Robust Stochastic Optimization via Gradient Quantile Clipping

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Abstract

We introduce a clipping strategy for Stochastic Gradient Descent (SGD) which uses quantiles of the gradient norm as clipping thresholds. We prove that this new strategy provides a robust and efficient optimization algorithm for smooth objectives (convex or non-convex), that tolerates heavy-tailed samples (including infinite variance) and a fraction of outliers in the data stream akin to Huber contamination. Our mathematical analysis leverages the connection between constant step size SGD and Markov chains and handles the bias introduced by clipping in an original way. For strongly convex objectives, we prove that the iteration converges to a concentrated distribution and derive high probability bounds on the final estimation error. In the non-convex case, we prove that the limit distribution is localized on a neighborhood with low gradient. We propose an implementation of this algorithm using rolling quantiles which leads to a highly efficient optimization procedure with strong robustness properties, as confirmed by our numerical experiments.

Keywords. robust methods, stochastic optimization, heavy-tailed data, outliers, generalization error

1 Introduction

Stochastic gradient descent (SGD) [72] is the core optimization algorithm at the origin of most stochastic optimization procedures [46, 23, 44]. SGD and its variants are ubiquitously employed in machine learning in order to train most models [47, 7, 48, 79, 13, 55]. The convergence properties of SGD are therefore subjects of major interest. The first guarantees [62, 30] hold under strong statistical assumptions which require data to follow light-tailed sub-Gaussian distributions and provide error bounds in expectation. With the recent resurgence of interest for robust statistics [37, 25, 49, 70], variants of SGD based on clipping are shown to be robust to heavy-tailed gradients [31, 81], where the gradient samples are only required to have a finite variance. The latter requirement has been further weakened to the existence of a q-th moment for some $q > 1$ in [77, 65]. In this paper, we go further and show that another variant of clipped SGD with proper thresholds is robust both to heavy tails *and* outliers in the data stream.

Robust statistics appeared in the 60s with the pioneering works of Huber, Tukey and others [82, 41, 39, 76, 32]. More recently, the field found new momentum thanks to a series of works about robust scalar mean estimation [18, 1, 43, 53] and the more challenging multidimensional case [35, 19, 52, 59, 22, 24, 50, 27]. These paved the way to the elaboration of a host of robust learning algorithms [34, 70, 49, 51, 66] which have to date overwhelmingly focused on the batch learning setting. We consider the setting of streaming stochastic optimization [12, 14, 57], which raises an additional difficulty coming from the fact that algorithms can see each sample only once and must operate under an $\mathcal{O}(d)$ memory and complexity constraint for d-dimensional optimization

problems. A limited number of papers [81, 60, 28] propose theoretical guarantees for robust algorithms learning from streaming data.

This work introduces such an algorithm that learns from data on the fly and is robust both to heavy tails and outliers, with minimal computational overhead and sound theoretical guarantees.

We consider the problem of minimizing a smooth objective

$$\min_{\theta\in\mathbb{R}^d} \mathcal{L}(\theta) := \mathbb{E}_\zeta[\ell(\theta,\zeta)] \tag{1}$$

using observations $G(\theta,\zeta_t)$ of the unknown gradient $\nabla\mathcal{L}(\theta)$, based on samples $(\zeta_t)_{t\geq 0}$ received sequentially that include corruptions with probability $\eta < 1/2$. Formulation (1) is common to numerous machine learning problems where ℓ is a loss function evaluating the fit of a model with parameters θ on a sample ζ, the expectation $\mathbb{E}$ is w.r.t the unknown uncorrupted sample distribution.

We introduce quantile-clipped SGD (QC-SGD) which uses the iteration

$$\theta_{t+1} = \theta_t - \alpha_{\theta_t}\beta G(\theta_t,\zeta_t) \quad \text{with} \quad \alpha_{\theta_t} = \min\Big(1, \frac{\tau_{\theta_t}}{\|G(\theta_t,\zeta_t)\|}\Big), \tag{2}$$

where $\beta > 0$ is a constant step size and α_{θ_t} is the clipping factor with threshold chosen as the p-th quantile $\tau_{\theta_t} = Q_p(\|\widetilde{G}(\theta_t,\zeta_t)\|)$ with $\widetilde{G}(\theta_t,\zeta_t)$ an uncorrupted sample of $\nabla\mathcal{L}(\theta_t)$ and $p \in (0,1)$ (details will follow). Quantiles are a natural choice of clipping threshold which allows to handle heavy tails [75, 11] and corrupted data. For instance, the trimmed mean offers a robust and computationally efficient estimator of a scalar expectation [53]. Since the quantile $Q_p(\|\widetilde{G}(\theta_t,\zeta_t)\|)$ is non-observable, we introduce a method based on rolling quantiles in Section 5 which keeps the procedure $\mathcal{O}(d)$ both memory and complexity-wise.

Contributions. Our main contributions are as follows:

- For small enough η and well-chosen p, we show that, whenever the optimization objective is smooth and strongly convex, QC-SGD converges *geometrically* to a limit distribution such that the deviation around the optimum achieves the *optimal* dependence on η.
- In the non-corrupted case $\eta = 0$ and with a strongly convex objective, we prove that a coordinated choice of β and p ensures that the limit distribution is sub-Gaussian with constant of order $\mathcal{O}(\sqrt{\beta})$. In the corrupted case $\eta > 0$, the limit distribution is sub-exponential.
- For a smooth objective (non-convex) whose gradient satisfies an identifiability condition, we prove that the total variation distance between QC-SGD iterates and its limit distribution vanishes sub-linearly. In this case, the limit distribution is such that the deviation of the objective gradient is optimally controlled in terms of η.
- Finally, we provide experiments to demonstrate that QC-SGD can be easily implemented by estimating $Q_p(\|\widetilde{G}(\theta_t,\zeta_t)\|)$ with rolling quantiles. In particular, we show that the iteration is indeed robust to heavy tails and corruption on multiple stochastic optimization tasks.

Our theoretical results are derived thanks to a modelling through Markov chains and hold under an L_q assumption on the gradient distribution with $q > 1$.

Related works. Convergence in distribution of the Markov chain generated by constant step size SGD, relatively to the Wasserstein metric, was first established in [29]. Another geometric convergence result was derived in [86] for non-convex, non-smooth, but quadratically growing objectives, where a convergence statement relatively to a weighted total variation distance is given

and a CLT is established. These papers do not consider robustness to heavy tails or outliers. Early works proposed stochastic optimization and parameter estimation algorithms which are robust to a wide class of noise of distributions [56, 67, 68, 71, 80, 21, 20, 61], where asymptotic convergence guarantees are stated for large sample sizes. Initial evidence of the robustness of clipped SGD to heavy tails was given by [87] who obtained results in expectation. Subsequent works derived high-confidence sub-Gaussian performance bounds under a finite variance assumption [31, 81] and later under an L_q assumption [77, 65] with $q > 1$. A similar SGD clipping scheme to (2) is presented in [78], however, in contrast to our work, they do not consider the robust setting and focus on experimental study while we also provide theoretical guarantees.

Robust versions of Stochastic Mirror Descent (SMD) are introduced in [60, 45]. For a proper choice of the mirror map, SMD is shown to handle infinite variance gradients without any explicit clipping [85]. Finally, [28] studY heavy-tailed and outlier robust streaming estimation algorithms of the expectation and covariance. On this basis, robust algorithms for linear and logistic regression are derived. However, the involved filtering procedure is hard to implement in practice and no numerical evaluation of the considered approach is proposed.

Agenda. In Section 2 we set notations, state the assumptions required by our theoretical results and provide some necessary background on continuous state Markov chains. In Section 3, we state our results for strongly convex objectives including geometric ergodicity of QC-SGD (Theorem 1), characterizations of the limit distribution and deviation bounds on the final estimate. In Section 4, we remove the convexity assumption and obtain a weaker ergodicity result (Theorem 2) and characterize the limit distribution in terms of the deviations of the objective gradient. Finally, we present a rolling quantile procedure in Section 5 and demonstrate its performance through a few numerical experiments on synthetic and real data.

2 Preliminaries

The model parameter space is $\mathbb{R}^d$ endowed with the Euclidean norm $\|\cdot\|$, $\mathcal{B}(\mathbb{R}^d)$ is the Borel σ-algebra of $\mathbb{R}^d$ and we denote by $\mathcal{M}_1(\mathbb{R}^d)$ the set of probability measures over $\mathbb{R}^d$. We assume throughout the paper that the objective $\mathcal{L}$ is smooth.

Assumption 1. *The objective $\mathcal{L}$ is L-Lipschitz-smooth, namely*

$$\mathcal{L}(\theta') \leq \mathcal{L}(\theta) + \langle \nabla \mathcal{L}(\theta), \theta' - \theta \rangle + \frac{L}{2}\|\theta - \theta'\|^2$$

with $L < +\infty$ for all $\theta, \theta' \in \mathbb{R}^d$.

The results from Section 3 below use the following

Assumption 2. *The objective $\mathcal{L}$ is μ-strongly convex, namely*

$$\mathcal{L}(\theta') \geq \mathcal{L}(\theta) + \langle \nabla \mathcal{L}(\theta), \theta' - \theta \rangle + \frac{\mu}{2}\|\theta - \theta'\|^2$$

with $\mu > 0$ for all $\theta, \theta' \in \mathbb{R}^d$.

An immediate consequence of Assumption 2 is the existence of a unique minimizer $\theta^\star = \operatorname{argmin}_{\theta \in \mathbb{R}^d} \mathcal{L}(\theta)$. The next assumption formalizes our corruption model.

Assumption 3 (η-corruption)**.** *The gradients $(G(\theta_t, \zeta_t))_{t \geq 0}$ used in Iteration (2) are sampled as $G(\theta_t, \zeta_t) = U_t \breve{G}(\theta_t) + (1 - U_t)\widetilde{G}(\theta_t, \zeta_t)$ where U_t are i.i.d Bernoulli random variables with parameter $\eta < 1/2$, $\breve{G}(\theta_t) \sim \mathcal{D}_{\mathcal{O}}(\theta_t)$ with $\mathcal{D}_{\mathcal{O}}(\theta_t)$ an arbitrary distribution and $\widetilde{G}(\theta_t, \zeta_t) \sim \mathcal{D}_{\mathcal{I}}(\theta_t)$ follows the true gradient distribution and is independent from the past given θ_t.*

Assumption 3 is an online analog of the Huber contamination model [38, 41] where corruptions occur with probability η and where the distribution of corrupted samples is not fixed and may depend on the current iterate θ_t. The next assumption requires the true gradient distribution to be unbiased and diffuse.

Assumption 4. *For all θ, non-corrupted gradient samples $\widetilde{G}(\theta, \zeta) \sim \mathcal{D}_{\mathcal{I}}(\theta)$ are such that*

$$\widetilde{G}(\theta, \zeta) = \nabla \mathcal{L}(\theta) + \varepsilon_\theta, \tag{3}$$

where ε_θ is a centered noise $\mathbb{E}[\varepsilon_\theta | \theta] = 0$ with distribution $\delta\nu_{\theta,1} + (1-\delta)\nu_{\theta,2}$ where $\delta > 0$ and $\nu_{\theta,1}, \nu_{\theta,2}$ are distributions over $\mathbb{R}^d$ such that $\nu_{\theta,1}$ admits a density h_θ w.r.t. the Lebesgue measure satisfying

$$\inf_{\|\omega\| \leq R} h_\theta(\omega) > \varkappa(R) > 0$$

for all $R > 0$, where $\varkappa(\cdot)$ is independent of θ.

Assumption 4 imposes a weak constraint, since it is satisfied, for example, as soon as the noise ε_θ admits a density w.r.t. Lebesgue's measure. Our last assumption formalizes the requirement of a finite moment for the gradient error.

Assumption 5. *There is $q > 1$ such that for $\widetilde{G}(\theta, \zeta) \sim \mathcal{D}_{\mathcal{I}}(\theta)$, we have*

$$\mathbb{E}\big[\|\varepsilon_\theta\|^q \mid \theta\big]^{1/q} = \mathbb{E}\big[\big\|\widetilde{G}(\theta, \zeta) - \nabla \mathcal{L}(\theta)\big\|^q \mid \theta\big]^{1/q} \leq A_q \|\theta - \theta^\star\| + B_q \tag{4}$$

for all $\theta \in \mathbb{R}^d$, where $A_q, B_q > 0$. When $\mathcal{L}$ is not strongly convex, we further assume that $A_q = 0$.

The bound (4) captures the case of arbitrarily high noise magnitude through the dependence on $\|\theta - \theta^\star\|$. This is consistent with common strongly convex optimization problems such as least squares regression. For non-strongly convex $\mathcal{L}$, we require $A_q = 0$ since $\theta^\star$ may not exist.

Definition 1. *If X is a real random variable, we say that X is K-sub-Gaussian for $K > 0$ if*

$$\mathbb{E} \exp(\lambda^2 X^2) \leq e^{\lambda^2 K^2} \quad \textit{for} \quad |\lambda| \leq 1/K. \tag{5}$$

We say that X is K-sub-exponential for $K > 0$ if

$$\mathbb{E} \exp(\lambda |X|) \leq \exp(\lambda K) \quad \textit{for all} \quad 0 \leq \lambda \leq 1/K. \tag{6}$$

The convergence results presented in this paper use the following formalism of continuous state Markov chains. Given a step size $\beta > 0$ and a quantile $p \in (0, 1)$, we denote by $P_{\beta,p}$ the Markov transition kernel governing the Markov chain $(\theta_t)_{t \geq 0}$ generated by QC-SGD, so that

$$\mathbb{P}(\theta_{t+1} \in A \mid \theta_t) = P_{\beta,p}(\theta_t, A)$$

for $t \geq 0$ and $A \in \mathcal{B}(\mathbb{R}^d)$. The transition kernel $P_{\beta,p}$ acts on probability distributions $\nu \in \mathcal{M}_1(\mathbb{R}^d)$ through the mapping $\nu \to \nu P_{\beta,p}$ which is defined, for all $A \in \mathcal{B}(\mathbb{R}^d)$, by $\nu P_{\beta,p}(A) = \int_A P_{\beta,p}(\theta, A) d\nu(\theta) = \mathbb{P}(\theta_{t+1} \in A \mid \theta_t \sim \nu)$. For $n \geq 1$, we similarly define the multi-step transition kernel $P^n_{\beta,p}$ which is such that $P^n_{\beta,p}(\theta_t, A) = \mathbb{P}(\theta_{t+n} \in A \mid \theta_t)$ and acts on probability distributions $\nu \in \mathcal{M}_1(\mathbb{R}^d)$ through $\nu P^n_{\beta,p} = (\nu P_{\beta,p}) P^{n-1}_{\beta,p}$. Finally, we define the total variation (TV) norm of a signed measure ν as

$$2\|\nu\|_{\mathrm{TV}} = \sup_{f : |f| \leq 1} \int f(\theta) \nu(d\theta) = \sup_{A \in \mathcal{B}(\mathbb{R}^d)} \nu(A) - \inf_{A \in \mathcal{B}(\mathbb{R}^d)} \nu(A).$$

In particular, we recover the TV *distance* between $\nu_1, \nu_2 \in \mathcal{M}_1(\mathbb{R}^d)$ as $d_{\mathrm{TV}}(\nu_1, \nu_2) = \|\nu_1 - \nu_2\|_{\mathrm{TV}}$.

3 Strongly Convex Objectives

We are ready to state our convergence result for the stochastic optimization of a strongly convex objective using QC-SGD with η-corrupted samples.

Theorem 1 (Geometric ergodicity)**.** *Let Assumptions 1-5 hold and assume there is a quantile $p \in [\eta, 1-\eta]$ such that*

$$\kappa := (1-\eta)p\mu - \eta L - (1-p)^{-\frac{1}{q}} A_q (1 - p(1-\eta)) > 0. \tag{7}$$

Then, for a step size β satisfying

$$\beta < \frac{1}{4} \frac{\kappa}{\mu^2 + 6L^2 + 16\eta^{-\frac{2}{q}} A_q^2}, \tag{8}$$

the Markov chain $(\theta_t)_{t\geq 0}$ generated by QC-SGD with parameters β and p converges geometrically to a unique invariant measure $\pi_{\beta,p}$: for any initial $\theta_0 \in \mathbb{R}^d$, there is $\rho < 1$ and $M < \infty$ such that after T iterations

$$\left\| \delta_{\theta_0} P_{\beta,p}^T - \pi_{\beta,p} \right\|_{\mathrm{TV}} \leq M \rho^T \big(1 + \|\theta_0 - \theta^\star\|^2\big),$$

where δ_{θ_0} is the Dirac measure located at θ_0.

The proof of Theorem 1 is given in Appendix D.2 and relies on the geometric ergodicity result of [58, Chapter 15] for Markov chains with a geometric drift property. A similar result for quadratically growing objectives was established by [86] and convergence w.r.t. Wasserstein's metric was shown in [29] assuming uniform gradient co-coercivity. However, robustness was not considered in these works. The restriction $p \in [\eta, 1-\eta]$ comes from the consideration that other quantiles are not estimable in the event of η-corruption. Condition (7) is best interpreted for the choice $p = 1 - \eta$ in which case it translates into $\eta^{1-1/q} \leq \mathcal{O}(\mu/(L + A_q))$ implying that it is verified for η small enough within a limit fixed by the problem conditioning. A similar condition with $q = 2$ appears in [28, Theorem E.9] which uses a finite variance assumption.

The constants M and ρ controlling the geometric convergence speed in Theorem 1 depend on the parameters β, p and the initial θ_0. Among choices fulfilling the convergence conditions, it is straightforward that greater step size β and θ_0 closer to $\theta^\star$ lead to faster convergence. However, the dependence in p is more intricate and should be evaluated through the resulting value of κ. We provide a more detailed discussion about the value of ρ in Appendix C.

The choice $p = 1 - \eta$ appears to be ideal since it leads to optimal deviation of the invariant distribution around the optimum $\theta^\star$ which is the essence of our next statement.

Proposition 1. *Assume the same as in Theorem 1 and condition* (7) *with the choice $p = 1 - \eta$. For step size β satisfying* (8)*, $q \geq 2$, and additionally*:

$$\beta \leq \eta^{2-2/q} / \kappa, \tag{9}$$

for $\theta \sim \pi_{\beta, 1-\eta}$, we have the following upper bound:

$$\mathbb{E}\|\theta - \theta^\star\|^2 \leq \Big(\frac{6\eta^{1-1/q} B_q}{\kappa} \Big)^2.$$

Proposition 1 is proven in Appendix D.3. An analogous result holds for $q \in (1, 2)$ but requires a different proof and can be found in Appendix D.4. Proposition 1 may be compared to [86, Theorem 3.1] which shows that the asymptotic estimation error can be reduced arbitrarily using a small step size. However, this is impossible in our case since we consider corrupted gradients.

The performance of Proposition 1 is best discussed in the specific context of linear regression where gradients are given as $G(\theta,(X,Y)) = X(X^\top\theta - Y)$ for samples $X, Y \in \mathbb{R}^d \times \mathbb{R}$ such that $Y = X^\top\theta^\star + \epsilon$ with ϵ a centered noise. In this case, a finite moment of order k for the data implies order $k/2$ for the gradient corresponding to an $\eta^{1-2/k}$ rate in Proposition 1. Since Assumption 5 does not include independence of the noise ϵ from X, this corresponds to the negatively correlated moments assumption of [2] being unsatisfied. Consequently, Proposition 1 is information-theoretically optimal in η based on [2, Corollary 4.2]. Nonetheless, the poor dimension dependence through B_q may still be improved. If the gradient is sub-Gaussian with constant K, we would have $B_q \lesssim K\sqrt{q}$ for $q \geq 1$ (see [84] for a reference), in which case, the choice $q = \log(1/\eta)$ recovers the optimal rate in $\eta\sqrt{\log(1/\eta)}$ for the Gaussian case.

We now turn to showing strong concentration properties for the invariant distribution $\pi_{\beta,p}$. For this purpose, we restrict the optimization to a bounded and convex set $\Theta \subset \mathbb{R}^d$ and replace Iteration (2) by the projected iteration

$$\theta_{t+1} = \Pi_\Theta\big(\theta_t - \alpha_{\theta_t}\beta G(\theta_t,\zeta_t)\big), \tag{10}$$

where Π_Θ is the projection onto Θ. Assuming that the latter contains the optimum $\theta^\star \in \Theta$, one can check that the previous results continue to hold thanks to the inequality

$$\|\Pi_\Theta(\theta) - \theta^\star\| = \|\Pi_\Theta(\theta) - \Pi_\Theta(\theta^\star)\| \leq \|\theta - \theta^\star\|,$$

which results from the convexity of Θ. The restriction of the optimization to a bounded set allows us to uniformly bound the clipping threshold τ_θ, which is indispensable for the following result.

Proposition 2. *In the setting of Theorem 1, consider projected QC-SGD* (10) *and let* $\overline{\tau} = \sup_{\theta\in\Theta}\tau_\theta$, $D = \operatorname{diam}(\Theta)$ *the diameter of* Θ *and* $\overline{B}_q = A_qD + B_q$.

- *Consider the non-corrupted case* $\eta = 0$ *and set the quantile* p *such that* $p \geq 1 - (\beta\mu)^{\frac{q}{2(q-1)}}$. *Then, for* $\theta \sim \pi_{\beta,p}$, *the variable* $\|\theta - \theta^\star\|$ *is sub-Gaussian in the sense of Definition 1 with constant*
$$K = 4\sqrt{\frac{2\beta(\overline{B}_q^2 + \overline{\tau}^2)}{p\mu}}.$$
- *Consider the corrupted case* $\eta > 0$, *and set the quantile* $p \in [\eta, 1-\eta]$ *such that Inequality* (7) *holds. Then, for* $\theta \sim \pi_{\beta,p}$, *the variable* $\|\theta - \theta^\star\|$ *is sub-exponential in the sense of Definition 1 with constant*
$$K = \frac{7\overline{\tau} + (1-p)^{1-1/q}\overline{B}_q}{p\mu}.$$

The proof can be found in Appendix D.5. The strong concentration properties given by Proposition 2 for the invariant distribution appear to be new. Still, the previous result remains asymptotic in nature. High confidence deviation bounds for an iterate θ_t can be derived by leveraging the convergence in Total Variation distance given by Theorem 1 leading to the following result.

Corollary 1. *In the setting of Proposition 2, in the absence of corruption* $\eta = 0$, *after* T *iterations, for* $\delta > 0$, *we have*

$$\mathbb{P}\bigg(\big\|\theta_T - \theta^\star\big\| > 4\sqrt{\overline{B}_q^2 + \overline{\tau}^2}\sqrt{\frac{2\beta\log(e/\delta)}{p\mu}}\bigg) \leq \delta + \rho^T M\big(1 + \|\theta_0 - \theta^\star\|^2\big).$$

Choosing a smaller step size β in Corollary 1 allows to improve the deviation bound. However, this comes at the cost of weaker confidence because of slower convergence due to a greater ρ. See Appendix C for a discussion including a possible compromise. Corollary 1 may be compared to the results of [31, 81, 77, 65] which correspond to $\beta \approx 1/T$ and have a similar dependence on the dimension through the gradient variance. Although their approach is also based on gradient clipping, they use different thresholds and proof methods. In the presence of corruption, the invariant distribution is not sub-Gaussian. This can be seen by considering the following toy Markov chain:

$$X_{t+1} = \begin{cases} \alpha X_t + \xi & \text{w.p.} \quad 1-\eta \\ X_t + \tau & \text{w.p.} \quad \eta \end{cases}$$

where $\alpha < 1, \tau > 0$ are constants and ξ is a positive random noise. Using similar methods to the proof of Theorem 1, one can show that $(X_t)_{t\geq 0}$ converges (for any initial X_0) to an invariant distribution whose moments can be shown to grow linearly, indicating a sub-exponential distribution and excluding a sub-Gaussian one. We provide additional details for the underlying argument in Appendix D.6. For the corrupted case, the sub-exponential property stated in Proposition 2 holds with a constant K of order $\overline{\tau}/\mu$, which is not satisfactory and leaves little room for improvement due to the inevitable bias introduced by corruption. Therefore, we propose the following procedure in order to obtain a high confidence estimate, similarly to Corollary 1.

Algorithm 1: Aggregation of cycling iterates

Input: Step size $\beta > 0$, quantile index $p \in (0,1)$, initial parameter $\theta_0 \in \Theta$, horizon T and
number of concurrent iterates $N \geq 1$.
Optimize multiple parameters $\theta_t^{(1)}, \dots, \theta_t^{(N)}$ starting from a common $\theta_0 = \theta_0^{(n)}$ for $n \in [\![N]\!] =: \{1, \dots, N\}$ and T steps $t = 0, \dots, T$ using the following cycling iteration:;

$$\theta_{t+1}^{(n)} = \begin{cases} \theta_t^{(n)} - \alpha_{\theta_t^{(n)}} \beta G\big(\theta_t^{(n)}, \zeta_t\big) & \text{if } t \equiv n-1 \bmod N, \\ \theta_t^{(n)} & \text{otherwise.} \end{cases} \tag{11}$$

Compute $r_{ij} = \big\|\theta_T^{(i)} - \theta_T^{(j)}\big\|$ for $i, j \in [\![N]\!]$.;
For $j \in [\![N]\!]$, let $r^{(j)} \in \mathbb{R}_+^N$ be the vector $r_{j,:} := [r_{j,1}, \dots, r_{j,N}]$ sorted in non decreasing order.;
Compute the aggregated estimator as $\widehat{\theta} = \theta_T^{(\widehat{i})}$ with $\widehat{i} = \operatorname{argmin}_{i \in [\![N]\!]} r_{N/2}^{(i)}$.;
return $\widehat{\theta}$

Algorithm 1 uses ideas from [37] (see also [59, 45]) and combines a collection of *weak* estimators (only satisfying L_2 bounds) into a strong one with sub-exponential deviation. The aggregated estimator $\widehat{\theta}$ satisfies the high probability bound given in the next result.

Corollary 2. *Assume the same as in Theorem 1 and Proposition 1. Consider $\widehat{\theta}$ given by Algorithm 1, with the assumption that the gradient sample sets used for each $\big(\theta_T^{(n)}\big)_{n \in [\![N]\!]}$ in Equation* (11) *are independent. For $\delta > 0$, if $N \geq 16 \log(1/\delta)$ and T satisfies*

$$T \geq N \log(15M(1 + \|\theta_0 - \theta^\star\|^2))/\log(1/\rho),$$

then, with probability at least $1-\delta$, we have

$$\big\|\widehat{\theta} - \theta^\star\big\| \leq \frac{27 \eta^{1-\frac{1}{q}} \overline{B}_q}{\kappa}. \tag{12}$$

We obtain a high confidence version of the bound in expectation previously stated in Proposition 1. As argued before, the above bound depends optimally on η. Similar bounds to (12) are obtained for $q = 2$ in [28] for streaming mean estimation, linear and logistic regression. Their results enjoy better dimension dependence but are less general than ours. In addition, the implementation of the associated algorithm is not straightforward whereas our method is quite easy to use (see Section 5).

4 Smooth Objectives

In this section, we drop Assumption 2 and consider the optimization of possibly non-convex objectives. Consequently, the existence of a unique optimum $\theta^\star$ and the quadratic growth of the objective are no longer guaranteed. This motivates us to use a uniform version of Assumption 5 with $A_q = 0$ since the gradient is no longer assumed coercive and its deviation moments can be taken as bounded. In this context, we obtain the following weaker (compared to Theorem 1) ergodicity result for QC-SGD.

Theorem 2 (Ergodicity)**.** *Let Assumptions 1, 3, 4 and 5 hold with $A_q = 0$ (uniformly bounded moments) and positive objective $\mathcal{L}$. Let $(\theta_t)_{t\geq 0}$ be the Markov chain generated by QC-SGD with step size β and quantile $p \in [\eta, 1-\eta]$. Assume that p and β are such that $3p(1-\eta)/4 > L\beta + \eta$ and that the subset of $\mathbb{R}^d$ given by*

$$\left\{\theta : \frac{1}{2}\big\|\nabla\mathcal{L}(\theta)\big\|^2 \leq \frac{B_q^2\big((1-p)^{-\frac{2}{q}}(L\beta+2\eta^2)+2\eta^{2-\frac{2}{q}}\big)}{p(1-\eta)(3p(1-\eta)/4-L\beta-\eta)}\right\} \tag{13}$$

is bounded. Then, for any initial $\theta_0 \in \mathbb{R}^d$, there exists $M < +\infty$ such that after T iterations

$$\big\|\delta_{\theta_0} P_{\beta,p}^T - \pi_{\beta,p}\big\|_{\mathrm{TV}} \leq \frac{M}{T}, \tag{14}$$

where $\pi_{\beta,p}$ is a unique invariant measure and where δ_{θ_0} is the Dirac measure located at θ_0.

The proof is given in Appendix D.9 and uses ergodicity results from [58, Chapter 13]. Theorem 2 provides convergence conditions for an SGD Markov chain on a smooth objective in a robust setting. We are unaware of anterior results of this kind in the literature. Condition (13) requires that the true gradient exceeds the estimation error at least outside of a bounded set. If this does not hold, the gradient would be dominated by the estimation error, leaving no hope for the iteration to converge. Observe that, for no corruption ($\eta = 0$), the condition is always fulfilled for some β and p. Note also that without strong convexity (Assumption 2), convergence occurs at a slower sublinear rate which is consistent with the optimization rate expected for a smooth objective (see [15, Theorem 3.3]).

As previously, we complement Theorem 1 with a characterization of the invariant distribution.

Proposition 3. *Under the conditions of Theorem 2, assume that the choice $p = 1-\eta$ is such that the set* (13) *is bounded. For step size $\beta \leq \eta^2/L$, the stationary measure $\theta \sim \pi_{\beta,1-\eta}$ satisfies*

$$\mathbb{E}\big\|\nabla\mathcal{L}(\theta)\big\|^2 \leq \frac{5\eta^{2-\frac{2}{q}}B_q^2}{p(1-\eta)\big(3p(1-\eta)/4-L\beta-\eta\big)}. \tag{15}$$

The statement of Proposition 3 is clearly less informative than Propositions 1 and 2 since it only pertains to the gradient rather than, for example, the excess risk. This is due to the weaker assumptions that do not allow to relate these quantities. Still, the purpose remains to find a critical point and is achieved up to $\mathcal{O}(\eta^{1-1/q})$ precision according to this result. Due to corruption, the estimation error on the gradient cannot be reduced beyond $\Omega(\eta^{1-1/q})$ [69, 36, 26]. Therefore, one may draw a parallel with a corrupted mean estimation task, in which case, the previous rate is, in fact, information-theoretically optimal.

5 Implementation and Numerical Experiments

The use of the generally unknown quantile $Q_p(\|\widetilde{G}(\theta_t, \zeta_t)\|)$ in QC-SGD constitutes the main obstacle to its implementation. For strongly convex objectives, one may use a proxy such as $a\|\theta_t - \theta_{\mathrm{ref}}\| + b$ with positive a, b and $\theta_{\mathrm{ref}} \in \mathbb{R}^d$ an approximation of $\theta^\star$ serving as reference point. This choice is consistent with Assumptions 1 and 5, see Lemma 2 in Appendix D. In the non-strongly convex case, a constant threshold can be used since the gradient is a priori uniformly bounded, implying the same for the quantiles of its deviations. In practice, we propose a simpler and more direct approach: we use a rolling quantile procedure, described in Algorithm 2. The latter stores the values $(\|G(\theta_{t-j}, \zeta_{t-j})\|)_{1 \le j \le S}$ in a buffer of size $S \in \mathbb{N}^*$ and replaces $Q_p(\|\widetilde{G}(\theta_t, \zeta_t)\|)$ in QC-SGD by an estimate $\widehat{Q}_p$ which is the $\lfloor pS \rfloor$-th order statistic in the buffer. Note that only the norms of previous gradients are stored in the buffer, limiting the memory overhead to $\mathcal{O}(S)$. The computational cost of $\widehat{Q}_p$ can also be kept to $\mathcal{O}(S)$ per iteration thanks to a bookkeeping procedure (see Appendix B).

Algorithm 2: Rolling QC-SGD

Input: Step size $\beta > 0$, quantile index
$p \in (0, 1)$, initial parameter $\theta_0 \in \mathbb{R}^d$, $\tau_{\mathrm{unif}} > 0$, buffer B of size S and horizon T.
Fill B with $S - 1$ values equal to τ_{unif}.
for $t = 0 \dots T - 1$ **do**
  Draw a sample $G(\theta_t, \zeta_t)$ and add $\|G(\theta_t, \zeta_t)\|$ to B.
  $\widehat{Q}_p \leftarrow \lfloor pS \rfloor$ rank element of B.
  $\theta_{t+1} \leftarrow \theta_t - \beta \mathrm{clip}(G(\theta_t, \zeta_t), \widehat{Q}_p)$
  Delete the oldest value in B.
end
return θ_T

We implement this procedure for a few tasks and compare its performance with relevant baselines. We do not include a comparison with [28] whose procedure has no implementation we are aware of and is difficult to use in practice due to its dependence on a number of unknown constants. Our experiments on synthetic data consider an infinite horizon, dimension $d = 128$, and a constant step size for all methods.

Linear regression. We consider least-squares linear regression and compare RQC-SGD with Huber's estimator [40] and clipped SGD (designated as CClip(λ)) with three clipping levels $\lambda \sigma_{\max} \sqrt{d}$ for $\lambda \in \{0.8, 1.0, 1.2\}$ where $\sigma_{\max}$ is a fixed data scaling factor. These thresholds provide a rough estimate of the gradient norm. We generate covariates X and labels Y both heavy-tailed and corrupted. Corruption in the data stream is generated according to Assumption 3 with outliers represented either by aberrant values or *fake* samples $Y = X^\top \theta_{\mathrm{fake}} + \epsilon$ using a false parameter θ_{fake}, see Appendix B for further details on data generation and fine tuning of the Huber parameter. All methods are run with constant step size and averaged results over 100 runs are displayed on Figure 1 (top row).

As anticipated, Huber's loss function is not robust to corrupted covariates. In contrast, using gradient clipping allows convergence to meaningful estimates. Although this holds true for a constant threshold, Figure 1 shows it may considerably slow the convergence if started away from the optimum. In addition, the clipping level also affects the final estimation precision and requires tuning. Both of the previous issues are well addressed by RQC-SGD whose adaptive clipping level allows fast progress of the optimization and accurate convergence towards a small neighborhood of the optimum.

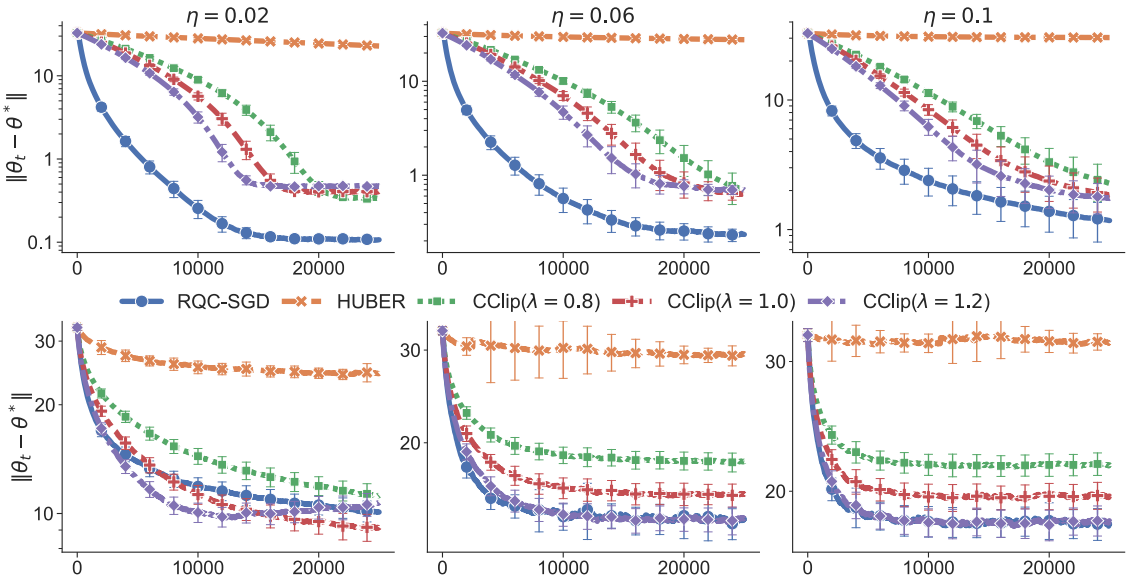

Figure 1: Evolution of $\|\theta_t - \theta^\star\|$ on the tasks of linear regression (top row) and logistic regression (bottom row) averaged over 100 runs at increasing corruption levels (error bars represent half the standard deviation). Estimators based on Huber's loss are strongly affected by data corruption. SGD with constant clipping thresholds is robust but slow to converge for linear regression and requires tuning for better final precision. RQC-SGD combines fast convergence with good final precision thanks to its adaptive clipping strategy.

Logistic regression. We test the same methods on logistic regression. Huber's baseline is represented by the modified Huber loss (also known as quadratic SVM [88]). We generate data similarly to the previous task except for the labels which follow $Y \sim \text{Bernoulli}(\sigma(X^\top \theta^\star))$ with σ the sigmoid function. Corrupted labels are either uninformative, flipped or obtained with a fake θ_{fake} (see details in Appendix B). Results are displayed on the bottom row of Figure 1.

As previously, Huber's estimator performs poorly with corruption. However, constant clipping appears to be better suited when the gradient is bounded, so that the optimization is less affected by its underestimation. We observe, nonetheless, that a higher clipping level may lead to poor convergence properties, even at a low corruption rate. Note also that the constant levels we use are based on prior knowledge about the data distribution and would have to be fine tuned in practice. Meanwhile, the latter issue is well addressed by quantile clipping. Finally, notice that no algorithm truly approaches the true solution for this task. This reflects the difficulty of improving upon Proposition 3 which only states convergence to a neighborhood where the objective gradient is comparable to the estimation error in magnitude.

Classification with shallow networks. Finally, we evaluate the performance on the task of training a single hidden layer neural network classifier on some real datasets which corresponds to a non-convex optimization problem. To handle multiclass data, we use the cross entropy loss and replace Huber's baseline with plain SGD for simplicity. We define constant clipping baselines using thresholds given by the quantiles of order $p = 0.25, 0.5$, and 0.75 of the norms of a batch of gradients at the beginning of the optimisation. Due to the greater sensitivity to corruption observed in this case, we set $\eta = 0.02$ and use $p = 0.9$ for RQC-SGD. We train all methods with one sample per iteration using equal step sizes and evaluate them through the test loss. We provide further results and experimental details in Appendix B. Results are displayed on Figure 2.

Unsurprisingly, standard SGD is not robust to corrupted samples and, while using a constant

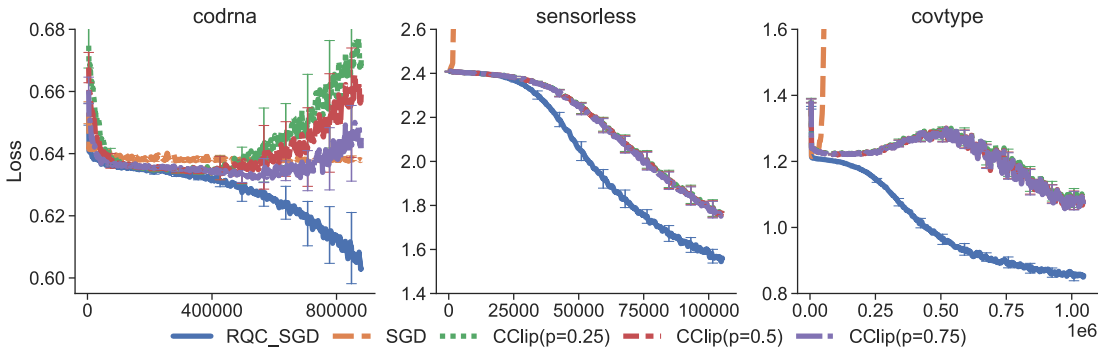

Figure 2: Evolution of the test loss (y-axis) against iteration t (x-axis) for the training of a single hidden layer network on different real world classification datasets (average over 20 runs). We observe more consistent and stable objective decrease for RQC-SGD whereas constant clipping baselines are slower and may fail to converge.

clipping level helps keep the optimisation on track, the experiments show that careful tuning may sometimes be necessary to prevent divergence. On the other hand, the adaptive clipping levels used by RQC-SGD allow to make the iteration faster *and* more resilient to corruption. This leads to an optimization path with a more consistent decrease of the objective. Moreover, we also observe that RQC-SGD allows for a better control of the asymptotic variance of the optimized parameter compared to constant clipping.

6 Conclusion

We introduced a new clipping strategy for SGD and proved that it defines a stochastic optimization procedure which is robust to both heavy tails and outliers in the data stream. We also provided an efficient rolling quantile procedure to implement it and demonstrated its performance through numerical experiments on synthetic and real data. Future research directions include improving the dimension dependence in our bounds, possibly by using sample rejection rules or by considering stochastic mirror descent [63, 5] clipped with respect to a non Euclidean norm. This may also procure robustness to higher corruption rates. Another interesting research track is the precise quantification of the geometric convergence speed of the Markov chain generated by constant step size SGD on a strongly convex objective.

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