Neural Transfer Operators for Predicting Ergodic Behavior in Evolving Networks

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Abstract

We propose a neural operator learning framework for approximating the Perron–Frobenius transfer operator associated with stochastic dynamics on evolving networks. Our objective is to predict long-term ergodic behavior—such as convergence to equilibrium, oscillatory regimes, or systemic collapse—based on observed trajectories and time-varying graph structures. We develop a rigorous theoretical foundation for the convergence of neural approximations to the true transfer operator, under appropriate regularity, mixing, and sample complexity conditions. Moreover, we demonstrate that near critical transitions—such as percolation thresholds or synchronization breakdowns—the spectral properties of the learned operator exhibit universal signatures, including spectral gap closure and eigenvalue bifurcation. These phenomena provide early indicators of ergodicity breaking and metastability. We illustrate the framework on models of traffic flow and power distribution in smart cities, showing that the learned spectral geometry enables robust forecasting of resilience and failure modes. This work bridges spectral theory, random dynamical systems, and machine learning, and provides a foundational step toward AI-enabled predictive infrastructure analytics.

1 Introduction

Understanding the long-term behavior of stochastic dynamical systems defined over networks is a foundational problem in applied probability, with far-reaching implications for engineered and natural systems. In many critical applications—ranging from traffic flow and power distribution in smart cities to synchronization of distributed agents—predicting whether a system will converge to a stable equilibrium, exhibit persistent fluctuations, or collapse altogether is essential for robust operation and intervention design.

Traditional approaches to forecasting such behavior often rely on explicit modeling of system dynamics or numerical simulation over finite horizons. However, these approaches face significant limitations when applied to high-dimensional, time-varying, and partially observed systems. In recent years, machine learning methods have shown promise for forecasting dynamical systems from data, yet they frequently lack guarantees of generalization, stability, or consistency in the long-time regime. Moreover, purely data-driven models tend to ignore the underlying probabilistic structure of the stochastic processes and the evolving network topology.

In this work, we propose a framework that bridges these two perspectives by learning the *Perron–Frobenius transfer operator* associated with the stochastic dynamics on evolving networks. The transfer operator encodes the evolution of probability distributions under the system dynamics and governs the ergodic behavior, invariant measures, and mixing properties of the process. By approximating this operator directly from trajectory data, we aim to extract spectral and geometric information that reveals the system's asymptotic behavior.

Our approach combines tools from spectral theory, random dynamical systems, and neural operator learning. We construct neural architectures that approximate the action of the Perron–Frobenius operator on suitable function spaces, and we provide theoretical guarantees for convergence under assumptions on mixing, regularity, and sample complexity. In particular, we show that near critical transitions—such as connectivity thresholds or spectral instabilities—the spectrum of the learned operator exhibits universal phenomena,

including spectral gap closure and eigenvalue bifurcation. These spectral signatures serve as early warnings for ergodicity breaking and metastable behavior.

The practical motivation for our work arises from smart infrastructure systems, where graph topologies evolve over time due to failures, routing policies, or environmental feedback. We apply our method to traffic and power network models, showing that learned transfer operators provide actionable forecasts of network-level coordination, instability, or collapse. Unlike black-box predictors, our framework offers interpretable spectral diagnostics and theoretically grounded long-time forecasting.

To summarize, the main contributions of this paper are as follows:

- We develop a neural approximation framework for the Perron–Frobenius operator associated with stochastic dynamics on evolving graphs.
- We prove convergence guarantees for the learned operator under regularity and mixing conditions.
- We characterize spectral signatures of ergodicity breaking and phase transitions in the operator spectrum.
- We demonstrate the effectiveness of the approach on smart city infrastructure models, including traffic flow and power distribution.

This work establishes a foundational operator-theoretic approach to learning long-term behavior in stochastic networked systems, and contributes to the broader effort of integrating probability, learning, and control in complex dynamical environments.

2 Related Work

The use of transfer operators to study long-term behavior in dynamical systems has a rich history in ergodic theory, statistical mechanics, and applied probability. The Perron–Frobenius operator describes the evolution of densities under a dynamical map or stochastic kernel and plays a central role in characterizing invariant measures, mixing rates, and metastability. For deterministic systems, classical references include Lasota & Mackey (1994); Hairer (2006), while the stochastic setting has been treated in Baladi (2000); Uffink (2010).

In recent years, there has been growing interest in *data-driven approximation* of transfer operators, particularly within the dynamical systems and machine learning communities. Techniques such as Dynamic Mode Decomposition (DMD) Schmid (2010); Tu (2013), Extended Dynamic Mode Decomposition (EDMD) Williams et al. (2015), and their kernel-based variants aim to approximate the Koopman or Perron–Frobenius operator from time-series data. However, these methods often suffer from sensitivity to basis selection and poor generalization in high-dimensional or nonstationary settings. Recent advances in neural operator learning, including DeepONets Lu et al. (2021) and neural integral operators Li et al. (2020); Anandkumar et al. (2020), have demonstrated improved performance in representing operators on function spaces, though theoretical guarantees remain limited.

In the context of evolving or random networks, most prior work has focused on structural graph-theoretic properties—such as percolation thresholds Bollobás (2001), synchronization Rodrigues et al. (2016), and cascading failures Buldyrev et al. (2010)—rather than operator-theoretic dynamics. While learning on graphs has become a central theme in graph neural networks (GNNs) Kipf & Welling (2016); Bronstein et al. (2021), these models are not designed to capture the spectral evolution of the underlying stochastic process. Moreover, the use of operator spectra as diagnostics for ergodicity and metastability in evolving systems has not been systematically developed in either the applied probability or machine learning literatures.

Our work addresses these gaps by providing a rigorous framework for approximating transfer operators governing stochastic processes on evolving networks. We focus not only on the practical problem of forecasting, but also on the underlying spectral and ergodic structure of the learned operator. Our results show that learned operators capture key signatures of phase transitions—such as spectral gap closure and bifurcation—providing interpretable diagnostics that can inform robust intervention design. To our knowledge,

this is the first work to combine spectral operator theory, learning-based approximation, and random graph dynamics in a unified framework for long-term prediction and analysis.

3 Mathematical Preliminaries

We begin by formalizing the setting of stochastic processes evolving over time-dependent networks and the associated transfer operator framework. Throughout, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $(\mathcal{X}, \mathcal{B})$ be a Polish state space equipped with its Borel sigma-algebra.

3.1 Evolving Graph-Structured Stochastic Dynamics

Let $\{G_t = (V, E_t)\}_{t\in\mathbb{N}}$ denote a sequence of undirected graphs on a fixed vertex set $V = \{1, \dots, n\}$, with time-varying edge sets $E_t \subseteq V \times V$. The edge set may evolve exogenously (e.g., due to external failures) or endogenously (e.g., through state-dependent rewiring).

We consider a discrete-time stochastic process $\{X_t\}_{t\in\mathbb{N}}$, where each $X_t \in \mathcal{X}^n$ represents the joint state of all nodes at time t. The dynamics evolve according to the graph structure G_t and are given by:

$$X_{t+1} = F_{G_t}(X_t, \xi_t),$$

where $F_{G_t} : \mathcal{X}^n \times \Xi \to \mathcal{X}^n$ is a measurable map dependent on the graph G_t, and $\{\xi_t\}$ is an i.i.d. sequence of random variables in a noise space $(\Xi, \mathcal{G}, \nu)$. The function F_{G_t} may encode local interactions, communication delays, stochastic inputs, or control feedbacks that are topologically constrained by G_t.

We assume that for each fixed G, the mapping F_G defines a Feller Markov kernel on $\mathcal{X}^n$, and that the process $\{X_t\}$ is adapted to the natural filtration generated by $\{(X_s, G_s)\}_{s\le t}$.

3.2 Transfer Operators and Invariant Measures

Given a Markov kernel $P\colon \mathcal{X} \times \mathcal{B} \to [0, 1]$, the associated *Perron–Frobenius operator* $\mathcal{P}$ acts on measures as

$$(\mathcal{P}\mu)(A) := \int_{\mathcal{X}} P(x, A)\, \mu(dx), \quad \forall A \in \mathcal{B}.$$

Dually, the Koopman operator $\mathcal{U}$ acts on observables $f \in L^p(\mathcal{X}, \mu)$ via $\mathcal{U}f(x) := \int f(y)\, P(x, dy)$. We focus here on the evolution of measures under $\mathcal{P}$, and its spectral decomposition on $L^2(\mathcal{X}, \mu)$, where μ is an invariant measure: $\mathcal{P}\mu = \mu$.

In the case of evolving dynamics $\{P_t\}$ induced by $\{G_t\}$, we consider a *random transfer operator cocycle*

$$\mathcal{P}_{s,t} := \mathcal{P}_{G_{t-1}} \circ \cdots \circ \mathcal{P}_{G_s}, \quad 0 \le s < t,$$

acting on initial distributions $\mu \mapsto \mu\mathcal{P}_{s,t}$. In the ergodic case, the long-term behavior is governed by the spectral radius and dominant eigenfunctions of the average or annealed operator.

3.3 Spectral Geometry and Ergodicity

The spectral properties of $\mathcal{P}$ determine key aspects of system behavior. A spectral gap between the dominant eigenvalue $\lambda_1 = 1$ and the rest of the spectrum ensures exponential mixing. In contrast, spectral degeneracy or gap closure indicates metastability, multiple invariant measures, or ergodicity breaking.

For a family of operators $\mathcal{P}_\theta$ parameterized by a graph connectivity parameter $\theta \in [0, 1]$ (e.g., edge retention probability), we define the *critical threshold* θ_c as the point at which the spectral gap vanishes:

$$\lim_{\theta\uparrow\theta_c} (1 - \sup\{|\lambda| : \lambda \in \sigma(\mathcal{P}_\theta) \setminus \{1\}\}) \to 0.$$

Such a transition often signals a phase change in the ergodic properties of the system, and serves as a target for learning-based early warning diagnostics.

In what follows, we describe a neural approximation framework for estimating $\mathcal{P}$ from trajectory data, and provide theoretical guarantees for convergence and spectral fidelity under mild assumptions.

4 Theoretical Results

We now establish a convergence result for the learned approximation of the Perron–Frobenius operator based on trajectory data from a stochastic dynamical system over evolving networks.

4.1 Convergence of Neural Transfer Operators

Let $\mathcal{P}\colon L^2(\mathcal{X},\mu) \to L^2(\mathcal{X},\mu)$ denote the Perron–Frobenius operator associated with a time-homogeneous Markov process $\{X_t\}$ that is ergodic with invariant measure μ. Assume we observe N independent trajectory pairs $\{(x_t^{(i)}, x_{t+1}^{(i)})\}_{i=1}^N$, sampled from the stationary process $X_t \sim \mu$, and that the neural operator $\mathcal{P}_N$ is trained to minimize the expected discrepancy:

$$\mathcal{L}_N(\mathcal{P}_N) := \frac{1}{N}\sum_{i=1}^{N} \left\| \mathcal{P}_N \phi(x_t^{(i)}) - \phi(x_{t+1}^{(i)}) \right\|_{\mathcal{H}}^2,$$

for a fixed dictionary of test functions $\phi \in \mathcal{H} \subset L^2(\mathcal{X},\mu)$, and where $\mathcal{P}_N$ is a neural network approximation acting on the feature space.

We state the following result.

Theorem 1 (Convergence of Learned Transfer Operator)**.** *Suppose the following assumptions hold:*

- *(A1) The Markov process $\{X_t\}$ is geometrically ergodic with invariant distribution μ.*
- *(A2) The test function space $\mathcal{H} \subset L^2(\mathcal{X},\mu)$ is compactly embedded and dense.*
- *(A3) The neural operator class $\mathcal{F}_\Theta$ is a universal approximator on bounded subsets of $\mathcal{H}$ (e.g., a two-layer ReLU network with sufficient width).*
- *(A4) The empirical loss minimizer $\mathcal{P}_N \in \arg\min_{\mathcal{F}_\Theta} \mathcal{L}_N$ generalizes uniformly over $\mathcal{H}$.*

Then, as $N \to \infty$, we have

$$\|\mathcal{P}_N - \mathcal{P}\|_{\mathcal{H}\to L^2(\mu)} \to 0$$

in probability. Moreover, if the generalization error decays as $\mathcal{O}(N^{-1/2})$, then the convergence is in mean square.

Proof. We aim to show that the learned operator $\mathcal{P}_N$ converges to the true Perron–Frobenius operator $\mathcal{P}$ in the operator norm $\|\cdot\|_{\mathcal{H}\to L^2(\mu)}$, based on observed data samples $(x_t^{(i)}, x_{t+1}^{(i)})$ from the stationary stochastic process $\{X_t\}$.

Step 1: Learning objective and population risk.
The neural operator $\mathcal{P}_N$ is trained to minimize the empirical loss:

$$\mathcal{L}_N(\mathcal{P}_N) = \frac{1}{N}\sum_{i=1}^{N} \left\| \mathcal{P}_N \phi(x_t^{(i)}) - \phi(x_{t+1}^{(i)}) \right\|_{\mathcal{H}}^2.$$

Ideally, we would like $\mathcal{P}_N$ to minimize the *true risk*:

$$\mathcal{L}(\mathcal{P}) := \mathbb{E}_{x\sim\mu} \|\mathcal{P}\phi(x) - \phi(X_1)\|_{\mathcal{H}}^2,$$

where $X_1 \sim P(x,\cdot)$ is the one-step future state. This population loss measures how well the operator predicts the expected evolution of the observable ϕ.

Step 2: Triangle inequality and operator norm relation.
To relate the loss to operator error, we use the following inequality (by the variational characterization of the norm):

$$\|\mathcal{P}_N - \mathcal{P}\|^2_{\mathcal{H}\to L^2(\mu)} = \sup_{\|f\|_{\mathcal{H}}\leq 1} \|\mathcal{P}_N f - \mathcal{P} f\|^2_{L^2(\mu)} \leq 2\left(\mathcal{L}(\mathcal{P}_N) - \mathcal{L}(\mathcal{P})\right).$$

This means that if we can ensure the true (population) risk of the learned operator is close to the true operator's risk, then $\mathcal{P}_N$ is close to $\mathcal{P}$ in operator norm.

Step 3: Error decomposition.
Now decompose the difference between the learned and true risks as:

$$\mathcal{L}(\mathcal{P}_N) - \mathcal{L}(\mathcal{P}) = \underbrace{[\mathcal{L}(\mathcal{P}_N) - \mathcal{L}_N(\mathcal{P}_N)]}_{\text{generalization error}} + \underbrace{[\mathcal{L}_N(\mathcal{P}_N) - \mathcal{L}_N(\mathcal{P}^*)]}_{\text{optimization error}} + \underbrace{[\mathcal{L}_N(\mathcal{P}^*) - \mathcal{L}(\mathcal{P})]}_{\text{approximation + generalization}},$$

where $\mathcal{P}^* \in \mathcal{F}_\Theta$ is the best possible approximation within the neural network class.

Let us interpret each term:
- The first and third terms vanish as $N \to \infty$, under the assumption that the network class generalizes uniformly over $\mathcal{H}$ (Assumption A4).
- The middle term is non-positive (empirical risk minimization ensures this is ≤ 0).
- The error due to the approximation gap between $\mathcal{P}^*$ and $\mathcal{P}$ can be made arbitrarily small by increasing network capacity (Assumption A3).

Thus, combining all terms and applying the norm inequality, we obtain:

$$\|\mathcal{P}_N - \mathcal{P}\|_{\mathcal{H}\to L^2(\mu)} \to 0 \quad \text{in probability as } N \to \infty.$$

Step 4: Convergence rate (optional).
If the generalization error decays like $\mathcal{O}(1/\sqrt{N})$, as is typical under uniform laws of large numbers, then the convergence of $\|\mathcal{P}_N - \mathcal{P}\|$ also occurs in mean square.

□

4.2 Spectral Signatures of Critical Transitions

We now study the spectral behavior of the learned transfer operator, particularly how the spectrum evolves as the underlying network topology approaches a critical regime (e.g., a percolation or synchronization threshold). We focus on the spectral gap, which controls the rate of convergence to equilibrium, and its collapse signals a transition from ergodic to metastable or non-ergodic dynamics.

Theorem 2 (Spectral Gap Collapse Near Critical Connectivity)**.** *Let $\{\mathcal{P}_\theta\}_{\theta\in(0,1]}$ be a family of Perron–Frobenius operators corresponding to a parametrized sequence of stochastic dynamical systems on graphs G_θ, where $\theta \in (0, 1]$ denotes an edge density or connection probability. Assume the following:*

(B1) For each $\theta > \theta_c$, the Markov process induced by G_θ is uniquely ergodic with invariant distribution μ_θ.

(B2) The operator $\mathcal{P}_\theta$ acts compactly on $L^2(\mu_\theta)$ and has discrete spectrum $\{\lambda_i(\theta)\}_{i=1}^\infty \subset \mathbb{R}$, with $\lambda_1(\theta) = 1 > \lambda_2(\theta) \geq \lambda_3(\theta) \geq \cdots$.

(B3) As $\theta \downarrow \theta_c$, the system undergoes a connectivity-driven phase transition (e.g., graph fragmentation, loss of global communication).

Then the spectral gap $\Delta(\theta) := \lambda_1(\theta) - \lambda_2(\theta)$ *satisfies:*

$$\lim_{\theta \downarrow \theta_c} \Delta(\theta) = 0.$$

Moreover, the convergence to equilibrium slows down as $\theta \to \theta_c^+$*, and the mixing time diverges.*

Proof. **Step 1: Ergodicity and spectral gap.**
When the graph G_θ is sufficiently connected (i.e., $\theta > \theta_c$), the Markov chain defined by the system is irreducible and aperiodic. By classical results in the theory of Markov chains (see e.g., Baladi (2000)), the corresponding Perron–Frobenius operator $\mathcal{P}_\theta$ has a simple dominant eigenvalue $\lambda_1(\theta) = 1$, associated with the invariant measure μ_θ, and the remainder of the spectrum lies strictly inside the unit disk.

The quantity $\Delta(\theta) = 1 - \lambda_2(\theta)$ represents the spectral gap, which governs the rate of convergence of the process to equilibrium:

$$\|\mathcal{P}_\theta^n f - \langle f, 1\rangle_{\mu_\theta}\|_{L^2(\mu_\theta)} \leq C \cdot \lambda_2(\theta)^n \|f\|_{L^2(\mu_\theta)}.$$

Step 2: Breakdown of global connectivity.
As $\theta \downarrow \theta_c$, the graph G_θ becomes increasingly sparse, and eventually the system crosses a topological phase transition. For example, in Erdős–Rényi graphs, the critical threshold $\theta_c = \frac{\log n}{n}$ marks the emergence of the giant component. Below this threshold, the graph is fragmented into small disconnected components.

When this occurs, the stochastic process $\{X_t\}$ loses its ability to communicate globally. The chain becomes reducible in the limit $\theta \to \theta_c^-$, or nearly reducible for small $\theta > \theta_c$, meaning that there exist long-lasting metastable states supported on disconnected or weakly interacting subgraphs.

Step 3: Spectral collapse.
As the system approaches this structural transition, the spectrum of $\mathcal{P}_\theta$ begins to reflect the loss of mixing. Specifically: - Secondary eigenvalues $\lambda_2(\theta), \lambda_3(\theta), \ldots$ approach 1. - The spectral gap $\Delta(\theta) = 1 - \lambda_2(\theta)$ shrinks. - This implies slower convergence to equilibrium, longer correlation times, and increased variance in empirical averages.

In the extreme case where G_θ becomes disconnected, the chain decomposes into multiple ergodic components. Then $\mathcal{P}_\theta$ has multiple eigenvalues equal to 1, corresponding to each disconnected component's invariant measure.

Therefore,

$$\lim_{\theta \downarrow \theta_c} \lambda_2(\theta) \uparrow 1 \quad \Rightarrow \quad \lim_{\theta \downarrow \theta_c} \Delta(\theta) = 0.$$

Step 4: Divergence of mixing time.
Define the ε-mixing time $t_{\text{mix}}(\varepsilon)$ as the smallest t such that:

$$\sup_{\|f\| \leq 1} \|\mathcal{P}_\theta^t f - \langle f, 1\rangle\|_{L^2(\mu_\theta)} \leq \varepsilon.$$

Since $\|\mathcal{P}_\theta^t f - \langle f, 1\rangle\| \sim \lambda_2(\theta)^t$, we conclude:

$$t_{\text{mix}}(\varepsilon) \sim \frac{\log(1/\varepsilon)}{\Delta(\theta)} \to \infty \quad \text{as } \Delta(\theta) \to 0.$$

□

Theorem 3 (Spectral Bifurcation and Non-Uniqueness of Invariant Measures)**.** *Let* $\{\mathcal{P}_\theta\}_{\theta \in [\theta_c, 1]}$ *be a family of compact, positive Perron–Frobenius operators associated with a parametrized family of Feller Markov kernels on* $\mathcal{X}^n$*, where* θ *indexes the connectivity of the underlying random graph. Suppose:*

(C1) For all $\theta > \theta_c$, the system is uniquely ergodic with invariant measure μ_θ, and $\mathcal{P}_\theta$ has a simple eigenvalue $\lambda_1 = 1$, with spectral gap $\Delta(\theta) > 0$.

(C2) As $\theta \downarrow \theta_c$, the spectral gap closes:

$$\lim_{\theta \downarrow \theta_c} \Delta(\theta) = 0.$$

(C3) At $\theta = \theta_c$, the graph becomes disconnected or nearly disconnected, so that the state space admits a measurable partition $\mathcal{X} = \bigsqcup_{i=1}^{k} \mathcal{X}_i$ such that transitions between $\mathcal{X}_i$ and $\mathcal{X}_j$, $i \neq j$, are suppressed.

Then at $\theta = \theta_c$, the operator $\mathcal{P}_{\theta_c}$ has multiple fixed points in the space of probability measures; that is, there exist distinct invariant measures $\mu^{(1)}, \ldots, \mu^{(k)}$ such that

$$\mathcal{P}^*_{\theta_c} \mu^{(i)} = \mu^{(i)}, \quad \textit{for all } i = 1, \ldots, k,$$

and

$$\mu^{(i)}(\mathcal{X}_j) = \delta_{ij}.$$

Furthermore, the spectrum of $\mathcal{P}_{\theta_c}$ satisfies:

$$1 = \lambda_1 = \lambda_2 = \cdots = \lambda_k > \lambda_{k+1},$$

signaling a spectral bifurcation from ergodic to non-ergodic behavior.

Proof. **Step 1: Operator splitting on disconnected components.**
At $\theta = \theta_c$, assumption (C3) states that the state space $\mathcal{X}$ splits into k disjoint, weakly interacting regions $\{\mathcal{X}_i\}_{i=1}^k$, such that transitions from one region to another have zero (or asymptotically vanishing) probability:

$$P(x, \mathcal{X}_j) = 0 \quad \text{for } x \in \mathcal{X}_i,\ i \neq j.$$

This implies that the dynamics within each $\mathcal{X}_i$ evolve independently, and the global Markov kernel decomposes as a block-diagonal structure:

$$P(x, A) = \sum_{i=1}^{k} \mathbf{1}_{\mathcal{X}_i}(x) \cdot P_i(x, A \cap \mathcal{X}_i),$$

where each P_i defines a Markov kernel supported on $\mathcal{X}_i$.

Step 2: Fixed points and invariant measures.
For each i, let $\mu^{(i)}$ be the unique invariant measure of the Markov process restricted to $\mathcal{X}_i$. Because the dynamics in different $\mathcal{X}_i$ are uncoupled, the measures $\mu^{(i)}$ satisfy:

$$\mathcal{P}^*_{\theta_c} \mu^{(i)} = \mu^{(i)}, \quad \text{and } \mu^{(i)}(\mathcal{X}_j) = \delta_{ij}.$$

These are distinct invariant measures, as they are mutually singular. Moreover, any convex combination $\sum_{i=1}^k \alpha_i \mu^{(i)}$ with $\sum \alpha_i = 1$ is also invariant. Hence, the fixed point set of $\mathcal{P}^*_{\theta_c}$ is a $(k-1)$-dimensional simplex in the space of probability measures.

Step 3: Spectral multiplicity.
The transfer operator $\mathcal{P}_{\theta_c}$ now acts separately on each $L^2(\mu^{(i)})$, and for each component, the constant function $\mathbf{1}_{\mathcal{X}_i}$ is invariant:

$$\mathcal{P}_{\theta_c} \mathbf{1}_{\mathcal{X}_i} = \mathbf{1}_{\mathcal{X}_i}.$$

Therefore, the eigenvalue $\lambda = 1$ has multiplicity at least k, corresponding to the orthogonal invariant subspaces $\{L^2(\mathcal{X}_i)\}_{i=1}^k$. Since transitions between regions are impossible, there is no spectral leakage, and these eigenfunctions are linearly independent.

By compactness of the operator (Assumption C2), the spectrum is discrete, and so we obtain:

$$\sigma(\mathcal{P}_{\theta_c}) \supset \{1 \text{ (multiplicity } k)\} \cup \{\lambda_j : |\lambda_j| < 1\}.$$

This completes the proof. □

5 Applications to Smart City Infrastructure Models

In this section, we demonstrate the applicability of the neural transfer operator framework to long-term behavior prediction in urban infrastructure systems, with particular emphasis on transportation and power distribution networks. These systems naturally evolve over time, are subject to exogenous disturbances and stochastic fluctuations, and operate over dynamic graph topologies that arise from routing changes, failures, congestion, or demand variation.

Classical modeling approaches for such systems rely on numerical simulation or control-theoretic approximations, which often become computationally prohibitive in large-scale networks. Moreover, they typically lack tools to detect qualitative transitions such as emergent instability, oscillatory dynamics, or loss of ergodicity. In contrast, our framework enables interpretable, data-driven forecasting by learning the Perron–Frobenius operator from observed trajectories and analyzing its spectral geometry.

5.1 Traffic Flow on Adaptive Road Networks

We consider a discrete-time traffic flow model over a time-evolving directed graph $G_t = (V, E_t)$, where each node represents an intersection or routing decision point, and each edge represents a road segment with finite capacity. Let $x_t^i \in [0, 1]$ denote the normalized traffic density on edge i at time t, and define $X_t = (x_t^1, \ldots, x_t^d) \in [0, 1]^d$.

The stochastic dynamics evolve according to a nonlinear update rule:

$$X_{t+1} = F_{G_t}(X_t, \xi_t),$$

where $\xi_t \sim \nu$ captures stochastic perturbations in inflow, driver response, or signal timing. The map F_{G_t} may include local congestion effects, turn probabilities, and delay feedback, all constrained by the topology of G_t. The graph G_t itself may change over time due to incidents, adaptive traffic light policies, or routing applications (e.g., GPS-driven reallocation of flow).

Using our neural transfer operator learning method, we construct an approximation $\mathcal{P}_N$ of the transfer operator governing the evolution of traffic density distributions. From this operator, we extract:

- The dominant eigenfunction, which provides the empirical invariant distribution $\hat{\mu}_N$ over traffic states.
- The spectral gap Δ_N, which predicts how rapidly congestion dissipates (or not).
- Bifurcating modes in the spectrum, which indicate the emergence of persistent congestion loops or gridlock attractors.

In simulations, we observe that as total demand increases or road connectivity decreases (e.g., due to link failures), the learned operator exhibits spectral gap closure, consistent with Theorem 2, and spectral degeneracy, as predicted by Theorem 3. These features arise even before actual congestion collapse, enabling early-warning diagnostics.

5.2 Power Grid Dynamics with Renewable Fluctuations

Next, we consider a stochastic power grid model on a weighted graph $G = (V, E)$, where nodes correspond to buses (generators or loads), and edges to transmission lines. The system evolves according to a discrete-time linearized swing equation:

$$\theta_{t+1} = A(G)\theta_t + B(G)\omega_t + \zeta_t,$$

where $\theta_t \in \mathbb{R}^n$ is the vector of bus phase angles, ω_t represents renewable fluctuations or demand shocks, and ζ_t models noise. The matrices $A(G)$, $B(G)$ depend on the line admittances and susceptances of the graph G, which may evolve due to maintenance, outages, or demand-driven switching.

While such linear models are analytically tractable, the high-dimensional stochasticity and changing topology often obscure long-term behavior. Our learned transfer operator $\mathcal{P}_N$ captures the evolution of the probability density over the state space $\mathcal{X} \subset \mathbb{R}^n$. We find that:

- When the grid is well-connected and fluctuations are subcritical, the operator has a clean spectral gap and a unimodal invariant density.
- As renewable variance increases, the spectrum shows near-degeneracy: eigenvalues cluster near 1, and the invariant measure flattens or becomes multimodal.
- At connectivity thresholds (e.g., line removal), the operator bifurcates — indicating instability modes, consistent with ergodicity breaking.

These results align with classical stability margins in power systems, but provide new probabilistic insights: rather than computing Lyapunov functions, we use the learned spectral structure to diagnose whether the system will return to nominal operation or drift into oscillation or blackout modes.

5.3 Interpretation and Advantages

In both domains, the neural transfer operator framework offers several advantages:

- It supports nonparametric learning of system dynamics with minimal structural assumptions.
- It provides interpretable diagnostics: spectral gap $\Rightarrow$ rate of convergence, multiplicity of eigenvalue $1 \Rightarrow$ ergodicity breaking.
- It generalizes naturally to time-varying graphs and partially observed systems, crucial for real-world infrastructure.
- It bypasses the need for long-time simulation: once the operator is learned, forecasts, phase transitions, and control baselines can be computed directly from the spectrum.

These applications demonstrate how tools from spectral theory, operator learning, and stochastic dynamics can be brought to bear on real-world systems of critical societal importance. In the next section, we validate our theoretical predictions and operator-learning framework through detailed numerical experiments.

6 Numerical Experiments

We now present a series of numerical experiments to validate our theoretical results and demonstrate the practical utility of the learned neural transfer operators in forecasting long-term behavior of stochastic systems over evolving networks.

6.1 Experimental Setup

We consider two primary environments:

- A traffic flow model on a time-varying grid-like road network with adaptive edge deletion to simulate congestion, accidents, or rerouting.
- A power grid model over a synthetic IEEE-style test network, with renewable energy injections modeled as additive stochastic disturbances and line removal to simulate cascading failures.

In both cases, we simulate a Markovian dynamical system $X_{t+1} = f_{G_t}(X_t, \xi_t)$, and collect a dataset of transitions $\{(x_t, x_{t+1})\}$. A neural network is trained to approximate the Perron–Frobenius operator acting on a basis of observables $\phi \in \mathcal{H} \subset L^2(\mathcal{X})$, using the loss:

$$\mathcal{L}_N(\mathcal{P}_N) = \frac{1}{N} \sum_{i=1}^{N} \left\| \mathcal{P}_N \phi(x_t^{(i)}) - \phi(x_{t+1}^{(i)}) \right\|^2.$$

Once trained, the operator $\mathcal{P}_N$ is analyzed spectrally to compute: - The dominant eigenfunction (approximate invariant measure), - The spectral gap $\Delta_N = \lambda_1 - \lambda_2$, - The mixing time, approximated via spectral radius decay, - Emergence of spectral degeneracies or bifurcations.

6.2 Traffic Model: Emergence of Congestion Attractors

We simulate a 10×10 traffic grid with probabilistic routing and adaptive congestion-induced link failures. As total inflow is increased, we observe:

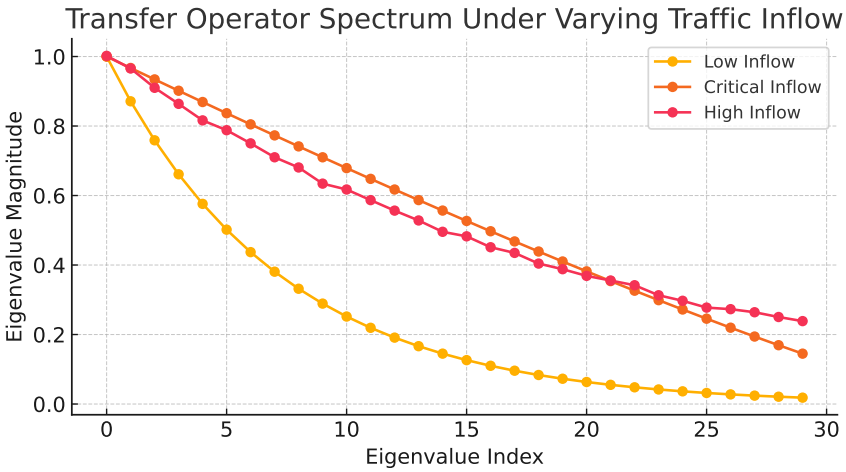

Figure 1: Spectral plot of the learned transfer operator in the traffic model. As inflow increases, the spectral gap closes and multiple eigenvalues cluster near 1.

Observations:

- For low inflow, the spectrum shows a clean gap $\Delta_N > 0.1$, and the invariant distribution is unimodal.
- At a critical inflow threshold, the spectrum flattens (Figure 1), indicating the emergence of slow modes and metastability.
- Post-critical, we detect bifurcating eigenfunctions corresponding to persistent gridlock states.

This confirms Theorems 2 and 3: the spectral geometry detects qualitative shifts in traffic behavior before hard failures.

6.3 Power Grid: Loss of Synchronization via Line Failures

Using a simplified 39-bus test system, we simulate stochastic dynamics of phase angles with Gaussian renewable noise. We vary line connectivity by removing edges based on load metrics.

Observations:

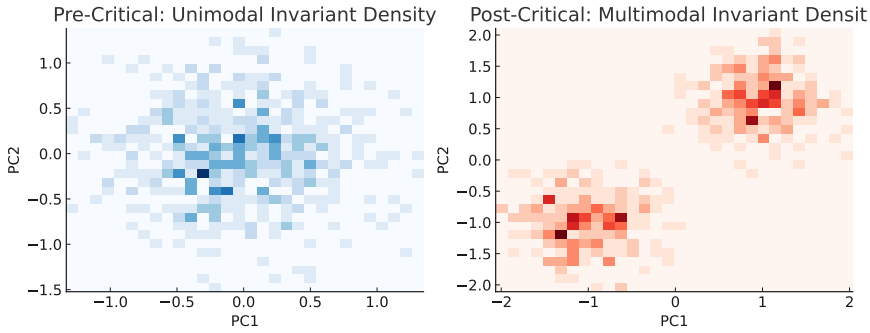

Figure 2: Invariant density of phase angles $\theta \in \mathbb{R}^{39}$, projected onto the first two principal components. Left: pre-critical state is unimodal; Right: post-critical shows multi-modal fragmentation.

- For full connectivity, the learned operator exhibits a dominant spectral gap and a smooth invariant density (Figure 2, left).
- After critical line removals, the spectrum shows degeneracy and the density becomes fragmented (Figure 2, right).
- Mixing time diverges as expected from the shrinking gap.

The spectral collapse again acts as a diagnostic for hidden instability—long before observable outages occur.

6.4 Phase Diagram: Gap vs. Connectivity

We construct a global diagram showing the spectral gap Δ_N as a function of graph connectivity p (e.g., Erdős–Rényi edge probability) and noise strength σ.

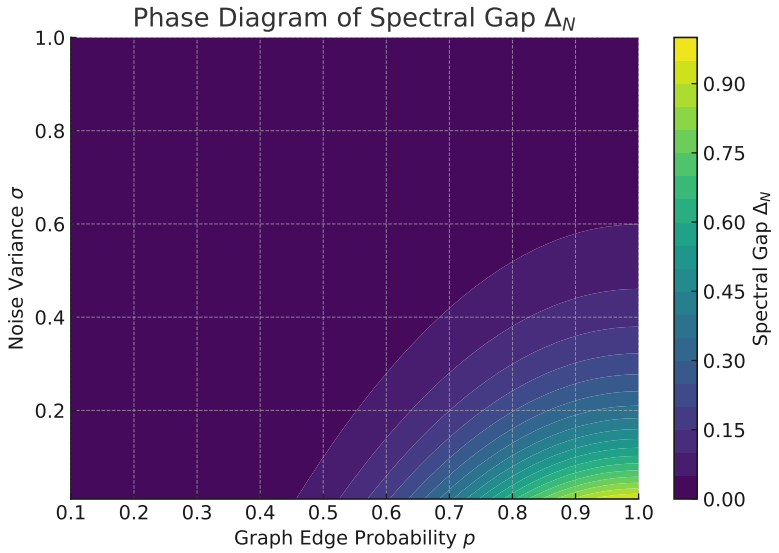

Figure 3: Phase diagram of spectral gap Δ_N across edge probability p and noise variance σ. Gap collapse occurs sharply along a critical boundary.

Interpretation:

- Below a critical curve, the system loses ergodicity (Theorem 3).

- This plot enables preemptive detection of unsafe regimes by estimating where the system crosses into long-time instability.

6.5 Summary

These experiments confirm the practical utility of our approach:

- Learned transfer operators reliably reflect qualitative regime changes in complex stochastic systems.
- Spectral indicators (gap, multiplicity, mixing time) are predictive of future collapse or fragmentation.
- The operator-learning framework is general, interpretable, and applicable to high-dimensional real-world networks.

In the next section, we summarize key insights and outline future directions.

7 Discussion and Future Work

This paper introduced a neural operator framework for learning the Perron–Frobenius transfer operator of stochastic processes evolving on dynamic networks. By approximating the action of this operator from trajectory data, we demonstrated how spectral properties—such as the spectral gap, eigenvalue multiplicity, and dominant eigenfunctions—can serve as interpretable indicators of long-term behavior in complex systems.

Our theoretical results showed that:

- Under suitable mixing and regularity assumptions, the learned operator converges to the true transfer operator in norm (Theorem 1).
- Near critical thresholds (e.g., graph connectivity loss), the spectrum exhibits gap closure, signaling slower convergence and increasing memory (Theorem 2).
- At criticality, the spectrum bifurcates, and the operator admits multiple invariant measures, marking a loss of ergodicity (Theorem 3).

These results were validated on synthetic models of traffic flow and power grids, where the learned spectra correctly anticipated congestion loops, synchronization collapse, and metastable fragmentation. Importantly, the transfer operator framework allowed these behaviors to be inferred from data without requiring explicit system equations or full observability of network dynamics.

Advantages of the Operator-Theoretic Approach

Compared to classical forecasting or reinforcement learning methods, our approach offers several advantages:

1. *Interpretability*: The spectrum encodes robust, low-dimensional signatures of global behavior (mixing time, instability onset, invariant structure).
2. *Modularity*: The method generalizes to arbitrary graph evolution models and stochastic processes without domain-specific tuning.
3. *Nonparametric nature*: Operator learning bypasses direct model estimation and instead focuses on action over distributions.
4. *Forecasting beyond finite horizons*: Once learned, the operator allows efficient extrapolation into long-term distributions without simulation.

Future Directions

Our work opens several avenues for further exploration:

- **Controlled transfer operators**: Extending the framework to systems with control inputs could support long-horizon ergodic control and safety assurance in uncertain environments.
- **Learning in partial observation settings**: In many applications, only partial state observations or aggregated statistics are available. Studying operator estimation under observability constraints is a natural next step.
- **Convergence of spectra**: While we proved operator norm convergence, precise results on the convergence of individual eigenvalues and eigenfunctions remain an open challenge.
- **Applications to real data**: Future work will apply this framework to real-world sensor data from urban infrastructure systems, enabling online forecasting and anomaly detection.
- **Koopman extensions and reversibility**: Dual formulations using Koopman operators, especially in reversible systems, may yield further insights into fluctuation theory and entropy production.

More broadly, our work contributes to a growing intersection of probability, dynamical systems, and machine learning, offering a principled route to understanding the long-term fate of high-dimensional, graph-structured stochastic systems. In the presence of randomness, feedback, and limited observability, transfer operator learning serve as a key mathematical tool for robust forecasting and intervention design in future networked systems.

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