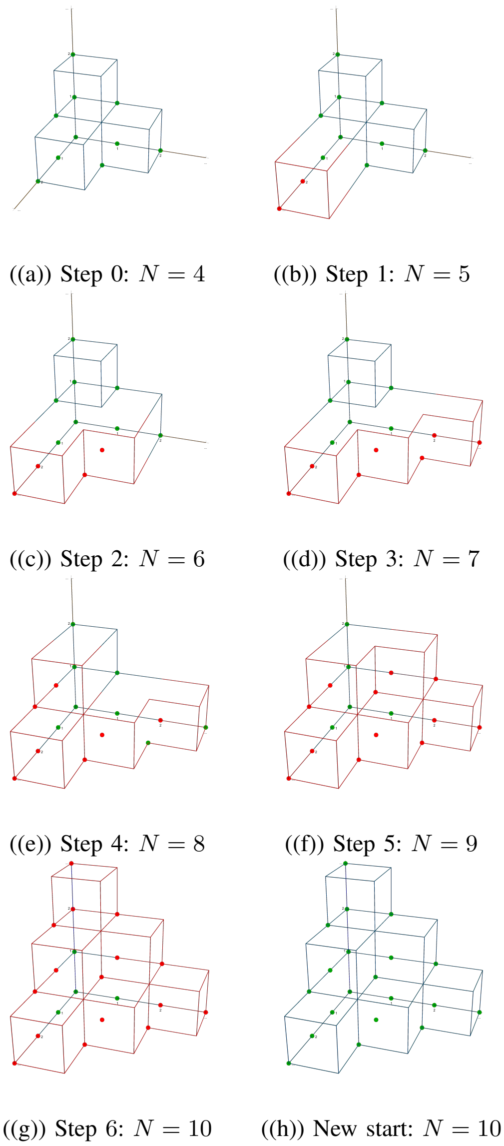
which we call an additional index set.

Definition III-.1. *Given an N-sparse exponential sum f, we define Prony-type polynomials (PTPs) as determinants of the following block matrices:*

$$P_N^{\mathbf{m}}(\mathbf{z}) = \frac{1}{\det \mathcal{T}_N} \left|\begin{array}{c|c} \mathcal{T}_N & \mathbf{f}_{N,\mathbf{m}} \\ \hline \widehat{\mathbf{z}}_N & \mathbf{z}^{\mathbf{m}} \end{array}\right|,$$

for all $\mathbf{m} \in D_N$. From the cardinality of D_N, it follows that there are exactly $l_3(N) + \binom{|\mathbf{l}(N)|+2}{|\mathbf{l}(N)|} + 1$ polynomials $P_N^{\mathbf{m}}$ for the N-sparse f. The Prony-type polynomials can have different total degrees.

Here, we want to underline the difference between the Prony-type polynomials and those proposed in [24]. The use of the Cantor tuple function is crucial for the Prony-type polynomials. The Cantor tuple function allows the construction of a set of polynomials with the parameters $\mathbf{z}_1, \ldots, \mathbf{z}_N$ as the set of their common zeros using a smaller number of samples than [24], as our main result shows below.



((a)) Step 0: $N = 4$ ((b)) Step 1: $N = 5$ ((c)) Step 2: $N = 6$ ((d)) Step 3: $N = 7$ ((e)) Step 4: $N = 8$ ((f)) Step 5: $N = 9$ ((g)) Step 6: $N = 10$ ((h)) New start: $N = 10$

Fig. 1: Illustration of sets of initial monomials for an N-sparse exponential sum for PTPs $P_N^{\mathbf{m}}, \mathbf{m} \in D_N$.

Theorem III-.1. *Let $f\colon \mathbb{Z}_+^3 \to \mathbb{C}$ be an N-sparse exponential sum of the form*

$$f(\mathbf{n}) = \sum_{j=1}^{N} a_j \exp\left(-\mathrm{i}\langle \boldsymbol{\omega}_j, \mathbf{n}\rangle\right) = \sum_{j=1}^{N} a_j \mathbf{z}_j^{\mathbf{n}},$$

with coefficients $a_j \in \mathbb{C}\backslash\{0\}$ and parameters $\mathbf{z}_j \in \mathbb{T}^3, j = 1, \ldots, N$. Besides, let D_N be the degree set as in (7) and $P_N^{\mathbf{m}}$ be the corresponding Prony-type *polynomials defined in* (III-.1) *for all* $\mathbf{m} \in D_N$. *If the parameters* $\mathbf{z}_j$ *are pairwise distinct, then the parameters* $\mathbf{z}_j$, $j = 1, \ldots, N$, *form the set of common zeros of the polynomial set* $\mathcal{P}_N = \{P_N^{\mathbf{m}} : \mathbf{m} \in D_N\}$.

Proof. Similarly to [30], using the multilinearity of determinants, one can show that the parameters $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_N$ belong to the set of common zeros of the Prony-type polynomials $P_N^{\mathbf{m}}$, $\mathbf{m} \in D_N$, i.e. $P_N^{\mathbf{m}}(\mathbf{z_j}) = 0$, for all $j=1, \ldots, N$, and all $\mathbf{m} \in D_N$. This shows that N is a lower bound for the number of common zeros of the polynomial set $\mathcal{P}_N$.

Therefore, to prove the statement of the theorem, we need to show that N is also an upper bound for the number of common zeros of the set of Prony-type polynomials $\mathcal{P}_N = \{P_N^{\mathbf{m}}(\mathbf{z}) \colon \mathbf{m} \in D_N\}$. We will employ the results from Gröbner basis theory and the residue rings [10] to prove this. Namely, to find an upper bound for the number of common zeros, one needs to look at the leading terms of the polynomials from $\mathcal{P}_N$ and collect all monomials that are less than leading terms of $\mathcal{P}_N$ according to the fixed monomial Grlex order, see Fig 1. We denote this set of monomials by $\mathcal{W}$. If the cardinality of $\mathcal{W}$, i.e., $\#\mathcal{W}$, is finite, then the set of common zeros of $\mathcal{P}_N$ is discrete and the number of common zeros of $\mathcal{P}_N$ is upper-bounded by $\#\mathcal{W}$, for more details see [10], [30]. ☐

Theorem III-.1 leads to the following PTP algorithm for N-sparse exponential sums, see Algorithm 1, which is situated in terms of the sampling budget between the methods proposed in [21], [22].

Algorithm 1: PTP algorithm

Data:
- Number of $3D$ frequencies $N \in \mathbb{N}$
- Samples $f(\mathbf{n})$, for $\mathbf{n} \in I_{PTP}(N) = I_N \cup I_N^+$

begin
- Set up $\mathcal{P}_N = \{P_N^{\mathbf{m}}(\mathbf{z}) : \ \mathbf{m} \in D_N\}$
- Compute common zeros of the PTP $\mathcal{V}(\mathcal{P}_N) = \{\mathbf{z}_1, \ldots, \mathbf{z}_N\}$

Result: Frequencies $\{\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_N\}$

IV. Numerical Experiments

In this section, we test numerically Algorithm 1 in case of noise corruption and compare the obtained results with the method [22]. The PTP algorithm was implemented in Wolfram Mathematica with a working precision of 50 digits. In our numerical computation, we used only four (e.g, the first four) Prony-type polynomials $P_N^{\mathbf{m}}$, $\mathbf{m} \in D_N^{\dagger} := \{\mathbf{l}(j) \colon j=N, \ldots, N+3\}$ in order to find the common zeros, which allows for more numerical efficiency. We compare the performance of this numerical PTP algorithm to the method of the *Minimal Number of Samples* (MNS) [22] with the sampling direction $\Delta = (1, 0, 0)$, the shift vector $\delta_1 = (0, 1, 0)$ and $\delta_2 = (0, 0, 1)$ (see [22]), and precision of 50 digits. For this, we consider the following two experiments.

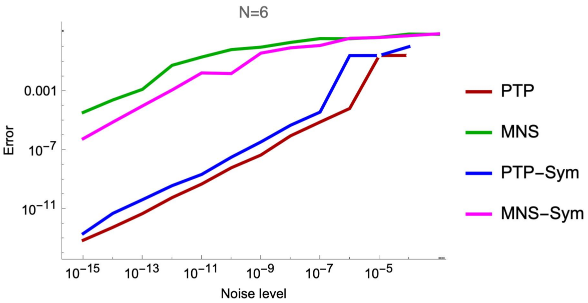


Fig. 2: The behavior of the reconstruction error for Experiment 1 (PTP and MNS) and Experiment 2 (PTP-Sym and MNS-Sym).

Experiment 1. For the first experiment, we have considered the 6-sparse exponential sum

$$f(\mathbf{n}) = \sum_{j=1}^{6} a_j \exp\left(-\mathrm{i}\langle \boldsymbol{\omega}_j, \mathbf{n}\rangle\right) + \varepsilon(\mathbf{n}), \quad \mathbf{n} \in \mathbb{Z}^3,$$

with complex coefficients $a_j \in \mathbb{C} \backslash \{0\}$, pairwise distinct frequency vectors $\boldsymbol{\omega}_j \in [0, 2\pi)^3$, $j=1, \ldots, 6$, and additive noise of the form $\varepsilon(\mathbf{n}) = \epsilon\, \mathrm{e}^{\mathrm{i}\varphi}$ with a random absolut value ϵ uniformly distributed in $[1 \times 10^{-\eta}, 9 \times 10^{-\eta}]$, $\eta=3, \ldots, 15$ and a random angle φ uniformly distributed in $[0, 2\pi)$. In our numerical experiments, we use sets of twenty randomly generated coefficients and frequency vectors $\{(a_j, \boldsymbol{\omega}_j), j=1, \ldots, 6\}$. We consider the ℓ_2-norm error, $\Delta\omega_j = \|\boldsymbol{\omega}_j - \tilde{\boldsymbol{\omega}}_j\|_2$, $j=1, \ldots, 6$, where $\tilde{\boldsymbol{\omega}}_j$ is the corresponding recovered frequency vector, and compute the average of the maximal deviations $\Delta = \max_{j=1,\ldots,6}(\Delta\omega_j)$ over twenty settings.

Experiment 2. For the experiment, we have considered the symmetric 6-sparse exponential sum

$$f(\mathbf{n}) = \sum_{|j|=1}^{3} a_j \exp\left(-\mathrm{i}\langle \boldsymbol{\omega}_j, \mathbf{n}\rangle\right) + \varepsilon(\mathbf{n}),$$

with complex coefficients $a_j \in \mathbb{C} \backslash \{0\}$, pairwise distinct frequency vectors $\boldsymbol{\omega}_j \in [0, \pi)^3$ and additive noise $\varepsilon(\mathbf{n})$ of the same nature as in Experiment 1. Here, we also used the data of twenty randomly generated collections of coefficients a_j and frequency vectors $\boldsymbol{\omega}_j$ with the property $a_{-j} = \overline{a_j}$ and $\boldsymbol{\omega}_{-j} = -\boldsymbol{\omega}_j$ for all $j=1, 2, 3$ and present the maximal deviation in terms of the ℓ_2-norm.

The results of Experiments 1 and 2 are shown in Figure 2. As we can observe, the PTP method enjoys more stability in both experimental settings.

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