Global Nash Equilibrium in a Class of Nonconvex N-player Games

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Abstract

We consider seeking the global Nash equilibrium (NE) in a class of nonconvex N-player games. The structured nonconvex payoffs are composited with canonical functions and quadratic operators, which are broadly investigated in various tasks such as robust network training and sensor network communication. However, the full-fledged development of nonconvex min-max games may not provide available help due to the interference of multiple players' coupled stationary conditions, and the existing results on convex games may also perform unsatisfactorily since they may be stuck in local NE or Nash stationary points, rather than the global NE. Here, we first make efforts to take a canonical conjugate transformation of the nonconvex N-player game, and cast the complementary problem into a variational inequality (VI) problem for the derivation of the global NE. Then we design a conjugate-based ordinary differential equation (ODE) for the solvable VI problem, and present the equilibrium equivalence and guaranteed convergence within the ODE. Furthermore, we provide a discretized algorithm based on the ODE, and discuss step-size settings and convergence rates in two typical nonconvex N-player games. At last, we conduct experiments in practical tasks to illustrate the effectiveness of our approach.

1 Introduction

Game theory plays an essential role in the leading edge nowadays such as adversarial training (Heusel et al., 2017; Song et al., 2018; Madry et al., 2018; Vlatakis-Gkaragkounis et al., 2021) and reinforcement learning (Busoniu et al., 2008; Lanctot et al., 2017; Dai et al., 2018). Therein, the Nash equilibrium (NE) (Nash, 1951) becomes popular in various fields like applied mathematics, computer sciences, and engineering, in addition to economy. However, seeking NE for a given game is not easy especially when there is underlying nonconvexity, because nonconvex problems cannot be solved by a common methodology. Up to now, some inspiring breakthroughs have been made for solving nonconvex two-player min-max games in different situations, including Polyak-Łojasiewicz cases (Nouiehed et al., 2019; Fiez et al., 2021), strongly-concave cases (Lin et al., 2020; Rafique et al., 2021), and general nonconvex nonconcave cases (Heusel et al., 2017; Daskalakis & Panageas, 2018; Adolphs et al., 2019; Jin et al., 2020).

Nevertheless, seeking NE in nonconvex N-player games needs to be deeply explored, which is not enough to merely concentrate on the nonconvexity in two-player games. Actually, various fields involve the interaction and interference of multiple players, such as smart grids (Saad et al., 2012), intelligent transportation (Saharan et al., 2020), and cloud computing (Pang et al., 2008). N-player models established over large-scale networks help analyze more expressive systems and reflect more realistic phenomenons. Meanwhile, advanced learning approaches in artificial intelligence are developed toward multi-agent, distributed, and federated frameworks (Yu et al., 2019; Li et al., 2019; Fan et al., 2021). Their common core is to utilize the enhanced ability of individual computational units, and sufficiently exploit their autonomy and evolvability in large-scale tasks. As one of the most popular schemes, adversarial learning is gradually generalized to multiple agents (Song et al., 2018; Zhao et al., 2020; Ferdowsi & Saad, 2020), no longer constrained in classic models with one generator or one discriminator.

Up to now, many theoretical results in N-player games have been built on fundamental convexity assumptions (Yi & Pavel, 2019; Facchinei & Kanzow, 2010; Chen et al., 2021). On this basis, NE seeking in many typical game models has been extensively studied, including aggregative games (Koshal et al., 2016) and potential games (Lei & Shanbhag, 2020). Despite many efficient tools within convex conditions leading to fruitful achievements, they may be far from enough when encountering nonconvexity in practical circumstances. Indeed, there have been pioneers paying attention to the importance of seeking NE in nonconvex N-player games. For instance, (Pang & Scutari, 2011) proposed a best-response scheme for Nash stationary points of a class of nonconvex games in signal processing, and then (Hao & Pang, 2020) extended this method in N-player bilevel games with nonconvex constraints. Moreover, (Raghunathan et al., 2019) introduced a gradient-based Nikaido-Isoda function to find Nash stationary points in a reformulated nonconvex game, while (Liu et al., 2020) designed a gradient-proximal algorithm for approximate NE in a class of nonconvex aggregative games. Actually, most of them revolve around seeking Nash stationary points or local NE, rather than global NE. It is the status quo that finding global NE in nonconvex N-player games is still a difficult and challenging task, which deserves further investigation.

In this view, we focus on global NE seeking in a typical class of nonconvex N-player games. The structured nonconvex payoffs are composited with canonical functions and quadratic operators, which are widely investigated in engineering tasks such as adversarial training and sensor network communication. For example, such payoffs are reified as Euclidian norms in sensor localization (Ke et al., 2017; Yang et al., 2018), while they are endowed with log-sum-exp forms in robust neural network training (Nouiehed et al., 2019; Deng & Mahdavi, 2021). Our goal is to propose a theoretically guaranteed algorithm for seeking the global NE in such an important class of nonconvex N-player games. The consequent results will help demystify the complicated interactions among players and provide trustworthy insights for large-scale problems afterward. Due to the nonconvexity therein, the classic gradient-based methods for convex games may be stuck in some NE stationary points when tracking along the pseudo-gradients, rather than reaching a global NE. On the other hand, with the interference of all players' decisions, the global stationary conditions are mutually coupled and can not be handled individually by each player. In this way, although similar nonconvex structures in the payoffs have been considered in previous optimization research (Chiang et al., 2007; Latorre & Gao, 2016; Liang & Cheng, 2019), their techniques may not be directly adopted in such nonconvex games. Thus, we need novel processes to overcome the bottleneck mentioned above.

Thereby, the contributions of this paper are summarized in the following. To deal with the nonconvexity in payoff functions, we first utilize the canonical duality theory (Gao et al., 2017) and obtain a canonical conjugate transformation. By compactly formulating the stationary conditions of the transformed problem as a continuous mapping, we then cast it into a variational inequality (VI) problem (Facchinei & Pang, 2003) to verify the global NE via the fixed points of this VI. At this point, we propose a conjugate-based ordinary differential equation (ODE) for solutions to the VI. The designed ODE evolves in the dual space, and the mapping from the dual space back to the primal space is enforced via conjugate gradients. Also, we show the derivation of global NE in the original nonconvex N-player game by the equilibrium of this conjugate-based ODE accompanied with the canonical duality relation, as well as the convergence analysis and the convergence rate of the ODE. Afterward, to make our approach implementable, we derive the discrete algorithm induced from the conjugate-based ODE, and further provide the step-size settings and the convergence rates in two typical nonconvex game models. Specifically, the convergence rate achieves $\mathcal{O}(1/k)$ in a class of N-player generalized monotone games (Facchinei & Kanzow, 2010; Koshal et al., 2016), while achieves $\mathcal{O}(1/\sqrt{k})$ in a class of N-player potential games (Ke et al., 2017; Yang et al., 2018). At last, experiments for practical tasks are illustrated to show the effectiveness of our approach.

2 NONCONVEX N-PLAYER GAMES

We begin our study of the nonconvex games with N players indexed by $\mathcal{I} = \{1, \cdots, N\}$. For $i \in \mathcal{I}$, the ith player has an action variable x_i in an action set $\Omega_i \subseteq \mathbb{R}^n$, where Ω_i is compact and convex, and $\boldsymbol{\Omega} = \prod_{i=1}^{N} \Omega_i$. Let $\boldsymbol{x} = \mathrm{col}\{x_1, ..., x_N\} \in \mathbb{R}^{nN}$ be the profile of all players' actions, while $\boldsymbol{x}_{-i}$ be the profile of all players' actions except for the ith player's. Moreover, the ith player has a payoff function $J_i(x_i, \boldsymbol{x}_{-i}) : \boldsymbol{\Omega} \to \mathbb{R}$, which is dependent on both x_i and $\boldsymbol{x}_{-i}$, and twice continuously differentiable in x_i. Given $\boldsymbol{x}_{-i}$, the ith player intends to solve the following problem

$$\min_{x_i} J_i\left(x_i, \boldsymbol{x}_{-i}\right), \quad \text{s.t. } x_i \in \Omega_i. \tag{1}$$

In this paper, we focus on a typical class of nonconvex N-player games, in which the ith player's payoff function is endowed with the following structure

$$J_i(x_i, \boldsymbol{x}_{-i}) = \Psi_i(\Lambda_i(x_i, \boldsymbol{x}_{-i})). \tag{2}$$

Here $\Lambda_i : \mathbb{R}^{Nn} \to \Theta_i \subseteq \mathbb{R}^{q_i}$ is a vector-valued nonlinear operator with $\Lambda_i = (\Lambda_{i,1}, \cdots, \Lambda_{i,q_i})^T$. For $k \in \{1, \cdots, q_i\}$, each $\Lambda_{i,k} : \mathbb{R}^{Nn} \to \mathbb{R}$ is quadratic in x_i, whose second-order partial derivative in x_i is both x_i-free and $\boldsymbol{x}_{-i}$-free, e.g., $\Lambda_{i,k} = x_i^T A_{i,k} x_i + \sum_{i \neq j} x_i^T B_{i,k} x_j$. Moreover, $\Psi_i : \Theta_i \to \mathbb{R}$ is a convex differential canonical function (Gao et al., 2017), whose gradient $\nabla \Psi_i : \Theta_i \to \Theta_i^*$ is a one-to-one mapping. Such nonconvex structures composited by canonical functions and quadratic operators emerge in broad applications, including robust network training (Nouiehed et al., 2019), senor network communication (Yang et al., 2018), and generative adversarial networks (Gidel et al., 2018). We provide specific examples in the following for intuition about the above nonconvex model.

Euclidian distance function

$$\Psi_i(\Lambda_i(x_i, \boldsymbol{x}_{-i})) = \sum\nolimits_{j \in \mathcal{N}_i} (\|x_i - x_j\|^2 - d_{i,j})^2, \tag{3}$$

where $\Psi_i = \sum_{j=1}^{\mathcal{N}_i} \Lambda_{i,j}^T \Lambda_{i,j}$ and $\Lambda_{i,j} = \|x_i - x_j\|^2 - d_{i,j}$. (3) usually serves as the payoffs in sensor localization tasks (Jia et al., 2013; Ke et al., 2017; Yang et al., 2018), in which $x_i \in \Omega_i$ is an anchor node, $\mathcal{N}_i$ is neighbors of node i, and $d_{i,j}$ is the distance parameter.

Log-sum-exp function

$$\Psi_i(\Lambda_i(x_i, \boldsymbol{x}_{-i})) = \beta_1 \log[1 + \exp(x_i^T x_i + \sum\nolimits_{j=1}^{N} x_i^T x_j - \beta_2^T x_i)], \tag{4}$$

where $\Psi_i = \beta_1 \log[1 + \exp \Lambda_i]$ and $\Lambda_i = x_i^T x_i + \sum_{j=1}^N x_i^T x_j - \beta_2^T x_i$. (4) usually appears in the tasks like robust neural network training (Nouiehed et al., 2019; Deng & Mahdavi, 2021), where x_i is the neural network parameter, x_j is the perturbation, and β_1, β_2 are training data.

Log-posynomial function

$$\Psi_i(\Lambda_i(x_i, \boldsymbol{x}_{-i})) = \log(x_i^T \mathbf{C}_i x_i + x_i^T \mathbf{D}_i \boldsymbol{x}_{-i})^{-1} \tag{5}$$

where $\Psi_i = \log(\Lambda_i)^{-1}$ and $\Lambda_i = x_i^T \mathbf{C}_i x_i + x_i^T \mathbf{D}_i \boldsymbol{x}_{-i}$. (5) usually occurs in resource allocation (Yang & Xie, 2019; Ruby et al., 2015; Chiang et al., 2007), where x_i stands for transmit resources, and matrices $\mathbf{C}_i$ and $\mathbf{D}_i$ represent the correlation coefficients.

For solving the nonconvex N-player game (1), we introduce the following important concept.

Definition 1 (global Nash equilibrium) *A strategy profile $\boldsymbol{x}^\diamond \in \boldsymbol{\Omega}$ is said to be a global Nash equilibrium (NE) of (1), if for all $i \in \mathcal{I}$,*

$$J_i(x_i^\diamond, \boldsymbol{x}_{-i}^\diamond) \leq J_i(x_i, \boldsymbol{x}_{-i}^\diamond), \quad \forall x_i \in \Omega_i. \tag{6}$$

The global NE above characterizes a strategy profile that each player adopts its globally optimal strategy. That is, given others' actions, no player can benefit from changing her/his action unilaterally. Actually, the conception of global NE here is indeed the concept of NE (Nash, 1951), and we emphasize *global* in the nonconvex formulation to tell the difference from *local* NE (Pang & Scutari, 2011; Nouiehed et al., 2019; Heusel et al., 2017). Also, we consider another mild but well-known concept to help characterize the solutions to (1).

Definition 2 (Nash stationary point) *A strategy profile $\boldsymbol{x}^\diamond$ is said to be a Nash stationary point of (1) if for all $i \in \mathcal{I}$,*

$$\mathbf{0}_n \in \nabla_{x_i} J_i(x_i^\diamond, \boldsymbol{x}_{-i}^\diamond) + \mathcal{N}_{\Omega_i}(x_i^\diamond). \tag{7}$$

It is not difficult to reveal that, if $\boldsymbol{x}^\diamond$ is a global NE, then it must be a NE stationary point, but not vice versa. Actually, as for convex games, most existing research computes global NE via investigating Nash stationary points (Facchinei & Kanzow, 2010; Koshal et al., 2016; Chen et al., 2021). However, considering the bumpy geometric structure of the nonconvex payoff function, one cannot expect to find a global NE of (1) merely via the Nash stationary conditions in (7). For instance, the classic gradient-based methods for convex games may be stuck in these stationary points rather than a global NE, see Fig. 1 for an illustration. To this end, we aim at obtaining a global NE of (1) and begin the exploration in the sequel.

(a) (b)

Figure 1: A nonconvex two-player game with log-sum-exp payoff functions described in (4). (a) The global NE distinguishes from Nash stationary points, shown on the surface plot of one player's nonconvex payoff; (b) The gradient-based method for convex games fails to converge to the global NE.

3 Derivation of global NE

Following the definition of canonical functions, for $i \in \mathcal{I}$, take $\xi_i = \Lambda_i(x_i, \boldsymbol{x}_{-i}) \in \Theta_i$ in the payoff function of (2), which is called canonical measure. Since $\Psi_i(\xi_i)$ is a convex canonical function, the one-to-one duality relation $\sigma_i = \nabla \Psi_i(\xi_i) : \Theta_i \to \Theta_i^*$ implies the existence of the conjugate function $\Psi_i^* : \Theta_i^* \to \mathbb{R}$, which can be uniquely described by the Legendre transformation (Rockafellar, 1974; Gao et al., 2017), that is,

$$\Psi_i^*(\sigma_i) = \xi_i^T \sigma_i - \Psi_i(\xi_i),$$

where $\sigma_i \in \Theta_i^*$ is a canonical dual variable. Thus, denote $\boldsymbol{\sigma} = \text{col}\{\sigma_1, \cdots, \sigma_N\}$ and $\boldsymbol{\Theta}^* = \prod_{i=1}^N \Theta_i^* \subseteq \mathbb{R}^q$ with $q = \sum_{i=1}^N q_i$. Then the complementary function $\Gamma_i : \boldsymbol{\Omega} \times \Theta_i^* \to \mathbb{R}$ referring to the canonical duality theory can be defined as

$$\Gamma_i(x_i, \sigma_i, \boldsymbol{x}_{-i}) = \xi_i^T \sigma_i - \Psi_i^*(\sigma_i) = \sigma_i^T \Lambda_i(x_i, \boldsymbol{x}_{-i}) - \Psi_i^*(\sigma_i). \quad (8)$$

The following lemma reveals the equivalency relationship of stationary points between (8) and (1).

Lemma 1 *A profile $\boldsymbol{x}^\diamondsuit$ is a Nash stationary point of (1) if there exists $\boldsymbol{\sigma}^\diamondsuit \in \boldsymbol{\Theta}^*$, such that for all $i \in \mathcal{I}$, $(x_i^\diamondsuit, \sigma_i^\diamondsuit)$ is a stationary point of complementarity function $\Gamma_i(x_i, \sigma_i, \boldsymbol{x}_{-i}^\diamondsuit)$.*

Lemma 1 shows that with the canonical transformation above, we can close the duality gap between the nonconvex original game and its canonical dual problem. Further, for $i \in \mathcal{I}$, the second-order partial derivative of $\Gamma_i(x_i, \sigma_i, \boldsymbol{x}_{-i})$ in x_i is defined as follows.

$$P_i(\sigma_i) = \nabla_{x_i}^2 \Gamma_i = \sum_{k=1}^{q_i} [\sigma_i]_k \nabla_{x_i}^2 \Lambda_{i,k}(x_i, \boldsymbol{x}_{-i}).$$

Recalling that $\Lambda_i : \boldsymbol{\Omega} \to \Theta_i$ is a quadratic operator and $\nabla_{x_i}^2 \Lambda_i$ is both x_i-free and $\boldsymbol{x_{-i}}$-free (see the cases in (3)-(5)), we can easily check that $P_i(\sigma_i)$ is indeed a linear combination of $[\sigma_i]_k$. On this basis, we introduce the following set of σ_i for $i \in \mathcal{I}$.

$$\mathscr{E}_i^+ = \Theta_i^* \cap \{\sigma_i : P_i(\sigma_i) \succeq \kappa_x \boldsymbol{I}_n\}, \quad \mathscr{E}^+ = \mathscr{E}_1^+ \times \cdots \times \mathscr{E}_N^+, \quad (9)$$

where the constant $\kappa_x > 0$.

The computation of $\mathscr{E}_i^+$ is actually not so hard in most practical cases. For example, take the payoff function in (3) with $i = 1, 2$ and $n = q_i = 1$. The complementary function is $\Gamma_i(x_i, \sigma_i, x_{3-i}) = \sigma_i((x_i - x_{3-i})^2 - d_{i,3-i}) - \sigma_i^2/4$, where $x_i \in \Omega_i = [a, b]$ and $\sigma_i \in \Theta_i^* = [-2d_{i,3-i}, 2(b-a)^2 - 2d_{i,3-i}]$. Thus, the subset $\mathscr{E}_i^+ = \{\sigma_i : 2\sigma_i \geq \kappa_x\} \cap \Theta_i^* = [\kappa_x/2, 2(b-a)^2 - 2d_{i,3-i}]$.

When $\sigma_i \in \mathscr{E}_i^+$, the positive definiteness of $P_i(\sigma_i)$ implies that $\Gamma_i(x_i, \sigma_i, \boldsymbol{x}_{-i})$ is convex with respect to x_i. Besides, the convexity of $\Psi_i(\xi_i)$ derives that its Legendre conjugate $\Psi_i^*(\sigma_i)$ is also convex. Hence, the complementary function $\Gamma_i(x_i, \sigma_i, \boldsymbol{x}_{-i})$ is concave in σ_i. This convex-concave property of Γ_i enables us to further investigate the optimality of the stationary points of (8), that is, the optimality of the Nash stationary point of (1). Moreover, with the interference of $\boldsymbol{x}_{-i}$, the transformed problem reflects a cluster of Γ_i with a mutual coupling of stationary conditions, rather than a deterministic one. Therefore, different from classic optimization works, the stationary points for player i cannot be calculated independently. We should consider the computation of all players' stationary point profile and discuss its optimality in an entire perspective.

To this end, variational inequalities (VI) help us to carry forward. By generalizing the coupled stationary conditions to continuous vector fields, these stationary conditions can be compactly formulated as a continuous mapping in a VI problem (Facchinei & Pang, 2003). Then, seeking all players' stationary point profile (or Nash stationary point) can be accomplished by verifying a fixed point of this continuous mapping. The seed of employing the VI idea in game problems dates back to (Harker & Pang, 1990), and has since found wide applications in various game models, for a survey, see (Giannessi et al., 2006) and the references therein. Specifically, denote $\boldsymbol{z} = \mathrm{col}\{\boldsymbol{x}, \boldsymbol{\sigma}\}$ and $\boldsymbol{\Xi} = \boldsymbol{\Omega} \times \mathscr{E}^+$. Take the following continuous mapping as the pseudo-gradient of (8).

$$F(\boldsymbol{z}) = \mathrm{col}\left\{ \mathrm{col}\{\sum_{k=1}^{q_i} [\sigma_i]_k \nabla_{x_i} \Lambda_{i,k}(x_i, \boldsymbol{x}_{-i})\}_{i=1}^N,\ \mathrm{col}\{-\Lambda_i(x_i, \boldsymbol{x}_{-i}) + \nabla\Psi_i^*(\sigma_i)\}_{i=1}^N \right\}.$$

Then (8) can be cast as a VI problem $\mathrm{VI}(\boldsymbol{\Xi}, F)$ to solve, i.e., finding $\boldsymbol{z}^\diamondsuit \in \boldsymbol{\Xi}$ such that

$$(\boldsymbol{z} - \boldsymbol{z}^\diamondsuit)^T F(\boldsymbol{z}^\diamondsuit) \geq 0, \quad \forall \boldsymbol{z} \in \boldsymbol{\Xi}. \tag{10}$$

Note that the interaction on all players' variables is displayed in mapping F, which is a joint function of the partial derivatives of all players' complementary functions (8). Together with Lemma 1, we have the following result for identifying the global NE of (1).

Theorem 1 *If $(\boldsymbol{x}^\diamondsuit, \boldsymbol{\sigma}^\diamondsuit)$ is a solution to $\mathrm{VI}(\boldsymbol{\Xi}, F)$ with $\sigma_i^\diamondsuit = \nabla\Psi_i(\xi_i) \mid_{\xi_i = \Lambda_i\left(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit\right)}$ for $i \in \mathcal{I}$, then $\boldsymbol{x}^\diamondsuit$ is the global NE of the nonconvex game (1).*

Proved in § D, this result underlines that once the solution of $\mathrm{VI}(\boldsymbol{\Xi}, F)$ is obtained, we can check whether the duality relation $\sigma_i^\diamondsuit = \nabla\Psi_i(\xi_i) \mid_{\xi_i = \Lambda_i\left(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit\right)}$ holds, so as to identify whether the solution of $\mathrm{VI}(\boldsymbol{\Xi}, F)$ is a global NE. This inspires us to explore approaches to solve $\mathrm{VI}(\boldsymbol{\Xi}, F)$ via its first-order conditions and employ this relation as a criterion for identifying the global NE.

We would like to remark that the foundation to realize the above idea is the nonempty set $\mathscr{E}_i^+$. It is possible to obtain an empty $\mathscr{E}_i^+$ in reality, provided by $P_i(\sigma_i) \succeq \kappa_x \boldsymbol{I}_n$ has no intersection with Θ_i^*, and these situations make the above duality theory approach unavailable. Thus, $\mathscr{E}_i^+$ should be effectively checked once the problem is formulated. Such a process has also been similarly employed in some classic optimization works to solve nonconvex problems (Zhu & Martínez, 2012; Liang & Cheng, 2019; Ren et al., 2021; Zheng et al., 2012; Latorre & Gao, 2016). In addition, this is why we cannot directly employ the standard Lagrange multiplier method and the associated KKT theory, because we need to confirm the domain of multiplier σ_i by utilizing canonical duality information (referring to Θ_i^*).

4 CONJUGATE-BASED ODE

Recalling the aforementioned process for nonconvexity, we have made a canonical transformation for game (1) by introducing the Legendre conjugate of Ψ_i and the canonical dual variable σ_i. With these assisted complementary information, we propose a scheme to seek the solutions to (10). In fact, ordinary differential equations (ODE) provide evolved dynamics, which help reveal how the primal variables and the canonical dual ones influence each other via conjugate gradient information. Meanwhile, the analysis techniques in modern calculus and nonlinear systems for theoretical guarantees of ODEs may lead to comprehensive results with mild assumptions. In this light, we intend to design a conjugate-based ODE to solve (10) in the following.

Surprisingly, we find that this conjugate-based idea not only helps solve the inherent nonconvexity in such a class of N-player games, but also brings us convenience to deal with local set constraints with specific structures. Particularly, the local set constraints of variables in the transformation, like Ω_i and $\mathscr{E}_i^+$ of (10), are usually equipped with specific structures such as the Euclidean sphere of parameter perturbation in (Deng & Mahdavi, 2021) and the unit simplex of soft-max output layers in (Daskalakis et al., 2018). Hence, it is worth dealing with these constraint structures in an efficient way. Enlightened by conjugate properties of the generating functions within Bregman divergence, we intend to employ conjugate mappings with different generating functions in the ODE design.

Table 1: Conjugate gradients with different generating functions.

	Feasible set	Generating function	Conjugate gradient
General convex set	Ω	$\frac{1}{2}\|x\|_2^2$	$\mathrm{argmin}_{x\in\Omega}\frac{1}{2}\|x-y\|^2$
Non-negative orthant	$\mathbb{R}_+^n$	$\sum_{l=1}^n x^l\log(x^l)-x^l$	$\exp(y^l)$
Unit square $[a,b]^n$	$\{x^l\in\mathbb{R}: a\leq x^l\leq b\}$	$\sum_{l=1}^n(x^l-a)\log(x^l-a)$ $+(b-x^l)\log(b-x^l)$	$\mathrm{col}\{\frac{a+b\exp(y^l)}{\exp(y^l)+1}\}_{l=1}^n$
Simplex Δ^n	$\{x\in\mathbb{R}_+^n: \sum_{l=1}^n x^l=1\}$	$\sum_{l=1}^n x^l\log(x^l)$	$\mathrm{col}\{\frac{\exp(y^l)}{\sum_{j=1}^n\exp(y^j)}\}_{l=1}^n$
Euclidean sphere $\mathbf{B}_\rho^n(w)$	$\{x\in\mathbb{R}^n: \|x-w\|_2^2\leq p\}$	$-\sqrt{p^2-\|x-w\|_2^2}$	$py[\sqrt{1+\|y\|_2^2}]^{-1}-w$

To this end, take $\phi_i(x_i)$ and $\varphi_i(\sigma_i)$ as two generating functions, where $\phi_i(x_i)$ is μ_x-strongly convex and L_x-smooth on Ω_i, and $\varphi_i(\sigma_i)$ is μ_σ-strongly convex and L_σ-smooth on $\mathscr{E}_i^+$. It follows from the Fenchel inequality (Diakonikolas & Orecchia, 2019) that the Legendre conjugate ϕ_i^* and φ_i^* are convex and differentiable, where for $y_i\in\mathbb{R}^n$, $\phi_i^*(y_i)\triangleq\min_{x_i\in\Omega_i}\{-x_i^Ty_i+\phi_i(x_i)\}$, and for $\nu_i\in\mathbb{R}^{q_i}$, $\varphi_i^*(\nu_i)\triangleq\min_{\sigma_i\in\mathscr{E}_i^+}\{-\sigma_i^T\nu_i+\varphi_i(\sigma_i)\}$. On this basis, the conjugate gradients satisfy

$$\nabla\phi_i^*(y_i)=\mathrm{argmin}_{x_i\in\Omega_i}\left\{-x_i^Ty_i+\phi_i(x_i)\right\}, \tag{11}$$

$$\nabla\varphi_i^*(\nu_i)=\mathrm{argmin}_{\sigma_i\in\mathscr{E}_i^+}\{-\sigma_i^T\nu_i+\varphi_i(\sigma_i)\}. \tag{12}$$

Indeed, these conjugate gradients can efficiently bring explicit map relations between dual spaces and primal spaces. Recalling Example 1, we can take $\phi_i(x_i)=(x_i-a)\log(x_i-a)+(b-x_i)\log(b-x_i)$ since the feasible set Ω_i has a unit-square form. See Table 1 without subscript i for more details.

Now, we design the conjugate-based ODE by processing two essential issues. One is to design the dynamics for $y_i(t)$ and $\nu_i(t)$ in dual spaces via the stationary conditions in (10), while the other is to update $x_i(t)$ and $\sigma_i(t)$ in primal spaces via the mapping from conjugate gradients of generating functions. Here, t represents continuous time, and in the following, we drop t in dynamics $x_i(t)$ and so on for concise expressions. Therefore, for $i\in\mathcal{I}$, the conjugate-based ODE for seeking a global NE of the nonconvex N-player game (1) is presented by

$$\begin{cases}\dot{y}_i=-\sigma_i^T\nabla_{x_i}\Lambda_i\left(x_i,\boldsymbol{x}_{-i}\right)+\nabla\phi_i(x_i)-y_i, & y_i(0)=y_{i0}\in\mathbb{R}^n,\\ \dot{\nu}_i=\Lambda_i\left(x_i,\boldsymbol{x}_{-i}\right)-\nabla\Psi_i^*\left(\sigma_i\right)+\nabla\varphi_i(\sigma_i)-\nu_i, & \nu_i(0)=\nu_{i0}\in\mathbb{R}^{q_i},\\ x_i=\nabla\phi_i^*\left(y_i\right), & x_i(0)=\nabla\phi_i^*(y_{i0}),\\ \sigma_i=\nabla\varphi_i^*\left(\nu_i\right), & \sigma_i(0)=\nabla\varphi_i^*(\nu_{i0}).\end{cases} \tag{13}$$

In (13), the terms about $-\sigma_i^T\nabla_{x_i}\Lambda_i(x_i,\boldsymbol{x}_{-i})$ and $\Lambda_i(x_i,\boldsymbol{x}_{-i})-\nabla\Psi_i^*(\sigma_i)$ represent the directions of gradient descent and ascent according to Γ_i. Besides, $\nabla\phi_i(x_i)$ and $\nabla\varphi_i(\sigma_i)$ are regarded as damping terms in ODE to avoid y_i and ν_i going to infinity (Nemirovskij & Yudin, 1983; Krichene et al., 2015). With the help of conjugate gradients $\nabla\phi_i^*$ and $\nabla\varphi_i^*$, the mapping from dual spaces back to primal spaces is implemented to obtain the output feedback in system updating. Particularly, these conjugate mappings are established based on valid generating functions rather than a conventional Euclidean norm, which makes the conjugate-based ODE (13) flexibly employed under diverse constraint conditions.

Next, we investigate the variables' trajectories of (13). Similarly to $\boldsymbol{x}$ and $\boldsymbol{\sigma}$, compactly denote $\boldsymbol{y}\in\mathbb{R}^{nN}$ and $\boldsymbol{\nu}\in\mathbb{R}^q$. The following lemma shows a relationship between the equilibrium of (13) and the global NE of (1).

Lemma 2 *Suppose that $(\boldsymbol{y}^\diamondsuit,\boldsymbol{\nu}^\diamondsuit,\boldsymbol{x}^\diamondsuit,\boldsymbol{\sigma}^\diamondsuit)$ is an equilibrium point of ODE (13). If $\sigma_i^\diamondsuit=\nabla\Psi_i\left(\xi_i\right)|_{\xi_i=\Lambda_i\left(x_i^\diamondsuit,\boldsymbol{x}_{-i}^\diamondsuit\right)}$ for $i\in\mathcal{I}$, then $\boldsymbol{x}^\diamondsuit$ is the global NE of (1).*

Then the subsequent theorem presents the main convergence result of (13).

Theorem 2 *If $\mathscr{E}_i^+$ is nonempty for $i\in\mathcal{I}$, then ODE (13) is bounded and convergent. Moreover, if the convergent point $(\boldsymbol{y}^\diamondsuit,\boldsymbol{\nu}^\diamondsuit,\boldsymbol{x}^\diamondsuit,\boldsymbol{\sigma}^\diamondsuit)$ satisfies $\sigma_i^\diamondsuit=\nabla\Psi_i\left(\xi_i\right)|_{\xi_i=\Lambda_i\left(x_i^\diamondsuit,\boldsymbol{x}_{-i}^\diamondsuit\right)}$ for $i\in\mathcal{I}$, then $\boldsymbol{x}^\diamondsuit$ is the global NE of (1).*

Proved in § E, Theorem 2 implies that global NE of game (1) can be found along the trajectory of conjugate-based ODE (13). Moreover, we give the convergence rate of (13) with its proof in § E.

Theorem 3 *If $\mathscr{E}_i^+$ is nonempty and $\Psi_i(\cdot)$ is $\frac{1}{\kappa_\sigma}$-smooth for $i \in \mathcal{I}$, then (13) converges at an exponential rate, i.e.,*

$$\|\boldsymbol{z}(t) - \boldsymbol{z}^\Diamond\| \le \sqrt{\frac{\tau}{\mu}}\|\boldsymbol{z}(0)\| \exp(-\frac{\kappa}{2\tau}\, t),$$

where $\mu = \min\{\mu_x/2, \mu_\sigma/2\}$, $\kappa = \min\{\kappa_\sigma, \kappa_x\}$, $\tau = \max\{L_x/2\mu_x, L_\sigma/2\mu_\sigma\}$.

5 Discretization

As the establishment of the ODE above, we utilize the conjugate gradient information from dual spaces and obtain global NE according to the guidelines in the VI perspective. Following these instructive ideas, we consider deriving the discretization from the conjugate-based ODE. Notice that each step in discrete algorithms can directly compute the minimum of a sub-problem, rather than walks along some trajectories of conjugate functions with explicit expressions in the ODE. Tapping into this advantage, the corresponding discrete algorithm is not obligated to resort to conjugate information like y_i and ν_i, which results in simplifying the algorithm iteration. Specifically, without computing the conjugate information of Ψ_i^*, we redefine an operator generated by Ψ_i on Θ_i as

$$\Pi_{\Theta_i}^{\Psi_i}(\sigma_i) = \operatorname{argmin}_{\xi_i \in \Theta_i}\{-\sigma_i^T \xi_i + \Psi_i(\xi_i)\}.$$

Similarly, redefine operators $\Pi_{\Omega_i}^{\phi_i}(\cdot) = \nabla\phi_i^*(\cdot)$ in (11) and $\Pi_{\mathscr{E}_i^+}^{\varphi_i}(\cdot) = \nabla\varphi_i^*(\cdot)$ in (12). With a step size α_k at discrete time k, we discretize the conjugate-based ODE (13) in the following Algorithm 1.

Algorithm 1

Input: Step size $\{\alpha_k\}$, proper generating functions ϕ_i on Ω_i and φ_i on $\mathscr{E}_i^+$.
Initialize: $x_i^0 \in \Omega_i, \sigma_i^0 \in \mathscr{E}_i^+$, $i \in \{1, \dots, N\}$.
1: **for** $k = 1, 2, \cdots$ **do**
2:     **for** $i \in \{1, \dots, N\}$ **do**
3:         compute the conjugate of Ψ_i:
        $\xi_i^k = \Pi_{\Theta_i}^{\Psi_i}(\sigma_i^k)$
4:         update the decision variable and the canonical dual variable:
        $x_i^{k+1} = \Pi_{\Omega_i}^{\phi_i}(\nabla\phi_i(x_i^k) - \alpha_k \sigma_i^{kT} \nabla_{x_i}\Lambda_i(x_i^k, \boldsymbol{x}_{-i}^k))$
        $\sigma_i^{k+1} = \Pi_{\mathscr{E}_i^+}^{\varphi_i}(\nabla\varphi_i(\sigma_i^k) + \alpha_k(\Lambda_i\left(x_i^k, \boldsymbol{x}_{-i}^k\right) - \xi_i^k))$
5:     **end for**
6: **end for**

In addition, we learn that the update of x_i^{k+1} in Algorithm 1 can be equivalently expressed as

$$x_i^{k+1} = \operatorname{argmin}_{x \in \Omega_i}\{\langle x, \sigma_i^{kT} \nabla_{x_i}\Lambda_i(x_i^k, \boldsymbol{x}_{-i}^k)\rangle + \frac{1}{\alpha_k} D_{\phi_i}(x, x_i^k)\},$$

where $D_{\phi_i}(x, x_i^k)$ is the Bregman divergence with generating function ϕ_i. A similar equivalent scheme can be found in σ_i^{k+1}. These equivalent iteration schemes reveal that, parts of the idea in Algorithm 1 derived from the conjugate-based ODE (13) actually coincide with the *mirror descent* method (Nemirovskij & Yudin, 1983). Therefore, after computing the conjugate of Ψ_i and plugging it into the update of σ_i referring to properties of canonical functions and VIs, readers may regard Algorithm 1 from mirror descent perspectives based on personal preference and convenience.

Hereinafter, we provide the step-size settings and the corresponding convergence rates of Algorithm 1 in two typical nonconvex N-player games.

N-player generalized monotone games

Monotone games stand for a broad category in game models, where the pseudo-gradients of all players' payoffs satisfy the properties of monotonicity (Facchinei & Kanzow, 2010; Koshal et al., 2016; Chen et al., 2021). The monotone property yields the equivalence between the weak and the strong solutions to VI problems (Minty, 1962), which makes most convex games solvable by the first conditions in VI. Analogously, we consider Algorithm 1 under a class of generalized monotone conditions (Giannessi et al., 2006), referring to the canonical complementary function (8), and are rewarded by the following results.

Theorem 4 *If $\mathscr{E}_i^+$ is nonempty and $\Pi_{\Theta_i}^{\Psi_i}(\cdot)$ is κ_σ-strongly monotone, then Algorithm 1 converges at a rate of $\mathcal{O}(1/k)$ with step size $\alpha_k = 2\kappa^{-1}/(k+1)$, i.e.,*

$$\|\boldsymbol{x}^k - \boldsymbol{x}^\diamond\|^2 + \|\boldsymbol{\sigma}^k - \boldsymbol{\sigma}^\diamond\|^2 \leq \frac{1}{k+1}\frac{M_1}{\mu^2\kappa^2},$$

where $\mu = \min\{\mu_x/2, \mu_\sigma/2\}$, $\kappa = \min\{\kappa_\sigma, \kappa_x\}$, and M_1 is a positive constant.

N-player potential games

Potential games also have a wide spectrum of applications such as power allocation (Yang et al., 2018), congestion control (Lei & Shanbhag, 2020), and multi-target tracking (Soto et al., 2009). In a potential game, there exists a unified potential function for all players such that the change in each players payoff is equivalent to the change in the potential function. Hence, the deviation in the payoff of each player in (2) can be concretely mapped to a uniformed canonical potential function, that is,

$$J_i\left(x_i', \boldsymbol{x}_{-i}\right) - J_i(\boldsymbol{x}) = \Psi(\Lambda(x_i', \boldsymbol{x}_{-i})) - \Psi(\Lambda(\boldsymbol{x})). \tag{14}$$

The complementary function is thereby obtained with a common canonical dual variable σ as

$$\Gamma_i(x_i, \sigma, \boldsymbol{x}_{-i}) = \Gamma(\boldsymbol{x}, \sigma) = \sigma^T \Lambda\left(\boldsymbol{x}\right) - \Psi^*\left(\sigma\right). \tag{15}$$

Also, the set $\mathscr{E}^+$ of σ is in a unified form similar to (9). Considering the weighted averages $\hat{\boldsymbol{x}}^k$ and $\hat{\sigma}^k$ in course of k iterations, we give the convergence rate of Algorithm 1 in the result below.

Theorem 5 *If $\mathscr{E}^+$ is nonempty and players' payoffs are subjected to the potential function in (14), then Algorithm 1 converges at a $\mathcal{O}(1/\sqrt{k})$ rate with step size $\alpha_k = 2{M_2}^{-1}\sqrt{\mu d\,/\,k}$, i.e.,*

$$\Gamma(\hat{\boldsymbol{x}}^k, \sigma^\diamond) - \Gamma(\boldsymbol{x}^\diamond, \hat{\sigma}^k) \leq \frac{1}{\sqrt{k}}\sqrt{\frac{d}{\mu}}M_2,$$

where $\mu = \min\{\mu_x/2, \mu_\sigma/2\}$, and d, M_2 are two positive constants.

6 Experiments

We examine the effectiveness of our approach for seeking global NE in the following tasks.

Robust neural network training

Consider a two-player game in adversarial training (Deng & Mahdavi, 2021; Nouiehed et al., 2019), where the payoffs for both players are given in (4) with regularizers $\frac{\lambda_1}{2}\|x_1\|^2$ and $-\frac{\lambda_2}{2}\|x_2\|^2$.

To show the convergence performance of our approach, we compare Algorithm 1 with several familiar methods based on stationary information, such as the classic gradient descent (GD), the optimistic mirror descent (OMD) (Daskalakis et al., 2018), and the extra-gradient method (EG) (Korpelevich, 1976)). To gain insight into the effectiveness of these methods together with ours, in Fig. 2, we take decision variables $x_1, x_2 \in \mathbb{R}$ and plot their trajectories in this task by starting from the same initial point. It follows from Fig. 2 that only Algorithm 1 advocated in this paper converges to the global NE, while other methods diverge to a Nash stationary point instead.

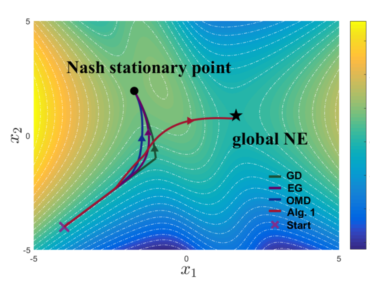


Figure 2: Performance of different methods for seeking global NE.

Sensor localization

Then we verify a class of nonconvex games in sensor localization with $N = 10$ anchor nodes (Jia et al., 2013; Yang et al., 2018). For $i \in \mathcal{I}$, the position strategy set Ω_i is equipped with a unit square form, and the payoff function defined in (3) for localization measurement is accompanied with a deviation of personal estimation $\|D_i x_i - e_i\|^2$. We reformulate this problem with a potential game model, which is similar to the previous works, and make a canonical transformation along the procedure in this paper to handle nonconvexity. As mentioned in (15), $\mathscr{E}^+$ is a polyhedron here due to a common σ after canonical transformation. In Fig. 3, we show the trajectories of all players strategies in Algorithm 1 with respect to one certain dimension. This reveals that all players find their appropriate localization on account of the convergence, which actually serves as the desired global NE.

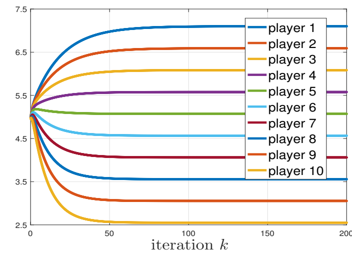
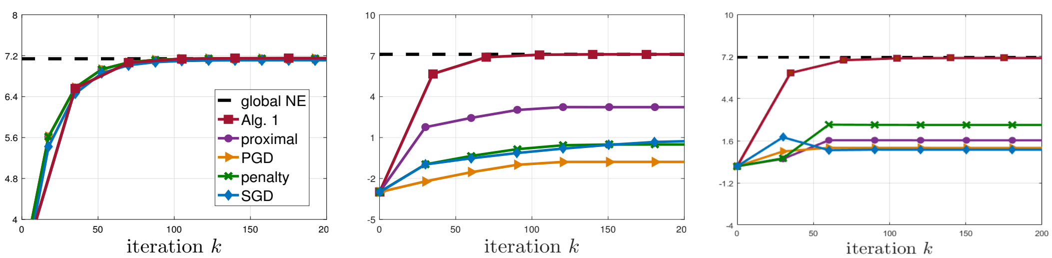


Figure 3: Convergence of all players decisions in Algorithm 1.

For further illustration in this task, we compare Algorithm 1 with several existing methods for solving such N-player games, including projected gradient descent (PGD) (Chen et al., 2021), penalty-based methods (Facchinei & Kanzow, 2010), stochastic gradient descent (SGD) (Mertikopoulos & Zhou, 2019), and gradient-proximal methods (Liu et al., 2020). We check the convergence results of these diverse approaches in the view of a fixed player's decision in Fig. 3. Under such nonconvex settings, an effective algorithm for seeking global NE should be insusceptible wherever the initial point lies. Hence, it follows from Fig. 4 that only our algorithm achieves the target, while others fail with varied initial points.



(a) $x_{11}(0) = 3$ (b) $x_{11}(0) = -3$ (c) $x_{11}(0) = -0.1$

Figure 4: Comparison of convergence results with different initial points.

7 Discussion

We have considered a typical class of nonconvex N-player games, and discussed how to seek their global NE. By virtue of canonical duality theory and VIs, we have proposed a conjugate-based ODE for obtaining the solution of a transformed VI problem, which actually induces the global NE of the original nonconvex game if the duality relation can be checked. After providing theoretical convergence guarantees of the ODE, we have derived the discretization, as well as the step-size settings and the corresponding convergence rates under two typical nonconvex conditions.

Our exploration does not cease to advance, because there will be abundant follow-up work for enhancement and elaboration on the basis of this work. In terms of the convergence rate, proper accelerate approaches may be combined for promising results; In terms of the N-player background, players' interaction may rely on a communication network in consideration of privacy and security, which may suggest the necessity for distributed or decentralized protocol.

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Supplementary Materials

A CANONICAL DUALITY THEORY

We begin the supplementary of this paper with the following fundamental concepts of canonical duality theory. A differentiable function $\Psi : \Theta \to \mathbb{R}$ is said to be a canonical function if its derivative $\nabla\Psi : \Theta \to \Theta^*$ is a one-to-one mapping. Besides, if Ψ is a convex canonical function, its conjugate function $\Psi^* : \Theta^* \to \mathbb{R}$ can be uniquely defined by the Legendre transformation, that is,

$$\Psi^*(\sigma) = \left\{ \xi^T\sigma - \Psi(\xi) \mid \sigma = \nabla\Psi(\xi) \right\},$$

where $\sigma \in \Theta^*$ is a canonical dual variable. On this basis, there are corresponding canonical duality relations holding on $\Theta \times \Theta^*$:

$$\begin{aligned} & \sigma = \nabla\Psi(\xi), \\ \Leftrightarrow \quad & \xi = \nabla\Psi^*(\sigma), \\ \Leftrightarrow \quad & \xi^T\sigma = \Psi(\xi) + \Psi^*(\sigma). \end{aligned}$$

Here, (ξ, σ) is called the Legendre canonical duality pair on $\Theta \times \Theta^*$.

B PROOF OF LEMMA 1

Lemma 1 investigates the relationship of stationary points between (8) and (1). Here we reclaim Lemma 1 for convenience.

Lemma A profile $\boldsymbol{x}^\lozenge$ is a Nash stationary point of (1) if there exists $\boldsymbol{\sigma}^\lozenge \in \boldsymbol{\Theta}^*$, such that for all $i \in \mathcal{I}$, $(x_i^\lozenge, \sigma_i^\lozenge)$ is a stationary point of complementarity function $\Gamma_i(x_i, \sigma_i, \boldsymbol{x}_{-i}^\lozenge)$.

Proof For a given strategy profile $\boldsymbol{x}^\lozenge$, if there exists $\boldsymbol{\sigma}^\lozenge \in \boldsymbol{\Theta}^*$ such that for all $i \in \mathcal{I}$, $(x_i^\lozenge, \sigma_i^\lozenge)$ is a stationary point of complementarity function $\Gamma_i(x_i, \sigma_i, \boldsymbol{x}_{-i}^\lozenge)$, then it satisfies the following first order conditions:

$$\mathbf{0}_n \in \sigma_i^{\lozenge T} \nabla_{x_i} \Lambda_i(x_i^\lozenge, \boldsymbol{x}_{-i}^\lozenge) + \mathcal{N}_{\Omega_i}(x_i^\lozenge), \tag{16a}$$

$$\mathbf{0}_{q_i} \in -\Lambda_i(x_i^\lozenge, \boldsymbol{x}_{-i}^\lozenge) + \nabla\Psi_i^*(\sigma_i^\lozenge) + \mathcal{N}_{\Theta_i^*}(\sigma_i^\lozenge), \tag{16b}$$

where $\mathcal{N}_{\Omega_i}(x_i^\lozenge)$ is the normal cone at point $x_i^\lozenge$ on set Ω_i, with a similar definition for the normal cone $\mathcal{N}_{\Theta_i^*}(\sigma_i^\lozenge)$. Following the definition of the convex canonical function Ψ_i, we can learn that its derivative $\nabla\Psi_i : \Theta_i \to \Theta_i^*$ is a one-to-one mapping from Θ_i to its range Θ_i^*. Thus, for given $\xi_i^\lozenge \in \Theta_i$ with $\xi_i^\lozenge = \Lambda_i(x_i^\lozenge, \boldsymbol{x}_{-i}^\lozenge)$, there exists a unique $\sigma_i^\lozenge \in \Theta_i^*$ such that

$$\sigma_i^\lozenge = \nabla\Psi_i(\xi_i^\lozenge).$$

Meanwhile, given this Legendre canonical duality pair $(\xi_i^\lozenge, \sigma_i^\lozenge)$ on $\Theta_i \times \Theta_i^*$, the duality relation holds that

$$\sigma_i^\lozenge = \nabla\Psi_i(\xi_i^\lozenge) \iff \xi_i^\lozenge = \nabla\Psi_i^*(\sigma_i^\lozenge).$$

With all this in mind, (16b) can be transformed into

$$\Lambda_i(x_i^\lozenge, \boldsymbol{x}_{-i}^\lozenge) = \nabla\Psi_i^*(\sigma_i^\lozenge). \tag{17}$$

Using the the duality relation again, (17) is equivalent to

$$\sigma_i^\Diamond = \nabla\Psi_i(\Lambda_i(x_i^\Diamond, \boldsymbol{x}_{-i}^\Diamond)). \tag{18}$$

By substituting (18) into (16a), we have

$$\mathbf{0}_n \in \nabla\Psi_i(\Lambda_i(x_i^\Diamond, \boldsymbol{x}_{-i}^\Diamond))^T \nabla_{x_i}\Lambda_i(x_i^\Diamond, \boldsymbol{x}_{-i}^\Diamond) + \mathcal{N}_{\Omega_i}(x_i^\Diamond). \tag{19}$$

According to the chain rule,

$$\nabla\Psi_i(\Lambda_i(x_i^\Diamond, \boldsymbol{x}_{-i}^\Diamond))^T \nabla_{x_i}\Lambda_i(x_i^\Diamond, \boldsymbol{x}_{-i}^\Diamond) = \nabla_{x_i} J_i(x_i^\Diamond, \boldsymbol{x}_{-i}^\Diamond).$$

Therefore, (19) is equivalent to

$$\mathbf{0}_n \in \nabla_{x_i} J_i(x_i^\Diamond, \boldsymbol{x}_{-i}^\Diamond) + \mathcal{N}_{\Omega_i}(x_i^\Diamond). \tag{20}$$

Since (20) is true for any player $i \in \mathcal{I}$, the profile $\boldsymbol{x}^\Diamond$ satisfies the Nash stationary condition, which completes the proof. □

C Variational inequality

Recall the following notations according to problem (8)

$$\boldsymbol{z} = \mathrm{col}\{\boldsymbol{x}, \boldsymbol{\sigma}\}, \quad \Xi = \boldsymbol{\Omega} \times \mathscr{E}^+ \subset \mathbb{R}^{nN+q}.$$

For the conjugate gradient of canonical function Ψ_i for $i \in \mathcal{I}$, denote

$$\nabla\Psi^*(\boldsymbol{\sigma}) = \mathrm{col}\{\nabla\Psi_i^*(\sigma_i)\}_{i=1}^N.$$

Also, denote the profile of all Λ_i by

$$\Lambda(\boldsymbol{x}) = \mathrm{col}\left\{\Lambda_i(x_i, \boldsymbol{x}_{-i})\right\}_{i=1}^N,$$

and the augmented partial derivative profile as

$$G(\boldsymbol{x}, \boldsymbol{\sigma}) = \mathrm{col}\left\{\sigma_i^T \nabla_{x_i}\Lambda_i(x_i, \boldsymbol{x}_{-i})\right\}_{i=1}^N.$$

In this way, the pseudo-gradient of (8) can be rewritten as

$$F(\boldsymbol{z}) \triangleq \begin{bmatrix} G(\boldsymbol{x}, \boldsymbol{\sigma}) \\ -\Lambda(\boldsymbol{x}) + \nabla\Psi^*(\boldsymbol{\sigma}) \end{bmatrix} = \begin{bmatrix} \mathrm{col}\{\sum_{k=1}^{q_i}[\sigma_i]_k \nabla_{x_i}\Lambda_{i,k}(x_i, \boldsymbol{x}_{-i})\}_{i=1}^N \\ \mathrm{col}\left\{-\Lambda_i(x_i, \boldsymbol{x}_{-i}) + \nabla\Psi_i^*(\sigma_i)\right\}_{i=1}^N \end{bmatrix}. \tag{21}$$

To proceed, the accordingly introduced variational inequality (VI) problem $\mathrm{VI}(\Xi, F)$ is defined as

$$\text{to find} \quad \boldsymbol{z} \in \Xi \quad \text{such that} \quad (\boldsymbol{z}' - \boldsymbol{z})^T F(\boldsymbol{z}) \geq 0, \quad \forall \boldsymbol{z}' \in \Xi. \tag{22}$$

The solution of this VI problem is denoted by $\mathrm{SOL}(\Xi, F)$. Moreover, since $F(\boldsymbol{z})$ is a continuous mapping and Ξ is a closed set, we have the following result referring to (Facchinei & Pang, 2003, Page 2-3).

Lemma C1 *The solution set* $\mathrm{SOL}(\Xi, F)$ *of* $\mathrm{VI}(\Xi, F)$ *in (22) is closed. Moreover, any profile* $\boldsymbol{z} \in \mathrm{SOL}(\Xi, F)$ *if and only if*

$$\mathbf{0}_{nN+q} \in F(\boldsymbol{z}) + \mathcal{N}_\Xi(\boldsymbol{z}).$$

D PROOF OF THEOREM 1

With the help of the preparation aforementioned about the canonical duality theory and the property in VI problems, we can give the proof of Theorem 1. Here, Theorem 1 reveals that solutions to VI(Ξ, F) yields the global NE of (1). Also, we reproduce Theorem 1 here for convenience.

Theorem If $(\boldsymbol{x}^\diamondsuit, \boldsymbol{\sigma}^\diamondsuit)$ is a solution to VI(Ξ, F) with $\sigma_i^\diamondsuit = \nabla\Psi_i\left(\xi_i\right)|_{\xi_i=\Lambda_i\left(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit\right)}$ for $i \in \mathcal{I}$, then $\boldsymbol{x}^\diamondsuit$ is the global NE of the nonconvex game (1).

Proof If there exists $\boldsymbol{\sigma}^\diamondsuit \in \mathscr{E}^+$ such that $\boldsymbol{z}^\diamondsuit = \mathrm{col}\{\boldsymbol{x}^\diamondsuit, \boldsymbol{\sigma}^\diamondsuit\}$ is a solution to VI(Ξ, F), then it follows from Lemma C1 that

$$\mathbf{0}_{nN+q} \in F(\boldsymbol{z}^\diamondsuit) + \mathcal{N}_{\boldsymbol{\Xi}}(\boldsymbol{z}^\diamondsuit), \tag{23}$$

which implies that for $i \in \mathcal{I}$,

$$\begin{aligned} &\mathbf{0}_n \in \sigma_i^{\diamondsuit T}\nabla_{x_i}\Lambda_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit) + \mathcal{N}_{\Omega_i}(x_i^\diamondsuit), \\ &\mathbf{0}_{q_i} \in -\Lambda_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit) + \nabla\Psi_i^*(\sigma_i^\diamondsuit) + \mathcal{N}_{\mathscr{E}_i^+}(\sigma_i^\diamondsuit), \end{aligned}$$

or equivalently described as

$$\begin{aligned} (\sigma_i^{\diamondsuit T}\nabla_{x_i}\Lambda_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit))^T(x_i - x_i^\diamondsuit) \geq 0, \quad \forall x_i \in \Omega_i, \\ (-\Lambda_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit) + \nabla\Psi_i^*(\sigma_i^\diamondsuit))^T(\sigma_i - \sigma_i^\diamondsuit) \geq 0, \quad \forall \sigma_i \in \mathscr{E}_i^+. \end{aligned} \tag{24}$$

Moreover, if $\sigma_i^\diamondsuit = \nabla\Psi_i\left(\xi_i\right)|_{\xi_i=\Lambda_i\left(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit\right)}$ for $i \in \mathcal{I}$, then the the canonical duality relation hold on $\Theta_i \times \mathscr{E}_i^+$ for $i \in \mathcal{I}$. This indicates that the solution to VI(Ξ, F) is a stationary point profile of (8) on $\Theta_i \times \Theta_i^*$.

Thus, similar to the chain rules employed in Lemma 1, we can further derive that

$$(\nabla_{x_i} J_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit))^T(x_i - x_i^\diamondsuit) \geq 0, \quad \forall x_i \in \Omega_i.$$

Moreover, when $\sigma_i \in \mathscr{E}_i^+$, the Hessian matrix satisfies

$$\nabla_{x_i}^2\Gamma_i(x_i, \sigma_i, \boldsymbol{x}_{-i}) = \sum\nolimits_{k=1}^{q_i}[\sigma_i]_k\nabla_{x_i}^2\Lambda_{i,k}(x_i, \boldsymbol{x}_{-i}) \succeq \kappa_x\boldsymbol{I}_n,$$

which indicates the convexity of $\Gamma_i(x_i, \sigma_i, \boldsymbol{x}_{-i})$ with respect to x_i. Besides, due to the convexity of Ψ_i, its Legendre conjugate Ψ_i^* is also convex (Yang Gao, 2000). Therefore, the total complementary function $\Gamma_i(x_i, \sigma_i, \boldsymbol{x}_{-i})$ is concave in canonical dual variable σ_i.

In this light, we can obtain the globally optimality of $(\boldsymbol{x}^\diamondsuit, \boldsymbol{\sigma}^\diamondsuit)$ on $\boldsymbol{\Omega} \times \mathscr{E}^+$, i.e., for $i \in \mathcal{I}$,

$$\Gamma_i(x_i^\diamondsuit, \sigma_i, \boldsymbol{x}_{-i}^\diamondsuit) \leq \Gamma_i(x_i^\diamondsuit, \sigma_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit) \leq \Gamma_i(x_i, \sigma_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit), \quad \forall x_i \in \Omega_i,\ \sigma_i \in \mathscr{E}_i^+.$$

The inequality relation above tells that

$$J_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit) \leq J_i(x_i, \boldsymbol{x}_{-i}^\diamondsuit), \quad \forall x_i \in \Omega_i, \quad \forall i \in \mathcal{I}.$$

This confirms that $\boldsymbol{x}^\diamondsuit$ is the global NE of (1), which completes the proof. □

E CONVERGENCE ANALYSIS OF THE CONJUGATE-BASED ODE

We first provide some preliminaries that are necessary in the convergence analysis of ODE (13). Here are some widely-accepted concepts in convex analysis. Take $h(z) : \Xi \to \mathbb{R}$ as a differentiable ω-strongly convex function on a closed convex set Ξ, which satisfies

$$h\left(\theta z+(1-\omega) z^{\prime}\right) \leq \theta h(z)+(1-\theta) h\left(z^{\prime}\right)-\frac{\omega}{2} \theta(1-\theta)\left\|z^{\prime}-z\right\|^{2}, \quad \forall z, z^{\prime} \in \Xi,\, \theta \in[0,1].$$

Additionally, h is said to be L-smooth if there exists a constant $L > 0$ such that ∇h is L-Lipschitz, i.e.,

$$\|\nabla h(z')-\nabla h(z)\| \leq L\|z-z'\|, \quad \forall z, z' \in \Xi,$$

which is equivalent to

$$h\left(z^{\prime}\right)-h(z) \leq\left(z^{\prime}-z\right)^{\mathrm{T}} \nabla h(z)+\frac{L}{2}\left\|z-z^{\prime}\right\|^{2}, \quad \forall z, z^{\prime} \in \Xi.$$

On the other hand, following the duality theory (Nemirovskij & Yudin, 1983), the conjugate function of h defined on the dual space Ξ^* is

$$h^*(s)=\sup_{z \in \Xi}\left\{z^T s-h(z)\right\},$$

where $s \in \Xi^*$ serves as a dual variable. Moreover, consider h as a differentiable and strongly convex function on a closed convex set Ξ. Then according to (Diakonikolas & Orecchia, 2019), h^* is also convex and differentiable on Ξ^*, and satisfies

$$h^*(s)=\min_{z \in \Xi}\left\{-z^T s+h(z)\right\}.$$

Moreover, the conjugate gradient $\nabla h^*(s)$ who maps Ξ^* to Ξ satisfies

$$\nabla h^*(s)=\operatorname{argmin}_{z \in \Xi}\left\{-z^T s+h(z)\right\}.$$

With these preliminaries at hand, we investigate the convergence of ODE (13). For simplicity, let us denote the following compact forms associated with the gradients therein

$$\nabla \boldsymbol{\phi}(\boldsymbol{x}) \triangleq \operatorname{col}\left\{\nabla \phi_i\left(x_i\right)\right\}_{i=1}^N, \quad \nabla \boldsymbol{\varphi}(\boldsymbol{\sigma})=\operatorname{col}\{\nabla \varphi_i(\sigma_i)\}_{i=1}^N;$$

$$\nabla \boldsymbol{\phi}^*(\boldsymbol{y})=\operatorname{col}\{\nabla \phi_i^*\left(y_i\right)\}_{i=1}^N, \quad \nabla \boldsymbol{\varphi}^*(\boldsymbol{\nu})=\operatorname{col}\{\nabla \varphi_i^*\left(\nu_i\right)\}_{i=1}^N.$$

Hence, together with the compact forms $G(\boldsymbol{x}, \boldsymbol{\sigma})$, $\Lambda(\boldsymbol{x})$, and $\nabla\Psi^*(\boldsymbol{\sigma})$ defined in (21), ODE (13) can be compactly presented by

$$\begin{cases} \dot{\boldsymbol{y}}=-G(\boldsymbol{x}, \boldsymbol{\sigma})+\nabla \boldsymbol{\phi}(\boldsymbol{x})-\boldsymbol{y}, \\ \dot{\boldsymbol{\nu}}=\Lambda(\boldsymbol{x})-\nabla \Psi^*(\boldsymbol{\sigma})+\nabla \boldsymbol{\varphi}(\boldsymbol{\sigma})-\boldsymbol{\nu}, \\ \boldsymbol{x}=\nabla \boldsymbol{\phi}^*(\boldsymbol{y}), \\ \boldsymbol{\sigma}=\nabla \boldsymbol{\varphi}^*(\boldsymbol{\nu}). \end{cases} \tag{25}$$

On this basis, we first show a relationship between the equilibrium in ODE (25) (or ODE (13)) and the global NE of game (1). Rewrite Lemma 2 here for conveniency.

Lemma If $(\boldsymbol{y}^\diamondsuit, \boldsymbol{\nu}^\diamondsuit, \boldsymbol{x}^\diamondsuit, \boldsymbol{\sigma}^\diamondsuit)$ is an equilibrium point of ODE (25) (or (13)) and $\sigma_i^\diamondsuit = \nabla\Psi_i(\xi_i)\,|_{\xi_i=\Lambda_i\left(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit\right)}$ for $i \in \mathcal{I}$, then $\boldsymbol{x}^\diamondsuit$ is the global NE of (1).

Proof If $(\boldsymbol{y}^\diamondsuit, \boldsymbol{x}^\diamondsuit, \boldsymbol{\nu}^\diamondsuit, \boldsymbol{\sigma}^\diamondsuit)$ is an equilibrium point of ODE (25), we have

$$\mathbf{0}_{nN} = -G(\boldsymbol{x}^\diamondsuit, \boldsymbol{\sigma}^\diamondsuit) + \nabla\boldsymbol{\phi}(\boldsymbol{x}^\diamondsuit) - \boldsymbol{y}^\diamondsuit, \tag{26a}$$
$$\mathbf{0}_q = \Lambda\left(\boldsymbol{x}^\diamondsuit\right) - \nabla\Psi^*\left(\boldsymbol{\sigma}^\diamondsuit\right) + \nabla\boldsymbol{\varphi}(\boldsymbol{\sigma}^\diamondsuit) - \boldsymbol{\nu}^\diamondsuit, \tag{26b}$$
$$\boldsymbol{x}^\diamondsuit = \nabla\boldsymbol{\phi}^*(\boldsymbol{y}^\diamondsuit), \tag{26c}$$
$$\boldsymbol{\sigma}^\diamondsuit = \nabla\boldsymbol{\varphi}^*(\boldsymbol{\nu}^\diamondsuit). \tag{26d}$$

It follows from $\boldsymbol{y}^\diamondsuit = -G(\boldsymbol{x}^\diamondsuit, \boldsymbol{\sigma}^\diamondsuit) + \nabla\boldsymbol{\phi}(\boldsymbol{x}^\diamondsuit)$ that (26c) becomes

$$\boldsymbol{x}^\diamondsuit = \nabla\boldsymbol{\phi}^*(-G(\boldsymbol{x}^\diamondsuit, \boldsymbol{\sigma}^\diamondsuit) + \nabla\boldsymbol{\phi}(\boldsymbol{x}^\diamondsuit)). \tag{27}$$

For $i \in \mathcal{I}$, (27) is equivalent to

$$x_i = \nabla\phi_i^*(-\sigma_i^T\nabla_{x_i}\Lambda_i\left(x_i, \boldsymbol{x}_{-i}\right) + \nabla\phi_i(x_i)).$$

Moreover, by recalling

$$\nabla\phi_i^*(y_i) = \operatorname{argmin}_{x_i\in\Omega_i}\left\{-x_i^T y_i + \phi_i(x_i)\right\},$$

and taking y_i as $-\sigma_i^T\nabla_{x_i}\Lambda_i\left(x_i, \boldsymbol{x}_{-i}\right) + \nabla\phi_i(x_i)$, we obtain the associated first order condition, expressed as the following compact form

$$\mathbf{0}_{nN} \in G(\boldsymbol{x}^\diamondsuit, \boldsymbol{\sigma}^\diamondsuit) + \mathcal{N}_{\boldsymbol{\Omega}}(\boldsymbol{x}^\diamondsuit). \tag{28}$$

Similarly, it follows from (26b) and (26d) that

$$\boldsymbol{\sigma}^\diamondsuit = \nabla\boldsymbol{\varphi}^*(\Lambda\left(\boldsymbol{x}^\diamondsuit\right) - \nabla\Psi^*\left(\boldsymbol{\sigma}^\diamondsuit\right) + \nabla\boldsymbol{\varphi}(\boldsymbol{\sigma}^\diamondsuit)),$$

which yields

$$\mathbf{0}_q \in -\Lambda\left(\boldsymbol{x}^\diamondsuit\right) + \nabla\Psi^*\left(\boldsymbol{\sigma}^\diamondsuit\right) + \mathcal{N}_{\mathscr{E}^+}(\boldsymbol{\sigma}^\diamondsuit). \tag{29}$$

Thus, combining (28) and (29), it follows from Lemma C1 that $\boldsymbol{z}^\diamondsuit = \mathrm{col}\{\boldsymbol{x}^\diamondsuit, \boldsymbol{\sigma}^\diamondsuit\}$ is a solution to $\mathrm{VI}(\boldsymbol{\Xi}, F)$. Moreover, due to Theorem 1, the solution of $\mathrm{VI}(\boldsymbol{\Xi}, F)$ derives a global NE of game (1) if $\sigma_i^\diamondsuit = \nabla\Psi_i\left(\xi_i\right)|_{\xi_i=\Lambda_i\left(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit\right)}$ for $i \in \mathcal{I}$, which completes the proof. □

Now, we are in a position to prove the convergence of conjugate-based ODE (13). Reclaim Theorem 2 here for convenience.

Theorem If $\mathscr{E}_i^+$ is nonempty for $i \in \mathcal{I}$, then ODE (13) is bounded and convergent. Moreover, if the convergent point $(\boldsymbol{y}^\diamondsuit, \boldsymbol{\nu}^\diamondsuit, \boldsymbol{x}^\diamondsuit, \boldsymbol{\sigma}^\diamondsuit)$ satisfies $\sigma_i^\diamondsuit = \nabla\Psi_i\left(\xi_i\right)|_{\xi_i=\Lambda_i\left(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit\right)}$ for $i \in \mathcal{I}$, then $\boldsymbol{x}^\diamondsuit$ is the global NE of (1).

Proof (i) We first prove that the trajectory $(\boldsymbol{y}(t), \boldsymbol{x}(t), \boldsymbol{\nu}(t), \boldsymbol{\sigma}(t))$ of (13) is bounded along ODE (13). Construct a Lyapunov candidate function as

$$V_1 = \sum_{i=1}^{N} D_{\phi_i^*}(y_i, y_i^\diamondsuit) + D_{\varphi_i^*}(\nu_i, \nu_i^\diamondsuit), \tag{30}$$

where Bregman divergences therein are expressed detailedly as

$$D_{\phi_i^*}(y_i, y_i^\diamondsuit) = \phi_i^*(y_i) - \phi_i^*(y_i^\diamondsuit) - \nabla\phi_i^*(y_i^\diamondsuit)^T(y_i - y_i^\diamondsuit),$$
$$D_{\varphi_i^*}(\nu_i, \nu_i^\diamondsuit) = \varphi_i^*(\nu_i) - \varphi_i^*(\nu_i^\diamondsuit) - \nabla\varphi_i^*(\nu_i^\diamondsuit)^T(\nu_i - \nu_i^\diamondsuit).$$

Consider the term $D_{\phi_i^*}(y_i, y_i^\Diamond)$ for $i \in \mathcal{I}$. Since $x_i = \nabla\phi_i^*(y_i)$ and $x_i^\Diamond = \nabla\phi_i^*\left(y_i^\Diamond\right)$, it follows from the expression of $\nabla\phi_i^*$ in (11) that

$$\phi_i^*(y_i) = x_i^T y_i - \phi_i(x_i), \quad \phi_i^*(y_i^\Diamond) = x_i^{\Diamond T} y_i^\Diamond - \phi_i(x_i^\Diamond). \tag{31}$$

Thus, by (31), we get

$$\begin{aligned} D_{\phi_i^*}(y_i, y_i^\Diamond) &= \phi_i^*(y_i) - \phi_i^*(y_i^\Diamond) - \nabla\phi_i^*(y_i^\Diamond)^T(y_i - y_i^\Diamond) \\ &= \phi_i(x_i^\Diamond) - \phi_i(x_i) - (x_i^\Diamond - x_i)^T y_i \\ &= \phi_i(x_i^\Diamond) - \phi_i(x_i) - (x_i^\Diamond - x_i)^T\nabla\phi(x_i) + (x_i^\Diamond - x_i)^T\nabla\phi(x_i) - (x_i^\Diamond - x_i)^T y_i. \end{aligned}$$

Since ϕ_i is μ_x-strongly convex on Ω_i, we can further derive

$$D_{\phi_i^*}(y_i, y_i^\Diamond) \geq \frac{\mu_x}{2}\left\|x_i - x_i^\Diamond\right\|^2 + (x_i^\Diamond - x_i)^T(\nabla\phi(x_i) - y_i),$$

which yields

$$\sum_{i=1}^N D_{\phi_i^*}(y_i, y_i^\Diamond) \geq \frac{\mu_x}{2}\left\|\boldsymbol{x} - \boldsymbol{x}^\Diamond\right\|^2 + \sum_{i=1}^N (x_i^\Diamond - x_i)^T\left(\nabla\phi_i\left(x_i\right) - y_i\right). \tag{32}$$

In fact, recall $\nabla\phi_i^*\left(y_i\right) = \operatorname{argmin}_{x\in\Omega_i}\left\{-x^T y_i + \phi_i(x)\right\}$. Due to the optimality of $\nabla\phi_i^*\left(y_i\right)$ and the convexity of ϕ_i, we have

$$(\nabla\phi_i^*(y_i))^T\left(\nabla\phi_i\left(\nabla\phi_i^*\left(y_i\right)\right) - y_i\right) \leq (\nabla\phi_i^*(y_i^\Diamond))^T\left(\nabla\phi_i\left(\nabla\phi_i^*\left(y_i\right)\right) - y_i\right). \tag{33}$$

Furthermore, in consideration of $x_i = \nabla\phi_i^*\left(y_i\right)$ and $x_i^\Diamond = \nabla\phi_i^*\left(y_i^\Diamond\right)$ again, (33) indicates

$$\begin{aligned} 0 &\leq \nabla\phi_i^*\left(y_i^\Diamond\right)^T\left(\nabla\phi_i\left(\nabla\phi_i^*\left(y_i\right)\right) - y_i\right) - \nabla\phi_i^*\left(y_i\right)^T\left(\nabla\phi_i\left(\nabla\phi_i^*\left(y_i\right)\right) - y_i\right) \\ &= (x_i^\Diamond - x_i)^T\left(\nabla\phi_i\left(x_i\right) - y_i\right). \end{aligned} \tag{34}$$

Thus, (32) becomes

$$\sum_{i=1}^N D_{\phi_i^*}(y_i, y_i^\Diamond) \geq \frac{\mu_x}{2}\left\|\boldsymbol{x} - \boldsymbol{x}^\Diamond\right\|^2.$$

The analogous analysis of the term $D_{\varphi_i^*}(\nu_i, \nu_i^\Diamond)$ in (30) can be carried on, which also indicates that

$$\sum_{i=1}^N D_{\varphi_i^*}(\nu_i, \nu_i^\Diamond) \geq \frac{\mu_\sigma}{2}\left\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^\Diamond\right\|^2 + \sum_{i=1}^N (\sigma_i^\Diamond - \sigma_i)^T\left(\nabla\varphi_i\left(\sigma_i\right) - \nu_i\right).$$

Besides, recall $\nabla\varphi_i^*\left(\nu_i\right) = \operatorname{argmin}_{\sigma_i\in\mathscr{E}_i^+}\{-\sigma_i^T\nu_i + \varphi_i(\sigma_i)\}$. Based on the convexity of φ_i and the optimality of $\nabla\varphi_i^*\left(\nu_i\right)$, we obtain

$$0 \leq (\sigma_i^\Diamond - \sigma_i)^T\left(\nabla\varphi_i\left(\sigma_i\right) - \nu_i\right), \tag{35}$$

which similarly leads to

$$\sum_{i=1}^N D_{\varphi_i^*}(\nu_i, \nu_i^\Diamond) \geq \frac{\mu_\sigma}{2}\left\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^\Diamond\right\|^2.$$

As a result, we obtain the lower bound of (30) that

$$V_1 \geq \mu(\|\boldsymbol{x} - \boldsymbol{x}^\Diamond\|^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}^\Diamond\|^2) \geq 0,$$

where $\mu = \min\{\mu_x/2, \mu_\sigma/2\}$. This means that V_1 is positive semi-definite, and $V_1 = 0$ if and only if $\boldsymbol{x} = \boldsymbol{x}^\diamond$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}^\diamond$. Moreover, V_1 is radially unbounded in $\boldsymbol{x}(t)$ and $\boldsymbol{\sigma}(t)$.

Next, we investigate the derivative of V_1 along ODE (13), that is,

$$\begin{aligned}
\frac{d}{dt}V_1(t) &= \frac{d}{dt}\sum_{i=1}^{N} D_{\phi_i^*}(y_i, y_i^\diamond) + D_{\varphi_i^*}(\nu_i, \nu_i^\diamond) \\
&= \frac{d}{dt}\sum_{i=1}^{N}(\phi_i^*(y_i) - \phi_i^*(y_i^\diamond) - (y_i - y_i^\diamond)^T \nabla\phi_i^*\,(y_i^*)) \\
&\qquad + \frac{d}{dt}\sum_{i=1}^{N}(\varphi_i^*(\nu_i) - \varphi_i^*(\nu_i^\diamond) - (\nu_i - \nu_i^\diamond)^T \nabla\varphi_i^*\,(\nu_i^*)) \\
&= \sum_{i=1}^{N}(\nabla\phi_i^*(y_i) - \nabla\phi_i^*(y_i^\diamond))^T \dot{y}_i(t) + \sum_{i=1}^{N}(\nabla\varphi_i^*(\nu_i) - \nabla\varphi_i^*(\nu_i^\diamond))^T \dot{\nu}_i(t) \\
&= \sum_{i=1}^{N}(x_i - x_i^\diamond)^T \dot{y}_i(t) + \sum_{i=1}^{N}(\sigma_i - \sigma_i^\diamond))^T \dot{\nu}_i(t).
\end{aligned}$$

Here we employ the compact form defined in (25) for a more concise statements below and derive

$$\begin{aligned}
\frac{d}{dt}V_1(t) = (\boldsymbol{x} - \boldsymbol{x}^\diamond)^T(-\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{\sigma}) + \nabla\boldsymbol{\phi}(\boldsymbol{x}) - \boldsymbol{y}) \\
+ \left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^\diamond\right)^T (\Lambda\,(\boldsymbol{x}) - \nabla\Psi^*\,(\boldsymbol{\sigma}) + \nabla\boldsymbol{\varphi}(\boldsymbol{\sigma}) - \boldsymbol{\nu}).
\end{aligned} \tag{36}$$

Meanwhile, by rearranging the terms in (36), we have

$$\begin{aligned}
\dot{V}_1 &= -(\boldsymbol{x} - \boldsymbol{x}^\diamond)^T \boldsymbol{G}(\boldsymbol{x}, \boldsymbol{\sigma}) - \left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^\diamond\right)^T (-\Lambda\,(\boldsymbol{x}) + \nabla\Psi^*\,(\boldsymbol{\sigma})) \\
&\qquad + (\boldsymbol{x} - \boldsymbol{x}^\diamond)^T(\nabla\boldsymbol{\phi}(\boldsymbol{x}) - \boldsymbol{y}) + \left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^\diamond\right)^T (\nabla\boldsymbol{\varphi}(\boldsymbol{\sigma}) - \boldsymbol{\nu}) \\
&= -(\boldsymbol{z} - \boldsymbol{z}^\diamond)^T F(\boldsymbol{z}) + (\boldsymbol{x} - \boldsymbol{x}^\diamond)^T(\nabla\boldsymbol{\phi}(\boldsymbol{x}) - \boldsymbol{y}) + (\boldsymbol{\sigma} - \boldsymbol{\sigma}^\diamond)^T (\nabla\boldsymbol{\varphi}(\boldsymbol{\sigma}) - \boldsymbol{\nu})\,,
\end{aligned} \tag{37}$$

where $\boldsymbol{z} = \mathrm{col}\{\boldsymbol{x}, \boldsymbol{\sigma}\}$ and $F(\boldsymbol{z}) = \mathrm{col}\{G(\boldsymbol{x}, \boldsymbol{\sigma}),\ -\Lambda\,(\boldsymbol{x}) + \nabla\Psi^*\,(\boldsymbol{\sigma})\}$ are defined in (21). Notice that (34) and (35) actually reveals that

$$(\boldsymbol{x} - \boldsymbol{x}^\diamond)^T(\nabla\boldsymbol{\phi}(\boldsymbol{x}) - \boldsymbol{y}) \leq 0, \quad (\boldsymbol{\sigma} - \boldsymbol{\sigma}^\diamond)^T (\nabla\boldsymbol{\varphi}(\boldsymbol{\sigma}) - \boldsymbol{\nu}) \leq 0. \tag{38}$$

Because $\boldsymbol{z}^\diamond$ is a solution to VI(Ξ, F),

$$\left(\boldsymbol{z} - \boldsymbol{z}^\diamond\right)^T F(\boldsymbol{z}^\diamond) \geq 0. \tag{39}$$

Thus, (37) yields the further scaling that

$$\begin{aligned}
\dot{V}_1 &= -(\boldsymbol{z} - \boldsymbol{z}^\diamond)^T F(\boldsymbol{z}) + (\boldsymbol{x} - \boldsymbol{x}^\diamond)^T(\nabla\boldsymbol{\phi}(\boldsymbol{x}) - \boldsymbol{y}) + (\boldsymbol{\sigma} - \boldsymbol{\sigma}^\diamond)^T (\nabla\boldsymbol{\varphi}(\boldsymbol{\sigma}) - \boldsymbol{\nu}) \\
&\leq -(\boldsymbol{z} - \boldsymbol{z}^\diamond)^T F(\boldsymbol{z}) \\
&= -(\boldsymbol{z} - \boldsymbol{z}^\diamond)^T (F(\boldsymbol{z}) - F(\boldsymbol{z}^\diamond)) - (\boldsymbol{z} - \boldsymbol{z}^\diamond)^T F(\boldsymbol{z}^\diamond) \\
&\leq -(\boldsymbol{z} - \boldsymbol{z}^\diamond)^T (F(\boldsymbol{z}) - F(\boldsymbol{z}^\diamond)),
\end{aligned} \tag{40}$$

where the first inequality is due to (38) and the second inequality is due to (39). Now, we consider the term $(\boldsymbol{z}-\boldsymbol{z}^{\diamondsuit})^T(F(\boldsymbol{z})-F(\boldsymbol{z}^{\diamondsuit}))$ with details.

$$\begin{aligned}
&(\boldsymbol{z}-\boldsymbol{z}^{\diamondsuit})^T(F(\boldsymbol{z})-F(\boldsymbol{z}^{\diamondsuit}))\\
&=(\boldsymbol{x}-\boldsymbol{x}^{\diamondsuit})^T(\boldsymbol{G}(\boldsymbol{x},\boldsymbol{\sigma})-\boldsymbol{G}(\boldsymbol{x}^{\diamondsuit},\boldsymbol{\sigma}^{\diamondsuit}))\\
&\qquad+\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}^{\diamondsuit}\right)^T\left(-\Lambda\left(\boldsymbol{x}\right)+\nabla\Psi^*\left(\boldsymbol{\sigma}\right)-\left(-\Lambda(\boldsymbol{x}^{\diamondsuit})+\nabla\Psi^*(\boldsymbol{\sigma}^{\diamondsuit})\right)\right)\\
&=(\boldsymbol{x}-\boldsymbol{x}^{\diamondsuit})^T(\boldsymbol{G}(\boldsymbol{x},\boldsymbol{\sigma})-\boldsymbol{G}(\boldsymbol{x}^{\diamondsuit},\boldsymbol{\sigma}^{\diamondsuit}))-\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}^{\diamondsuit}\right)^T\left(\Lambda(\boldsymbol{x})-\Lambda(\boldsymbol{x}^{\diamondsuit})\right)\\
&\qquad+(\boldsymbol{\sigma}-\boldsymbol{\sigma}^{\diamondsuit})^T(\nabla\Psi^*(\boldsymbol{\sigma})-\nabla\Psi^*(\boldsymbol{\sigma}^{\diamondsuit})).
\end{aligned}$$

Due to the convexity of Ψ_i for $i\in\mathcal{I}$, the Legendre conjugate Ψ_i^* is also convex (Yang Gao, 2000), which indicates

$$(\boldsymbol{\sigma}-\boldsymbol{\sigma}^{\diamondsuit})^T(\nabla\Psi^*(\boldsymbol{\sigma})-\nabla\Psi^*(\boldsymbol{\sigma}^{\diamondsuit}))\geq 0.$$

Hence,

$$\begin{aligned}
&(\boldsymbol{z}-\boldsymbol{z}^{\diamondsuit})^T(F(\boldsymbol{z})-F(\boldsymbol{z}^{\diamondsuit}))\\
&\geq(\boldsymbol{x}-\boldsymbol{x}^{\diamondsuit})^T(\boldsymbol{G}(\boldsymbol{x},\boldsymbol{\sigma})-\boldsymbol{G}(\boldsymbol{x}^{\diamondsuit},\boldsymbol{\sigma}^{\diamondsuit}))-\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}^{\diamondsuit}\right)^T\left(\Lambda(\boldsymbol{x})-\Lambda(\boldsymbol{x}^{\diamondsuit})\right).
\end{aligned}\tag{41}$$

Expanding the expression in (41),

$$\begin{aligned}
&(\boldsymbol{x}-\boldsymbol{x}^{\diamondsuit})^T(\boldsymbol{G}(\boldsymbol{x},\boldsymbol{\sigma})-\boldsymbol{G}(\boldsymbol{x}^{\diamondsuit},\boldsymbol{\sigma}^{\diamondsuit}))-\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}^{\diamondsuit}\right)^T\left(\Lambda(\boldsymbol{x})-\Lambda(\boldsymbol{x}^{\diamondsuit})\right)\\
&=\sum_{i=1}^{N}(x_i-x_i^{\diamondsuit})^T\left(\sum_{k=1}^{q_i}[\sigma_i]_k\nabla_{x_i}\Lambda_{i,k}(x_i,\boldsymbol{x}_{-i})-\sum_{k=1}^{q_i}[\sigma_i^{\diamondsuit}]_k\nabla_{x_i}\Lambda_{i,k}(x_i^{\diamondsuit},\boldsymbol{x}_{-i}^{\diamondsuit})\right)\\
&\qquad-\sum_{i=1}^{N}\sum_{k=1}^{q_i}([\sigma_i]_k-[\sigma_i^{\diamondsuit}]_k)\left(\Lambda_{i,k}\left(x_i,\boldsymbol{x}_{-i}\right)-\Lambda_{i,k}(x_i^{\diamondsuit},\boldsymbol{x}_{-i}^{\diamondsuit})\right).
\end{aligned}\tag{42}$$

Rearranging (42), we have

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{k=1}^{q_i}\left([\sigma_i]_k(x_i-x_i^{\diamondsuit})^T\nabla_{x_i}\Lambda_{i,k}(x_i,\boldsymbol{x}_{-i})-[\sigma_i^{\diamondsuit}]_k(x_i-x_i^{\diamondsuit})^T\nabla_{x_i}\Lambda_{i,k}(x_i^{\diamondsuit},\boldsymbol{x}_{-i}^{\diamondsuit})\right)\\
&\qquad-\sum_{i=1}^{N}\sum_{k=1}^{q_i}([\sigma_i]_k-[\sigma_i^{\diamondsuit}]_k)\left(\Lambda_{i,k}\left(x_i,\boldsymbol{x}_{-i}\right)-\Lambda_{i,k}(x_i^{\diamondsuit},\boldsymbol{x}_{-i}^{\diamondsuit})\right).
\end{aligned}\tag{43}$$

By merging like terms in (42), we have

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{k=1}^{q_i}\left((x_i-x_i^{\diamondsuit})^T([\sigma_i]_k\nabla_{x_i}\Lambda_{i,k}(x_i,\boldsymbol{x}_{-i}))+[\sigma_i]_k\Lambda_{i,k}(x_i^{\diamondsuit},\boldsymbol{x}_{-i}^{\diamondsuit})-[\sigma_i]_k\Lambda_{i,k}(x_i,\boldsymbol{x}_{-i})\right)\\
&\quad+\sum_{i=1}^{N}\sum_{k=1}^{q_i}\left((x_i^{\diamondsuit}-x_i)^T([\sigma_i^{\diamondsuit}]_k\nabla_{x_i}\Lambda_{i,k}(x_i^{\diamondsuit},\boldsymbol{x}_{-i}^{\diamondsuit}))+[\sigma_i^{\diamondsuit}]_k\Lambda_{i,k}(x_i,\boldsymbol{x}_{-i})-[\sigma_i^{\diamondsuit}]_k\Lambda_{i,k}(x_i^{\diamondsuit},\boldsymbol{x}_{-i}^{\diamondsuit})\right).
\end{aligned}\tag{44}$$

Recalling the definition in (9) that for $i\in\mathcal{I}$,

$$\sigma_i,\sigma_i^{\diamondsuit}\in\mathscr{E}_i^{+}=\{\sigma_i\in\Theta_i^*:\ \sum_{k=1}^{q_i}[\sigma_i]_k\nabla_{x_i}^2\Lambda_{i,k}(x_i,\boldsymbol{x}_{-i})\succeq\kappa_x\boldsymbol{I}_n\}.$$

Hence, (44) satisfies

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{k=1}^{q_i}\left((x_i-x_i^{\diamondsuit})^T([\sigma_i]_k\nabla_{x_i}\Lambda_{i,k}(x_i,\boldsymbol{x}_{-i}))+[\sigma_i]_k\Lambda_{i,k}(x_i^{\diamondsuit},\boldsymbol{x}_{-i}^{\diamondsuit})-[\sigma_i]_k\Lambda_{i,k}(x_i,\boldsymbol{x}_{-i})\right)\\
&\quad+\sum_{i=1}^{N}\sum_{k=1}^{q_i}\left((x_i^{\diamondsuit}-x_i)^T([\sigma_i^{\diamondsuit}]_k\nabla_{x_i}\Lambda_{i,k}(x_i^{\diamondsuit},\boldsymbol{x}_{-i}^{\diamondsuit}))+[\sigma_i^{\diamondsuit}]_k\Lambda_{i,k}(x_i,\boldsymbol{x}_{-i})-[\sigma_i^{\diamondsuit}]_k\Lambda_{i,k}(x_i^{\diamondsuit},\boldsymbol{x}_{-i}^{\diamondsuit})\right)\\
&\geq\kappa_x\|\boldsymbol{x}-\boldsymbol{x}^{\diamondsuit}\|^2.
\end{aligned}$$

which further yields

$$(\boldsymbol{z}-\boldsymbol{z}^{\diamond})^T(F(\boldsymbol{z})-F(\boldsymbol{z}^{\diamond}))\geq\kappa_x\|\boldsymbol{x}-\boldsymbol{x}^{\diamond}\|^2.$$

In this view, we can accordingly get

$$\dot{V}_1\leq-\kappa_x\|\boldsymbol{x}-\boldsymbol{x}^{\diamond}\|^2\leq 0. \tag{45}$$

Since V_1 is radially unbounded in $\boldsymbol{x}(t)$ and $\boldsymbol{\sigma}(t)$, this implies that the trajectories of $\boldsymbol{x}(t)$ and $\boldsymbol{\sigma}(t)$ are bounded along the conjugate-based ODE (13).

Secondly, we show that $\boldsymbol{y}(t)$ and $\boldsymbol{\nu}(t)$ are bounded. Take another Lyapunov candidate function as

$$V_2=\frac{1}{2}\|\boldsymbol{y}\|^2,$$

which is radially unbounded in $\boldsymbol{y}$. Along the trajectories of (25), the derivative of V_2 satisfies

$$\dot{V}_2\leq\boldsymbol{y}^T(-\boldsymbol{G}(\boldsymbol{x},\boldsymbol{\sigma})+\nabla\Phi(\boldsymbol{x}))-\|\boldsymbol{y}\|^2.$$

It is clear that

$$\begin{aligned}\dot{V}_2&\leq-\|\boldsymbol{y}\|^2+p_1\|\boldsymbol{y}\|\\&=-2V_2+p_1\sqrt{2V_2},\end{aligned}$$

with a positive constant p_1, which is because $\boldsymbol{x}$, $\boldsymbol{\sigma}$ have been proved to be bounded. Analogously, take a third Lyapunov candidate function as

$$V_3=\frac{1}{2}\|\boldsymbol{\nu}\|^2,$$

which is radially unbounded in $\boldsymbol{\sigma}$. Along the trajectories of (25), the derivative of V_3 satisfies

$$\begin{aligned}\dot{V}_3&\leq\boldsymbol{\nu}^T(\Lambda\left(\boldsymbol{x}\right)-\nabla\Psi^*\left(\boldsymbol{\sigma}\right)+\nabla\varphi(\boldsymbol{\sigma})-\boldsymbol{\nu})-\|\boldsymbol{\nu}\|^2\\&\leq-\|\boldsymbol{\nu}\|^2+p_2\|\boldsymbol{\nu}\|\\&=-2V_3+p_2\sqrt{2V_3},\end{aligned}$$

with a positive constant p_2. Hence, it can be easily verified that V_2 and V_3 are bounded, so are $\boldsymbol{y}(t)$ and $\boldsymbol{\nu}(t)$.

(ii) Now let us investigate the set

$$Q\triangleq\left\{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\sigma},\boldsymbol{\nu}):\frac{d}{dt}V_1=0\right\},$$

and take set I_v as its largest invariant subset. It follows from the invariance principle (Haddad & Chellaboina, 2011, Theorem 2.41) that $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\sigma},\boldsymbol{\nu})\to I_v$ as $t\to\infty$, and I_v is a positive invariant set. Then it follows from the derivation in (45) that

$$I_v\subseteq\left\{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\sigma},\boldsymbol{\nu}):\boldsymbol{x}=\boldsymbol{x}^{\diamond}\right\}.$$

This indicates that any trajectory along ODE (13) results in the convergence of variable $\boldsymbol{x}$, that is, $\boldsymbol{x}(t)\to\boldsymbol{x}^{\diamond}$ as $t\to\infty$. Moreover, if $\sigma_i^{\diamond}=\nabla\Psi_i\left(\xi_i\right)|_{\xi_i=\Lambda_i\left(x_i^{\diamond},\boldsymbol{x}_{-i}^{\diamond}\right)}$ for $i\in\mathcal{I}$, then the convergent point $\boldsymbol{x}^{\diamond}$ is indeed a global NE. So far, we have accomplished the proof. □

Based on the proof of Theorem 2, we further show the convergence rate of ODE (13). Here, we also reproduce Theorem 3 below for convenience:

Theorem If $\mathscr{E}_i^{+}$ is nonempty and $\Psi_i\left(\cdot\right)$ is $\frac{1}{\kappa_\sigma}$-smooth for $i \in \mathcal{I}$, then (13) converges at an exponential rate, i.e.,

$$\|\boldsymbol{z}(t)-\boldsymbol{z}^\Diamond\| \leq \sqrt{\frac{\tau}{\mu}}\|\boldsymbol{z}(0)\|\exp(-\frac{\kappa}{2\tau}\,t),$$

where $\mu=\min\{\mu_x/2,\mu_\sigma/2\}$, $\kappa=\min\{\kappa_\sigma,\kappa_x\}$, $\tau=\max\{L_x/2\mu_x, L_\sigma/2\mu_\sigma\}$.

Proof Take the same Lyapunov function as in Theorem 2:

$$V_1=\sum_{i=1}^{N} D_{\phi_i^*}(y_i,y_i^\Diamond)+D_{\varphi_i^*}(\nu_i,\nu_i^\Diamond).$$

Recalling the analysis in Theorem 2, we have

$$V_1 \geq \mu\left(\left\|\boldsymbol{x}-\boldsymbol{x}^\Diamond\right\|^2+\left\|\boldsymbol{\sigma}-\boldsymbol{\sigma}^\Diamond\right\|^2\right)=\mu\left\|\boldsymbol{z}-\boldsymbol{z}^\Diamond\right\|^2, \tag{46}$$

where $\mu=\min\{\mu_x/2,\mu_\sigma/2\}$.Based on the standard duality relations, the μ_x-strong convexity of generating function ϕ_i on Ω_i implies that its conjugate gradient $\nabla\phi_i^*$ is continuously differentiable on $\mathbb{R}^n$ with $1/\mu_x$-Lipschitz continuous gradient (Ben-Tal et al., 2001). Thus,

$$\begin{aligned}\sum_{i=1}^{N} D_{\phi_i^*}(y_i,y_i^\Diamond) &= \sum_{i=1}^{N} \phi_i^*(y_i)-\phi_i^*(y_i^\Diamond)-\nabla\phi_i^*(y_i^\Diamond)^T(y_i-y_i^\Diamond)\\ &\leq \frac{1}{2\mu_x}\sum_{i=1}^{N}\|y_i-y_i^\Diamond\|^2.\end{aligned} \tag{47}$$

Moreover, with the duality relation $\nabla\phi_i(x_i)=y_i$ and $\nabla\phi_i(x_i^\Diamond)=y_i^\Diamond$, (47) yields

$$\begin{aligned}\sum_{i=1}^{N} D_{\phi_i^*}(y_i,y_i^\Diamond) &\leq \frac{1}{2\mu_x}\sum_{i=1}^{N}\|y_i-y_i^\Diamond\|^2\\ &= \frac{1}{2\mu_x}\sum_{i=1}^{N}\|\nabla\phi_i(x_i)-\nabla\phi_i(x_i^\Diamond)\|^2\\ &\leq \frac{L_x}{2\mu_x}\|\boldsymbol{x}-\boldsymbol{x}^\Diamond\|^2,\end{aligned} \tag{48}$$

where the last inequality is due to the L_x-smooth of generating function ϕ_i. Analogously, $\nabla\varphi_i^*$ is $1/\mu_\sigma$-Lipschitz because of the μ_σ-strongly convexity of generating function φ_i on $\mathscr{E}_i^+$. With $\nabla\varphi_i(\sigma_i) = \nu_i$ and $\nabla\varphi_i(\sigma_i^\Diamond) = \nu_i^\Diamond$, we also have

$$\begin{aligned}
\sum_{i=1}^N D_{\varphi_i^*}(\nu_i, \nu_i^\Diamond) &= \sum_{i=1}^N \varphi_i^*(\nu_i) - \varphi_i^*(\nu_i^\Diamond) - \nabla\varphi_i^*(\nu_i^\Diamond)^T(\nu_i - \nu_i^\Diamond) \\
&\leq \frac{1}{2\mu_\sigma} \sum_{i=1}^N \|\nu_i - \nu_i^\Diamond\|^2 \\
&= \frac{1}{2\mu_\sigma} \sum_{i=1}^N \|\nabla\varphi_i(\sigma_i) - \nabla\varphi_i(\sigma_i^\Diamond)\|^2 \\
&\leq \frac{L_\sigma}{2\mu_\sigma} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}^\Diamond\|^2,
\end{aligned}$$

where the last inequality is due to the L_σ-smooth of generating function φ_i. Therefore,

$$V_1 \leq \sum_{i=1}^N D_{\phi_i^*}(y_i, y_i^\Diamond) + D_{\varphi_i^*}(\nu_i, \nu_i^\Diamond) \leq \tau\|\boldsymbol{z} - \boldsymbol{z}^\Diamond\|^2,$$

where $\tau = \max\{L_x/2\mu_x, L_\sigma/2\mu_\sigma\}$. Moreover, following the proof of Theorem 2, the derivate of V_1 satisfies

$$\dot{V}_1 \leq -(\boldsymbol{z} - \boldsymbol{z}^\Diamond)^T(F(\boldsymbol{z}) - F(\boldsymbol{z}^\Diamond)).$$

Since $\Psi_i(\cdot)$ is $\frac{1}{\kappa_\sigma}$-smooth, we can derive the κ_σ-strongly convexity of Ψ_i^* by using duality relation (Beck, 2017). Then we have

$$\begin{aligned}
&(\boldsymbol{z} - \boldsymbol{z}^\Diamond)^T(F(\boldsymbol{z}) - F(\boldsymbol{z}^\Diamond)) \\
&= (\boldsymbol{x} - \boldsymbol{x}^\Diamond)^T(\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{\sigma}) - \boldsymbol{G}(\boldsymbol{x}^\Diamond, \boldsymbol{\sigma}^\Diamond)) \\
&\qquad + \left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^\Diamond\right)^T\left(-\Lambda\left(\boldsymbol{x}\right) + \nabla\Psi^*\left(\boldsymbol{\sigma}\right) - \left(-\Lambda\left(\boldsymbol{x}^\Diamond\right) + \nabla\Psi^*\left(\boldsymbol{\sigma}^\Diamond\right)\right)\right) \\
&= (\boldsymbol{x} - \boldsymbol{x}^\Diamond)^T(\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{\sigma}) - \boldsymbol{G}(\boldsymbol{x}^\Diamond, \boldsymbol{\sigma}^\Diamond)) - \left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^\Diamond\right)^T\left(\Lambda\left(\boldsymbol{x}\right) - \Lambda(\boldsymbol{x}^\Diamond)\right) \\
&\qquad + (\boldsymbol{\sigma} - \boldsymbol{\sigma}^\Diamond)^T(\nabla\Psi^*(\boldsymbol{\sigma}) - \nabla\Psi^*(\boldsymbol{\sigma}^\Diamond)) \\
&\geq \kappa_x\|\boldsymbol{x} - \boldsymbol{x}^\Diamond\|^2 + \kappa_\sigma\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^\Diamond\|^2 \\
&\geq \kappa\|\boldsymbol{z} - \boldsymbol{z}^\Diamond\|^2,
\end{aligned}$$

where $\kappa = \min\{\kappa_\sigma, \kappa_x\}$. Therefore,

$$\dot{V}_1 \leq -\kappa\|\boldsymbol{z} - \boldsymbol{z}^\Diamond\|^2. \tag{49}$$

It follows from (46) and (49) that

$$\dot{V}_1 \leq -\kappa\|\boldsymbol{z} - \boldsymbol{z}^\Diamond\|^2 \leq -\frac{\kappa}{\tau}V_1,$$

which actually yields the exponential convergence rate. In other words,

$$\begin{aligned}
\mu\|\boldsymbol{z}(t) - \boldsymbol{z}^\Diamond\|^2 &\leq V_1(\boldsymbol{z}(t)) \\
&\leq V_1(\boldsymbol{z}(0))\exp(-\frac{\kappa}{\tau}t) \\
&\leq \tau\|\boldsymbol{z}(0)\|^2\exp(-\frac{\kappa}{\tau}t).
\end{aligned}$$

Thus, we can also obtain

$$\|\boldsymbol{z}(t) - \boldsymbol{z}^{\Diamond}\| \leq \sqrt{\frac{\tau}{\mu}}\|\boldsymbol{z}(0)\| \exp(-\frac{\kappa}{2\tau}\, t),$$

which implies this conclusion. □

F BREGMAN DIVERGENCE

After the analysis of ODE (13), we turn to investigate the discrete algorithm 1 induced from ODE (13). Also, we provide some auxiliary results that are needed in the following contents.

First of all, the Bregman divergence associated to a generating function $h : \Xi \to \mathbb{R}$ is defined as

$$D_h(z', z) = h(z') - h(z) - (z' - z)^T \nabla h(z),\ \forall z, z' \in \Xi.$$

In what follows, we give basic bounds on the Bregman divergence. Firstly, the basic ingredient for these bounds is a generalization of the (Euclidean) law of cosines, which is known in the literature as the "three-point identity" (Chen & Teboulle, 1993):

Lemma F1 *Let the continuously differentiable generating function h be ω-strongly convex on set Ξ. For z, z', z^+ in Ξ, there holds*

$$D_h(z', z^+) + D_h(z^+, z) = D_h(z', z) + \langle z' - z^+, \nabla h(z) - \nabla h(z^+)\rangle. \tag{50}$$

Proof It follows that the definition of the Bregman divergence,

$$\begin{aligned} D_h(z', z^+) &= h(z') - h\left(z^+\right) - \left(z' - z^+\right)^T \nabla h(z^+), \\ D_h(z^+, z) &= h(z^+) - h(z) - \left(z^+ - z\right)^T \nabla h(z), \\ D_h(z', z) &= h(z') - h\left(z\right) - (z' - z)^T \nabla h(z). \end{aligned}$$

This lemma then follows by adding the first two equalities and subtracting the last one. □

Secondly, with the identity above, we have the following upper bound:

Lemma F2 *Let the continuously differentiable generating function h be ω-strongly convex on set Ξ. For z, z' in Ξ, and $z^+ = \Pi_{\Xi}^{h}(g) = \operatorname{argmin}_{z\in\Xi}\left\{-z^T g + h(z)\right\}$, there holds*

$$D_h(z', z^+) \leq D_h(z', z) - D_h(z^+, z) + (g - \nabla h(z))^T (z^+ - z'). \tag{51}$$

Proof Based on the three-point identity (50), we obtain

$$D_h(z', z^+) + D_h(z^+, z) = D_h(z', z) + (z^+ - z')^T (\nabla h(z^+) - \nabla h(z)).$$

Rearranging these terms gives

$$D_h(z', z^+) = D_h(z', z) - D_h(z^+, z) + (z^+ - z')^T (\nabla h(z^+) - \nabla h(z)). \tag{52}$$

Moreover, with the fact that $z^+ = \Pi_{\Xi}^{h}(g) = \operatorname{argmin}_{z\in\Xi}\left\{-z^T g + h(z)\right\}$, we learn from the optimality of z^+ and the convexity of h that

$$(-g + \nabla h(z^+))^T z^+ \le (-g + \nabla h(z^+))^T z',$$

which implies

$$(z^+ - z')^T \nabla h(z^+) \le (z^+ - z')^T g. \tag{53}$$

Thus, (51) holds by plugging (53) into (52). □

G PROOF OF THEOREM 4

Before the proof, we introduce two classic inequalities in the following.

Lemma F3 (Fenchel's inequality) *Take f as a continuous function on set C. Then the Fenchel conjugate f^* in dual space C^* is $f^*(b) = \sup_{a\in C}\left\{a^T b - f(a)\right\}$, which results in the following inequality*

$$a^T b \le f(a) + f^*(b).$$

Lemma F4 (Jensen's inequality) *Take f as a convex function on a convex set U, then*

$$f(\sum_{l=1}^{k}\gamma_l x_l) \le \sum_{l=1}^{k}\gamma_l f(x_l),$$

where $x_1, \cdots, x_k \in U$ and $\gamma_1, \cdots, \gamma_k > 0$ with $\gamma_1 + \cdots + \gamma_k = 1$.

With help of the basis mentioned above, we show the convergence analysis of Algorithm 1 on a class of N-player generalized monotone games. Here, we reproduce Theorem 4 below for convenience:

Theorem If $\mathscr{E}_i^{+}$ is nonempty and $\Pi_{\Theta_i}^{\Psi_i}(\cdot)$ is κ_σ-strongly monotone, then Algorithm 1 converges at a rate of $\mathcal{O}(1/k)$ with step size $\alpha_k = 2\kappa^{-1}/(k+1)$, i.e.,

$$\|\boldsymbol{x}^k - \boldsymbol{x}^\Diamond\|^2 + \|\boldsymbol{\sigma}^k - \boldsymbol{\sigma}^\Diamond\|^2 \le \frac{1}{k+1}\frac{M_1}{\mu^2\kappa^2},$$

where $\mu = \min\{\mu_x/2, \mu_\sigma/2\}$, $\kappa = \min\{\kappa_\sigma, \kappa_x\}$, and M_1 is a positive constant.

Proof Take the collection of the Bregman divergence as

$$\Delta(\boldsymbol{z}^\Diamond, \boldsymbol{z}^{k+1}) \triangleq \sum_{i=1}^{N} D_{\phi_i}(x_i^\Diamond, x_i^{k+1}) + D_{\varphi_i}(\sigma_i^\Diamond, \sigma_i^{k+1}), \tag{54}$$

where

$$D_{\phi_i}(x_i^\Diamond, x_i^{k+1}) = \phi_i(x_i^\Diamond) - \phi_i(x_i^{k+1}) - \nabla\phi_i(x_i^{k+1})^T(x_i^\Diamond - x_i^{k+1}),$$
$$D_{\varphi_i}(\sigma_i^\Diamond, \sigma_i^{k+1}) = \varphi_i(\sigma_i^\Diamond) - \varphi_i(\sigma_i^{k+1}) - \nabla\varphi_i(\sigma_i^{k+1})^T(\sigma_i^\Diamond - \sigma_i^{k+1}).$$

Because ϕ_i is μ_x-strongly convex and φ_i is μ_σ-strongly convex for $i \in \mathcal{I}$, we obtain that

$$\begin{aligned}\Delta(\boldsymbol{z}^\Diamond, \boldsymbol{z}^{k+1}) &\ge \frac{\mu_x}{2}\sum_{i=1}^{N}\left\|x_i^{k+1} - x_i^\Diamond\right\|^2 + \frac{\mu_\sigma}{2}\sum_{i=1}^{N}\left\|\sigma_i^{k+1} - \sigma_i^\Diamond\right\|^2 \\ &\ge \mu\|\boldsymbol{z}^{k+1} - \boldsymbol{z}^\Diamond\|^2,\end{aligned} \tag{55}$$

where $\mu = \min\{\mu_x/2, \mu_\sigma/2\}$. Then, consider the term $D_{\phi_i}(x_i^\diamondsuit, x_i^{k+1})$ in (54). By employing three-point identity in Lemma F1, we obtain that

$$D_{\phi_i}(x_i^\diamondsuit, x_i^{k+1}) = D_{\phi_i}(x_i^\diamondsuit, x_i^k) - D_{\phi_i}(x_i^{k+1}, x_i^k) + (\nabla\phi_i(x_i^{k+1}) - \nabla\phi_i(x_i^k))^T(x_i^{k+1} - x_i^\diamondsuit). \tag{56}$$

Denote

$$g_i = \nabla\phi_i(x_i^k) - \alpha_k\sigma_i^{kT}\nabla_{x_i}\Lambda_i(x_i^k, \boldsymbol{x}_{-i}^k).$$

According to Algorithm 1,

$$x_i^{k+1} = \Pi_{\Omega_i}^{\phi_i}(g_i) = \mathrm{argmin}_{x\in\Omega_i}\left\{-x^T g_i + \phi_i(x)\right\},$$

which implies that

$$\begin{aligned}0 &\le \left(\nabla\phi_i\left(x_i^{k+1}\right) - g_i\right)^T x_i^\diamondsuit - \left(\left(\nabla\phi_i\left(x_i^{k+1}\right)\right) - g_i\right)^T x_i^{k+1} \\ &= \left(\nabla\phi_i\left(x_i^{k+1}\right) - g_i\right)^T (x_i^\diamondsuit - x_i^{k+1}).\end{aligned}$$

In addition,

$$(\nabla\phi_i(x_i^{k+1})^T(x_i^{k+1} - x_i^\diamondsuit) \le (\nabla\phi_i(x_i^k) - \alpha_k\sigma_i^{kT}\nabla_{x_i}\Lambda_i(x_i^k, \boldsymbol{x}_{-i}^k))^T(x_i^{k+1} - x_i^\diamondsuit).$$

Then (56) becomes

$$D_{\phi_i}(x_i^\diamondsuit, x_i^{k+1}) \le D_{\phi_i}(x_i^\diamondsuit, x_i^k) - D_{\phi_i}(x_i^{k+1}, x_i^k) - \alpha_k(\sigma_i^{kT}\nabla_{x_i}\Lambda_i(x_i^k, \boldsymbol{x}_{-i}^k))^T(x_i^{k+1} - x_i^\diamondsuit). \tag{57}$$

Similarly, as for the term $D_{\varphi_i}(\sigma_i^\diamondsuit, \sigma_i^{k+1})$ in (54), we get

$$D_{\varphi_i}(\sigma_i^\diamondsuit, \sigma_i^{k+1}) \le D_{\varphi_i}(\sigma_i^\diamondsuit, \sigma_i^k) - D_{\varphi_i}(\sigma_i^{k+1}, \sigma_i^k) - \alpha_k(-\Lambda_i\left(x_i^k, \boldsymbol{x}_{-i}^k\right) + \xi_i^k)^T(\sigma_i^{k+1} - \sigma_i^\diamondsuit), \tag{58}$$

where $\xi_i^k = \Pi_{\Theta_i}^{\Psi_i}(\sigma_i^k)$. To proceed, combining (57) and (58) gives

$$\begin{aligned}\Delta(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^{k+1}) &= \sum_{i=1}^N D_{\phi_i}(x_i^\diamondsuit, x_i^{k+1}) + D_{\varphi_i}(\sigma_i^\diamondsuit, \sigma_i^{k+1}) \\ &\le \sum_{i=1}^N D_{\phi_i}(x_i^\diamondsuit, x_i^k) - D_{\phi_i}(x_i^{k+1}, x_i^k) - \alpha_k(\sigma_i^{kT}\nabla_{x_i}\Lambda_i(x_i^k, \boldsymbol{x}_{-i}^k))^T(x_i^{k+1} - x_i^\diamondsuit) \\ &\quad + \sum_{i=1}^N D_{\varphi_i}(\sigma_i^\diamondsuit, \sigma_i^k) - D_{\varphi_i}(\sigma_i^{k+1}, \sigma_i^k) - \alpha_k(-\Lambda_i\left(x_i^k, \boldsymbol{x}_{-i}^k\right) + \xi_i^k)^T(\sigma_i^{k+1} - \sigma_i^\diamondsuit), \\ &= \Delta(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^k) - \alpha_k F(\boldsymbol{z}^k)^T(\boldsymbol{z}^{k+1} - \boldsymbol{z}^\diamondsuit) - \Delta(\boldsymbol{z}^{k+1}, \boldsymbol{z}^k).\end{aligned}$$

Hence,

$$\begin{aligned}&\Delta(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^{k+1}) \\ &\le \Delta(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^k) - \alpha_k F(\boldsymbol{z}^k)^T(\boldsymbol{z}^{k+1} - \boldsymbol{z}^\diamondsuit) - \Delta(\boldsymbol{z}^{k+1}, \boldsymbol{z}^k) \\ &= \Delta(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^k) - \alpha_k F(\boldsymbol{z}^k)^T(\boldsymbol{z}^k - \boldsymbol{z}^\diamondsuit) + \alpha_k F(\boldsymbol{z}^k)^T(\boldsymbol{z}^k - \boldsymbol{z}^{k+1}) - \Delta(\boldsymbol{z}^{k+1}, \boldsymbol{z}^k) \\ &\le \Delta(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^k) - \alpha_k F(\boldsymbol{z}^k)^T(\boldsymbol{z}^k - \boldsymbol{z}^\diamondsuit) + \alpha_k F(\boldsymbol{z}^k)^T(\boldsymbol{z}^k - \boldsymbol{z}^{k+1}) - \mu\|\boldsymbol{z}^k - \boldsymbol{z}^{k+1}\|^2,\end{aligned}$$

where the last inequality is due to the similar property in (55). On this basis, by additionally employing Fenchel's inequality and subsituting f in Lemma F3 with

$\frac{1}{2}\|\cdot\|$, we derive that

$$\begin{aligned}\alpha_k F(\boldsymbol{z}^k)^T(\boldsymbol{z}^k-\boldsymbol{z}^{k+1}) &\le \frac{(2\mu)}{2}\|\boldsymbol{z}^k-\boldsymbol{z}^{k+1}\|^2+\frac{(2\mu)^{-1}}{2}\alpha_k^2\|F(\boldsymbol{z}^k)^T\|_*^2\\ &=\mu\|\boldsymbol{z}^k-\boldsymbol{z}^{k+1}\|^2+\frac{1}{4\mu}\alpha_k^2\|F(\boldsymbol{z}^k)^T\|_*^2\\ &=\mu\|\boldsymbol{z}^k-\boldsymbol{z}^{k+1}\|^2+\frac{1}{4\mu}\alpha_k^2\|F(\boldsymbol{z}^k)^T\|^2,\end{aligned}$$

where the last equality follows from the fact that the conjugate norm of ℓ_2 norm is also ℓ_2 norm itself.

Hence, we can make further scaling so that

$$\begin{aligned}&\Delta(\boldsymbol{z}^\diamond,\boldsymbol{z}^{k+1})\\ &\le \Delta(\boldsymbol{z}^\diamond,\boldsymbol{z}^k)-\alpha_k F(\boldsymbol{z}^k)^T(\boldsymbol{z}^k-\boldsymbol{z}^\diamond)+\alpha_k F(\boldsymbol{z}^k)^T(\boldsymbol{z}^k-\boldsymbol{z}^{k+1})-\mu\|\boldsymbol{z}^k-\boldsymbol{z}^{k+1}\|^2\\ &\le \Delta(\boldsymbol{z}^\diamond,\boldsymbol{z}^k)-\alpha_k F(\boldsymbol{z}^k)^T(\boldsymbol{z}^k-\boldsymbol{z}^\diamond)+\mu\|\boldsymbol{z}^k-\boldsymbol{z}^{k+1}\|^2+\frac{\alpha_k^2}{4\mu}\|F(\boldsymbol{z}^k)^T\|^2-\mu\|\boldsymbol{z}^k-\boldsymbol{z}^{k+1}\|^2\\ &=\Delta(\boldsymbol{z}^\diamond,\boldsymbol{z}^k)-\alpha_k F(\boldsymbol{z}^k)^T(\boldsymbol{z}^k-\boldsymbol{z}^\diamond)+\frac{\alpha_k^2}{4\mu}\|F(\boldsymbol{z}^k)\|^2\\ &=\Delta(\boldsymbol{z}^\diamond,\boldsymbol{z}^k)-\alpha_k(F(\boldsymbol{z}^k)-F(\boldsymbol{z}^\diamond))^T(\boldsymbol{z}^k-\boldsymbol{z}^\diamond)-\alpha_k F(\boldsymbol{z}^\diamond)^T(\boldsymbol{z}^k-\boldsymbol{z}^\diamond)+\frac{\alpha_k^2}{4\mu}\|F(\boldsymbol{z}^k)\|^2\\ &\le \Delta(\boldsymbol{z}^\diamond,\boldsymbol{z}^k)-\alpha_k(F(\boldsymbol{z}^k)-F(\boldsymbol{z}^\diamond))^T(\boldsymbol{z}^k-\boldsymbol{z}^\diamond)+\frac{\alpha_k^2}{4\mu}\|F(\boldsymbol{z}^k)\|^2,\end{aligned} \tag{59}$$

where the last inequality is true because $\boldsymbol{z}^\diamond$ is a solution to $\mathrm{VI}(\Xi,F)$. Moreover, with κ_σ-strongly monotonicity of operator $\Pi_{\Theta_i}^{\Psi_i}(\cdot)$. Hence, there holds the inequality.

$$(F(\boldsymbol{z}^k)-F(\boldsymbol{z}^\diamond))^T(\boldsymbol{z}^k-\boldsymbol{z}^\diamond)\ge\kappa\|\boldsymbol{z}^k-\boldsymbol{z}^\diamond\|^2,$$

where $\kappa=\min\{2\kappa_x,\kappa_\sigma\}$. Then, it derives that

$$\Delta(\boldsymbol{z}^\diamond,\boldsymbol{z}^{k+1})\le\Delta(\boldsymbol{z}^\diamond,\boldsymbol{z}^k)-\alpha_k\kappa\|\boldsymbol{z}^k-\boldsymbol{z}^\diamond\|^2+\frac{\alpha_k^2}{4\mu}\|F(\boldsymbol{z}^k)\|^2.$$

Denote $\eta_k=\kappa\alpha_k$ with $\eta_0=1$. We can verify that

$$\frac{1-\eta_{k+1}}{\eta_{k+1}^2}\le\frac{1}{\eta_k^2},\quad\forall k\ge 0.$$

Then, with the subsitute above,

$$\Delta(\boldsymbol{z}^\diamond,\boldsymbol{z}^{k+1})\le\Delta(\boldsymbol{z}^\diamond,\boldsymbol{z}^k)-\eta_k\|\boldsymbol{z}^k-\boldsymbol{z}^\diamond\|^2+\frac{\eta_k^2}{4\kappa^2\mu}\|F(\boldsymbol{z}^k)\|^2. \tag{60}$$

On the one hand, recalling the property of the Bregman divergence (Nedic & Lee, 2014), we have

$$\Delta(\boldsymbol{z}^\diamond,\boldsymbol{z})\le\frac{1}{2}\|\boldsymbol{z}-\boldsymbol{z}^\diamond\|^2\le\|\boldsymbol{z}-\boldsymbol{z}^\diamond\|^2,\quad\forall\boldsymbol{z},\boldsymbol{z}^\diamond\in\boldsymbol{\Xi}.$$

On the other hand, since $\mathscr{E}_i^+$ is nonempty and the stationary points are with finite values, the set $\mathscr{E}_i^+$ for $i\in\mathcal{I}$ can be regarded as bounded without loss of generality.

Then together with the compactness of the feasible set Ω_i for $i \in \mathcal{I}$, there exists a finite constant $M_1 > 0$ such that $\|F(\boldsymbol{z})\|^2 \leq M_1$. On this basis, we obtain

$$\begin{aligned}\Delta(\boldsymbol{z}^\Diamond, \boldsymbol{z}^{k+1}) &\leq \Delta(\boldsymbol{z}^\Diamond, \boldsymbol{z}^k) - \eta_k \Delta(\boldsymbol{z}^\Diamond, \boldsymbol{z}^k) + \frac{\eta_k^2}{4\kappa^2\mu}\|F(\boldsymbol{z}^k)\|^2 \\ &\leq (1-\eta_k)\Delta(\boldsymbol{z}^\Diamond, \boldsymbol{z}^k) + \frac{\eta_k^2}{4\kappa^2\mu}\|F(\boldsymbol{z}^k)\|^2 \\ &\leq (1-\eta_k)\Delta(\boldsymbol{z}^\Diamond, \boldsymbol{z}^k) + \frac{\eta_k^2}{4\kappa^2\mu}M_1.\end{aligned} \tag{61}$$

Multiplying both sides of the relation above by $1/\eta_k^2$, and recalling the property $\frac{1-\eta_{k+1}}{\eta_{k+1}^2} \leq \frac{1}{\eta_k^2}$, we have

$$\begin{aligned}\frac{1}{\eta_k^2}\Delta(\boldsymbol{z}^\Diamond, \boldsymbol{z}^{k+1}) &\leq \frac{1-\eta_k}{\eta_k^2}\Delta(\boldsymbol{z}^\Diamond, \boldsymbol{z}^k) + \frac{M_1}{4\kappa^2\mu} \\ &\leq \frac{1}{\eta_{k-1}^2}\Delta(\boldsymbol{z}^\Diamond, \boldsymbol{z}^k) + \frac{M_1}{4\kappa^2\mu}.\end{aligned}$$

Hence, take the sum of these inequalities over $k, \cdots, 1$ with $\eta_0 = 1$, that is,

$$\frac{1}{\eta_k^2}\Delta(\boldsymbol{z}^\Diamond, \boldsymbol{z}^{k+1}) \leq \Delta(\boldsymbol{z}^\Diamond, \boldsymbol{z}^1) + k\frac{M_1}{4\kappa^2\mu}.$$

Additionally, by taking $k = 1$ and $\eta_0 = 1$ in (61), $\Delta(\boldsymbol{z}^\Diamond, \boldsymbol{z}^1) \leq \frac{\eta_k^2 M_1}{4\kappa^2\mu}$. Therefore, recalling the step size setting $\eta_k = \kappa\alpha_k = 2/(k+1)$, for all $k \geq 1$, we get

$$\begin{aligned}\Delta(\boldsymbol{z}^\Diamond, \boldsymbol{z}^{k+1}) &\leq \eta_k^2(k+1)\frac{M_1}{4\kappa^2\mu} \\ &= \frac{1}{k+1}\frac{M_1}{\mu\kappa^2}.\end{aligned} \tag{62}$$

Recall

$$\Delta(\boldsymbol{z}^\Diamond, \boldsymbol{z}^{k+1}) \geq \mu\|\boldsymbol{z}^{k+1} - \boldsymbol{z}^\Diamond\|^2 = \mu(\|\boldsymbol{x}^k - \boldsymbol{x}^\Diamond\|^2 + \|\boldsymbol{\sigma}^k - \boldsymbol{\sigma}^\Diamond\|^2).$$

Therefore, we are finally rewarded by

$$\|\boldsymbol{x}^k - \boldsymbol{x}^\Diamond\|^2 + \|\boldsymbol{\sigma}^k - \boldsymbol{\sigma}^\Diamond\|^2 \leq \frac{1}{k+1}\frac{M_1}{\mu^2\kappa^2},$$

which completes the proof. □

H PROOF OF THEOREM 5

As mentioned in §5, the nonconvex N-player potential game in (14) is endowed with a unified complementary function in (15), that is,

$$\Gamma(x_i, \sigma, \boldsymbol{x}_{-i}) = \sigma^T \Lambda(x_i, \boldsymbol{x}_{-i}) - \Psi^*(\sigma).$$

Thus, we can employ the gradient information of this unified complementary function in algorithm iterations, so as to reduce the computational cost in Algorithm 1.

Accordingly, we can rewrite Algorithm 1 for potential games in the following Algorithm 2.

Algorithm 2

Input: Step size $\{\alpha_k\}$, proper generating functions ϕ_i on Ω_i and φ on $\mathscr{E}^+$.
Initialize: Set $\sigma^0 \in \mathscr{E}^+, x_i^0 \in \Omega_i,\ i \in \{1,\ldots,N\}$,
1: **for** $k = 1, 2, \cdots$ **do**
2: compute the unified conjugate of Ψ: $\xi^k = \Pi_\Theta^\Psi(\sigma^k)$
3: update the unified canonical dual variable:
$\sigma^{k+1} = \Pi_{\mathscr{E}^+}^{\varphi}(\nabla\varphi(\sigma^k) + \alpha_k(\Lambda\left(x_i^k, \boldsymbol{x}_{-i}^k\right) - \xi^k))$
4: **for** $i = 1, \cdots N$ **do**
5: update the decision variable of player i:
$x_i^{k+1} = \Pi_{\Omega_i}^{\phi_i}(\nabla\phi_i(x_i^k) - \alpha_k \sigma^{kT}\nabla_{x_i}\Lambda\left(x_i^k, \boldsymbol{x}_{-i}^k\right))$
6: **end for**
7: **end for**

Similarly, define $\boldsymbol{z} = \operatorname{col}\{\boldsymbol{x}, \sigma\}$, and the simplified pseudo-gradient of (15) as

$$F(\boldsymbol{z}) \triangleq \left[\begin{array}{c} \operatorname{col}\left\{\sigma^T\nabla_{x_i}\Lambda\left(x_i, \boldsymbol{x}_{-i}\right)\right\}_{i=1}^N \\ -\Lambda\left(x_i, \boldsymbol{x}_{-i}\right) + \nabla\Psi^*\left(\sigma\right) \end{array}\right] \triangleq \left[\begin{array}{c} G(\boldsymbol{x}, \sigma) \\ -\Lambda\left(\boldsymbol{x}\right) + \nabla\Psi^*\left(\sigma\right) \end{array}\right].$$

Consider the weighted averaged iterates in course of k iterates as

$$\hat{\boldsymbol{x}}^k = \frac{\sum_{j=1}^k \alpha_j \boldsymbol{x}^j}{\sum_{j=1}^k \alpha^j}, \quad \hat{\sigma}^k = \frac{\sum_{j=1}^k \alpha_j \sigma^j}{\sum_{j=1}^k \alpha^j}.$$

Then we show the convergence rate of Algorithm 2 (or Algorithm 1 in potential games). We rewrite Theorem 5 below for convenience:

Theorem If $\mathscr{E}^+$ is nonempty and players' payoffs are subjected to the potential function in (14), then Algorithm 1 converges at a $\mathcal{O}(1/\sqrt{k})$ rate with step size $\alpha_k = 2{M_2}^{-1}\sqrt{\mu d\,/\,k}$, i.e.,

$$\Gamma(\hat{\boldsymbol{x}}^k, \sigma^\diamond) - \Gamma(\boldsymbol{x}^\diamond, \hat{\sigma}^k) \leq \frac{1}{\sqrt{k}}\sqrt{\frac{d}{\mu}}M_2,$$

where $\mu = \min\{\mu_x/2, \mu_\sigma/2\}$, and d, M_2 are two positive constants.

Proof Take another collection of the Bregman divergence as

$$\widetilde{\Delta}(\boldsymbol{z}^\diamond, \boldsymbol{z}) \triangleq D_\varphi(\sigma^\diamond, \sigma) + \sum_{i=1}^N D_{\phi_i}(x_i^\diamond, x_i). \tag{63}$$

Working as in the proof of Theorem 4, we obtain the following inequality by three-point identity and Fenchel's inequality:

$$
\begin{aligned}
&\widetilde{\Delta}(\boldsymbol{z}^{\Diamond}, \boldsymbol{z}^{k+1}) \\
&\leq \widetilde{\Delta}(\boldsymbol{z}^{\Diamond}, \boldsymbol{z}^{k}) - \alpha_k F(\boldsymbol{z}^k)^T(\boldsymbol{z}^{k+1} - \boldsymbol{z}^{\Diamond}) - \widetilde{\Delta}(\boldsymbol{z}^{k+1}, \boldsymbol{z}^{k}) \\
&\leq \widetilde{\Delta}(\boldsymbol{z}^{\Diamond}, \boldsymbol{z}^{k}) - \alpha_k F(\boldsymbol{z}^k)^T(\boldsymbol{z}^{k} - \boldsymbol{z}^{\Diamond}) + \alpha_k F(\boldsymbol{z}^k)^T(\boldsymbol{z}^{k} - \boldsymbol{z}^{k+1}) - \widetilde{\Delta}(\boldsymbol{z}^{k+1}, \boldsymbol{z}^{k}) \\
&\leq \widetilde{\Delta}(\boldsymbol{z}^{\Diamond}, \boldsymbol{z}^{k}) - \alpha_k F(\boldsymbol{z}^k)^T(\boldsymbol{z}^{k} - \boldsymbol{z}^{\Diamond}) + \alpha_k F(\boldsymbol{z}^k)^T(\boldsymbol{z}^{k} - \boldsymbol{z}^{k+1}) - \mu\|\boldsymbol{z}^k - \boldsymbol{z}^{k+1}\|^2 \\
&\leq \widetilde{\Delta}(\boldsymbol{z}^{\Diamond}, \boldsymbol{z}^{k}) - \alpha_k F(\boldsymbol{z}^k)^T(\boldsymbol{z}^{k} - \boldsymbol{z}^{\Diamond}) + \alpha_k F(\boldsymbol{z}^k)^T(\boldsymbol{z}^{k} - \boldsymbol{z}^{k+1}) - \mu\|\boldsymbol{z}^k - \boldsymbol{z}^{k+1}\|^2 \\
&\leq \widetilde{\Delta}(\boldsymbol{z}^{\Diamond}, \boldsymbol{z}^{k}) - \alpha_k F(\boldsymbol{z}^k)^T(\boldsymbol{z}^{k} - \boldsymbol{z}^{\Diamond}) + \frac{\alpha_k^2}{4\mu}\|F(\boldsymbol{z}^k)\|^2.
\end{aligned}
\tag{64}
$$

Moreover, according to $\sigma \in \mathscr{E}^+$ in (9),

$$
(\boldsymbol{x}^{\Diamond} - \boldsymbol{x}^k)^T G(\boldsymbol{x}^k, \sigma^k) \leq \sigma^{kT}(\Lambda\left(\boldsymbol{x}^{\Diamond}\right) - \Lambda\left(\boldsymbol{x}^k\right)).
$$

As a result,

$$
\begin{aligned}
\left\langle F(\boldsymbol{z}^k), \boldsymbol{z}^{\Diamond} - \boldsymbol{z}^k \right\rangle &= (\boldsymbol{x}^{\Diamond} - \boldsymbol{x}^k)^T G(\boldsymbol{x}^k, \sigma^k) + (\sigma^{\Diamond} - \sigma^k)^T(-\Lambda\left(\boldsymbol{x}^k\right) + \nabla\Psi^*\left(\sigma^k\right)) \\
&\leq \sigma^{kT}\Lambda\left(\boldsymbol{x}^{\Diamond}\right) - \Psi^*\left(\sigma^k\right) - (\sigma^{\Diamond T}\Lambda\left(\boldsymbol{x}^{kT}\right) - \Psi^*\left(\sigma^{\Diamond}\right)) \\
&= \Gamma(\boldsymbol{x}^{\Diamond}, \sigma^k) - \Gamma(\boldsymbol{x}^k, \sigma^{\Diamond}).
\end{aligned}
\tag{65}
$$

By substituting (65) into (64) and rearranging the terms therein, we have

$$
\begin{aligned}
\alpha_k(\Gamma(\boldsymbol{x}^k, \sigma^{\Diamond}) - \Gamma(\boldsymbol{x}^{\Diamond}, \sigma^k)) &\leq \alpha_k F(\boldsymbol{z}^k)^T(\boldsymbol{z}^k - \boldsymbol{z}^{\Diamond}) \\
&\leq \widetilde{\Delta}(\boldsymbol{z}^{\Diamond}, \boldsymbol{z}^k) - \widetilde{\Delta}(\boldsymbol{z}^{\Diamond}, \boldsymbol{z}^{k+1}) + \frac{\alpha_k^2}{4\mu}\|F(\boldsymbol{z}^k)\|^2.
\end{aligned}
$$

Meanwhile, since $\mathscr{E}^+$ is nonempty and the stationary points are with finite values, the set $\mathscr{E}^+$ can be regarded as bounded without loss of generality. Together with the compactness of Ω_i for $i \in \mathcal{I}$, there exists finite constants $d > 0$ and $M_2 > 0$ such that $\widetilde{\Delta}(\boldsymbol{z}^{\Diamond}, \boldsymbol{z}^1) \leq d$ and $\|F(\boldsymbol{z})\|^2 \leq M_2$. Then it follows from the sum of the above inequalities over $1, \cdots, k$ that

$$
\sum_{j=1}^{k} \alpha_j \left(\Gamma(\boldsymbol{x}^j, \sigma^{\Diamond}) - \Gamma(\boldsymbol{x}^{\Diamond}, \sigma^j)\right) \leq \widetilde{\Delta}(\boldsymbol{z}^{\Diamond}, \boldsymbol{z}^1) + \frac{\sum_{j=1}^{k} \alpha_j^2 M_2^2}{4\mu}. \tag{66}
$$

For more intuitive presentation, we denote the weight by $\lambda_j = \frac{\alpha_j}{\sum_{l=1}^k \alpha_l}$. Then (66) yields

$$
\begin{aligned}
&\frac{\widetilde{\Delta}(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^1) + (4\mu)^{-1} M_2^2 \sum_{j=1}^k \alpha_j^2}{\sum_{j=1}^k \alpha_j} \\
&\geq \sum_{j=1}^k \frac{\alpha_j}{\sum_{l=1}^k \alpha_l} \left(\Gamma(\boldsymbol{x}^j, \sigma^\diamondsuit) - \Gamma(\boldsymbol{x}^\diamondsuit, \sigma^j)\right) \\
&= \sum_{j=1}^k \lambda_j \left(\Gamma(\boldsymbol{x}^j, \sigma^\diamondsuit) - \Gamma(\boldsymbol{x}^\diamondsuit, \sigma^j)\right) \\
&\geq \Gamma(\sum_{j=1}^k \lambda_j \boldsymbol{x}^j, \sigma^\diamondsuit) - \Gamma(\boldsymbol{x}^\diamondsuit, \sum_{j=1}^k \lambda_j \boldsymbol{\sigma}^j) \\
&= \Gamma(\hat{\boldsymbol{x}}^k, \sigma^\diamondsuit) - \Gamma(\boldsymbol{x}^\diamondsuit, \hat{\sigma}^k),
\end{aligned}
$$

where the last inequality is true due to Jensen's inequality. Since the step size satisfies $\alpha_k = 2\sqrt{\mu d}/M_2\sqrt{k}$, we finally derive that

$$
\Gamma(\hat{\boldsymbol{x}}^k, \sigma^\diamondsuit) - \Gamma(\boldsymbol{x}^\diamondsuit, \hat{\sigma}^k) \leq \frac{1}{\sqrt{k}} \sqrt{\frac{d}{\mu}} M_2,
$$

which indicates the conclusion. □