Second-Order Beurling Approximations and Super-Resolution from Bandlimited Functions

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***Abstract*—The Beurling–Selberg extremal approximation problems are classics in functional analysis and have found applications in numerous areas of mathematics. Of particular interest, optimal solutions to those problems can be exploited to provide sharp bounds on the condition number of Vandermonde matrices with nodes on the unit circle, which is of great interest to many inverse problems, including super-resolution. However, those solutions have non-derivable Fourier transforms, which impedes their use in a stability analysis of the super-resolution problem. We propose novel second-order extensions to Beurling–Selberg problems, where the approximation residual to functions of bounded variation (BV) is constrained to faster decay rates in the asymptotic, ensuring the smoothness of their Fourier transforms. We harness the properties of those second-order approximants by establishing a link between the norms of the residuals and the minimal eigenvalue of the Fisher information matrix (FIM) of the super-resolution problem. This enables the derivation of a simple and computable minimal resolvable distance for the super-resolution problem, depending only on the properties of the point-spread function, above which stability can be guaranteed.**

I. Introduction

In the late 1930s, Beurling considered the problem of finding a function $F(t)$ that majorizes the signum function $\operatorname{sgn}(t)$[1] and with a Fourier transform having compact support in the interval $[-1, 1]$, while minimizing the integral of the residual $F(t) - \operatorname{sgn}(t) \geq 0$. He successfully showed that the function

$$B_0(t) = (1 + 2t)\operatorname{sinc}^2(\pi t) + \sum_{k=-\infty}^{\infty} \operatorname{sgn}(k)\operatorname{sinc}^2(\pi(t-k)), \quad (1)$$

also called the *Beurling majorant*, is *extremal* in the sense that any bandlimited majorant F of the signum function verifies

$$\int_{-\infty}^{\infty}(F(t) - \operatorname{sgn}(t))\mathrm{d}t \geq \int_{-\infty}^{\infty}(B_0(t) - \operatorname{sgn}(t))\mathrm{d}t = 1. \quad (2)$$

Furthermore, equality in the above inequation happens if and only if $F(t) = B_0(t)$. Although Beurling's study was originally motivated by proving uniform bounds on the derivatives of almost-periodic functions, the Beurling majorant is a versatile functional analysis tool that found usage in numerous areas of mathematics, including probability, dynamical systems, combinatorics, and sphere packing, and sampling theory (see [1] and references therein).

More relevant to the context of this paper, Selberg used Beurling's extremal function to construct extremal lower and upper bandlimited approximations of the rectangle function [2], [3]. This was later extended to arbitrary functions of bounded variation (BV) [4], at the cost of a possible loss of extremality. Those results pave the way for tight derivations of the *large sieve* inequalities [5], [6], which, in short, frame the total energy of a periodic function with the energy of its non-uniform samples. Among other applications, the large sieve inequalities provide sharp bounds on the extremal singular values of Vandermonde matrices with nodes on the unit circle. The conditioning of such matrices plays a critical role in the sampling, observability, and identification of shift-invariant systems and in the super-resolution of complex exponentials [7], also known as the *line spectral estimation problem* [8], [9].

However, the lack of derivability of the Fourier transform of the solutions to the Beurling–Selberg problems prevents their application to a perturbation analysis of super-resolution when dealing with noisy observations, as infinitesimal variations have to be considered. In this paper, we palliate this issue by introducing higher-order extensions to the Beurling–Selberg problems. We explore their properties and leverage them to study the stability of super-resolution under Gaussian noise, which we quantify in terms of the minimal eigenvalue of its associated Fisher information matrix (FIM) [10].

A. Contributions and Organization of the Paper

In Section II, novel extensions to the classical Beurling–Selberg extremal approximations problems are defined, which we call of *higher-order*, and where the approximation residuals are required to have faster decay rates, ensuring the smoothness of their Fourier transform. We specifically focus on the second-order case. Theorem 1 proposes an approximant of the signum function with a twice differentiable Fourier transform. Building on this first result, Theorem 3 proposes a construction of bandlimited minorants and majorants of arbitrary BV functions. Additionally, the approximation error of the solutions is controlled by a quantity that depends on the total variation of the approximated BV function and of the approximation bandwidth. Section III recalls the formulation of the super-resolution problem from a bandlimited point-spread function (PSF). Considering the case where the minimal separation between the sources is inversely proportional to the number of acquired moments, we investigate the stability of the problem in the lense of the degeneracy of its FIM [10], when the minimal eigenvalue of the FIM is *not* asymptotically vanishing. We leverage the properties of the higher-order approximants constructed in Section II to provide a lower bound on the minimal eigenvalue of the FIM and show it remains bounded

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[1]Herein, we adopt the convention $\operatorname{sgn}(0) = 0$.

away from 0 as long as the separation parameter is greater than a quantity that depends only on the autocorrelation of the PSF. This result provides a novel, simple, and insightful relationship between the BV norms of the Fourier transform of the autocorrelation and the associated stable resolution limit. Numerical experiments are presented to highlight our theoretical findings. Conclusions and future works are drawn in Section IV.

B. Notation and Definitions

Vectors of $\mathbb{C}^N$ and matrices of $\mathbb{C}^{N\times r}$ are denoted by boldface letters $\boldsymbol{a}$ and capital boldface letters $\boldsymbol{A}$, respectively. The minimal (resp. maximal) eigenvalues of a Hermitian matrix $\boldsymbol{A}$ are denoted $\lambda_{\min}(\boldsymbol{A})$ and $\lambda_{\max}(\boldsymbol{A})$, respectively. For any function $F \in L_2(\mathbb{R})$, we denote by $\widehat{F}$ its *continuous time Fourier transform*, defined almost everywhere as

$$\widehat{F}(u) = \int_{\mathbb{R}} F(t) e^{-i2\pi u t} \mathrm{d}t, \quad \forall u \in \mathbb{R}. \tag{3}$$

A function $F \in L_2(\mathbb{R})$ is said to be β*-bandlimited* if for all u such that $|u| > \beta$, we have $\widehat{F}(u) = 0$. For any $\beta > 0$, we write $F_\beta(t) = \beta F(\beta t)$. We highlight that if F is 1-bandlimited, then F_β is β-bandlimited. $F \in \mathrm{BV}$ means that F is of bounded variation. In that case, we denote by $\mathrm{d}F$ its derivative in the weak sense, and write by $V_F = \int_{-\infty}^{\infty} |\mathrm{d}F|$ its total variation over $\mathbb{R}$.

We let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the *unidimensional torus*. When $N = 2n+1$ is an odd number, we write by $\boldsymbol{v}_0(\tau), \boldsymbol{v}_1(\tau) \in \mathbb{C}^N$ the vectors given by

$$\boldsymbol{v}_0(\tau) = \frac{1}{\sqrt{N}} \left[e^{-i2\pi(-n)\tau}, \ldots, e^{-i2\pi n\tau} \right]^\top \tag{4a}$$

$$\boldsymbol{v}_1(\tau) = \frac{1}{\sqrt{N}} \left[-i2\pi(-n) e^{-i2\pi(-n)\tau}, \ldots, -i2\pi n e^{-i2\pi n\tau} \right]^\top, \tag{4b}$$

for any $\tau \in \mathbb{T}$. For any vector $\boldsymbol{\tau} \in \mathbb{T}^r$, we define by $\boldsymbol{V}_0(\boldsymbol{\tau}), \boldsymbol{V}_1(\boldsymbol{\tau}) \in \mathbb{C}^{N\times r}$, the *generalized Vandermonde matrices*

$$\boldsymbol{V}_0(\boldsymbol{\tau}) = \left[\boldsymbol{v}_0(\tau_1), \ldots, \boldsymbol{v}_0(\tau_r) \right], \tag{5a}$$

$$\boldsymbol{V}_1(\boldsymbol{\tau}) = \left[\boldsymbol{v}_1(\tau_1), \ldots, \boldsymbol{v}_1(\tau_r) \right], \tag{5b}$$

and write $\boldsymbol{W}(\boldsymbol{\tau}) = [\boldsymbol{V}_0(\boldsymbol{\tau}), \boldsymbol{V}_1(\boldsymbol{\tau})] \in \mathbb{C}^{N\times 2r}$ their concatenation. Finally, the *wrap-around distance* $\Delta(\boldsymbol{\tau})$ is defined by the minimal distance between two pairs of points in $\boldsymbol{\tau}$ over the torus, *i.e.* $\Delta(\boldsymbol{\tau}) \triangleq \min_{\ell\neq\ell'} \inf_{j\in\mathbb{Z}} |\tau_\ell - \tau_{\ell'} + j|$.

II. Higher-Order Beurling–Selberg Approximation

A. Higher-Order Beurling Majorant

Although the function B_0 realizes the best possible 1-bandlimited majorization of the signum function in the L_1-sense, the residual $R_0(t) = B_0(t) - \mathrm{sgn}(t)$ decays quite slowly as $\mathcal{O}(|t|^{-1})$ in the limit $|t| \to \infty$. One consequence is that the Fourier transform $\widehat{R_0}$ and $\widehat{B_0}$ will not be differentiable.

In this work, we define the *mth-order Beurling approximation problem* as the problem of majorizing the signum function with a 1-bandlimited function B_m for which the

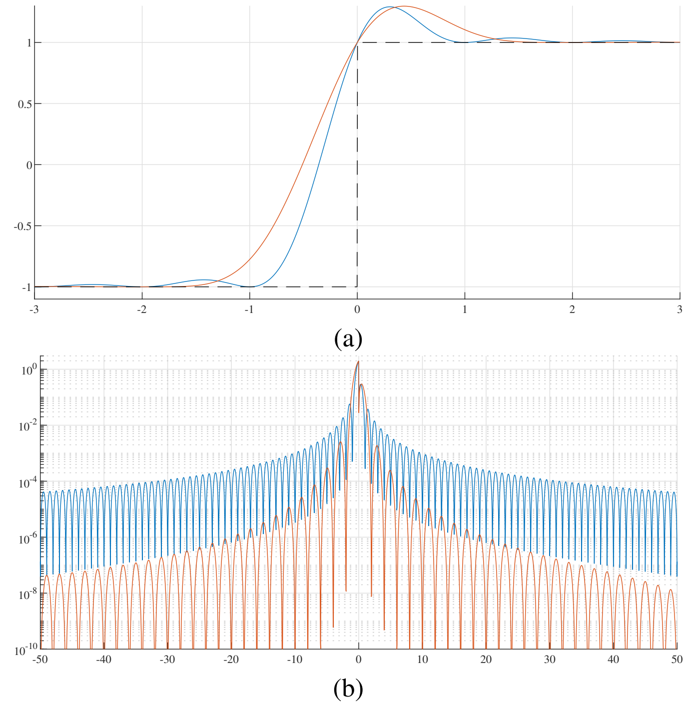

Figure 1. (a) The graphs of the functions $B_0(t)$ (in blue) and $B_2(t)$ (in red). (b): The graphs of the residuals functions $B_0(t) - \mathrm{sgn}(t)$ (in blue) and $B_2(t) - \mathrm{sgn}(t)$ (in red).

Fourier transform of the residual $R_m(t) = B_m(t) - \mathrm{sgn}(t)$ has an at least m-times differentiable Fourier transform $\widehat{R_m}$. The sequel focuses on the case of $m = 2$, and we define the *2nd order Beurling majorant set* as follows.

Definition 1 (2nd order Beurling majorant set)**.** We let $\mathcal{B}_2$ the set of functions $F : t \mapsto F(t)$ satisfying the following properties:

1) F is 1-bandlimited;
2) F majorizes sgn, *i.e.* $F(t) \geq \mathrm{sgn}(t)$ for all $t \in \mathbb{R}$;
3) $\int_{-\infty}^{\infty} 4\pi^2 t^2 \left(F(t) - \mathrm{sgn}(t) \right) \mathrm{d}t < \infty$.

The set $\mathcal{B}_2$ differs from Beurling's original formulation only in the third assumption, which implies the residual is at least twice differentiable. Additionally, it is easy to verify $B_0 \notin \mathcal{B}_2$ from (1). The central question is to find functions F that belong to $\mathcal{B}_2$ and realize a "sufficiently good" approximation of the signum function. The following result expresses one function $B_2 \in \mathcal{B}_2$.

Theorem 1 (2nd order Beurling majorant)**.** *Define the two auxiliary functions $K(t)$ and $L(t)$ as*

$$K(t) = \mathrm{sinc}^4\left(\frac{\pi t}{2}\right) \tag{6a}$$

$$L(t) = \left(\frac{\pi^2 + 3}{9} t + \frac{\pi^2}{6} t^3 \right) K(t) + \sum_{k=-\infty}^{\infty} \mathrm{sgn}(2k) \left(1 + \frac{\pi^2}{6} (t - 2k)^2 \right) K(t - 2k), \tag{6b}$$

and let $B_2(t) = K(t) + L(t)$*. Then,* $B_2 \in \mathcal{B}_2$ *and*

$$\int_{-\infty}^{\infty} (B_2(t) - \mathrm{sgn}(t))\,\mathrm{d}t = \frac{4}{3} \tag{7a}$$

$$\int_{-\infty}^{\infty} 4\pi^2 t^2 (B_2(t) - \mathrm{sgn}(t))\,\mathrm{d}t = 16. \tag{7b}$$

The proof of the theorem is skipped for conciseness. Nonetheless, we highlight that the crux in the construction of the function B_2 leverages higher-order Whitaker–Shannon interpolation formulas for bandlimited functions involving their uniform samples of their p first derivatives [11], [12]. The following Lemma specifies this formula when $p = 3$.

Lemma 2 (Higher-order interpolation formula [12])**.** *If* F *is a continuous* 1*-bandlimited then for any* $t \in \mathbb{R}$ *we have that*

$$\begin{aligned} F(t) = \sum_{k\in\mathbb{Z}} \Bigg(& F(2k) + F'(2k)(t-2k) \\ & + \left(F''(2k) + \frac{\pi^2}{3}F(2k)\right)\frac{(t-2k)^2}{2} \\ + \left(F'''(2k) + \frac{\pi^2}{3}F'(2k)\right) & \frac{(t-2k)^3}{6}\Bigg) \mathrm{sinc}^4\left(\frac{\pi}{2}(t-2k)\right). \end{aligned} \tag{8}$$

Identifying the terms in (8) with the expression of B_2 proposed in Theorem 1 yields $B(0) = 0$ and

$$B(2k) = \mathrm{sgn}(2k), \qquad B^{(\ell)}(2k) = 0. \tag{9}$$

for $\ell = 1, 2, 3$ and $k \in \mathbb{Z}\backslash\{0\}$. The previous relations suggest that $B_2(t)$ interpolates the signum function with vanishing derivatives up to the third order at every non-zero even integer. This foresees the quality of the one-sided approximation of the signum function realized by $B_2(t)$. The graphs of the function $B_2(t)$ and of its residual with the signum function $R_2(t)$ are plotted in Figure 1. As shown, the decay rate of the residual $R_2(t)$ has a faster rate than the one of $R_0(t)$. However, this benefit comes at a price of slightly greater L_1-norm, as confirmed by comparing the error metrics (2) and (7). Investigations on the extremality of $B_2(t)$ within the set $\mathcal{B}_2$ are left for future work.

B. One-Sided Approximations of BV Functions

Selberg used the Beurling majorant to construct extremal bandlimited majorization and minorization of the rectangular function [2], [3]. This result was further generalized to arbitrary BV functions [4]. Likewise, this subsection is dedicated to higher-order bandlimited one-sided approximations of arbitrary BV functions by exploiting the function B_2 constructed in Theorem 1. We start by defining the approximation sets of interest in the following.

Definition 2 (2nd-order minorant and majorant set)**.** Given a function $G \in$ BV and $\beta > 0$, we let $\mathcal{E}_\beta^-(G)$ and $\mathcal{E}_\beta^+(G)$ the second-order minorant and majorant sets, respectively, which we define as follows.
Minorant set: $F \in \mathcal{E}_\beta^-(G)$ if and only if

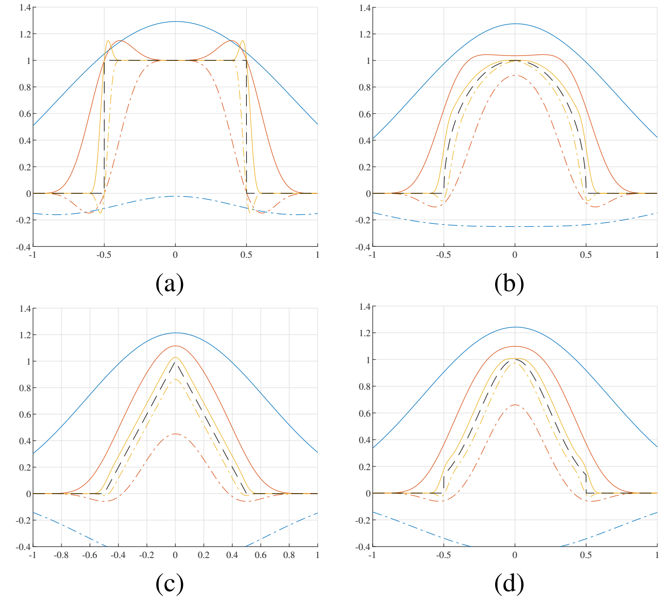

Figure 2. Graphs of the bandlimited majorants and minorants proposed by Theorem 3 for different BV functions: (a) the rectangle function; (b) the semi-circle function $t \mapsto \sqrt{1-4t^2}$ over $[-\frac{1}{2}, \frac{1}{2}]$; (c) the triangle function $t \mapsto \max\{0, 1-2|t|\}$; (d) the Gaussian function $t \mapsto \exp(-t^2/2)$, for different values of the bandlimit β. In blue: $\beta = 1$; In red: $\beta = 4$; In yellow: $\beta = 16$. Majorants are in plain lines, minorants are in dot-dashed lines, and the target function is in a dashed black line.

1) F is β-bandlimited;
2) F minorizes G, i.e. $F(t) \le G(t)$ for all t;
3) $\int_{-\infty}^{\infty} 4\pi^2 t^2 (G(t) - F(t))\,\mathrm{d}t < \infty$.

Majorant set: $F \in \mathcal{E}_\beta^+(G)$ if and only if

1) F is β-bandlimited;
2) F majorizes G, i.e. $F(t) \ge G(t)$ for all t;
3) $\int_{-\infty}^{\infty} 4\pi^2 t^2 (F(t) - G(t))\,\mathrm{d}t < \infty$.

Similarly to Definition 1 for the set $\mathcal{B}_2$, the third assumption implies that the residual functions of elements in $\mathcal{E}_\beta^-(G)$ and $\mathcal{E}_\beta^+(G)$ have a Fourier transform that is at least twice differentiable. In the sequel, we denote $J(t) = \frac{1}{2}L'(t)$ where, L is the auxiliary function defined in (6b). The next theorem proposes a generic construction of a pair of functions in those two sets by harnessing the properties of the function B_2 constructed in Theorem 1.

Theorem 3 (Bandlimited approximation of BV functions)**.** *Let* $G \in$ BV*. For any* $\beta > 0$ *the two functions*

$$G_\beta^-(t) = G * J_\beta(t) - (2\beta)^{-1}(\mathrm{d}V_G) * K_\beta(t) \tag{10a}$$

$$G_\beta^+(t) = G * J_\beta(t) + (2\beta)^{-1}(\mathrm{d}V_G) * K_\beta(t), \tag{10b}$$

are well-defined, β*-bandlimited, and* $G_\beta^-(g) \in \mathcal{E}_\beta^-(G)$ *and* $G_\beta^+ \in \mathcal{E}_\beta^+(G)$*. Moreover, the two integral identities,*

$$\int_{-\infty}^{\infty} \left(G_\beta^+(t) - G(t)\right)\mathrm{d}t = \int_{-\infty}^{\infty} \left(G(t) - G_\beta^-(t)\right)\mathrm{d}t = \left(\frac{3}{2}\beta\right)^{-1} V_G, \tag{11a}$$

$$\int_{-\infty}^{\infty} 4\pi^2 t^2 \left(G_\beta^+(t) - G(t)\right) \mathrm{d}t$$
$$= \int_{-\infty}^{\infty} 4\pi^2 t^2 \left(G(t) - G_\beta^-(t)\right) \mathrm{d}t = \left(\frac{1}{8}\beta\right)^{-1} V_{t\mapsto 4\pi^2 t^2 G(t)}. \tag{11b}$$

hold for any $\beta > 0$.

The proof of the above is skipped due to space constraints and is inspired by [4, Theorem 11]. Of particular importance in the previous is that both functions G_β^- and G_β^+ converge to G when we allow the bandwidth β of the approximants to tend to infinity. Figure 2 shows the graphs of the approximants proposed by Theorem 3 for four different BV functions G and for different approximation bandwidth β. It shows that increasing approximation bandwidth β results in tighter approximation which corroborates with the results.

III. Application to the Stability of Super-Resolution

Super-resolution is a fundamental signal processing problem consisting of recovering a stream of point-sources from their convolution with a known point-spread function, assumed to be $\frac{1}{2}$-bandlimited, up to a rescaling. A classical approach [8], [9], [13], [14] supposes measurements to be taken in the Fourier domain, either because of the physical nature of the measurements [15]–[17], or through transforms posterior to acquisition [8], [18]. We assume an odd number $N = 2n + 1$ of spectral measurements and a ground truth composed of r many sources with $r \leq n$ to ensure the uniqueness of the solution with r components in the absence of noise [19]. Super-resolution boils down to recovering the amplitudes $\boldsymbol{c} \in \mathbb{C}^r$ and the locations $\boldsymbol{\tau} \in \mathbb{T}^r$ of the point-sources from the noisy observation $\boldsymbol{y} \in \mathbb{C}^N$ of the form

$$\boldsymbol{y} = \operatorname{diag}(\boldsymbol{h}) \sum_{\ell=1}^{r} c_\ell \boldsymbol{v}_0(\tau_\ell) + \boldsymbol{w} = \operatorname{diag}(\boldsymbol{h}) \boldsymbol{V}_0(\boldsymbol{\tau}) \boldsymbol{c} + \boldsymbol{w}, \tag{12}$$

where $h_k = H\left(\frac{k}{N}\right)$ corresponds to the trigonometric moments of the PSF. The noise $\boldsymbol{w} \sim \mathcal{N}\left(\boldsymbol{0}, \sigma^2 \boldsymbol{I}_N\right)$ is assumed to be white Gaussian noise with variance σ^2. Additionally, up to scaling, it is assumed that $1 \leq |c_\ell| \leq \kappa$ of any $\ell = 1, \ldots, r$, where κ is the *dynamic range* of the problem.

We write $G = |H|^2$ and let $g_k = G(\frac{k}{N})$. As the vectors $\boldsymbol{c}$ and $\boldsymbol{\tau}$ are of different units, and since the statistical error of an estimator of τ is expected to be inversely proportional to the number of measurement N, we denote by $\widetilde{\boldsymbol{\tau}} = \boldsymbol{\tau}/\widehat{G}(0)$ the normalized location of the points. We seek to recover, without loss of generality, the set of parameters $\boldsymbol{\theta} = \{\boldsymbol{c}, \widetilde{\boldsymbol{\tau}}\}$.

A. Stability of Super-Resolution

An important statistical problem for super-resolution is to guarantee its stability. There are various definitions in the literature to characterize the stability of the super-resolution problem. Some stability criteria are *algorithm-specific*. In particular several super-resolution algorithms such as MUSIC [20], [21], ESPRIT [22], matrix-pencil [3], [23], or total-variation minimization methods [24]–[27] comes with provable stability guarantees. On the other hand, stability can be defined under various *statistical* and *algorithm-independent* metrics [28]–[30]. Nonetheless, all are related to the separation parameter $N\Delta(\boldsymbol{\tau})$ between the sources, as empirically established by Rayleigh (see *e.g.* [10], [31]). Here, we relate stability with the property of the Fisher information matrix (FIM) $\boldsymbol{J}(\boldsymbol{\theta})$ that its smallest eigenvalue is strictly bounded away from 0, by extending the FIM stability definition introduced in [10] to the case of an arbitrary PSF.

Definition 3 (Stability of the Fisher Information Matrix)**.** The super-resolution problem is said to be *stable* for a separation parameter β and a dynamic range κ if and only if there exists a constant $C_G(\beta, \kappa) > 0$ independent of N such that for any set of parameters $\boldsymbol{\theta}$ with $N\Delta(\boldsymbol{\tau}) \geq \beta$ and $1 \leq \min\{\boldsymbol{c}\} \leq \max\{\boldsymbol{c}\} \leq \kappa$

$$\lambda_{\min}\left(\boldsymbol{J}(\boldsymbol{\theta})\right) \geq \sigma^{-2} C_G(\beta, \kappa), \tag{13}$$

where the FIM $\boldsymbol{J}(\boldsymbol{\theta})$ under observations (12) reads [32], [33],

$$\boldsymbol{J}(\boldsymbol{\theta}) = \sigma^{-2} \operatorname{diag}(\boldsymbol{1}, \boldsymbol{c})^{\mathsf{H}} \boldsymbol{W}(\boldsymbol{\tau})^{\mathsf{H}} \operatorname{diag}(\boldsymbol{g}) \boldsymbol{W}(\boldsymbol{\tau}) \operatorname{diag}(\boldsymbol{1}, \boldsymbol{c}). \tag{14}$$

The Beurling extremal function, and in particular Selberg approximation of the rectangle, can be used to bound, in a fairly elegant manner, the singular values of Vandermonde matrices of the kind $\boldsymbol{V}_0(\boldsymbol{\tau})$ for any $\boldsymbol{\tau}$ as a sole function of the separation parameter $N\Delta(\boldsymbol{\tau})$ [3], [7]. In our setting, the FIM (14) reads as the Gramian of a weighted generalized Vandermonde matrix $\boldsymbol{W}(\boldsymbol{\tau})$, which cannot directly be tackled with classical Beurling–Selberg approximants. The next theorem concludes on the stability of super-resolution by building a relationship between the bandlimited approximations proposed in Theorem 1 and smallest eigenvalue of the FIM (14).

Theorem 4 (Conditioning of the FIM via Bandlimited Functions)**.** *Suppose that* $G \in \mathrm{BV}$*. If the separation parameter* $N\Delta(\boldsymbol{\tau})$ *verifies*

$$N\Delta(\boldsymbol{\tau}) > \max\left\{\frac{2V_G}{3\int_{-\infty}^{\infty} G(t)\mathrm{d}t}, \frac{8V_{t\mapsto 4\pi^2 t^2 G(t)}}{\int_{-\infty}^{\infty} 4\pi^2 t^2 G(t)\mathrm{d}t}\right\} \tag{15}$$

then super-resolution is stable in the sense of Definition 3.

Theorem 4 provides a generic and simple evaluation criterion to check the stability of the FIM of the super-resolution problem (12). Additionally, this result enables the derivation of lower bounds on the quantity $\lambda_{\min}\left(\boldsymbol{J}(\boldsymbol{\theta})\right)$ as a function of the Fourier transform of autocorrelation function G of the PSF, of the dynamic range κ, and of the separation parameter $\beta = N\Delta(\boldsymbol{\tau})$. Figure 3 plots this quantity and the associated stability bounds when the PSF is an ideal-low pass filter, a circular low-pass filter, and a triangular low-pass filter.

IV. Conclusion and Future Work

In the present work, we introduce an extension of the Beurling–Selberg bandlimited approximation problem by building one-sided approximants of BV functions with smooth

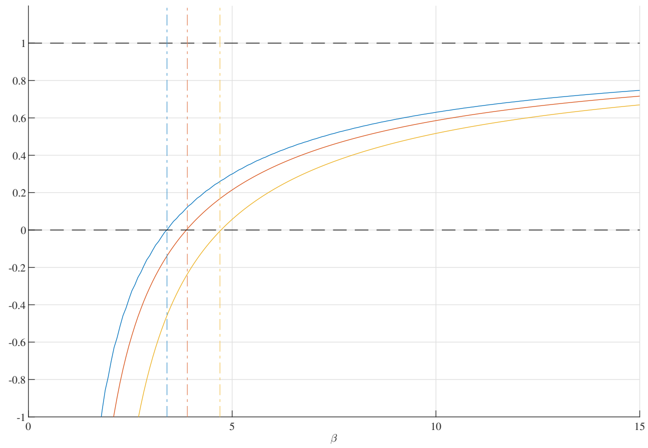

Figure 3. Theoretical lower bounds on the quantity $\lambda_{\min}(\boldsymbol{J}_G(\boldsymbol{\theta}))$ for three different PSF: In blue: ideal-low pass; In red: circular low-pass; In yellow: triangular low-pass. The lower bounds become non-negative when $\beta = N\Delta(\boldsymbol{\tau})$ is greater than 3.4, 3.9, and 4.7, respectively, up to a 10^{-1} imprecision. Herein, we set the dynamic range $\kappa = 1$.

Fourier transforms. We propose an application of our theoretical findings to the stability of the Fisher information matrix of the super-resolution problem from an *arbitrary* PSF whose Fourier transform is of bounded variation. In the regime where the separation between the sources is inversely proportional to the number of measurements, the results reveal the existence of the separation parameter, depending on the mass and second-order moments of the variations of the PSF, above which super-resolution is stable in a Fisher sense. We leave for an extended version of this work a complete demonstration of our results, a study of the extremality of the function B_2, and an exploration of the applicability of our higher-order bandlimited approximation theory to system identification. Additionally, extensions of the presented Beurling approximation problems with an arbitrary order m will be investigated.

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