A route to higher-order Kuramoto models through phase reduction theory

Keywords: phase reduction, synchronization, Kuramoto model, hypergraphs, simplicial complexes

Extended Abstract

Synchronization is a widespread emergent phenomenon where interacting units behave in unison [**1**]. Understanding its mechanisms is crucial, especially since interaction structure strongly influences the transition to synchronization. Phase models, which describe oscillators solely by their phase, are useful when coupling is weak [**2**]. The classic Kuramoto model captures synchronization via pairwise interactions, but many natural systems involve higher-order (many-body) interactions [**3**]. Moreover, higher-order interactions naturally emerge when phase reduction is performed beyond the first order [**4**]. Extensions of the Kuramoto model incorporating these interactions can lead to explosive synchronization or collective chaos [**5**, **6**]. Even the simplest minimal extension of Kuramoto-type phase model with higher-order interactions exhibits rich dynamics, as we showed in a previous work [**7**].

In this work, we provide a general theory of phase reduction for systems with higher-order interactions, yielding higher-order Kuramoto-like models. We start from a population of oscillators, whose isolated dynamics is given by $\dot{X}_j = F_j(X_j)$, coupled through 3-body interactions, i.e.,

$$\dot{X}_j = F_j(X_j) + \varepsilon \sum_{k,l=1}^{N} A_{jkl} g_{jkl}(X_k, X_l, X_j), \tag{1}$$

where A_{klj} is the adjacency tensor encoding the 3-body interactions, i.e., the hypergraph topology, and g_{jkl} is the 3-body interactions coupling function. We show that, although higher-order connection topology is preserved in the phase reduced model, the interaction topology generally changes, due to the fact that the hypergraph generally turns into a simplicial complex in the reduced Kuramoto-type model. Furthermore, we show that, when the oscillators have certain symmetries, even couplings are irrelevant for the dynamics at the first order. The changes in the topology are shown in Fig. 1.

Lastly, we show the applicability of the theory and its usefulness by allowing a detailed analysis of the dynamics of the SL oscillator, a paradigmatic model in the study of synchronization dynamics, because it is the normal form of the supercritical Hopf-Andronov bifurcation. For this purpose we perform the phase reduction for a population of N SL oscillators coupled through an all-to-all identical pairwise and 3-body interactions chosen as:

$$g_j = \frac{K_1}{N} \sum_{k=1}^{N} (x)_k - x_j 0 + \frac{K_2}{N^2} \sum_{k,l=1}^{N} (x)_k x_l x_j - x_j^3 0. \tag{2}$$

The absence of rotational symmetry in the coupling makes an analytical study extremely difficult and the SL model can be analyzed only numerically. However, the phase reduced model

$$\dot{\theta}_j \quad = \quad \omega + \frac{\varepsilon K_1}{2N} \sum_{k=1}^{N} [\sin(\theta_k - \theta_j) - \beta \cos(\theta_k - \theta_j)] \tag{3}$$

$$+\frac{\varepsilon K_2}{8N^2}\sum_{k,l=1}^{N}[\sin(\theta_k+\theta_l-2\theta_j)-\beta\cos(\theta_k+\theta_l-2\theta_j)]-\frac{\varepsilon\beta K_2}{8N^2}\sum_{k,l=1}^{N}[2\cos(\theta_k-\theta_l)-3]$$

can be analyzed analytically and the results give a good approximation of the behavior of the SL model.

References

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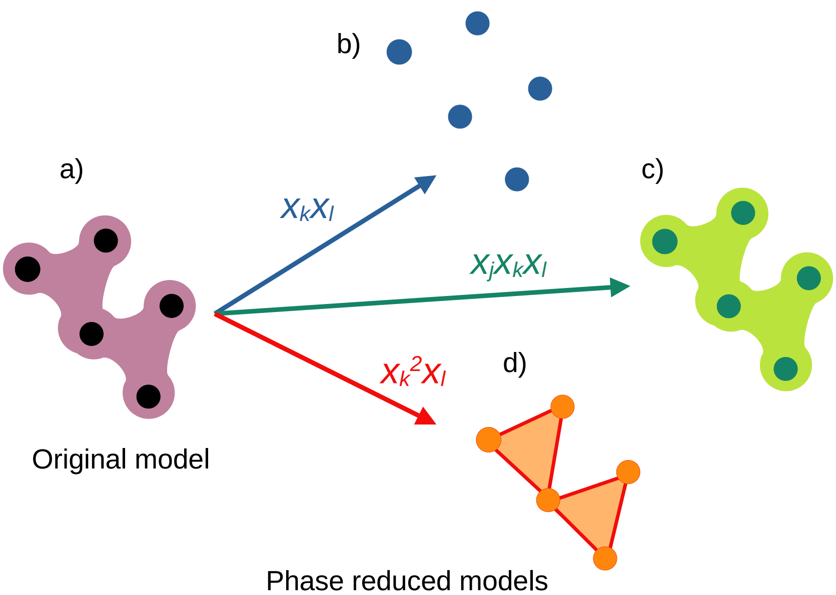

Figure 1: **Effect of phase reduction at the first order on the topology (a) for three different interactions:** (b) completely antisymmetric - no interaction, (c) completely symmetric - same interaction, (d) symmetric in one of the variables - from a hypergraph to a simplicial complex.