Lefschetz exceptional collections in S_k-equivariant categories of $(\mathbb{P}^n)^k$

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Abstract

We consider the bounded derived category of S_k-equivariant coherent sheaves on $(\mathbb{P}^n)^k$. The goal of this paper is to construct in this category a rectangular Lefschetz exceptional collection when this is possible, or a minimal Lefschetz exceptional collection when a rectangular one does not exist. The main results of the paper include the construction of a rectangular Lefschetz exceptional collection in the case $k = 3$ and in the case $n = 1$ when $\gcd(n+1, k) = 1$. We also construct a minimal Lefschetz exceptional collection for $n = 1$ and even k, and for $n = 2$ and $k = 3$.

Contents

1 Introduction

The bounded derived category of coherent sheaves is the main homological invariant of an algebraic variety which captures the most essential geometric information. It stands in the focus of many recent research papers. One of the ways to describe it is via an exceptional collection.

Recall that an object E in a $\mathbb{C}$-linear triangulated category $\mathcal{T}$ is *exceptional* if $\operatorname{Ext}^0(E,E) = \mathbb{C}$ and $\operatorname{Ext}^i(E,E) = 0$ for $i \neq 0$. Furthermore, a collection $E_1, \dots, E_r$ of objects in $\mathcal{T}$ is an *exceptional collection* if each E_i is an exceptional object and $\operatorname{Ext}^\bullet(E_i, E_j) = 0$ for $i > j$. An exceptional collection is *full* if the smallest full triangulated subcategory of $\mathcal{T}$ containing all E_i coincides with $\mathcal{T}$.

Recently a special class of exceptional collections attracted much attention. Recall that an exceptional collection $E_1, \dots, E_r$ in the bounded derived category of coherent sheaves $\mathcal{D}(X)$ of a smooth projective variety X is *Lefschetz* with respect to a line bundle $\mathcal{L}$ if there is a partition $r = r_0 + r_1 + \cdots + r_d$ with $r_0 \geq r_1 \geq \cdots \geq r_d$ such that

$$E_{r_0+r_1+\cdots+r_{i-1}+t} \cong E_t \otimes \mathcal{L}^i \qquad \text{for all } 1 \leq t \leq r_i \qquad \text{and } 1 \leq i \leq d.$$

In other words, if the objects of the collection are obtained by $\mathcal{L}$-twists from the subcollection of the first r_0 objects according to the pattern provided by the partition.

As it is clear from the definition, a Lefschetz collection with respect to a given line bundle $\mathcal{L}$ is determined by its *starting block* $E_1, \dots, E_{r_0}$ and the partition $(r_0, r_1, \dots, r_d)$. It is less evident, but is still true, that if a Lefschetz collection is full, then the partition is itself determined by the starting block of the collection [6, Lemma 4.5]. Thus, extendability to a Lefschetz collection is just a property of an exceptional collection $E_1, \dots, E_{r_0}$.

It follows that there is a natural partial order on the set of all Lefschetz collections in $\mathcal{D}(X)$ with respect to a given line bundle $\mathcal{L}$ — a Lefschetz collection with a starting block $E_1, \dots, E_{r_0}$ is *smaller* than a Lefschetz collection with a starting block $E'_1, \dots, E'_{s_0}$ if $E_1, \dots, E_{r_0}$ is a subcollection in $E'_1, \dots, E'_{s_0}$, see [10, Definition 1.4].

A Lefschetz collection $E_1, \dots, E_r$ with partition $r_0, r_1, \dots, r_d$ is called *rectangular of length* $d+1$, if $r_0 = r_1 = \cdots = r_d$ (equivalently, if the Young diagram representing the partition is a rectangle of length $d+1$). Of course, a necessary condition for the existence of a rectangular Lefschetz collection in $\mathcal{D}(X)$ is a factorization

$$\operatorname{rk}\left(K_0(\mathcal{D}(X))\right) = r_0(d+1) \tag{1.1}$$

for the rank of the Grothendieck group of X. On the other hand, if a rectangular Lefschetz decomposition in $\mathcal{D}(X)$ exists, and if its length $d+1$ has the property that $\mathcal{L}^{d+1} \cong \omega_X^{-1}$ where ω_X is the canonical bundle of X, that is $d+1$ equals the *index* of X with respect to $\mathcal{L}$, then this collection is automatically minimal (this follows easily from Serre duality, see [10, Subsection 2.1]).

Lefschetz collections have many nice properties and are very important for homological projective duality and categorical resolutions of singularities [7], [9]. Especially nice and important are rectangular (resp. minimal) Lefschetz collections. So, the following problem is very interesting.

Problem 1.1. *Given a smooth projective variety X and a line bundle $\mathcal{L}$, construct a full rectangular Lefschetz collection in $\mathcal{D}(X)$ with respect to $\mathcal{L}$ of length equal to the index of X, or, if the above is impossible, a minimal Lefschetz collection.*

There are many varieties X for which the above problem was solved. Among these are projective spaces, most of the Grassmannians, and some other homogeneous spaces [2]. In this paper we discuss Problem 1.1 for a very simple variety

$$X = X_k^n := \underbrace{\mathbb{P}^n \times \mathbb{P}^n \times \cdots \times \mathbb{P}^n}_{k \text{ copies}},$$

but replace the category $\mathcal{D}(X_k^n)$ with the equivariant derived category $\mathcal{D}_{S_k}(X_k^n)$ with respect to the natural action of the symmetric group S_k (by permutation of factors). Note that this category can be considered as the derived category of the *quotient stack* $[X_k^n/S_k]$. The line bundle $\mathcal{L}$ here is, of course, the ample generator $\mathcal{O}(1,1,\ldots,1)$ of the invariant Picard group $\mathrm{Pic}(X_k^n)^{S_k}$. Note that the index of X_k^n with respect to $\mathcal{L}$ is equal to $n+1$, so the goal of the paper can be formulated as follows.

Problem 1.2. *Find a full rectangular Lefschetz collection of length $n+1$ in $\mathcal{D}_{S_k}(X_k^n)$ with respect to the line bundle $\mathcal{O}(1,1,\ldots,1)$ or a minimal Lefschetz collection if the above is impossible.*

Note that without passing to the equivariant category the problem becomes trivial. To construct a rectangular Lefschetz collection in $\mathcal{D}(X_k^n)$ one can just choose any full exceptional collection in $\mathcal{D}(X_{k-1}^n)$ and consider its pullback to X_k^n as the starting block. Using the projective bundle formula it is elementary to check that it extends to a rectangular Lefschetz collection of length $n+1$. However, the S_k-symmetry in this construction is broken, and it cannot be performed in the equivariant category.

For $k=1$ the Problem 1.2 is trivial (the desired collection is just the Beilinson exceptional collection $\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n)$ of line bundles on $\mathbb{P}^n$). Furthermore, for $k=2$ the Problem 1.2 was essentially solved in [11].

The main result of our paper is a partial solution to the Problem 1.2.

First, we construct in Theorem 3.5 a S_k-invariant Lefschetz exceptional collection of line bundles in $\mathcal{D}(X_k^n)$ whose cardinality equals the rank of the Grothendieck group of X_k^n (by Elagin's Theorem, see Theorem 2.4, this gives an exceptional collection in the equivariant category, whose length equals the rank of its Grothendieck group). This collection is rectangular in case of coprime k and $n+1$. The first block of the collection is defined as $\langle \mathcal{O}(e)\rangle_{e\in\mathbb{E}_k^n}$, where

$$\mathbb{E}_k^n = \left\{ S_k \cdot e \;\middle|\; e_1 \geq \cdots \geq e_k = 0 \text{ and } e_i \leq \frac{(n+1)(k-i)}{k} \right\} \subset \mathbb{Z}^k. \tag{1.2}$$

So, it is natural to expect that this collection is full and (in the coprime case) gives a solution to Problem 1.2. However, in general we could not prove its fullness.

Our second main result is a proof of fullness of the above collection for $k=3$ and $n=3p$ or $n=3p+1$ (this ensures that k and $n+1$ are coprime).

We also perform a first step in the direction of non-coprime k and $n+1$ by constructing a minimal S_3-invariant Lefschetz exceptional collection in $\mathcal{D}(X_3^2)$ (including a proof of its fullness).

Besides that we also solve Problem 1.2 for $n=1$, that is, construct a rectangular S_k-invariant Lefschetz collection of length 2 in $\mathcal{D}(X_k^1)$ when k is odd, and a minimal Lefschetz collection when k is even. However, this case is much more simple than the case $k=3$ discussed above.

An interesting feature of the Lefschetz collections that we construct in Theorem 3.5 is that they resemble very much the minimal Lefschetz collections in the derived categories of the Grassmannians $\mathrm{Gr}(k, n+1+k)$ constructed by Anton Fonarev, see [2]. It would be very interesting to understand the relations between these, since on one hand, this suggests a possible solution to the Problem 1.2 for other values of k (by considering analogues of Fonarev's collections), and on the other hand, a solution to the Problem 1.2 can help in dealing with the Grassmannians $\mathrm{Gr}(k,n)$ when k and n are not coprime (in this case there is no rectangular collection on the Grassmannian, and a minimal collection is not quite known).

This paper is organized as follows. In Section 2 we recall the definitions of full exceptional collections, Lefschetz and rectangular decompositions, and Elagin's Theorem. In Section 3 we construct an S_k-invariant exceptional collection in $\mathcal{D}(X_k^n)$ and discuss numerical restrictions for the existence of a rectangular Lefschetz collection and some numerical bounds for a minimal Lefschetz collection. Finally, in Section 4 we prove fullness of the constructed collections for X_k^1, X_3^{3p}, X_3^{3p+1} and X_3^2 respectively.

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2 Preliminaries

Given an algebraic variety X we denote the bounded derived category $\mathcal{D}^b(\mathrm{coh}(X))$ of coherent sheaves on X by $\mathcal{D}(X)$. In this paper we concentrate on the case when X is a power of a projective space

$$X = X_k^n = (\mathbb{P}^n)^k,$$

In some cases, we will omit the indices k and n and write $\mathcal{D}(X)$ instead $\mathcal{D}(X_k^n)$.

We work over an algebraically closed field of characteristic zero.

2.1 Exceptional collections in $\mathcal{D}(X_k^n)$

Clearly, X_k^n is a smooth projective variety with $\dim(X) = kn$. Its Picard group is isomorphic to $\mathrm{Pic}(X_k^n) \cong \mathbb{Z}^k$ and has a basis consisting of the pullbacks of hyperplane classes of the factors. For $a = (a_1, \ldots, a_k) \in \mathbb{Z}^k$ we write

$$\mathcal{O}(a) = \mathcal{O}(a_1, \ldots, a_k) = \mathcal{O}(a_1) \boxtimes \cdots \boxtimes \mathcal{O}(a_k)$$

for the corresponding line bundle on X_k^n. We note that by the Künneth formula

$$\mathrm{Ext}^\bullet(\mathcal{O}(a), \mathcal{O}(b)) \cong \bigotimes_{i=1}^{k} \mathrm{Ext}^\bullet(\mathcal{O}(a_i), \mathcal{O}(b_i)). \tag{2.1}$$

In particular, any line bundle on X_k^n is exceptional, and the line bundles $\mathcal{O}(a)$ and $\mathcal{O}(b)$ are semiorthogonal, i.e., $\mathrm{Ext}^\bullet(\mathcal{O}(a), \mathcal{O}(b))$ is equal to 0, if and only if the pair $(\mathcal{O}(a_i), \mathcal{O}(b_i))$ on $\mathbb{P}^n$ is semiorthogonal for at least one i. In view of Bott's formula for the cohomology of line bundles on a projective space, we can rewrite the semiorthogonality condition as

$$\mathrm{Ext}^\bullet(\mathcal{O}(a), \mathcal{O}(b)) = 0 \text{ if and only if } 0 < a_i - b_i \leq n \text{ for some } 1 \leq i \leq k. \tag{2.2}$$

This property allows to verify easily semiorthogonality of collections of line bundles. For fullness, the following observations are useful.

For a subset $I \subset \{1, \ldots, k\}$ of indices define the set $[0, n]^I \subset \mathrm{Pic}(X_k^n)$ as

$$[0, n]^I = \left\{ a \in \mathbb{Z}^k \mid a_i \in [0, n] \text{ if } i \in I \text{ and } a_i = 0 \text{ if } i \notin I \right\}.$$

If $I = \{1, \ldots, k\}$, then denote $[0, n]^I$ by $[0, n]^k$.

Similarly, we define $\mathbb{Z}^I \subset \mathbb{Z}^k$ as

$$\mathbb{Z}^I = \left\{ a \in \mathbb{Z}^k \mid a_i = 0 \text{ if } i \notin I \right\}.$$

Theorem 2.1. *The collection* $\{\mathcal{O}(a)\}_{a\in[0,n]^k}$ *(lexicographically ordered) is a full exceptional collection in* $\mathcal{D}(X_k^n)$.

Proof. Semiorthogonality of the collection follows easily from (2.2). For fullness we refer to [12]. □

We will also need the following simple consequence of the fullness of the above collection.

Corollary 2.2. *Let* $\mathcal{T}$ *be a triangulated subcategory of* $\mathcal{D}(X_k^n)$. *Assume that for some subset* $I \subset \{1,\dots,k\}$ *and some* $a \in \mathrm{Pic}(X_k^n)$ *one has* $\mathcal{O}(a+b) \in \mathcal{T}$ *for any* $b \in [0,n]^I$. *Then the same holds true for any* $b \in \mathbb{Z}^I$.

Proof. First assume $a = 0$. Then the collection $\{\mathcal{O}(b)\}_{b\in[0,n]^I}$ is just the pullback of the full exceptional collection in

$$X_I^n = \prod_{i\in I} \mathbb{P}^n$$

with respect to the natural projection $X_k^n \to X_I^n$. Consequently, by Theorem 2.1 the category $\mathcal{T}$ contains the pullback of any line bundle on X_I^n, and this is just the claim of the lemma in this case.

For arbitrary a just note that $\{\mathcal{O}(a+b)\}_{b\in[0,n]^I}$ is the twist of $\{\mathcal{O}(b)\}_{b\in[0,n]^I}$ by $\mathcal{O}(a)$. Since a line bundle twist is an autoequivalence of $\mathcal{D}(X_k^n)$, the general claim follows. □

2.2 Semiorthogonal and Lefschetz decompositions

In some cases it is slightly more convenient to work with semiorthogonal decompositions than with exceptional collections. Here, we remind the corresponding definitions.

Definition 2.3. Suppose $\mathcal{A}_0, \dots, \mathcal{A}_d$ are full triangulated subcategories of $\mathcal{T}$ such that $\mathrm{Hom}(\mathcal{A}_i, \mathcal{A}_j) = 0$ for all $i > j$. We say that $\mathcal{A}_0, \dots, \mathcal{A}_d$ form a *semiorthogonal decomposition* of $\mathcal{T}$ if the smallest full triangulated subcategory of $\mathcal{T}$ containing $\mathcal{A}_i$ for all i coincides with $\mathcal{T}$.

We will denote a semiorthogonal decomposition by

$$\langle \mathcal{A}_0, \dots, \mathcal{A}_d \rangle = \mathcal{T}.$$

Assume that $\mathcal{T} = \mathcal{D}(X)$ and a line bundle $\mathcal{L}$ on X is given. For an object F in $\mathcal{D}(X)$ we denote

$$F(i) := F \otimes \mathcal{L}^i,$$

the image of F under the autoequivalence of $\mathcal{T}$ given by the $\mathcal{L}^i$-twist, and for a subcategory $\mathcal{A} \subset \mathcal{T}$ we denote

$$\mathcal{A}(i) := \{F(i) \mid F \in \mathcal{A}\} \subset \mathcal{T},$$

the image of $\mathcal{A}$ under this autoequivalence.

A semiorthogonal decomposition

$$\mathcal{D}(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \ldots, \mathcal{A}_d(d) \rangle \tag{2.3}$$

is called *Lefschetz decomposition* if $\mathcal{A}_{i+1} \subset \mathcal{A}_i$ for all $0 \le i < d$.

We say that a Lefschetz decomposition (2.3) is *rectangular* if $\mathcal{A}_0 = \cdots = \mathcal{A}_d$. A rectangular decomposition can be simply written as

$$\mathcal{D}(X) = \langle \mathcal{A}, \mathcal{A}(1), \ldots, \mathcal{A}(d) \rangle, \tag{2.4}$$

where $\mathcal{A} = \mathcal{A}_0$.

2.3 Exceptional collections in equivariant derived categories

Assume a finite group G acts on a smooth projective variety X. The following result of Alexei Elagin gives a way to construct an exceptional collection in the equivariant derived category $\mathcal{D}_G(X)$.

Theorem 2.4 ([1, Theorem 2.3]). *Assume that $E_1, \ldots, E_r$ is a full G-invariant exceptional collection in $\mathcal{D}(X)$, that is, the G-action induces a permutation of objects of the collection. Assume s is the number of G-orbits on $\{E_1, \ldots, E_r\}$ and let $E_{i_1}, \ldots E_{i_s}$, $i_1 < \cdots < i_s$ be their representatives. For each $1 \le t \le s$ let H_t be the stabilizer of E_{i_t} and assume that for each t the object E_{i_t} admits an H_t-equivariant structure. Then there exists a full exceptional collection of the equivariant category*

$$\mathcal{D}_G(X) = \langle \bar{E}_{i_1}^{(1)}, \ldots, \bar{E}_{i_1}^{(m_1)}, \ldots, \bar{E}_{i_s}^{(1)}, \ldots, \bar{E}_{i_s}^{(m_s)} \rangle.$$

Here $\bar{E}_{i_t}^{(j)} = E_{i_t} \otimes V_t^{(j)}$, where $V_t^{(1)}, \ldots, V_t^{(m_t)}$ are all irreducible representations of H_t up to isomorphism, and we consider the natural G-equivariant structure on $\bar{E}_{i_t}^{(j)}$.

We note that any line bundle on $X = X_k^n$ has a natural equivariant structure with respect to the subgroup of S_k that stabilizes it. Indeed, for this it is enough to note that the line bundle $\mathcal{O}(i, i, \dots, i)$ is S_k-equivariant for each i. Thus, the above theorem applies to any exceptional collection formed by line bundles on X_k^n as soon as it is S_k-invariant. To ensure that the resulting collection in the equivariant category is Lefschetz we will use the following evident observation.

Corollary 2.5. *Assume that $\mathcal{L}$ is a G-equivariant line bundle on X and $E_1, \dots, E_r$ is a Lefschetz exceptional collection with respect to $\mathcal{L}$ which satisfies the assumptions of Theorem* 2.4*. Then the corresponding exceptional collection in the equivariant category is also Lefschetz. Moreover, if the original collection is rectangular then so is the equivariant one with the same number of blocks.*

Proof. Let $E_1, \dots, E_{r_0}$ be the starting block of the original Lefschetz collection and let s_0 be the number of G-orbits in the block $E_1, \dots, E_{r_0}$.

Then it is straightforward to check that $\bar{E}_{i_1}^{(1)}, \dots, \bar{E}_{i_1}^{(m_1)}, \dots, \bar{E}_{i_{s_0}}^{(1)}, \dots, \bar{E}_{i_{s_0}}^{(m_{s_0})}$ can serve as the starting block of a Lefschetz collection in $\mathcal{D}_G(X)$. From the equivariance of $\mathcal{L}$ it is also clear that the property of being rectangular is preserved by this construction. □

Thus, to construct a (rectangular) Lefschetz collection in $\mathcal{D}_{S_k}(X_k^n)$ it is enough to construct a (rectangular) S_k-invariant Lefschetz collection in $\mathcal{D}(X)$ consisting of line bundles. This is what we do in the next sections.

3 A Lefschetz collection and numerical minimality

In this section we construct a Lefschetz S_k-invariant exceptional collection on X_k^n and find some numerical conditions for minimality of a Lefschetz exceptional collection. In what follows we always denote

$$h := n + 1.$$

3.1 A Lefschetz collection

We denote by $\mathsf{Y}_{h,k}$ the set of Young diagrams inscribed in the rectangle of size $h \times k$. We identify them with nonincreasing integer sequences $(e_1, \dots, e_k)$ such that $n \geq e_1 \geq \dots \geq e_k \geq 0$. We will use lexicographical order on this set of sequences.

Sometimes it is convenient to use other combinatorial interpretations of Young diagrams. One of these is an integer path going from the lower left corner to the upper right corner of the rectangle that goes only rightward and upward. Another is a binary sequence (i.e., a sequence of 0 and 1) of length $h+k$ containing 1 exactly k times.

There is a natural action of the group $\mathbb{Z}/(h+k)\mathbb{Z}$ on $\mathsf{Y}_{h,k}$. If $t_1, t_2, \dots, t_{h+k}$ is a binary sequence and g is a generator of $\mathbb{Z}/(h+k)\mathbb{Z}$, then

$$g\colon (t_1, t_2, \dots, t_{h+k}) \mapsto (t_2, \dots, t_{h+k}, t_1).$$

Concatenation of sequences is an operation

$$Y_{h_1,k_1} \times Y_{h_2,k_2} \to Y_{(h_1+h_2),(k_1+k_2)}$$

which we will denote by $(e', e'') \mapsto e' * e''$. Explicitly,

$$(e'_1, \dots, e'_{k_1}) * (e''_1, \dots, e''_{k_2}) = (e'_1 + h_2, \dots, e'_{k_1} + h_2, e''_1, \dots, e''_{k_2}).$$

Note that

$$g^{h_1+k_1}(e' * e'') = e'' * e'.$$

In particular, the Young diagrams $e' * e'', e'' * e' \in Y_{(h_1+h_2),(k_1+k_2)}$ are in the same orbit of the $\mathbb{Z}/(h_1+h_2+k_1+k_2)\mathbb{Z}$-action.

We remind the following definition from [2].

Definition 3.1. A diagram $(e_1, \dots, e_k) \in \mathsf{Y}_{h,k}$ is called *upper triangular*, if it (considered as a path in the rectangle) lies above the diagonal of the rectangle going from the upper right to the lower left corner. Equivalently,

$$e_i \le \frac{h(k-i)}{k}$$

for every $1 \le i \le k$.

Lemma 3.2 ([2, Lemma 3.2])**.** *Every orbit of the $\mathbb{Z}/(h+k)\mathbb{Z}$-action on $\mathsf{Y}_{h,k}$ contains an upper triangular element. If $\gcd(h,k) = 1$ then each orbit has length $h+k$.*

For reader's convenience we remind the proof.

Proof. Take any diagram $e \in \mathsf{Y}_{h,k}$ and extend the corresponding path periodically in both directions. This is the same as extending periodically the corresponding binary sequence. Draw all the lines with slope k/h passing through all the vertices of the extended path. As the path is $(h+k)$-periodic, one will get at most $h+k$ distinct lines and the path will lie above the lowest of them. Now draw the $h \times k$ rectangle putting the upper right corner in any of the vertices lying on the intersection of the lowest line with the path. That new diagram in the rectangle is upper triangular and lays in the orbit of e.

Now suppose that $\gcd(h,k) = 1$. Consider any $e \in \mathsf{Y}_{h,k}$ and the binary sequence $t_1, t_2, \ldots, t_{h+k}$ corresponding to e. Denote the length of orbit of e by $l(e)$. Then sequence $t_1, t_2, \ldots, t_{h+k}$ is a concatenation of $\frac{h+k}{l(e)} \in \mathbb{N}$ copies of the sequence $t_1, t_2, \ldots, t_{l(e)}$. Since $t_1, t_2, \ldots, t_{h+k}$ contains h zeros and k ones, we conclude $(\frac{h+k}{l(e)})|h$ and $(\frac{h+k}{l(e)})|k$. Since $\gcd(h,k) = 1$ we get that $\frac{h+k}{l(e)} = 1$ and $l(e) = h + k$. $\square$

For every orbit of $\mathbb{Z}/(h+k)\mathbb{Z}$-action on $\mathsf{Y}_{h,k}$ pick an upper triangular element. If the orbit contains several upper triangular elements, pick the smallest one with respect to lexicographical order. Denote this set of orbits representatives by $\mathbb{E}^n_k$. As we will see below this set will index (up to the S_k-action) objects in the first block of the exceptional collection. As before for every $e \in \mathbb{E}^n_k$ denote the number of elements in the $\mathbb{Z}/(h+k)\mathbb{Z}$-orbit of e by $l(e)$.

Notice that for every $e = (e_1, \ldots, e_k) \in \mathbb{E}^n_k$ we have

$$e_1 \geq \cdots \geq e_k = 0 \text{ and } e_i \leq \frac{h(k-i)}{k}, \tag{3.1}$$

since e is upper triangular.

Next, we describe the index sets for objects of the other blocks of the Lefschetz collection. For this we consider the following filtration of the set $S_k \cdot \mathbb{E}^n_k$:

$$\bar{E}_i = \left\{ S_k \cdot e \;\middle|\; e \in \mathbb{E}^n_k \text{ and } l(e) > i\,\frac{h+k}{h} \right\} \subset \mathbb{Z}^k. \tag{3.2}$$

Note that we have a chain of inclusions $S_k \cdot \mathbb{E}^n_k = \bar{E}_0 \supset \bar{E}_1 \supset \cdots \supset \bar{E}_n \supset \bar{E}_{n+1} = \varnothing$, and the subset $\mathbb{E}^n_k \subset \bar{E}_0$ is just the set of non-increasing representatives in the S_k-orbits on $\bar{E}_0 \subset \mathbb{Z}^k$.

We denote

$$\mathcal{A}_i := \langle \mathcal{O}(e) \rangle_{e \in \bar{E}_i} \subset \mathcal{D}(X^n_k), \tag{3.3}$$

the category generated by the corresponding line bundles. Then we have

$$\mathcal{A}_1 \supset \mathcal{A}_2 \supset \cdots \supset \mathcal{A}_n \supset 0.$$

We will show below that this gives a Lefschetz decomposition of a subcategory of $\mathcal{D}(X^n_k)$ (actually, we expect it to be a decomposition of the whole category). We start by checking that each component $\mathcal{A}_i$ is generated by an exceptional collection.

Lemma 3.3. *For every i the collection $\langle \mathcal{O}(e) \rangle_{e \in \bar{E}_i}$ (lexicographically ordered) is an exceptional collection in $\mathcal{D}(X)$.*

Proof. Notice that for every $e = (e_1, \dots e_k) \in \bar{E}_i$ we have $0 \leq e_j \leq n$ for every j by definition of $\mathbb{E}_k^n$. The collection $\langle \mathcal{O}(e) \rangle_{e \in \bar{E}_i}$ is exceptional since it is a subcollection of the exceptional collection $\langle \mathcal{O}(a) \rangle_{a \in [0,n]^k}$ (see Theorem 2.1). $\square$

The next observation shows when this collection is rectangular.

Lemma 3.4. *If* $\gcd(h, k) = 1$*, then* $\bar{E}_0 = \dots = \bar{E}_n$ *and* $\mathcal{A}_0 = \dots = \mathcal{A}_n$*.*

Proof. Notice that if $\gcd(h, k) = 1$ then $l(e) = h + k$ for every e by Lemma 3.2. Thus in that case we have

$$\bar{E}_i = \{ S_k \cdot e \mid e \in \mathbb{E}_k^n \} \tag{3.4}$$

for every $0 \leq i \leq n$. $\square$

The main result of this section is the following theorem.

Theorem 3.5. *For any* $n \geq i > j \geq 0$ *we have* $\operatorname{Hom}(\mathcal{A}_i(i), \mathcal{A}_j(j)) = 0$*.*

In particular, the category

$$\mathcal{T} := \langle \mathcal{A}_0, \mathcal{A}_1(1) \dots \mathcal{A}_n(n) \rangle \subset \mathcal{D}(X_k^n) \tag{3.5}$$

is generated by an S_k*-invariant Lefschetz collection.*

The proof of Theorem 3.5 is similar to the proof of Theorem 4.3 in [2].

Proof. By Lefschetz property, it is enough to prove the theorem for $j = 0$, $i > 0$. In other words, it is enough to prove that for any $a \in \bar{E}_i$, $b \in \bar{E}_0$ we have

$$\operatorname{Hom}(\mathcal{O}(a(i)), \mathcal{O}(b)) = 0.$$

Furthermore, by S_k-invariance of the sets $\bar{E}_0$ and $\bar{E}_i$, we can assume that $a \in \mathbb{E}_k^n$ and thus $a_1 \geq \dots \geq a_k = 0$. Suppose

$$\operatorname{Hom}(\mathcal{O}(a(i)), \mathcal{O}(b)) \neq 0, \tag{3.6}$$

where recall $a(i) = (a_i + i, a_2 + i, \dots, a_k + i)$.

We start by rewriting (3.6) in a more convenient form. First, note that $a_1 + i \geq h$. Indeed, otherwise we would have $0 < a_t + i < h$ for all t. On the other hand, since $b \in \bar{E}_0 = S_k \cdot \mathbb{E}_k^n$, by property (3.1) we have $b_t = 0$ for some t. Then $0 < (a_t + i) - b_t = a_t + i \leq n$, hence we have $\operatorname{Hom}(\mathcal{O}(a(i)), \mathcal{O}(b)) = 0$ by (2.2), which contradicts (3.6).

Let r be the maximal index such that

$$a_r + i \geq h \qquad \text{and} \qquad a_{r+1} + i < h. \tag{3.7}$$

Since $a_k + i = i < h$, we have $1 \le r < k$.

Using (2.2) we rewrite (3.6). If $a_j+i \ge h > b_j \ge 0$ then $\mathrm{Hom}(\mathcal{O}(a_j+i), \mathcal{O}(b_j)) \ne 0$ is equivalent to $b_j + h \le a_j + i$ by (2.2). If $0 \le a_j + i < h$, $0 \le b_j < h$ then $\mathrm{Hom}(\mathcal{O}(a_j + i), \mathcal{O}(b_j)) \ne 0$ is equivalent to $b_j \ge a_j + i$ by (2.2). Thus

$$b_j + h \le a_j + i \qquad \text{for} \qquad j \le r, \tag{3.8}$$

$$b_j \ge a_j + i \qquad \text{for} \qquad j > r. \tag{3.9}$$

Next, we replace b by an element in the same S_k-orbit such that

$$b_{r+1} \ge b_{r+2} \ge \cdots \ge b_k \ge b_1 \ge b_2 \ge \cdots \ge b_r \tag{3.10}$$

holds. Indeed, suppose $b_{j_1} > b_{j_2}$ for $j_1 \le r < j_2$. From inequalities (3.8) and (3.9) we see that we can exchange b_{j_1} and b_{j_2} without violating (3.6). Thus, replacing b appropriately, we can assume that $b_{j_1} < b_{j_2}$ for any $j_1 \le r < j_2$.

Similarly, suppose $b_{j_1} < b_{j_2}$ for $j_1 < j_2 \le r$. Since

$$b_{j_1} + h < b_{j_2} + h \le a_{j_2} + i,$$
$$b_{j_2} + h \le a_{j_2} + i \le a_{j_1} + i,$$

we see that we can also exchange b_{j_1} and b_{j_2} without violating (3.6). Thus, replacing b appropriately, we can assume that $b_1 \ge b_2 \ge \cdots \ge b_r$.

Finally, suppose $b_{j_1} < b_{j_2}$ for $r < j_1 < j_2$. Since

$$b_{j_1} > a_{j_1} + i \ge a_{j_2} + i,$$
$$b_{j_2} > b_{j_1} \ge a_{j_1} + i,$$

we see that we can also exchange b_{j_1} and b_{j_2} without violating (3.6). Thus, replacing b appropriately, we can assume that $b_{r+1} \ge b_{r+2} \ge \cdots \ge b_k$. Altogether, we conclude that if (3.6) holds then there is another b in the same S_k-orbit such that (3.10) and (3.6) (or its reformulations (3.8) and (3.9)) still hold. This implies that

$$\bar{b} := (b_{r+1}, \ldots, b_k, b_1, \ldots, b_r) \in \mathbb{E}^n_k.$$

We will use these assumptions from now on. We will need the following

Lemma 3.6. *If* (3.8), (3.9), *and* (3.10) *hold then we have*

$$a_r = \frac{h(k-r)}{k} \qquad \text{and} \qquad b_k = i = \frac{hr}{k}.$$

Proof. The first of inequalities (3.7) defining r implies

$$i \geq h - a_r \geq h - h(k-r)/k = hr/k. \tag{3.11}$$

On the other hand, by (3.1) we have

$$b_k = \bar{b}_{k-r} \leq hr/k.$$

Since $a_k = 0$, we deduce from (3.9) that

$$i \leq b_k \leq hr/k.$$

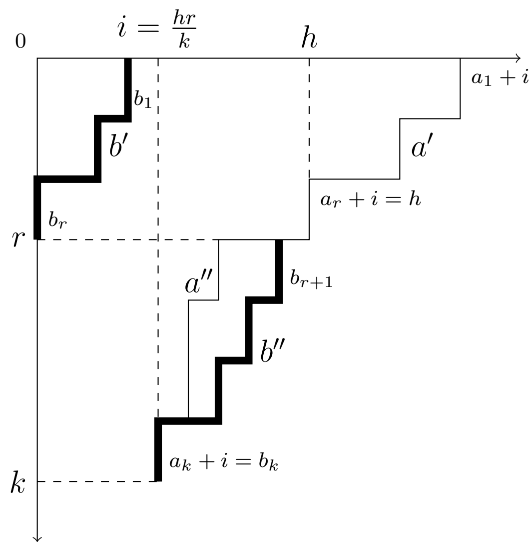
Combining this with (3.11), we conclude that

$$b_k = i = h - a_r = hr/k,$$

hence the lemma. □

We get the following picture:

Figure 1: Picture for Theorem 3.5



We split the Young diagrams $a(i)$ and b into two pieces each, that correspond to the paths in the rectangles on the picture. The path a' is contained in the rectangle

$[h, h+i] \times [0, r]$, the path b' is contained in the rectangle $[0, i] \times [0, r]$. The paths a'' and b'' are contained in the rectangle $[i, h] \times [r, k]$. Explicitly,

$$\begin{aligned} a' &= (a_1 - h(k-r)/k, \dots, a_r - h(k-r)/k), \\ a'' &= (a_{r+1}, \dots, a_k), \\ b' &= (b_1, \dots, b_r), \\ b'' &= (b_{r+1} - hr/k, \dots, b_k - hr/k). \end{aligned}$$

Then inequalities (3.8) and (3.9) combined with Lemma 3.6 give

$$b' \leq a', \qquad \text{and} \qquad a'' \leq b''. \tag{3.12}$$

Further, from Lemma 3.6 we see that $a', b' \in Y_{\frac{r}{k}(h+k), r}$, while $a'', b'' \in Y_{\frac{k-r}{k}(h+k), k-r}$, and all these Young diagrams are upper-triangular since $a, \bar{b} \in \mathbb{E}_k^n$ are upper-triangular by definition of $\mathbb{E}_k^n$. Note also that a and $\bar{b}$ can be represented as concatenations

$$a = a' * a'', \qquad \bar{b} = b'' * b'.$$

Moreover, $a' * a''$ and $a'' * a'$ are both upper-triangular, and as we noticed in the beginning of the section, they are in the same $\mathbb{Z}/(h+k)\mathbb{Z}$-orbit. Similarly $b' * b''$ and $b'' * b'$ are both upper-triangular and lie in the same $\mathbb{Z}/(h+k)\mathbb{Z}$-orbit. Thus, by definition of $\mathbb{E}_k^n$ we have:

$$a' * a'' \leq a'' * a', \tag{3.13}$$

$$b'' * b' \leq b' * b''. \tag{3.14}$$

Using these inequalities, we can make the last step of the proof.

Lemma 3.7. *Inequalities* (3.12) *are equalities.*

Proof. Suppose (3.12) are strict inequalities, that is $b' < a'$ and $a'' < b''$. From that and inequality (3.13) we get:

$$b' * b'' < a' * a'' \leq a'' * a' < b'' * b',$$

which contradicts inequality (3.14).

Thus we have either $a' = b'$ or $a'' = b''$.

Case 1. Suppose we have $a' = b'$ and $a'' < b''$. We replace b'' by smaller element a'' on both sides of inequality (3.14). On the left side of (3.14) substring b'' is closer to the beginning of the string, than on on the right side. Thus by lexicographical order the inequality remains true and becomes strict. So we get:

$$a'' * b' < b' * a''.$$

Since $b' = a'$ we replace b' by a':

$$a'' * a' < a' * a'',$$

which contradicts inequality (3.13).

Case 2. Suppose we have $b' < a'$ and $a'' = b''$. We replace b' by greater element a' on both sides of inequality (3.14). On the left side of (3.14) substring b' is further from the beginning of the string, than on on the right side. Thus by lexicographical order the inequality remains true and becomes strict. So we get:

$$b'' * a' < a' * b'',$$

Since $b'' = a''$ we replace b'' by a'':

$$a'' * a' < a' * a'',$$

which contradicts inequality (3.13).

Therefore we have $a' = b'$ and $a'' = b''$. $\square$

From Lemma 3.7 and (3.13), (3.14) we get:

$$a' * a'' \leq a'' * a' = b'' * b' \leq b' * b'' = a' * a'', \tag{3.15}$$

thus $a' * a'' = a'' * a'$. This means that

$$g^{r(h+k)/h}(a) = a,$$

where g is the generator of the $\mathbb{Z}/(h+k)\mathbb{Z}$-action. Consequently, the length $l(a)$ of the $\mathbb{Z}/(h+k)\mathbb{Z}$-orbit of a divides both $r(h+k)/h$ and $h+k$, hence

$$l(a) \leq \frac{r(h+k)}{k} = \frac{hr}{k}\frac{h+k}{h}.$$

By Lemma 3.6 the right side is equal to $i\frac{h+k}{h}$, and so the above inequality contradicts (3.2). This completes the proof of Theorem 3.5. $\square$

Conjecture 3.8. *The category $\mathcal{T}$ is equal to $\mathcal{D}(X_k^n)$ and the collection* (3.5) *is a minimal Lefschetz collection.*

Later we will prove that the category $\mathcal{T}$ defined by (3.5) is equal to $\mathcal{D}(X_k^n)$ in case $n = 1$ and any k (Subection 4.1), $n \neq 2 \bmod 3$, $k = 3$ (Subection 4.2), and in the case $n = 2$, $k = 3$ (Subsection 4.3).

3.2 Numerical restrictions

We keep the notation $h = n + 1$ and let V be a vector space of dimension h, so that $\mathbb{P}(V) = \mathbb{P}^n$. Denote by

$$K_{\mathbb{C}} := K_0(\mathbb{P}(V)) \otimes \mathbb{C},$$

the complexified Grothendieck group of coherent sheaves on $\mathbb{P}(V)$. It is also a vector space of dimension h. Moreover, we have

$$K_0(X_k^n) \otimes \mathbb{C} = K_0(\mathbb{P}(V)^k) \otimes \mathbb{C} \cong K_{\mathbb{C}}^{\otimes k}.$$

The group $\mathrm{GL}(K_{\mathbb{C}})$ acts naturally on the vector space $K_{\mathbb{C}}^{\otimes k}$, and the group S_k acts on $K_{\mathbb{C}}^{\otimes k}$ by permutation of factors (this action is induced by the action of S_k on X_k^n). These two actions commute, therefore $K_{\mathbb{C}}^{\otimes k}$ is a $(\mathrm{GL}(K_{\mathbb{C}}), S_k)$-bimodule. In the next lemma we describe the decomposition of $K_{\mathbb{C}}^{\otimes k}$ into a direct sum of irreducible representations, provided by the Schur–Weyl duality.

We denote by $\rho(h, k)$ the set of all Young diagrams of k boxes with at most h rows, by $\Sigma^\lambda K_{\mathbb{C}}$ the irreducible representation of $\mathrm{GL}(K_{\mathbb{C}})$ corresponding to the Young diagram λ, (it is also known as the Schur functor assoicated with λ), and by R_{λ^T} the irreducible representation of S_k corresponding to the transposed Young diagram λ^T.

Lemma 3.9 (Schur–Weyl duality, [4])**.** *There exists an isomorphism of* $\mathrm{GL}(K_{\mathbb{C}}) \times S_k$ *representations:*

$$K_{\mathbb{C}}^{\otimes k} = \bigoplus_{\lambda \in \rho(h,k)} \Sigma^\lambda K_{\mathbb{C}} \otimes R_{\lambda^T}.$$

In other words, the decomposition of $K_{\mathbb{C}}^{\otimes k}$ into a direct sum of irreducible S_k-representations contains $\dim(\Sigma^\lambda K_{\mathbb{C}})$ *copies of the irreducible representation R_{λ^T}.*

The above decomposition allows to give a simple necessary condition for the existence of a rectangular S_k-invariant Lefschetz collection in $\mathcal{D}(X_k^n)$. In what follows we call it the *divisibility criterion.*

Corollary 3.10. *If a rectangular S_k-invariant Lefschetz decomposition of length h of $\mathcal{D}(X_k^n)$ exists, then h divides* $\dim \Sigma^\lambda K_{\mathbb{C}}$ *for all $\lambda \in \rho(h, k)$.*

Proof. Assume $\mathcal{D}(X_k^n) = \langle \mathcal{A}_0, \mathcal{A}_0(1), \ldots, \mathcal{A}_0(n) \rangle$ is a rectangular S_k-invariant Lefschetz decomposition. Then we have

$$K_{\mathbb{C}}^{\otimes k} = K_0(X_k^n) \otimes \mathbb{C} = (K_0(\mathcal{A}_0) \otimes \mathbb{C})^{\oplus h}.$$

Since $\mathcal{A}_0$ is S_k-invariant, $K_0(\mathcal{A}_0) \otimes \mathbb{C} \subset K_0(X_k^n) \otimes \mathbb{C}$ is an S_k-subrepresentation, so the above equality shows that the multiplicity of each irreducible S_k-summand of $K_{\mathbb{C}}^{\otimes k}$ is divisible by h. □

The same argument as above gives the following bound for the ranks of the Grothendieck groups of components of an arbitrary S_k-invariant Lefschetz decomposition of $\mathcal{D}(X_k^n)$. Denote by $\lfloor t \rfloor$ and $\lceil t \rceil$ the lower and upper integral parts of t.

Corollary 3.11. *Suppose $\mathcal{D}(X_k^n) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \ldots, \mathcal{A}_n(n) \rangle$ is a Lefschetz S_k-invariant decomposition. Let r_i be the rank of $K_0(\mathcal{A}_i)$. Then*

$$r_0 \geq \sum_{\lambda \in \rho(h,k)} \left\lceil \frac{\dim \Sigma^\lambda K_{\mathbb{C}}}{h} \right\rceil \dim R_{\lambda^T} \quad \textit{and} \quad r_n \leq \sum_{\lambda \in \rho(h,k)} \left\lfloor \frac{\dim \Sigma^\lambda K_{\mathbb{C}}}{h} \right\rfloor \dim R_{\lambda^T}.$$

Proof. Suppose that

$$K_0(\mathcal{A}_i) \otimes \mathbb{C} = \bigoplus_{\lambda \in \rho(h,k)} R_{\lambda^T}^{\oplus a_i^\lambda}$$

is the decomposition into S_k-irreducibles with multiplicities. From Lemma 3.9 we get that $\sum\limits_{0 \leq i \leq n} a_i^\lambda = \dim \Sigma^\lambda K_{\mathbb{C}}$ for any λ. Since $\mathcal{A}_j \subset \mathcal{A}_i$ for any $i < j$, we have $a_j^\lambda \leq a_i^\lambda$ for any λ and $i < j$. Thus

$$a_0^\lambda \geq \left\lceil \frac{\dim \Sigma^\lambda K_{\mathbb{C}}}{h} \right\rceil \quad \text{and} \quad a_n^\lambda \leq \left\lfloor \frac{\dim \Sigma^\lambda K_{\mathbb{C}}}{h} \right\rfloor.$$

This completes the proof. □

As an example we consider the case $n = 1$ and $k = 2m$.

Corollary 3.12. *Suppose $\mathcal{D}(X_{2m}^1)) = \langle \mathcal{A}_0, \mathcal{A}_1(1) \rangle$ is a Lefschetz S_{2m}-invariant decomposition. Let r_i be the rank of $K_0(\mathcal{A}_i)$. Then $r_0 - r_1 \geq \binom{2m}{m}$.*

In Subsection 4.1 we will provide an example of $\mathcal{A}_0, \mathcal{A}_1$ such that $r_0 - r_1 = \binom{2m}{m}$.

Proof. Any diagram in $\rho(2, 2m)$ is of the shape

$$\lambda(l) := (2m - l, l).$$

for some $0 \leq l \leq m$. By Weyl dimension formula we have

$$\dim(\Sigma^{\lambda(l)} K_{\mathbb{C}}) = 2m - 2l + 1,$$

and by the hook-length formula

$$\dim R_{\lambda(l)^T} = \frac{2m!\,(2m - 2l + 1)}{(2m - l + 1)!\,l!} = \frac{2m - 2l + 1}{2m + 1} \binom{2m + 1}{l}.$$

For each l we have $\lceil \frac{2m-2l+1}{2} \rceil - \lfloor \frac{2m-2l+1}{2} \rfloor = 1$, hence by Corollary 3.11, we have

$$\begin{aligned} r_0 - r_1 \geq \sum_{l=0}^{m} \dim R_{\lambda(l)^T} &= \sum_{l=0}^{m} \frac{2m-2l+1}{2m+1} \binom{2m+1}{l} \\ &= \sum_{l=0}^{m} \binom{2m+1}{l} - 2 \sum_{l=0}^{m} \binom{2m}{l-1}. \end{aligned}$$

The first sum is equal to 2^{2m}, and the second is equal to $2^{2m} - \binom{2m}{m}$, so we conclude that $r_0 - r_1 \geq \binom{2m}{m}$. $\square$

If we replace S_k-invariant Lefschetz semiorthogonal decompositions by S_k-invariant Lefschetz collections, the inequalities of Corollary 3.11 can be, in general, improved, because in this case each $K_0(\mathcal{A}_i)$ is a *permutation representation* of S_k.

As an example, we consider the case $k = 3$, $n = 2$ (so that $h = 3$). In this case the set $\rho(3,3)$ consists of three Young diagrams: (3), $(2,1)$, and $(1,1,1)$, with

$$\dim \Sigma^{(3)} K_{\mathbb{C}} = 10, \qquad \dim \Sigma^{(2,1)} K_{\mathbb{C}} = 8, \qquad \dim \Sigma^{(1,1,1)} K_{\mathbb{C}} = 1,$$

while

$$\dim R_{(3)^T} = 1, \qquad \dim R_{(2,1)^T} = 2, \qquad \dim R_{(1,1,1)^T} = 1.$$

Consequently, if $\mathcal{D}(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \mathcal{A}_2(2) \rangle$ is an S_k-invariant Lefschetz decomposition and r_i is the rank of $K_0(\mathcal{A}_i)$, then Corollary 3.11 gives bounds

$$r_0 \geq \left\lceil \frac{10}{3} \right\rceil \cdot 1 + \left\lceil \frac{8}{3} \right\rceil \cdot 2 + \left\lceil \frac{1}{3} \right\rceil \cdot 1 = 4 + 6 + 1 = 11$$

and

$$r_2 \leq \left\lfloor \frac{10}{3} \right\rfloor \cdot 1 + \left\lfloor \frac{8}{3} \right\rfloor \cdot 2 + \left\lfloor \frac{1}{1} \right\rfloor \cdot 1 = 3 + 4 + 0 = 7.$$

On the other hand, we can prove the following result.

Proposition 3.13. *Assume $\langle \mathcal{A}_0, \mathcal{A}_1(1), \mathcal{A}_2(2) \rangle = \mathcal{D}(X_3^2)$ is a Lefschetz decomposition, such that each component $\mathcal{A}_i$ is generated by an S_3-invariant exceptional collection $\{E_{i,j}\}_{j=1}^{r_i}$. Then $r_0 \geq 13$ and $r_2 \leq 7$.*

Proof. The classes of exceptional objects $E_{i,j}$ form a basis of the Grothendieck group $K_0(\mathcal{A}_i)$. Since the collection is S_3-invariant, this basis is permuted by the group action, i.e., $K_0(\mathcal{A}_i) \otimes \mathbb{C}$ is a sum of permutation representations. There are three such representations:

$$\mathbb{C}[S_3] \cong R_{(3)^T} \oplus R_{(2,1)^T}^{\oplus 2} \oplus R_{(1,1,1)^T}, \quad \mathbb{C}[S_3/S_2] \cong R_{(3)^T} \oplus R_{(2,1)^T}, \quad \mathbb{C}[S_3/S_3] \cong R_{(3)^T}.$$

Note that $R_{(1,1,1)^T}$ only appears as a summand of $\mathbb{C}[S_3]$.

On the other hand, by Lemma 3.9 we have

$$K_0(\mathcal{A}_0) \oplus K_0(\mathcal{A}_1) \oplus K_0(\mathcal{A}_2) \cong K_0(X_3^2) \cong R_{(3)^T}^{\oplus 10} \oplus R_{(2,1)^T}^{\oplus 8} \oplus R_{(1,1,1)^T}.$$

Finally, by the Lefschetz property, we have $K_0(\mathcal{A}_2) \subset K_0(\mathcal{A}_1) \subset K_0(\mathcal{A}_0)$. This means that $R_{(1,1,1)}$ has to be a direct summand of $K_0(\mathcal{A}_0)$, hence $K_0(\mathcal{A}_0)$ contains the entire regular representation $\mathbb{C}[S_3]$, and implies $r_0 \geq r_1 + 6 \geq r_2 + 6$. Therefore

$$3r_0 \geq r_0 + (r_1 + 6) + (r_2 + 6) = 27 + 6 + 6 = 39,$$

and hence $r_0 \geq 13$.

Since

$$2r_2 \leq r_1 + r_2 = 27 - r_0 \leq 27 - 13 = 14,$$

we have $r_2 \leq 7$. $\square$

In Section 4.3 we will prove that the collection constructed in Theorem 3.5 for $k = 3$, $n = 2$ is a full S_3-invariant Lefschetz exceptional collection in $\mathcal{D}(X_3^2)$ with $(r_0, r_1, r_2) = (13, 7, 7)$.

3.3 Verifications of divisibility

To check divisibility of the dimensions of $\Sigma^\lambda K_{\mathbb{C}}$ the following corollary of Littlewood–Richardson rule is useful.

Lemma 3.14 ([4]). *Let $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ be a Young diagram. Then*

$$\Lambda^{\mu_1} K_{\mathbb{C}} \otimes \Lambda^{\mu_2} K_{\mathbb{C}} \otimes \cdots \otimes \Lambda^{\mu_m} K_{\mathbb{C}} \cong \Sigma^{\mu^T} K_{\mathbb{C}} \oplus \left(\bigoplus_{\lambda \prec \mu^T} (\Sigma^\lambda K_{\mathbb{C}})^{\oplus c(\lambda,\mu)} \right),$$

where $\prec$ stands for the dominance order [3, Section 2.2]*, and $c(\lambda, \mu)$ are nonnegative integers.*

The next proposition gives some necessary and sufficient conditions for divisibility.

Proposition 3.15. (1) *If h divides k, then* $\dim \Sigma^\lambda K_\mathbb{C}$ *is not divisible by h for some Young diagram $\lambda \in \rho(h,k)$.*
(2) *If k is not divisible by h and for any integer r such that $1 \leq r \leq \min(k, h-1)$ the binomial coefficient $\binom{h}{r}$ is divisible by h, then $\dim(\Sigma^\lambda K_\mathbb{C})$ is divisible by h for any Young diagram $\lambda \in \rho(h,k)$.*

Proof. (1) Suppose $k = ht$. Consider the Young diagram ξ with t columns of height h. Then $\Sigma^\xi K_\mathbb{C} \cong (\det K_\mathbb{C})^{\otimes t}$, hence $\dim(\Sigma^\xi K_\mathbb{C}) = 1$. Since $h = n+1 \geq 2$, we see that $\dim(\Sigma^\xi K_\mathbb{C})$ is not divisible by h.

(2) We use ascending induction on Young diagrams in $\rho(h,k)$ with respect to the dominance order.

Base. Suppose $k = ht + r$, where $t \in \mathbb{Z}_{\geq 0}$ and $1 \leq r \leq h-1$. It is clear that the smallest diagram $\omega \in \rho(h,k)$ is the diagram with t columns of height h and one column of height r. Then $\Sigma^\omega K_\mathbb{C} \cong (\det K_\mathbb{C})^{\otimes t} \otimes \Lambda^r K_\mathbb{C}$, hence $\dim(\Sigma^\omega K_\mathbb{C}) = \binom{h}{r}$, which is divisible by h by the assumption of the proposition.

Induction step. Consider a diagram μ such that $\mu^T \in \rho(h,k)$. Suppose that for any $\lambda \prec \mu^T$, $\lambda \in \rho(h,k)$, the dimension $\dim(\Sigma^\lambda K_\mathbb{C})$ is divisible by h. Let us prove that $\dim(\Sigma^{\mu^T} K_\mathbb{C})$ is also divisible by h. Using Lemma 3.14, we get

$$\dim(\Sigma^{\mu^T} K_\mathbb{C}) = \prod_{i=1}^m \dim(\Lambda^{\mu_i} K_\mathbb{C}) - \dim\left(\bigoplus_{\lambda \prec \mu^T} (\Sigma^\lambda K_\mathbb{C})^{\oplus c(\lambda)}\right) =$$
$$= \prod_{i=1}^m \binom{h}{\mu_i} - \sum_{\lambda \prec \mu^T} c(\lambda,\mu) \cdot \dim(\Sigma^\lambda K_\mathbb{C}).$$

By induction hypothesis, $\sum\limits_{\lambda \prec \mu^T} c(\lambda,\mu) \cdot \dim(\Sigma^\lambda K_\mathbb{C})$ is divisible by h. Since k is not divisible by h we see that there exist i such that $1 \leq \mu_i \leq h-1$. Clearly, $\mu_i \leq k$. Therefore, $1 \leq \mu_i \leq \min(k, h-1)$. Thus $\binom{h}{\mu_i}$ is divisible by h by the assumption of the proposition. Hence $\prod\limits_{i=1}^m \binom{h}{\mu_i}$ and consequently $\dim(\Sigma^{\mu^T} K_\mathbb{C})$ is divisible by h. □

Note that to prove the inductive step we need only one $\binom{h}{\mu_i}$ to be divisible by h for each $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ with $\mu^T \in \rho(h,k)$. This suggests that the assumption of Proposition 3.15(2) can be weakened.

Next, we discuss the divisibility criterion of Corollary 3.10 in the case $k = 3$.

Proposition 3.16. *If $n = 3p+2$, then the category $\mathcal{D}(X_3^n)$ does not have a rectangular S_3-invariant Lefschetz decomposition of length $n+1$.*

Proof. If $n = 3p + 2$, then $n > 1$ and $h = n + 1 \geq 3$. Thus for $k = 3$ all Young diagrams of three boxes are in $\rho(h, k)$. These diagrams are $(1, 1, 1)$, $(2, 1)$, and (3). The dimensions of the corresponding Schur functors are given by

$$\begin{aligned}
\dim \Sigma^{(3)} K_{\mathbb{C}} &= \tfrac{(h+2)(h+1)h}{6}, \\
\dim \Sigma^{(2,1)} K_{\mathbb{C}} &= \tfrac{(h+1)h(h-1)}{3}, \\
\dim \Sigma^{(1,1,1)} K_{\mathbb{C}} &= \tfrac{h(h-1)(h-2)}{6}
\end{aligned}$$

(see for instance the dimension formula from [4, Exercise 6.4]).

Thus, a necessary condition for the existence of a rectangular S_3-invariant Lefschetz decomposition of length h is that the three numbers above are divisible by h. This is equivalent to the integrality of the fractions

$$\frac{(h+2)(h+1)}{6}, \qquad \frac{(h+1)(h-1)}{3}, \quad \text{and} \quad \frac{(h-1)(h-2)}{6}.$$

It is easy to see that this condition holds if and only if h is not divisible by 3. Since $h = n + 1$ we obtain that this condition holds if and only if $n \neq 3p + 2$. $\square$

In other words, we can expect the existence of the desired rectangular decomposition only if $n = 3p$ or $n = 3p + 1$. In the Subsection 4.2 we prove that the desired rectangular decomposition exists in these cases.

4 Fullness

In this section we prove that the S_k-invariant Lefschetz collection (3.5) generates the category $\mathcal{D}(X_k^n)$ when $n = 1$ and any k (Subection 4.1), $k = 3$ and $n \neq 2 \mod 3$ (Subsection 4.2) and $k = 3$, $n = 2$ (Subsection 4.3) and moreover provides a minimal S_k-invariant Lefschetz collection in it.

4.1 Minimal Lefschetz decomposition for $\mathcal{D}(X_k^1)$

First, we consider the case $n = 1$. Recall the definition (3.3) and (3.2) of S_k-invariant subcategories $\mathcal{A}_1 \subset \mathcal{A}_0 \subset \mathcal{D}(X_k^1)$. In this case it can be rewritten as $\mathcal{A}_i = \langle \mathcal{O}(a) \rangle_{a \in \bar{E}_i}$, where

$$\bar{E}_i = \{a \in [0, 1]^k \mid \mathrm{Card}\,\{j \mid a_j = 0\} \geq \frac{k + i}{2}\}, \tag{4.1}$$

and Card stands for the cardinality of a set. If k is odd, $\mathcal{A}_0 = \mathcal{A}_1$.

Theorem 4.1. *We have S_k-invariant Lefschetz decomposition*

$$\mathcal{D}(X_k^1) = \langle \mathcal{A}_0, \mathcal{A}_1(1) \rangle$$

Moreover, this is a minimal Lefschetz collection.

Proof. By definition both subcategories $\mathcal{A}_1$ and $\mathcal{A}_0$ are generated by S_k-invariant exceptional collections. Moreover, by Theorem 3.5 they are semiorthogonal. Thus for the first part of the theorem it is enough to show that $\mathcal{A}_0$ and $\mathcal{A}_1(1)$ generate $\mathcal{D}(X_k^1)$. For this we show that

$$\mathcal{O}(b) \in \mathcal{A}_1(1) \qquad \text{if } b \in [0,2]^k \text{ and } \operatorname{Card}\{j \mid b_j = 1\} \geq \left\lfloor \frac{k}{2} \right\rfloor + 1. \tag{4.2}$$

Indeed, by definition of $\mathcal{A}_1(1)$ we have

$$\mathcal{O}(b) \in \mathcal{A}_1(1) \qquad \text{if } b \in [1,2]^k \text{ and } \operatorname{Card}\{j \mid b_j = 1\} \geq \left\lfloor \frac{k}{2} \right\rfloor + 1. \tag{4.3}$$

Note that $\mathcal{O}(1,\dots,1) \in \mathcal{A}_1(1)$. We apply Corollary 2.2 to $a = (1,\dots,1)$ and any I of cardinality $\lfloor \frac{k}{2} \rfloor$. It proves that for any $b \in [0,2]^k$ such that $b_j = 1$ for $j \notin I$ we have $\mathcal{O}(b) \in \mathcal{A}_0(1)$. This proves (4.2).

Combining (4.2) with the definition of $\mathcal{A}_0$, we deduce that all line bundles $\mathcal{O}(a)$ with $a \in [0,1]^k$ are contained in the subcategory of $\mathcal{D}(X_k^1)$ generated by $\mathcal{A}_0$ and $\mathcal{A}_1(1)$. By Theorem 2.1 this proves the first part of Theorem 4.1.

It remains to show the minimality of the constructed Lefschetz collection. For odd k the collection is rectangular of length $d = 2$, hence minimal (see [10, Subsection 2.1]), so there is nothing to prove. For even k we note that the ranks of the Grothendieck groups of $\mathcal{A}_0$ and $\mathcal{A}_1$ are given by

$$r_0 = 2^{2m} + \frac{1}{2}\binom{2m}{m} \qquad \text{and} \qquad r_1 = 2^{2m} - \frac{1}{2}\binom{2m}{m}$$

respectively. In particular, $r_0 - r_1 = \binom{2m}{m}$, hence the collection is minimal by Corollary 3.12. □

4.2 Lefschetz decompositions for $\mathcal{D}(X_3^{3p})$ and $\mathcal{D}(X_3^{3p+1})$

In this subsection we prove the following

Theorem 4.2. *Let $n = 3p$ or $n = 3p+1$, $k = 3$. The collection constructed in Theorem 3.5 gives S_3-invariant rectangular Lefschetz decomposition*

$$\mathcal{D}(X_3^n) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \dots, \mathcal{A}_n(n) \rangle.$$

The proof takes the rest of the section.

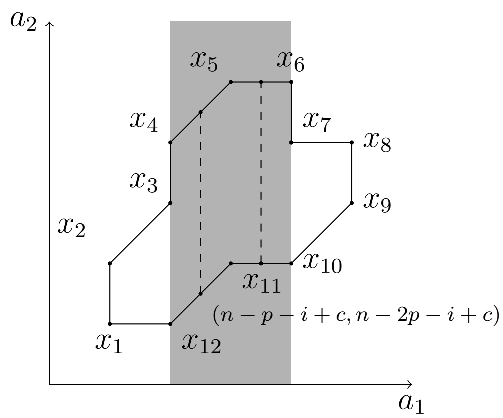
Note that by Lemma 3.4 we have $\mathcal{A} := \mathcal{A}_0 = \cdots = \mathcal{A}_n$. As in the case of Theorem 3.5 we denote by $\mathcal{T}$ the triangulated subcategory of $\mathcal{D}(X_3^n)$ generated by the above Lefschetz collection. Note that $\mathcal{T}$ is S_3-invariant. By subsequent applications of Corollary 2.2 we will show that many other line bundles are contained in $\mathcal{T}$, until in the end we have $\mathcal{O}(a) \in \mathcal{T}$ for all $a \in [0,n]^3$ and conclude by Theorem 2.1.

We will prove the statement of Theorem 4.2 for $n = 3p$ and $n = 3p+1$ in parallel. Denote by T the set of all $a \in \mathbb{Z}^3$ such that $\mathcal{O}(a) \in \mathcal{T}$. Note that T is S_3-invariant.

Proposition 4.3. *For each $i \in [n-p, n]$ and $a \in \mathbb{Z}^3$ with $a_3 = i$, we have $a \in T$.*

Proof. Let us fix $i \in [n-p, n]$. Consider a plane and mark on it all integral points (a_1, a_2) such that $(a_1, a_2, i) \in T$. By definition (3.2) all integral points of the polygon in Figure 2 are marked. The coordinates of its vertices $x_1, \ldots, x_{12}$ are listed in the table below.

Figure 2: Illustration for Step 1 of Proposition 4.3.



	$n=3p$	$n=3p+1$
x_1	$(i-2p,i-2p)$	$(i-2p-1,i-2p-1)$
x_2	$(i-2p,i-p)$	$(i-2p-1,i-p-1)$
x_3	$(i-p,i)$	$(i-p,i)$
x_4	$(i-p,i+p)$	$(i-p,i+p+1)$
x_5	$(i,i+2p)$	$(i,i+2p+1)$
x_6	$(i+p,i+2p)$	$(i+p,i+2p+1)$
x_7	$(i+p,i+p)$	$(i+p,i+p)$
x_8	$(i+2p,i+p)$	$(i+2p+1,i+p)$
x_9	$(i+2p,i)$	$(i+2p+1,i)$
x_{10}	$(i+p,i-p)$	$(i+p+1,i-p)$
x_{11}	$(i,i-p)$	$(i,i-p)$
x_{12}	$(i-p,i-2p)$	$(i-p-1,i-2p-1)$

Our goal is to show that all integral points of the plane are in T. We do this in several steps.

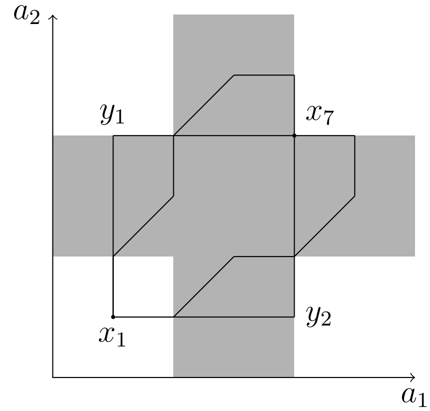
Step 1. We apply Corollary 2.2 with a any integral point on the union of the edges $[x_{10},x_{11}]$ and $[x_{11},x_{12}]$ of the polygon in Figure 2, i.e., with $a=(i+c,i-p,i)$, $c\in[0,p]$, $I=\{2\}$ or $a=(i+p+c,i-2p+c,i)$, $c\in[0,p]$, $I=\{2\}$. Each dashed segment in Figure 2 contains n integral points in T.

By Corollary 2.2 we conclude that all points $(i+c,t,i)$, $(i+p+c,t,i)$ are in T for any $t\in\mathbb{Z}$. In other words, all points in the grey vertical stripe in Figure 2 are in T.

Step 2. Using S_3-symmetry of T we conclude that all points in the horizontal grey stripe on Figure 3 are in T.

Step 3. Combining the results of Step 1 and Step 2 above, we see that $a\in T$ for any a such that $(a_1,a_2)\in[i+p-n,i+p]^2$, $a_3=i$. In other words, all points in the square with vertices x_1, y_1, x_7, y_2 in Figure 3 are in T, where $y_1=(i+p-n,i+p)$ and $y_2=(i+p,i+p-n)$. Therefore we can apply Corollary 2.2 with $a=(i+p-n,i+p-n,i)$ and $I=\{1,2\}$. We conclude that if $a_3=i$, then $a\in T$.

Figure 3: Illustration for Steps 2–3 of Proposition 4.3.

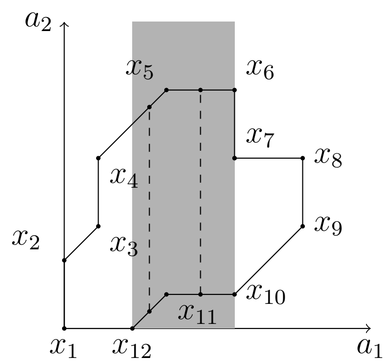


This completes the proof of Proposition 4.3. $\square$

Proposition 4.4. *For any $i \in [p, n-p-1]$, $a \in \mathbb{Z}^3$ such that $a_3 = i$, we have $a \in T$.*

Proof. Let us fix $i \in [p, n-p-1]$. Consider a plane and mark on it all integral points (a_1, a_2) such that $(a_1, a_2, i) \in T$. By definition of (3.2) all integral points of the polygon in Figure 4 are marked. The coordinates of its vertices $x_1, \dots, x_{12}$ are listed in the table below.

Figure 4: Illustration for Step 1 of Proposition 4.4.



	$n=3p$	$n=3p+1$
x_1	$(0,0)$	$(0,0)$
x_2	$(0,p)$	$(0,p)$
x_3	$(i-p,i)$	$(i-p,i)$
x_4	$(i-p,i+p)$	$(i-p,i+p+1)$
x_5	$(i,i+2p)$	$(i,i+2p+1)$
x_6	$(i+p,i+2p)$	$(i+p,i+2p+1)$
x_7	$(i+p,i+p)$	$(i+p,i+p)$
x_8	$(i+2p,i+p)$	$(i+2p+1,i+p)$
x_9	$(i+2p,i)$	$(i+2p+1,i)$
x_{10}	$(i+p,i-p)$	$(i+p+1,i-p)$
x_{11}	$(i,i-p)$	$(i,i-p)$
x_{12}	$(p,0)$	$(p,0)$

Our goal is to show that all integral points of the plane are in T. We do this in several steps.

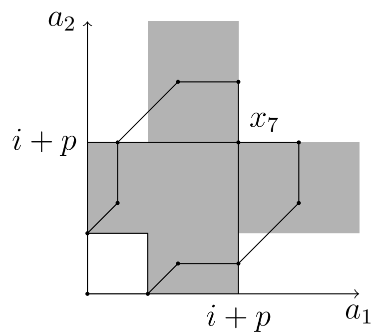
Step 1. We apply Corollary 2.2 with a any integral point on the union of the edges $[x_{10},x_{11}]$ and $[x_{11},x_{12}]$ of the polygon in Figure 4, i.e., with $a=(i+c,i-p,i)$, $c\in[0,i-p]$, $I=\{2\}$ or $a=(p+c,c,n-i)$, $c\in[0,i-p]$, $I=\{2\}$. Each dashed segment in Figure 4 contains n integral points in $\mathcal{T}$.

By Corollary 2.2 we conclude that all points $(p+c,t,i)$, $(i+c,t,i)$ are in T for any $t\in\mathbb{Z}$. In other words, all points in the grey vertical stripe in Figure 4 are in T.

Step 2. Using S_3-symmetry of T we conclude that all points in the horizontal grey stripe on Figure 5 are in T.

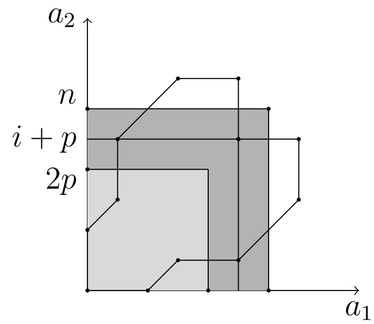
Step 3. Combining the results of Step 1 and Step 2 above, we see that $a\in T$ for any a such that $(a_1,a_2)\in[0,i+p]^2, a_3=i$. Since $i\in[p,n-p-1]$, we have $i+p\geq 2p$.

Figure 5: Illustration for Step 2 of Proposition 4.4.



Note that by Proposition 4.3 and S_3-symmetry of T we have $a \in T$ if $a_1 \in [n-p, n]$ or $a_2 \in [n-p, n]$. Using Step 2 and the inequality $n - p \leq 2p + 1$ we get that $a \in T$ for any a such that $(a_1, a_2) \in [0, n]^2, a_3 = i$. Therefore we can apply Corollary 2.2 with $a = (0, 0, i)$ and $I = \{1, 2\}$. We conclude that if $a_3 = i$, then $a \in T$.

Figure 6: Illustration for Step 3 of Proposition 4.4.

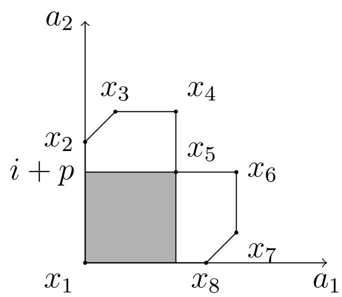


This completes the proof of Proposition 4.4. $\square$

Proposition 4.5. *For any $i \in [0, p-1]$ and $a \in \mathbb{Z}^3$ with $a_3 = i$, we have $a \in T$.*

Proof. Let us fix $i \in [0, p-1]$. Consider a plane and mark on it all integral points (a_1, a_2) such that $(a_1, a_2, i) \in T$. By definition of (3.2) all integral points of the polygon in Figure 7 are marked. The coordinates of its vertices $x_1, \ldots, x_8$ are listed in the table below.

Figure 7: Illustration for Proposition 4.5.



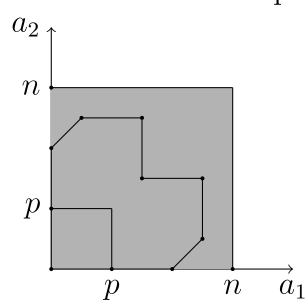
	$n=3p$	$n=3p+1$
x_1	$(0,0)$	$(0,0)$
x_2	$(0,2p)$	$(0,2p+1)$
x_3	$(i,i+2p)$	$(i,i+2p+1)$
x_4	$(i+p,i+2p)$	$(i+p,i+2p+1)$
x_5	$(i+p,i+p)$	$(i+p,i+p)$
x_6	$(i+2p,i+p)$	$(i+2p+1,i+p)$
x_7	$(i+2p,i)$	$(i+2p+1,i)$
x_8	$(2p,0)$	$(2p+1,0)$

Our goal is to show that all integral points of the plane are in T.

We see that $a \in T$ for any a such that $(a_1,a_2) \in [0,i+p]^2, a_3 = i$. Since i is in $[0,p-1]$, we have $i+p \geq p$.

Note that by Propositions 4.3 and 4.4 and S_3-symmetry of T we have $a \in T$ if $a_1 \in [p,n]$ or $a_2 \in [p,n]$. Thus we get that $a \in T$ for any a such that (a_1,a_2) is in $[0,n]^2, a_3 = i$. In other words, all points in the grey square in Figure 8 are in T. Therefore we can apply Corollary 2.2 with $a=(0,0,i)$ and $I=\{1,2\}$. We conclude that if $a_3 = i$, then $a \in T$.

Figure 8: Illustration for Proposition 4.5.



This completes the proof of Proposition 4.5. $\square$

Proof of Theorem 4.2. We combine Propositions 4.3–4.5 to conclude that if a_3 belongs to $[0,n]$, then $a \in T$. Therefore we can apply Corollary 2.2 with $a=(0,0,0)$ and $I=\{1,2,3\}$. This concludes the proof of Theorem 4.2. $\square$

4.3 Minimal Lefschetz decomposition for $\mathcal{D}(X_3^2)$

Consider the case $n=2$, $k=3$. We have $h=n+1=3$, $\dim K_{\mathbb{C}} = 3$. By Proposition 3.16, there is no rectangular S_3-invariant Lefschetz decomposition of $\mathcal{D}(X_3^2)$.

In this section we prove that the collection constructed in Theorem 3.5 gives a minimal (non-rectangular) S_3-invariant Lefschetz decomposition of $\mathcal{D}(X_3^2)$. In particular, we prove its fullness. The same method was used for proving fullness for any $n \neq 2 \bmod 3$.

The collection constructed in Theorem 3.5 has the following components:

$\mathcal{A}_0$	$\mathcal{A}_1(1)$	$\mathcal{A}_2(2)$	Cardinality of the S_3-orbit
$\mathcal{O}(0,0,0)$	$\mathcal{O}(1,1,1)$	$\mathcal{O}(2,2,2)$	1
$\mathcal{O}(1,0,0)$	$\mathcal{O}(2,1,1)$	$\mathcal{O}(3,2,2)$	3
$\mathcal{O}(1,1,0)$	$\mathcal{O}(2,2,1)$	$\mathcal{O}(3,3,2)$	3
$\mathcal{O}(2,1,0)$			6

The starting component $\mathcal{A}_0$ is generated by $1+3+3+6=13$ line bundles, while the other two components are generated by $1+3+3=7$ line bundles.

Theorem 4.6. *The categories $\mathcal{A}_0$, $\mathcal{A}_1(1)$ and $\mathcal{A}_2(2)$ described above generate a minimal S_3-invariant Lefschetz collection in $\mathcal{D}(X_3^2)$. In particular,*

$$\mathcal{D}(X_3^2) = \mathcal{T} := \langle \mathcal{A}_0, \mathcal{A}_1(1), \mathcal{A}_2(2) \rangle. \tag{4.4}$$

Proof. From Theorem 3.5 we know that $(\mathcal{A}_0, \mathcal{A}_1(1), \mathcal{A}_2(2))$ is semiorthogonal and S_3-invariant. Let us show that it generates $\mathcal{D}(X_3^2)$.

Applying Corollary 2.2 several times we will show that more line bundles are contained in $\mathcal{T}$. We note that $\mathcal{T}$ is S_3-invariant, so as soon as a line bundle is proved to be contained in $\mathcal{T}$, its entire S_3-orbit is also contained in $\mathcal{T}$.

Step 1. We note that $\mathcal{O}(2,2,1)$, $\mathcal{O}(2,2,2)$, and $\mathcal{O}(2,2,3)$ are all in $\mathcal{T}$ (the first is in $\mathcal{A}_1(1)$, while the other two are in $\mathcal{A}_2(2)$). Applying Corollary 2.2 with $a=(2,2,1)$ and $I=\{3\}$ we conclude that all line bundles $\mathcal{O}(2,2,t)$ are in $\mathcal{T}$. In particular,

$$\mathcal{O}(2,2,0) \in \mathcal{T}.$$

Step 2. We note that $\mathcal{O}(1,2,0)$, $\mathcal{O}(1,2,1)$, and $\mathcal{O}(1,2,2)$ are in $\mathcal{T}$ (the first is in $\mathcal{A}_0$, while the other two are in $\mathcal{A}_1(1)$). Applying Corollary 2.2 with $a=(1,2,0)$ and $I=\{3\}$ we conclude that all line bundles $\mathcal{O}(1,2,t)$ are in $\mathcal{T}$. In particular,

$$\mathcal{O}(1,2,3) \in \mathcal{T}.$$

Step 3. We note that $\mathcal{O}(3,2,1)$, $\mathcal{O}(3,2,2)$, and $\mathcal{O}(3,2,3)$ are in $\mathcal{T}$ (for the first of them we use the result of Step 2). Applying Corollary 2.2 with $a=(3,2,1)$ and $I=\{3\}$ we conclude that all line bundles $\mathcal{O}(3,2,t)$ are in $\mathcal{T}$. In particular,

$$\mathcal{O}(3,2,0) \in \mathcal{T}.$$

Step 4. We note that $\mathcal{O}(2,0,1)$, $\mathcal{O}(2,0,2)$, and $\mathcal{O}(2,0,3)$ are in $\mathcal{T}$ (for the last two of them we use the results of Step 1 and Step 3 and S_3-invariance of $\mathcal{T}$). Applying Corollary 2.2 with $a = (2,0,1)$ and $I = \{3\}$ we conclude that all line bundles $\mathcal{O}(2,0,t)$ are in $\mathcal{T}$. In particular,

$$\mathcal{O}(2,0,0) \in \mathcal{T}.$$

Combining the original collection with the results of Steps 1–4 above and S_3-invariance, we see that all line bundles $\mathcal{O}(a)$ with $a \in [0,2]^3$ are contained in $\mathcal{T}$. Therefore, by Theorem 2.1 we have $\mathcal{T} = \mathcal{D}(X_3^2)$.

Finally, the minimality of the constructed Lefschetz collection follows from Proposition 3.13. □

In this case we can also compute the residual category.

Definition 4.7 ([10, Definition 1.3])**.** Let $\langle \mathcal{B}_0, \mathcal{B}_1(1), \dots, \mathcal{B}_d(d)\rangle$ be a Lefschetz decomposition of a triangulated category $\mathcal{T}$. Its *rectangular part* is the subcategory $\langle \mathcal{B}_d, \mathcal{B}_d(1), \dots, \mathcal{B}_d(d)\rangle$ of $\mathcal{T}$, and its *residual category* is the orthogonal

$$\mathcal{R}_{\mathcal{B}_\bullet} = \langle \mathcal{B}_d, \mathcal{B}_d(1), \dots, \mathcal{B}_d(d)\rangle^{\perp}$$

to the rectangular part.

Theorem 4.8. *The residual category of the Lefschetz decomposition* (4.4) *is generated by the S_3-orbit of the line bundle $\mathcal{O}(1,-1,0)$. Thus the residual category in the equivariant category is generated by a single exceptional object.*

Proof. The pullback from the first factor $\mathbb{P}^2$ of X_3^2 of the Koszul complex

$$0 \to \mathcal{O}(-1,1,0) \to \mathcal{O}(0,1,0)^{\oplus 3} \to \mathcal{O}(1,1,0)^{\oplus 3} \to \mathcal{O}(2,1,0) \to 0$$

implies that $\mathcal{O}(-1,1,0)$ is right orthogonal both to $\mathcal{A}_1(1) = \mathcal{A}_2(1)$ and $\mathcal{A}_2(2)$. It is also easy to see that it is right orthogonal to $\mathcal{A}_2$, hence it is contained in the residual category $\mathcal{R}$. Using S_3-invariance of the category $\mathcal{A}_2$, we conclude that the whole S_3-orbit of $\mathcal{O}(-1,1,0)$ is contained in $\mathcal{R}$.

On the other hand, the same Koszul complex shows that together with the rectangular part of (4.4), the S_3-orbit of $\mathcal{O}(-1,1,0)$ generates the same subcategory of $\mathcal{D}(X_3^2)$ as the right side of (4.4) does, hence by Theorem 4.6 it generates the whole category. This proves that the S_3-orbit of $\mathcal{O}(-1,1,0)$ generates the residual category of (4.4). □

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