# IBCL: Zero-shot Model Generation under Stability-Plasticity Trade-offs

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## Abstract

Algorithms that balance the stability-plasticity trade-off are well studied in the Continual Learning literature. However, only a few focus on obtaining models for specified trade-off preferences. When solving the problem of continual learning under specific trade-offs (CLuST), state-of-the-art techniques leverage rehearsal-based learning, which requires retraining when a model corresponding to a new trade-off preference is requested. This is inefficient, since there potentially exist a significant number of different trade-offs, and a large number of models may be requested. As a response, we propose Imprecise Bayesian Continual Learning (IBCL), an algorithm that tackles CLuST efficiently. IBCL replaces retraining with a constant-time convex combination. Given a new task, IBCL (1) updates the knowledge base as a convex hull of model parameter distributions, and (2) generates one Pareto-optimal model per given trade-off via convex combination without additional training. That is, obtaining models corresponding to specified trade-offs via IBCL is zero-shot. Experiments whose baselines are current CLuST algorithms show that IBCL improves by at most 45% on average per task accuracy, and by 43% on peak per task accuracy while maintaining a near-zero to positive backward transfer. In addition, its training overhead, measured by the number of batch updates, remains constant at every task, regardless of the number of preferences requested. Details can be found at: `https://github.com/ibcl-anon/ibcl`.

## 1 Introduction

Continual Learning (CL), also known as lifelong machine learning, is a special case of multi-task learning, where tasks arrive in temporal sequence one-by-one (Thrun, 1998; Ruvolo & Eaton, 2013; Chen & Liu, 2016; Parisi et al., 2019). Two key properties matter for CL algorithms: stability and plasticity (De Lange et al., 2021). Here, stability means the ability to maintain performance on previous tasks, not forgetting what the model has learned, and plasticity refers to the ability to adapt to a new task. Unfortunately, these two properties are conflicting due to the multi-objective optimization nature of CL (Kendall et al., 2018; Sener & Koltun, 2018). For years, researchers have been balancing the stability-plasticity trade-off. However, few have discussed the problem of learning models for specifically given trade-off points. In this paper, we focus on such a problem, which we denote as CL under specific trade-offs (CLuST).

Why is CLuST important? First, in certain scenarios, it is important to explicitly specify *how much* stability and plasticity are needed to obtain a customized model for each trade-off preference. Second, when there exists a large number of preferences, the training *efficiency* of every customized model matters. Otherwise, the training cost accumulates on all preferences and becomes prohibitive. Therefore, we are not only looking for a solution to the CLuST problem, but also an efficient one.

**Motivating Example.** Consider an example of a movie recommendation system. The model is first trained to rate movies in the sci-fi genre. Then, the movie company adds a new genre, e.g., documentaries. The model needs to learn how to rate documentaries while not forgetting how to rate sci-fis. Training this model boils down to a CL problem. The company now wants to build a recommendation system that adapts to users' tastes in movies. For example, Alice has equal preferences over sci-fis and documentaries. Bob, however, wants to watch only documentaries and has no interest in sci-fis at all. Consequently, the company aims to

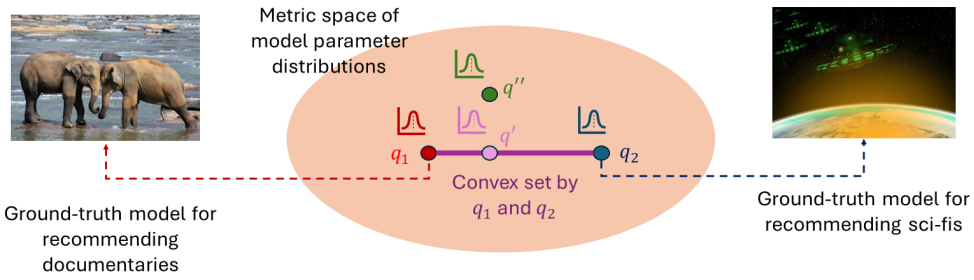


Figure 1: A Bayesian view of a Pareto-optimal parameter distribution $q'$ and a non-Pareto-optimal parameter distribution $q''$.

train two customized models for Alice and Bob, respectively, to predict how likely a sci-fi or a documentary are to be recommended. Based on individual preferences, Alice's personal model should balance between the accuracy in rating sci-fis and rating documentaries, while Bob's model allows for compromising on the accuracy in rating sci-fis to achieve a high accuracy in rating documentaries. As new genres are added, users should be able to input their preferences on all available genres to obtain customized models. Since there could be many different users, and each user's taste in movie genres could vary over time, the movie company should implement models that adapt to *a significant number of preferences*. The costs would be prohibitive if the company had to train one model per distinct preference.

**The CLuST Problem and its Challenges.** To formalize the CLuST problem, we take a Bayesian perspective, where learnable model parameters are viewed as random variables (Farquhar & Gal, 2019; Kessler et al., 2023; Nguyen et al., 2018). As illustrated in Figure 1, we consider all parameter distributions living in a metric space. This metric can be any valid metric for distributions, such as the 2-Wasserstein distance (Deza & Deza, 2013). The figure shows an example of two sequential tasks, with ground-truth parameter distributions $q_1$ and $q_2$, respectively. From this setup, a distribution that emphasizes stability (in task 2) is a distribution closer to $q_1$ than $q_2$, and a distribution that prioritizes plasticity is closer to $q_2$ than $q_1$. Notice that irrespective of the desired stability-plasticity trade-off, we want the distribution to be *Pareto-optimal*, which loosely means that there is no way to improve such distribution by making it closer to *both* $q_1$ and $q_2$. We can see that Pareto-optimality is equivalent to being inside the convex set enclosed by $q_1$ and $q_2$. For example, $q'$ in the figure is a Pareto-optimal distribution, while $q''$ is not. With this setting, we can specify a trade-off point using a *preference vector* (Mahapatra & Rajan, 2020; 2021) $\bar{w} = (w_1, w_2)$, where $w_1, w_2 \geq 0$ and $w_1 + w_2 = 1$. The preferred Pareto-optimal distribution is, therefore, a convex combination $w_1 q_1 + w_2 q_2$.

So far, researchers have already proposed the use of preference vectors to specify trade-off points in multi-task and continual learning (Gupta et al., 2021; Lin et al., 2019; 2020; Ma et al., 2020). However, instead of using them as coefficients of distributions, state-of-the-art techniques use them as coefficients for loss functions in *rehearsal-based methods*. That is, existing algorithms memorize some data $d_i$ for each task $i$ (for "rehearsal"), and let the loss at task $i$ be $l_i = \sum_{j=1}^{i} w_j l(d_j)$, with $l$ being a generic loss function like cross-entropy. There are at least two drawbacks to this approach. First, rehearsals must retrain the entire model whenever we have a new trade-off preference. **In plain words, these methods have a training overhead proportional to the number of preferences at each task.** As numerous possible preferences exist, this boils down to an efficiency issue when there is a large number of preferences, such as many users in the movie recommendation example. It would be desirable if we could obtain the preferred models using training-free constant-time operations instead of retraining. Moreover, rehearsals must cache data, and stable performance on previous tasks depends on which data can be memorized.

**The IBCL Algorithm.** To overcome these shortcomings faced by CLuST algorithms, we propose Imprecise Bayesian Continual Learning (IBCL), whose workflow is illustrated in Figure 3. In step 1, upon arrival of the training data for a new task, IBCL updates its *knowledge base* (that is, all information shared across tasks) in the form of a convex set of distributions with finitely many extreme elements (the elements that cannot be written as convex combinations of one another), called *finitely generated credal set* (FGCS) (Caprio et al.,

2024a). This is done by variational inference from the learned distribution of the previous task, and the learned distributions serve as extreme elements of the FGCS. Each point in the FGCS corresponds to one Pareto-optimal distribution on the trade-off polytope of all tasks so far. Then, at step 2, given any preference vector $\bar{w}$, IBCL selects the preferred distribution by convex combination. A parameter region is obtained as a highest density region (HDR) of the selected distribution, which is the smallest parameter set that contains the ground-truth model with high probability.

IBCL addresses the identified shortcomings as follows. **First, IBCL replaces retraining in state-of-the-art with constant-time, zero-shot convex combination to generate models. It has a constant training overhead per task (to update the FGCS), independent of the number of preferences.** Additionally, no data cache is required, and therefore the stability of our model does not depend on the data memorized. Experiments on image classification and NLP benchmarks support the effectiveness of IBCL. We find that IBCL improves on the baselines by at most 45% in average per task accuracy, and by 43% in peak per task accuracy, while maintaining a near-zero to positive backward transfer, with a constant training overhead regardless of the number of preferences. Most importantly, IBCL costs only 19% to 29% training cost measured in number of batch updates. We also show that IBCL has a sublinear memory growth along the number of tasks.

**Contributions.** In general, we have the following contributions:

1. We are the first to formulate the problem of Continual Learning under Specific Trade-offs (CLuST) from a Bayesian perspective. This problem requests one model per trade-off preference, and therefore demands efficiency due to a potentially large number of preferences (Section 3).
2. We propose Imprecise Bayesian Continual Learning (IBCL), a Bayesian CL algorithm that solves CLuST. IBCL leverages data structures from Imprecise Probability, and therefore is able to generate models to address arbitrary number of preferences at each task with a fixed training cost (Section 4).
3. We experiment IBCL on standard image classification and NLP CL benchmarks, with at most 45% improvement in average per task accuracy, 43% in peak per task accuracy, almost zero catastrophic forgetting, and most importantly, only 19% to 29% training cost measured in number of batch updates (Section 5).

## 2 Background

### 2.1 Imprecise Probability

Our algorithm hinges upon the concepts of finitely generated credal set (FGCS) from Imprecise Probability (IP) theory (Augustin et al., 2014; Caprio et al., 2024a; Caprio & Seidenfeld, 2023; Caprio, 2025a;b; Walley, 1991).

**Definition 2.1** (Finitely Generated Credal Set)**.** Given a finite set of probability distributions $\{q^j\}_{j=1}^m$, a finitely generated credal set (FGCS) is the convex set

$$\mathcal{Q} = \left\{ q : q = \sum_{j=1}^{m} \beta^j q^j,\ \beta^j \geq 0\ \forall j\ ,\ \sum_{j=1}^{m} \beta^j = 1 \right\}. \tag{1}$$

In other words, FGCS $\mathcal{Q}$ is the convex hull $\mathrm{CH}(\{q^j\}_{j=1}^m)$ of finitely many distributions $\{q^j\}_{j=1}^m$. That is, given a finite collection of distributions $\{q^j\}_{j=1}^m$ (that we call the extreme elements of the credal sets, and denote by $\mathrm{ex}[\mathcal{Q}]$), $\mathcal{Q}$ is the collection of all probability distributions $q$ that can be written as a convex combination of the $q^j$'s. If the state space is finite, then the $q^j$'s can be seen as probability vectors, whose entries represent the probability mass assigned by distribution $q^j$ to the elements of the state space.[1]

Next, we borrow the idea of highest density region (HDR) (Coolen, 1992).

[1]Working with sets of probabilities allows to mitigate problems ensuing from distribution misspecification and/or drift (Kaur et al., 2023; Lin et al., 2024).

**Definition 2.2** (Highest Density Region)**.** Let $\theta \in \Theta$ be a continuous random variable following a probability density function (pdf) $q$, with $\Theta$ being a set of interest.[2] Given a significance level $\alpha \in [0, 1]$, the $(1-\alpha)$-HDR is a subset $\Theta_q^\alpha \subset \Theta$, such that

$$\int_{\Theta_q^\alpha} q(\theta) d\theta \geq 1 - \alpha, \text{ and } \int_{\Theta_q^\alpha} d\theta \text{ is minimal.} \tag{2}$$

In equation 2, requiring that $\int_{\Theta_q^\alpha} d\theta$ is minimal corresponds to requiring that $\Theta_q^\alpha$ has the smallest possible cardinality (i.e., the least possible number of elements). Indeed, if $\Theta$ is finite, it can be replaced by "$|\Theta_q^\alpha|$ is minimal". Since we consider the most general case (in which set $\Theta$ may be uncountable), we must use the integral notion instead of cardinality, as pointed out in previous research (Coolen, 1992). In turn, equation 2 tells us that $\Theta_q^\alpha$ is the set having the smallest number of elements that also satisfies $\Pr_{\theta \sim q}[\theta \in \Theta_q^\alpha] \geq 1 - \alpha$, provided that $\theta \sim q$.[3]

To further explain HDR, an equivalent definition is as follows (Hyndman, 1996).

**Definition 2.3** (Highest Density Region, Alternative)**.** Let $\Theta$ be a set of interest, and consider a significance level $\alpha \in [0, 1]$. Suppose that a (continuous) random variable $\theta \in \Theta$ has probability density function (pdf) $q$.

The *$\alpha$-level Highest Density Region (HDR)* $\Theta_q^\alpha$ is the subset of $\Theta$ such that

$$\Theta_q^\alpha = \{\theta \in \Theta : q(\theta) \geq q^\alpha\}, \tag{3}$$

where $q^\alpha$ is a constant value. In particular, $q^\alpha$ is the largest constant such that $\Pr_{\theta \sim q}[\theta \in \Theta_q^\alpha] \geq 1 - \alpha$.

Some scholars indicate HDRs as the Bayesian counterpart to the frequentist concept of confidence intervals. In dimension 1, $\Theta_q^\alpha$ can be interpreted as the narrowest interval – or union of intervals – in which the value of the (true) parameter falls with probability of at least $1 - \alpha$ according to distribution $q$. We give a simple visual example in Figure 2.

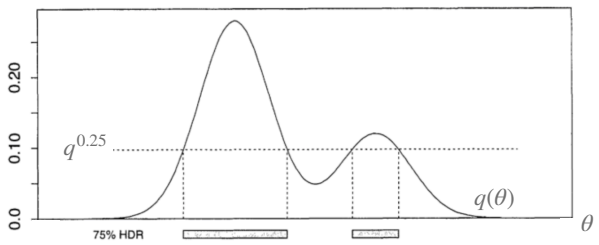


Figure 2: The 0.25-HDR for a Normal Mixture density. This is a replica of Hyndman (1996, Figure 1).

## 2.2 Continual Learning

Continual Learning, also known as lifelong learning, is a special case of multitask learning, where tasks arrive sequentially rather than simultaneously (Thrun, 1998; Ruvolo & Eaton, 2013). In this paper, we leverage Bayesian inference in the knowledge base update (Ebrahimi et al., 2019). Like generic multi-task learning, continual learning also faces the stability-plasticity trade-off (De Lange et al., 2021), which balances between performance on new tasks and resistance to catastrophic forgetting (Kirkpatrick et al., 2017). Current methods identify models to address trade-off preferences by techniques such as loss regularization (Servia-Rodriguez et al., 2021), which means that at least one model must be trained per preference.

Researchers in CL have proposed various approaches to retain knowledge while updating a model on new tasks. These include modified loss landscapes for optimization (Farajtabar et al., 2020), preservation of

[2]Here, for ease of notation, we do not distinguish between a random variable and its realization.

[3]As we can see, the HDR is close in spirit to the concept of Conformal Prediction Region (Angelopoulos & Bates, 2021; Caprio et al., 2024b).

critical pathways via attention (Abati et al., 2020), memory-based methods (Lopez-Paz & Ranzato, 2017), shared representations (Lee et al., 2019), and dynamic representations (Bulat et al., 2020).

One sub-category of CL is Bayesian Continual learning (BCL), which leverages probabilistic methods, defined as follows (Nguyen et al., 2018).

**Definition 2.4** (Bayesian Continual Learning)**.** BCL is a class of CL procedures, which starts with a prior distribution $q_0$. At a task $i \in \{1, 2, ...\}$, we are given i.i.d. training data $\{(x_s, y_s)\}_{s=1}^{n_i} \subset \mathcal{X} \times \mathcal{Y}$ of inputs and outputs (the $x$'s and $y$'s, respectively). The Bayesian model is updated from prior distribution $q_{i-1}$ to posterior $q_i$ using the labeled data. Then, $q_i$ is used as a prior in task $i + 1$.

In BCL (Nguyen et al., 2018; Ebrahimi et al., 2019), each task is associated with a data generating process parameterized by $\theta$. The latter is postulated to be a random quantity, which at the beginning of the analysis has a prior distribution, $\theta \sim q_0$. After training on the available data, the prior distribution is turned into posterior, $\theta \sim q_1$ via Bayes' theorem. The posterior $q_1$ is the revised parameter distribution after having learned from the data pertaining to the first task to complete. It is then used as a prior for the next task.

### 2.3 Multitask and Continual Learning under Preferences

Learning for Pareto-optimal models under task performance trade-offs has been studied by researchers in multitask and continual learning (Caruana, 1997; Sener & Koltun, 2018). Various techniques have been applied to obtain models that address particular trade-off points (Lin et al., 2019; 2020; Ma et al., 2020; Gupta et al., 2021). The idea of preferences on the trade-off points is introduced in multi-objective optimization (Lin et al., 2020; Sener & Koltun, 2018), and a preference can guide learning algorithms to search for a particular model. We borrow the formalization of preferences from (Mahapatra & Rajan, 2020), where a preference is given by a vector of non-negative real weights $\bar{w}$, with each entry $w_i$ corresponding to task $i$. That is, $w_i \geq w_j \iff i \succeq j$. This means that if $w_i \geq w_j$, then task $i$ is preferred to task $j$. However, state-of-the-art algorithms require training one model per preference, imposing large overhead when there is a large number of preferences.

## 3 Formulating the CLuST Problem

In this section we formalize the CLuST problem. We consider domain-incremental learning (Van de Ven & Tolias, 2019; Shi & Wang, 2023) for classification models, with an unbounded number of stability-plasticity trade-off preferences at each task. The goal is to construct a learning algorithm with training overhead independent of the number of preferences, and that enjoys performance guarantees.

### 3.1 Assumptions

Let $\mathcal{X}$ be the space of inputs, and $\mathcal{Y}$ be the space of labels. In a typical classification problem, $\mathcal{X}$ will be a subset of a Euclidean space, and $\mathcal{Y}$ a finite set. In a typical regression problem, $\mathcal{Y}$ will too be a subset of a Euclidean space. In general, we do not limit ourselves to either scenario. As a consequence, we let the input and the output spaces be generic sets. Call $\Delta_{\mathcal{XY}}$ the space of all possible distributions on $\mathcal{X} \times \mathcal{Y}$. A task $i$ is associated with a distribution $p_i \in \Delta_{\mathcal{XY}}$, from which labeled data can be drawn i.i.d.

A common assumption in CL is *task similarity*, which researchers formalize as closeness in data distributions (Wang et al., 2024). Here, we have the same assumption. To define task similarity, we first recall the concept of 2-Wasserstein metric (Deza & Deza, 2013) on the data distributions.

**Definition 3.1** (2-Wasserstein Metric on $\Delta_{\mathcal{XY}}$)**.** The 2-Wasserstein metric is a distance $||\cdot||_{W_2}$ that measures the dissimilarity between two probability distributions $p$ and $p' \in \Delta_{\mathcal{XY}}$, with

$$\|p - p'\|_{W_2} := \Big( \inf_{\gamma \in \Gamma(p,p')} \mathbb{E}_{((x_1,y_1),(x_2,y_2))\sim\gamma}[d((x_1,y_1),(x_2,y_2))^2] \Big)^{\frac{1}{2}}, \tag{4}$$

where

1. $\Gamma(p, p')$ is the set of all couplings of $p$ and $p'$. A coupling $\gamma$ is a joint probability measure on $(\mathcal{X}\times\mathcal{Y})\times(\mathcal{X}\times\mathcal{Y})$ whose marginals are $p$ and $p'$ on the first and second factors, respectively, and
2. $d$ is the product metric endowed to $\mathcal{X} \times \mathcal{Y}$.[4]

With Definition 3.1, we have the following assumption.

**Assumption 3.2** (Task Similarity)**.** *For all task $i$, $p_i \in \mathcal{F}$, where $\mathcal{F}$ is a convex subset of $\Delta_{\mathcal{X}\mathcal{Y}}$. Also, we assume that the diameter of $\mathcal{F}$ is some $r > 0$, that is, $\sup_{p,q\in\mathcal{F}} \|p - q\|_{W_2} \leq r$, where $\|\cdot\|_{W_2}$ denotes the 2-Wasserstein distance.*

Assumption 3.2 states that the true data-generating processes about different tasks are not too distant. In addition, such a notion of "being not too distant" is entirely in the hands of the user, via the choice of radius $r$ and of the metric to endow $\Delta_{\mathcal{X}\mathcal{Y}}$. This assumption means that we do not expect very dissimilar tasks. That is, we do not consider e.g. a situation in which a robot is able to fold our clothes (task 1) and then deliver a payload in combat zone (task 2). Details of this assumption, including its importance, and why 2-Wasserstein metric is chosen, are explained in Section 3.2.

Next, we assume the parameterization of class $\mathcal{F}$.

**Assumption 3.3** (Parameterization of Task Distributions)**.** *Every distribution $F$ in $\mathcal{F}$ is parameterized by $\theta$, a parameter belonging to a parameter space $\Theta$.*

Let us give an example of a parameterized family $\mathcal{F}$. Suppose that we have one-dimensional data points and labels. At each task $i$, the marginal on $\mathcal{X}$ of $p_i$ is a Gaussian $\mathcal{N}(\mu, 1)$, while the conditional distribution of label $y \in \mathcal{Y}$ given data point $x \in \mathcal{X}$ is a categorical $\text{Cat}(\vartheta)$. Hence, the parameter for $p_i$ is $\theta = (\mu, \vartheta)$, and it belongs to $\Theta = \mathbb{R} \times \mathbb{R}^{|\mathcal{Y}|}$. In this situation, an example of a family $\mathcal{F}$ that satisfies Assumptions 3.2 and 3.3 is the convex hull of distributions that can be decomposed as we just described, and whose distance according to the 2-Wasserstein metric does not exceed some $r > 0$.

Notice that all tasks share the same input space $\mathcal{X}$ and label space $\mathcal{Y}$, and we do not have task id's as an additional input, so learning is domain-incremental (Van de Ven & Tolias, 2019).

Preferences over stability-plasticity trade-offs is also an established concept (Mahapatra & Rajan, 2020; Servia-Rodriguez et al., 2021). We formalize it as follows.

**Definition 3.4** (Stability-plasticity Trade-off Preferences over Tasks)**.** *Consider $k$ tasks with underlying data distributions $p_1, p_2, \ldots, p_k$. We express a stability-plasticity trade-off preference (or simply, a preference) over them through a probability vector $\bar{w} = (w_1, w_2, \ldots, w_k)^\top$. That is, $w_i \geq 0$ for all $i \in \{1, \ldots, k\}$, and $\sum_{i=1}^k w_i = 1$.*

Based on Definition 3.4, given a preference $\bar{w}$ over all $k$ tasks encountered, the personalized model for the user aims to learn the distribution $p_{\bar{w}} := \sum_{i=1}^k w_i p_i$. Since $p_{\bar{w}}$ is the convex combination of $p_1, \ldots, p_k$, thanks to Assumptions 3.2 and 3.3, we have $p_{\bar{w}} \in \mathcal{F}$, and therefore it is also parameterized by some $\theta \in \Theta$.

Like in existing BCL literature (Nguyen et al., 2018; Kessler et al., 2023; Servia-Rodriguez et al., 2021), we assume that the learning procedure is Bayesian domain-incremental learning. That is, the learning follows BCL as in Definition 2.4, and all data and label distributions are similar, as per Assumption 3.2, without any knowledge of task id's. At any task $k$, we are given at least one user preference $\bar{w}$ over the $k$ tasks so far. The data drawn for task $k + 1$ will not be available until we have finished learning models for all preferences on task $k$.

### 3.2 Details of Assumption 3.2

We need Assumption 3.2 in light of the results in Kessler et al. (2023), where it is shown that misspecified models can suffer from catastrophic forgetting even when Bayesian inference is carried out exactly. By requiring that $\text{diam}(\mathcal{F}) = r$, we control the amount of misspecification via $r$. In Kessler et al. (2023), the authors design a new approach – called Prototypical Bayesian Continual Learning, or ProtoCL – that allows

[4]We denote by $d_{\mathcal{X}}$ and $d_{\mathcal{Y}}$ the metrics endowed to $\mathcal{X}$ and $\mathcal{Y}$, respectively.

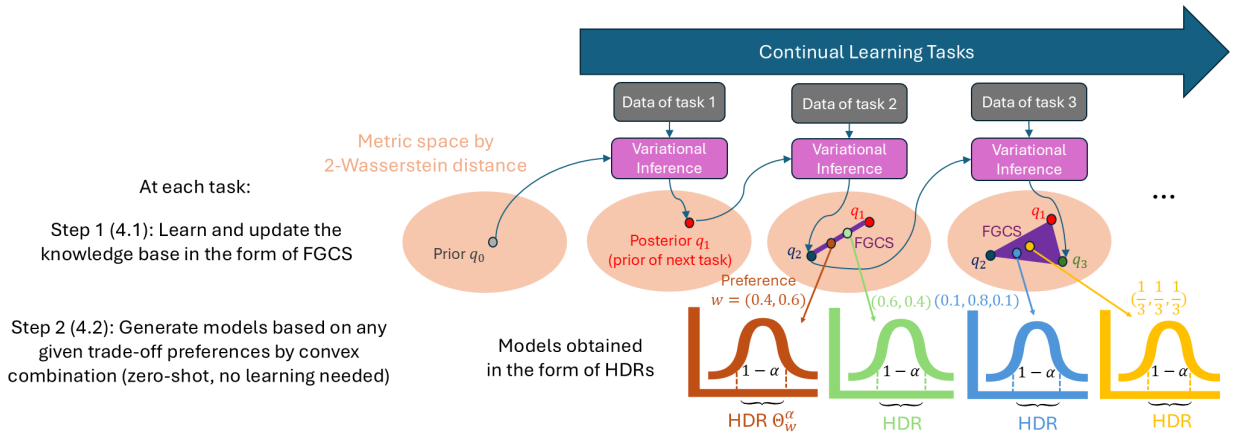


Figure 3: The workflow of Imprecise Bayesian Continual Learning (IBCL). Here, we start from 1 prior, but in practice, there may be more than 1 to reduce epistemic uncertainty (Hüllermeier & Waegeman, 2021).

dropping Assumption 3.2 while retaining the Bayesian benefit of remembering previous tasks. Because the main goal of this paper is to come up with a procedure that allows the designer to express preferences over the tasks, we retain Assumption 3.2, and we work in the classical framework of Bayesian Continual Learning. In the future, we plan to generalize our results by operating with ProtoCL.[5]

We choose the 2-Wasserstein distance for the ease of computation. In practice, when all distributions are modeled by Bayesian neural networks with independent Gaussian weights and biases, we have

$$\|q_1 - q_2\|_{W_2}^2 = \|\mu_{q_1}^2 - \mu_{q_2}^2\|_2^2 + \|\sigma_{q_1}^2 \mathbf{1} - \sigma_{q_2}^2 \mathbf{1}\|_2^2, \tag{5}$$

where $\|\cdot\|_2$ denotes the Euclidean norm, $\mathbf{1}$ is a vector of all 1's, and $\mu_q$ and $\sigma_q$ are respectively the mean and standard deviation of a multivariate normal distribution $q$ with independent dimensions, $q = \mathcal{N}(\mu_q, \sigma_q^2 I)$, $I$ being the identity matrix. Therefore, computing the $W_2$-distance between two distributions is equivalent to computing the difference between their means and variances.

### 3.3 Main Problem

We aim to design a domain-incremental learning algorithm that generates one model per preference over tasks, with an unbounded number of preferences over a finite number of tasks. Given a significance level $\alpha \in [0, 1]$, in any task $k$, the algorithm should satisfy:

1. **Zero-shot preferred model generation**. A fixed training cost is needed at each task, regardless of the number of preferences. In other words, we only need to train a small fixed number of models per task, and after that, model generation for any preference is zero-shot.
2. **Probabilistic Pareto-optimality**. Let $\hat{q}_{\bar{w}}$ denote the convex combination of the estimated parameter distributions for tasks $1, \ldots, k$ using preference weights $\bar{w}$. IBCL should be able to identify the smallest subset of model parameters, $\Theta_{\hat{q}_{\bar{w}}}^\alpha \subset \Theta$ (written as $\Theta_{\bar{w}}^\alpha$ for notational convenience from now on), that contains the Pareto-optimal parameter $\theta_{\bar{w}}^\star$ with high probability. Formally, $\Theta_{\bar{w}}^\alpha$ is the minimal set that satisfies $\Pr_{\theta_{\bar{w}}^\star \sim \hat{q}_{\bar{w}}}[\theta_{\bar{w}}^\star \in \Theta_{\bar{w}}^\alpha] \geq 1 - \alpha$.
3. **Sublinear buffer growth**. The memory overhead accumulated by IBCL throughout the tasks should grow sublinearly in the number of tasks.

[5]In Kessler et al. (2023), the authors also show that if there is a task dataset imbalance, then the model can forget under certain assumptions. To avoid complications, in this work we tacitly assume that task datasets are balanced.

# 4 Imprecise Bayesian Continual Learning

Figure 3 shows that IBCL performs two steps in each task. First, it updates the knowledge base as a FGCS (Section 4.1). Second, it uses a convex combination of the extreme elements of the FGCS, instead of retraining, to zero-shot generate models under given preferences (Section 4.2).

## 4.1 FGCS Knowledge Base Update

As discussed in the Introduction, we take a Bayesian Continual Learning (BCL) approach, that is, the parameter $\theta$ of the distribution $p_k$ related to task $k$ is viewed as a random variable distributed according to some distribution $q$.

At the beginning of the analysis, we specify $m$ many such distributions, $\text{ex}[\mathcal{Q}_0] = \{q_0^1, \ldots, q_0^m\}$. They are the ones that the designer deems plausible – a priori – for the parameter $\theta$ of the task 1. Upon observing data from task 1, we learn a set $\mathcal{Q}_1^{tmp}$ of posterior parameter distributions and buffer them as extreme elements $\text{ex}[\mathcal{Q}_1]$ of the FGCS $\mathcal{Q}_1$ corresponding to task 1. We proceed in a similar way for successive tasks $i \geq 2$.

**Algorithm 1** FGCS Knowledge Base Update

1: **Input:** Current knowledge base in the form of FGCS extreme elements $\text{ex}[\mathcal{Q}_{i-1}] = \{q_{i-1}^1, \ldots, q_{i-1}^m\}$, observed labeled data $(\bar{x}_i, \bar{y}_i)$ at task $i$, and distribution distance threshold $d \geq 0$
2: **Output:** Updated extreme elements $\text{ex}[\mathcal{Q}_i]$
3: $\mathcal{Q}_i^{tmp} \leftarrow \emptyset$
4: **for** $j \in \{1, \ldots, m\}$ **do**
5: $\quad q_i^j \leftarrow \mathsf{variational\_inference}(q_{i-1}^j, \bar{x}_i, \bar{y}_i)$
6: $\quad d_i^j \leftarrow \min_{q \in \text{ex}[\mathcal{Q}_{i-1}]} \|q_i^j - q\|_{W_2}$
7: $\quad$ **if** $d_i^j \geq d$ **then**
8: $\quad\quad \mathcal{Q}_i^{tmp} \leftarrow \mathcal{Q}_i^{tmp} \cup \{q_i^j\}$ {Store distribution $q_i^j$}
9: $\quad$ **else**
10: $\quad\quad q_i^j \leftarrow \arg\min_{q \in \text{ex}[\mathcal{Q}_{i-1}]} \|q_i^j - q\|_{W_2}$ {Fetch the stored distribution with minimal distance to $q_i^j$, and overwrite $q_i^j$ with a pointer to that distribution}
11: $\quad\quad \mathcal{Q}_i^{tmp} \leftarrow \mathcal{Q}_i^{tmp} \cup \{q_i^j\}$ {Only a pointer is stored}
12: $\quad$ **end if**
13: **end for**
14: $\text{ex}[\mathcal{Q}_i] \leftarrow \text{ex}[\mathcal{Q}_{i-1}] \cup \mathcal{Q}_i^{tmp}$

In Algorithm 1, we use notation $(\bar{x}_i, \bar{y}_i)$ to denote vectors of inputs and outputs pertaining to task $i$. In task $i$, we approximate $m$ posteriors $q_i^1, \ldots q_i^m$ by variational inference from buffered priors $q_{i-1}^1, \ldots q_{i-1}^m$ one-by-one (line 3). Variational inference is a standard Bayesian learning procedure that minimizes the evidence lower bound (ELBO) loss to infer a posterior distribution from a prior and observed data (Nguyen et al., 2018). However, if a posterior is very similar to an existing prior in the cache, it would give estimations with negligible differences to that prior. In this case, buffering this new posterior would be a waste in space. Therefore, we use a distance threshold $d$ to exclude the posteriors that are similar to the distributions that are already buffered (lines 4 - 10). When distributions similar to $q_i^j$ (within threshold $d$) are found in the knowledge base, we store a pointer to the distribution with minimal distance in place of $q_i^j$, and do not memorize $q_i^j$ (lines 8-9). The posteriors that are sufficiently different from the already buffered distributions are then appended to the knowledge base (line 12).

Notice that the memory overhead of Algorithm 1 is remembering at most $m$ distributions into $\mathcal{Q}_i^{tmp}$ at line 6. In practice, $m$ is a small constant (we choose $m = 3$ in our experiments). Therefore, the memory complexity is $O(1)$. Moreover, some newly memorized distributions may be discarded and replaced by a previous distribution in cache at line 8. With larger threshold $d$ at line 5, more distributions are discarded at lines 8-9. The amortized memory complexity analysis under different threshold $d$'s is discussed in our ablation studies: see Section 5.1.

The time complexity of Algorithm 1 is dominated by variational inference at line 3. Every variational inference costs a non-negligible training time, which we denote as $O(v)$. There are a total of $m$ variational inferences computed for each task. Since $m$ is a constant, the overall time complexity remains $O(v)$.

### 4.2 Zero-shot Generation of User Preferred Models

Next, after having updated the FGCS extreme elements for task $i$, we are given a set of user preferences. For each preference $\bar{w}$, we need to identify the Pareto-optimal parameter $\theta^\star_{\bar{w}}$ for the preferred data distribution $p_{\bar{w}}$. This procedure can be divided into two steps as follows.

First, we find the parameter distribution $\hat{q}_{\bar{w}}$ via a convex combination of the extreme elements in the knowledge base, whose weights correspond to the entries of preference vector $\bar{w} = \{w_1, \ldots, w_i\}$ over the $i$ tasks so far. That is,

$$\hat{q}_{\bar{w}} = \sum_{k=1}^{i} \sum_{j=1}^{m_k} \beta_k^j q_k^j, \text{ where } \sum_{j=1}^{m_k} \beta_k^j = w_k, \text{ and } \beta_k^j \geq 0, \text{ for all } j \text{ and all } k. \tag{6}$$

Here, $q_k^j$ is a buffered extreme point of FGCS $\mathcal{Q}_k$, i.e. the $j$-th parameter posterior of task $k$. The weight $\beta_k^j$ of this extreme point is decided by preference vector entry $\bar{w}_j$. In implementation, if we have $m_k$ extreme elements stored for task $k$, we can choose equal weights $\beta_k^1 = \cdots = \beta_k^m = w_k/m_k$. For example, if we have preference $\bar{w} = (0.8, 0.2)^\top$ on two tasks so far, and we have two extreme elements per task stored in the knowledge base, we can use $\beta_1^1 = \beta_1^2 = 0.8/2 = 0.4$ and $\beta_2^1 = \beta_2^2 = 0.2/2 = 0.1$.

The following proposition ensures us that it is equivalent to express preferences over tasks $k$, or over the parameter distributions $q_k^j$ associated with each task, thus justifying the definition of $\hat{q}_{\bar{w}}$ in equation 6.

**Proposition 4.1** (Selection Equivalence)**.** *Let $q_k^j$ be an extreme point posterior of $\mathcal{Q}_i$ learned from the $j$-th prior at task $k \in \{1, \ldots, i\}$. For any preference $\bar{w} = (w_1, \ldots, w_i)^\top$ on tasks $\{1, \ldots, i\}$, there exists a probability vector $\bar{\beta} = (\beta_1^1, \ldots, \beta_1^{m_1}, \ldots, \beta_i^1, \ldots, \beta_i^{m_i})^\top$, with $\sum_{j=1}^{m_k} \beta_k^j = w_k$, for all $k \in \{1, \ldots, i\}$, such that*

$$\hat{q}_{\bar{w}} = \sum_{k=1}^{i} \sum_{j=1}^{m_k} \beta_k^j q_k^j.$$

*In other words, selecting a precise distribution $\hat{q}_{\bar{w}}$ from $\mathcal{Q}_i$ is equivalent to specifying a preference weight vector $\bar{w}$ on tasks $\{1, \ldots, i\}$.*

We refer to Appendix B for the proof.

Second, we compute the HDR $\Theta^\alpha_{\bar{w}} \subset \Theta$ from $\hat{q}_{\bar{w}}$. This is implemented using a standard procedure that locates the smallest region in the parameter space whose enclosed probability mass is (at least) $1-\alpha$, according to $\hat{q}_{\bar{w}}$. This procedure can be routinely implemented, e.g., in R, using package HDInterval (Juat et al., 2022). As a result, we locate the smallest set of parameters $\Theta^\alpha_{\bar{w}} \subset \Theta$ associated with the preference $\bar{w}$. This subroutine is formalized in Algorithm 2. Notice that this computation is simply a convex combination, i.e., a weighted sum of all distributions in ex$[\mathcal{Q}_i]$. The summation is defined under 2-Wasserstein metric. As explained in Section 3.2, the convex combination has a computational complexity proportional to the parameterization size of distributions. In practice, we first extract features from the data and use a relatively small parameterization for distributions on top of the extracted features. Please see Section 5 for details. Furthermore, this algorithm does not produce any memory overhead.

### 4.3 Overall IBCL Algorithm and Analysis

From the two subroutines in Sections 4.1 and 4.2, we construct the overall IBCL algorithm as in Algorithm 3.

For each task, in line 3, we use Algorithm 1 to update the knowledge base by learning $m$ posteriors from the current priors. In lines 5-7, according to a user-given preference over all tasks so far, we obtain the HDR of the model associated with preference $\bar{w}$ in zero-shot via Algorithm 2. Notice that this HDR computation

**Algorithm 2** Preference HDR Computation

1: **Input:** Knowledge base $\text{ex}[\mathcal{Q}_i]$ with $m_k$ extreme elements saved for task $k \in \{1, \dots, i\}$, preference $\bar{w}$ on the $i$ tasks, significance level $\alpha \in [0, 1]$
2: **Output:** HDR $\Theta^{\alpha}_{\bar{w}} \subset \Theta$
3: **for** $k = 1, \dots, i$ **do**
4: $\quad \beta^1_k = \cdots = \beta^m_k \leftarrow w_k / m_k$
5: **end for**
6: $\hat{q}_{\bar{w}} = \sum_{k=1}^{i} \sum_{j=1}^{m_k} \beta^j_k q^j_k$
7: $\Theta^{\alpha}_{\bar{w}} \leftarrow \mathsf{hdr}(\hat{q}_{\bar{w}}, \alpha)$

**Algorithm 3** Imprecise Bayesian Continual Learning

1: **Input:** Prior distributions $\text{ex}[\mathcal{Q}_0] = \{q^1_0, \dots, q^m_0\}$, hyperparameters $\alpha$ and $d$
2: **Output:** HDR $\Theta^{\alpha}_{\bar{w}}$ for each given preference $\bar{w}$ at each task $i$
3: **for** task $i = 1, 2, \dots$ **do**
4: $\quad \bar{x}_i, \bar{y}_i \leftarrow$ sample $n_i$ labeled data points i.i.d. from $p_i$
5: $\quad \text{ex}[\mathcal{Q}_i] \leftarrow \mathsf{fgcs\_update}(\text{ex}[\mathcal{Q}_{i-1}], \bar{x}_i, \bar{y}_i, d)$ {Algorithm 1}
6: $\quad$ **while** user has a new preference **do**
7: $\quad\quad \bar{w} \leftarrow$ user input
8: $\quad\quad \Theta^{\alpha}_{\bar{w}} \leftarrow \mathsf{preference\_hdr\_computation}(\text{ex}[\mathcal{Q}_i], \bar{w}, \alpha)$ {Algorithm 2}
9: $\quad$ **end while**
10: **end for**

does not require the initial priors $\text{ex}[\mathcal{Q}_0]$, so we can discard them once the posteriors $\mathcal{Q}_1$ are learned in the first task.

The overall time complexity is dominated by the $O(v)$ variational inference in Algorithm 1, used as a subroutine in line 3. Compared to variational inference, the $O(1)$ preferred model generation via convex combination in Algorithm 2 in line 7 is negligible. Therefore, the overall time complexity for $n$ tasks is $O(nv)$, regardless of preferred model generation. Moreover, as the memory complexity at each task is contributed by $O(1)$ memorization of posteriors by Algorithm 1, the total memory complexity is $O(n)$. Some of these posteriors will be discarded, as discussed in Section 4.1. Therefore, in the amortized case, Algorithm 3 ensures **sublinear buffer growth**.

The following proposition ensures that IBCL locates the user-preferred Pareto-optimal model with high probability.

**Proposition 4.2** (Probabilistic Pareto-optimality)**.** *Pick any $\alpha \in [0, 1]$. The Pareto-optimal parameter $\theta^{\star}_{\bar{w}}$, i.e., the ground-truth parameter for $p_{\bar{w}}$, belongs to $\Theta^{\alpha}_{\bar{w}}$ with probability at least $1 - \alpha$ under distribution $\hat{q}_{\bar{w}}$. In formulas, $\Pr_{\theta^{\star}_{\bar{w}} \sim \hat{q}_{\bar{w}}} [\theta^{\star}_{\bar{w}} \in \Theta^{\alpha}_{\bar{w}}] \geq 1 - \alpha$.*

Proposition 4.2 gives us a $(1 - \alpha)$-guarantee in obtaining Pareto-optimal models for given task trade-off preferences. In other words, the Pareto-optimal parameter $\theta^{\star}_{\bar{w}}$ is guaranteed to belong to the Highest Density Region $\Theta^{\alpha}_{\bar{w}}$ that we build, with high probability. Our algorithm does not find the parameter $\theta^{\star}_{\bar{w}}$ itself, but instead the narrowest region $\Theta^{\alpha}_{\bar{w}}$ that contains it with high probability. In spirit, this result is very similar to what conformal prediction does (for predicted outputs, rather than parameters of interest (Angelopoulos & Bates, 2021)). Consequently, the IBCL algorithm enjoys the **probabilistic Pareto-optimality** targeted by our main problem. Please refer to Appendix B for the proof.

## 5 Experiments

### 5.1 Setup

**Baselines.** Although there are many baseline methods for CL, only a few baselines for CLuST exist. The following CLuST baselines are selected for comparison.

1. **Convex Combination of Deterministic Models.** This is the deterministic version of our approach. It trains one deterministic model per task and combine the model weights using the preference vectors.
2. **Rehearsal-based Deterministic Models.** This is the state-of-the-art technique for CLuST (Lin et al., 2019). These methods memorize a subset of training data for every task encountered. Task preferences are then given as weights to regularize the loss on each task's memorized data. We choose (i) GEM (Lopez-Paz & Ranzato, 2017), (ii) A-GEM (Chaudhry et al., 2018), (iii) DER, and (iv) DER++ (Buzzega et al., 2020) as baselines.
3. **Rehearsal-based Bayesian Models.** We also compare IBCL with a Bayesian technique, VCL (Nguyen et al., 2018). We equip VCL with episodic memory to make it rehearsal-based and to be able to specify a preference, an approach that has been used in (Servia-Rodriguez et al., 2021).
4. **Prompt-based.** *Prompt-based CL has never been used for CLuST and, therefore, is not state-of-the-art.* Still, they are considered efficient modern CL techniques. Therefore, we attempted to specify preferences in L2P (Wang et al., 2022), a prompt-based method, by training a learnable prompt prefix per task and using a preference-weighted sum of the prompts at inference time.

**Datasets.** We experiment on four standard continual learning benchmarks, including three image classification and one NLP. The same datasets or datasets of the same scale has been used by the baseline papers (Lopez-Paz & Ranzato, 2017; Chaudhry et al., 2018; Buzzega et al., 2020). These include

1. 5 tasks in 20 News Group (Lang, 1995) (news related to computers vs. not related to computers).
2. 10 tasks in Split CIFAR-100 (Zenke et al., 2017) (animals vs. non-animals),
3. 10 tasks in Tiny ImageNet (Le & Yang, 2015) (animals vs. non-animals), and
4. 15 tasks in CelebA (Liu et al., 2015) (with vs. without attributes).

The features are first extracted by ResNet-18 (He et al., 2016) for the first three image benchmarks. For 20 News Group, features are extracted by TF-IDF (Aizawa, 2003). For each benchmark, all tasks share the same input and label space. There is no task id at training or inference time, so the algorithm does not know which task each data point comes from. Therefore, all experiments are domain-incremental according to Van de Ven & Tolias (2019).

**Evaluation metrics.** To evaluate how well a model addresses preferences, we randomly generate $n_{\text{prefs}}$ preferences per task, except for task 1, whose preference is always a scalar 1. Formally, at each task $i > 1$, we have preferences $\bar{w}_i^k = (w_{i1}^k, \dots, w_{ii}^k)$ for $k = 1, \dots n_{\text{prefs}}$.

Like all continual learning evaluations, after training on a task $i$, we first evaluate the accuracy $acc_{ij}$ of the current model on the testing sets of all tasks $j = 1, \dots, i$ encountered so far. To do so, the method (a baseline or IBCL) computes an $i$-dimensional accuracy vector for each preference. That is, we have

$$\bar{acc}_i^k = (acc_{i1}^k, \dots, acc_{ii}^k), \text{ for } k = 1, \dots, n_{\text{prefs}}. \tag{7}$$

Then, to compute the accuracy $acc_{ij}$ that takes account of all preferences, we do a weighted sum of each $acc_{ij}^k$ for all $k$. Since the preference indicates how important a task is considered, it is computed as

$$acc_{ij} = \frac{1}{\sum_{k=1}^{n_{\text{prefs}}} w_{ij}^k} \sum_{k=1}^{n_{\text{prefs}}} w_{ij}^k acc_{ij}^k. \tag{8}$$

For example, at task $i = 2$, suppose we have $n_{\text{prefs}} = 3$, with preferences $(0.5, 0.5)$, $(0.1, 0.9)$ and $(0.8, 0.2)$. The method evaluates on the testing data of task 1 with accuracies $a$, $b$ and $c$, respectively for the 3 preferences. We therefore have $acc_{21} = \frac{1}{0.5+0.1+0.8}(0.5a + 0.1b + 0.8c)$.

In the experiments, we set $n_{\text{prefs}} = 10$. After obtaining $acc_{ij}$, we use state-of-the-art continual learning metrics to evaluate performance with

1. Average per task accuracy: $\frac{1}{i}\sum_{j=1}^{i} acc_{ij}$, and
2. Peak per task accuracy: $\max_{j=1,\ldots,i} acc_{ij}$.

Starting from task $i = 2$, resistance to catastrophic forgetting is evaluated by

3. Backward transfer (Díaz-Rodríguez et al., 2018): $\frac{1}{i-1}\sum_{j=1}^{i-1}(acc_{ij} - acc_{i-1,j})$, with a more positive value indicating higher resistance, and a more negative value indicating higher forgetting.

**System.** Experiments are run on Intel(R) Core(TM) i7-8550U CPU @ 1.80GHz.

Detailed experiment configurations can be found in Appendix C.

### 5.2 Main Results

We present the three metrics (average per task accuracy, peak per task accuracy, and backward transfer) on the four datasets. The results of 20 News Group and Split CIFAR-100 are illustrated in Figure 4, and Tiny ImageNet and CelebA in Figure 5. Our results support the claim that IBCL not only achieves high performance by probabilistic Pareto-optimality, but is also efficient with zero-shot generation of models.

From Figures 4 and 5, we can see that IBCL overall generates the model with top performance (high accuracy) in all cases, while maintaining little catastrophic forgetting (near zero to positive backward transfer). This is due to the probabilistic Pareto-optimality guarantee. Statistically, IBCL improves on baselines by at most 45% on average per task accuracy, and by 43% on peak per task accuracy (compared to convex combination of deterministic models in 20 News Group). So far, to our knowledge, there is no discussion on how to specify a task trade-off preference in prompt-based continual learning, and we only make an attempt for L2P, which generally works poorly.

As illustrated in the figures, IBCL has a slightly negative backward transfer at first, but then this value converges to near-zero or positive. This shows that although IBCL may slightly forget the knowledge learned from the first task in the second task, it steadily retains knowledge afterward.

Although some baselines, such as VCL and DER, have backward transfer higher than IBCL's in the first few tasks, they eventually decrease to similar to or smaller than that of IBCL. This happens at the 5th task of 20 News Group, 8th task of Split CIFAR-100, 6th task of Tiny ImageNet, and 5th task of CelebA.

We measure training overhead in terms of # of batch updates required at a task in Table 1. Here, $n_i$: # of training data points at task $i$, $n_{\text{prefs}}$: # of preferences per task, $n_{\text{mem}}$: # of data points memorized per task in rehearsal, $n_{\text{priors}}$: # of priors in IBCL, $e$: # of epochs and $b$: batch size. Notice that the overhead of rehearsal-based methods is proportional to $n_{\text{prefs}}$, which is potentially a large number.

Table 1: Training overhead comparison, with hyperparameters setup in Appendix C.

| | # batch updates at task $i$ | # batch updates at last task | | | |
|---|---|---|---|---|---|
| | | CelebA | CIFAR100 | TImgNet | 20News |
| Convex Comb | $n_{\text{prefs}} \times n_i \times e/b$ | 95384 | 12500 | 9380 | 29063 |
| GEM<br>A-GEM<br>DER<br>DER++<br>VCL | $n_{\text{prefs}} \times (n_i + (i-1) \times n_{\text{mem}}) \times e/b$ | 99747 | 19532 | 13594 | 35313 |
| L2P | $n_i \times e/b$ | 9538 | 1250 | 938 | 2907 |
| IBCL (ours) | $n_{\text{priors}} \times n_i \times e/b$ | 28614 | 3750 | 2814 | 8721 |

Table 1 shows the training overhead comparison measured in number of batch updates per task. We can see how IBCL's overhead is independent of the number of preferences $n_{\text{prefs}}$ because it only requires training for

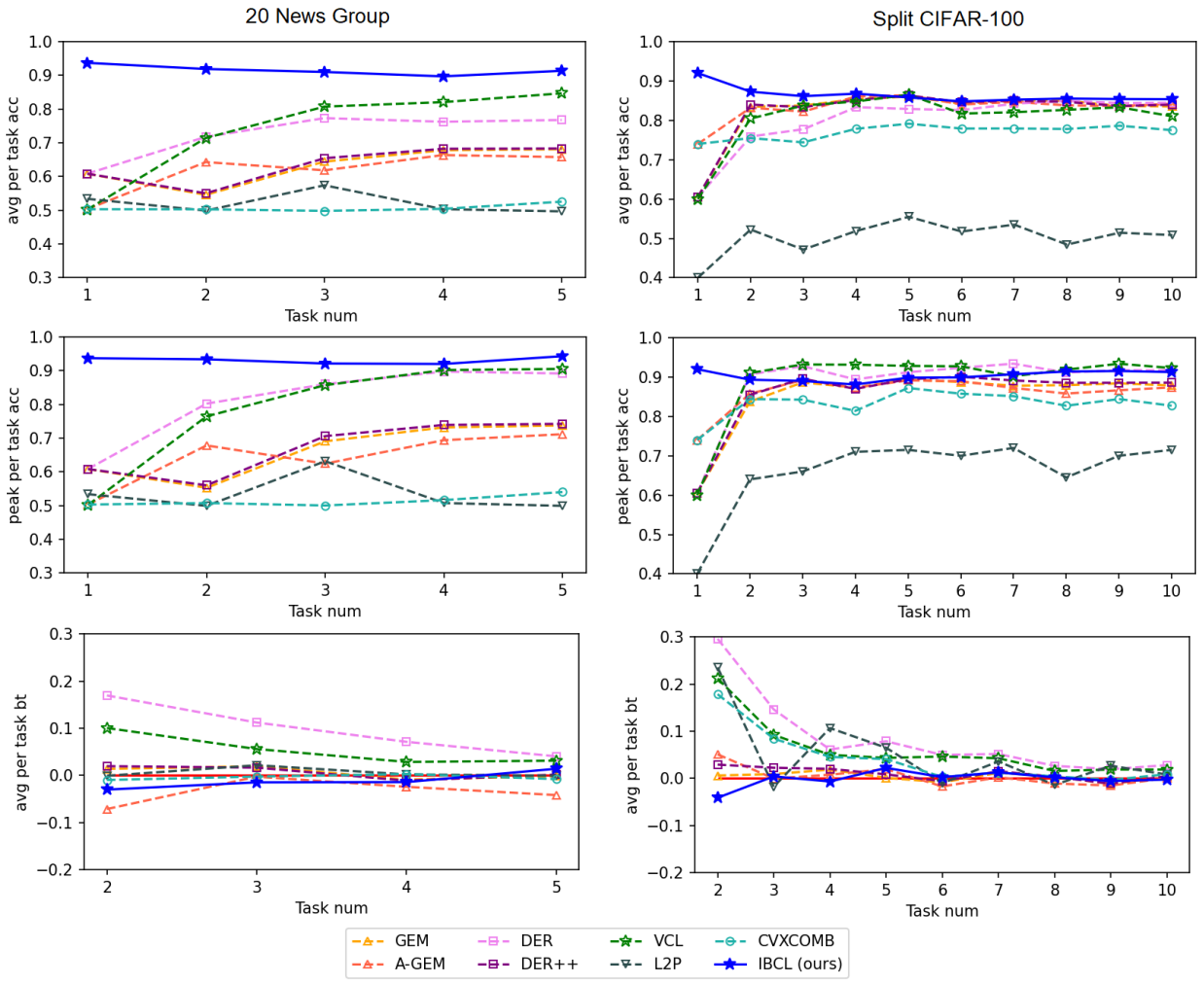


Figure 4: Results of 20 News Group (left column) and Split CIFAR-100 (right column).

the FGCS but not for the preferred models. From this table, we see that in terms of batch updates, IBCL costs at least 19% as the rehearsal baselines (CelebA) and at most 29% (Split CIFAR-100). **Consequently, our experiments show that IBCL is able to maintain a constant training overhead per task, regardless of $n_{\text{prefs}}$ while achieving high performance. Although L2P also has this constant overhead, its performance is too poor to be acceptable.**

### 5.3 Abalation Studies

The main experiments are conducted with $\alpha = 0.01$, $d = 0.002$, prior distributions specified in Appendix C, and equal-sized $\beta$. Here, we conduct ablation studies on these hyperparameters.

#### 5.3.1 Different $\alpha$'s

Here, we evaluate the effects of choosing different significance level $\alpha$. We experiment on 20 News Group and Split CIFAR100. The variations include

1. $\alpha = 0.01$, same as the main experiments.

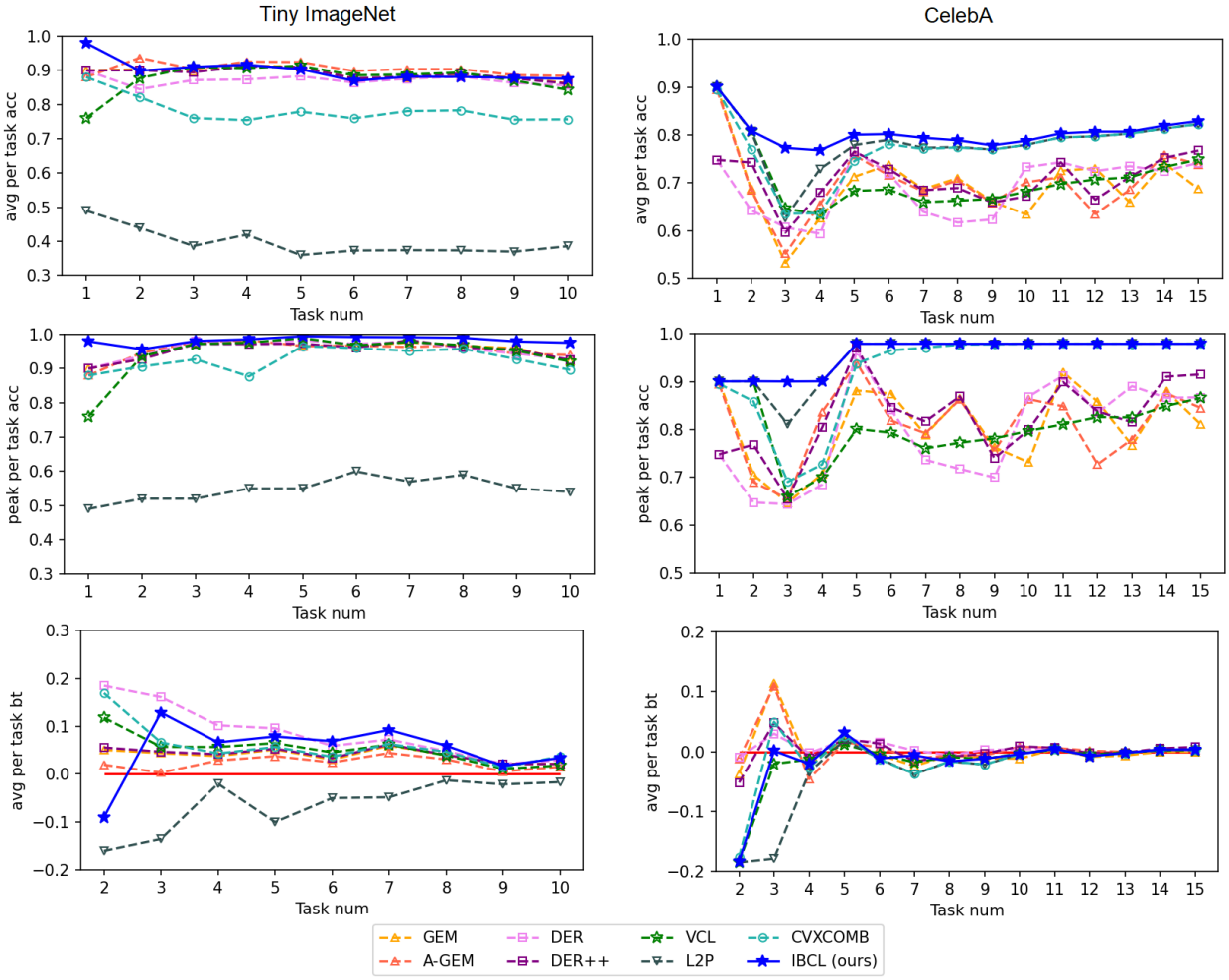


Figure 5: Results of Tiny ImageNet (left column) and CelebA (right column).

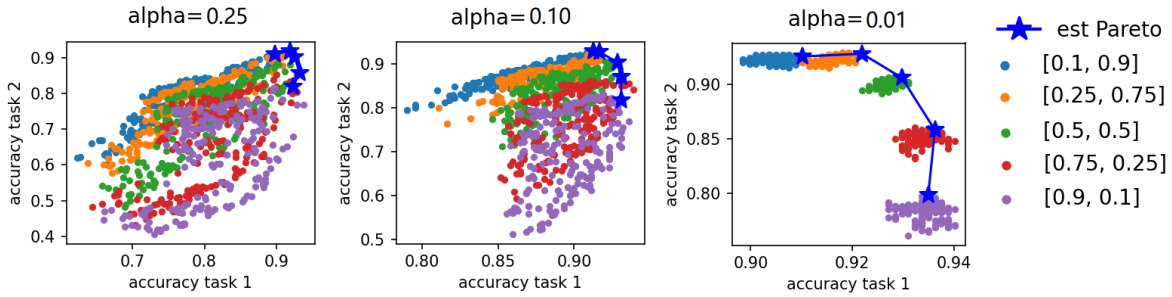


Figure 6: Different $\alpha$'s on different preferences over the first two tasks in 20 News Group.

2. $\alpha = 0.1$.
3. $\alpha = 0.25$.

In Figure 6, we evaluate testing accuracy on three different $\alpha$'s over five different preferences (from $[0.1, 0.9]$ to $[0.9, 0.1]$) on the first two tasks of 20 News Group. For each preference, we uniformly sample 200 deterministic

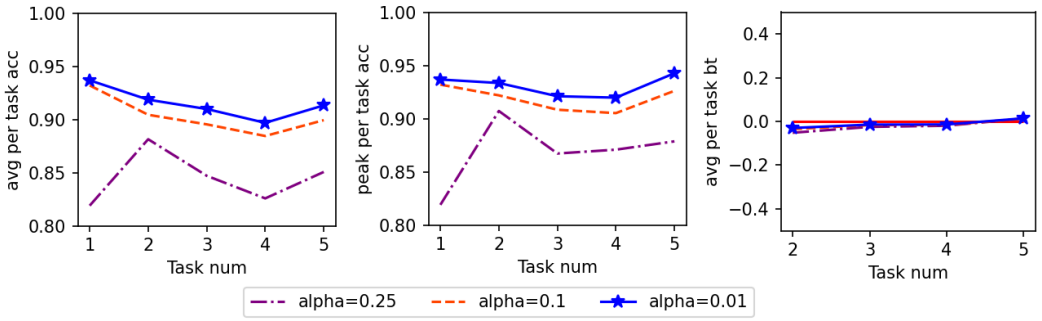


Figure 7: Different $\alpha$'s on randomly generated preferences over all tasks in 20 News Group.

models from the HDR. We use the sampled model with the maximum L2 sum of the two accuracies to estimate the Pareto optimality under a preference. We can see that, as $\alpha$ approaches 0, we tend to sample closer to the Pareto front. This is because, with a smaller $\alpha$, HDRs become wider and we have a higher probability to sample Pareto-optimal models according to Proposition 4.2. For instance, when $\alpha = 0.01$, we have a probability of at least 0.99 that the Pareto-optimal solution is contained in the HDR. Figure 7 shows that the performance drops as $\alpha$ increases, because we are more likely to sample poorly performing models from the HDR.

### 5.3.2 Different $d$'s

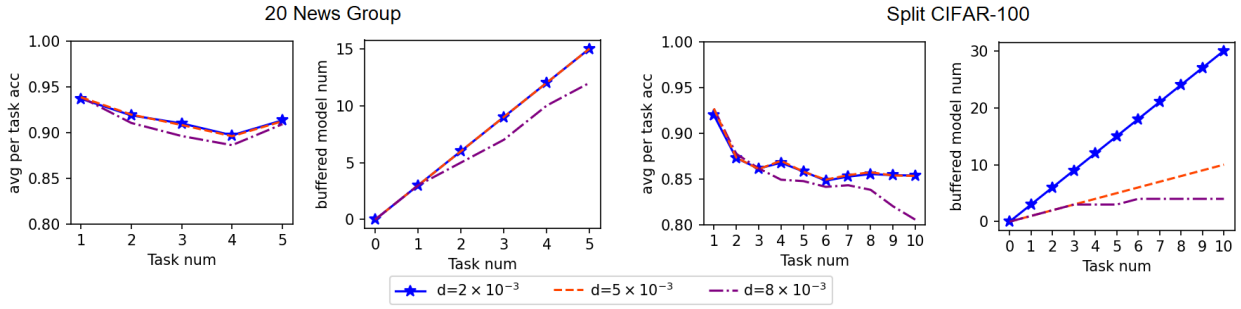


Figure 8: Different $d$'s on 20 News Group and Split CIFAR100.

Here, we evaluate the effects of choosing different thresholds $d$. We experiment on 20 News Group and Split CIFAR100. The variations include

1. $d = 2 \times 10^{-3}$, same as in the main experiments.
2. $d = 5 \times 10^{-3}$.
3. $d = 8 \times 10^{-3}$.

As $d$ increases, we are allowing more posteriors in the knowledge base to be reused. This will lead to memory efficiency at the cost of a performance drop. Figure 8 supports this trend. With an appropriately selected $d$, we can guarantee that the model's performance will not be overly affected, and that we save buffer memory. For Split CIFAR100, when $d = 8 \times 10^{-3}$ , the buffer stops growing after task 6.

### 5.3.3 Different Priors

Here, we evaluate the effects of different priors. We experiment on 20 News Group and Split CIFAR100. We first evaluate different sizes of prior standard deviations (stds).

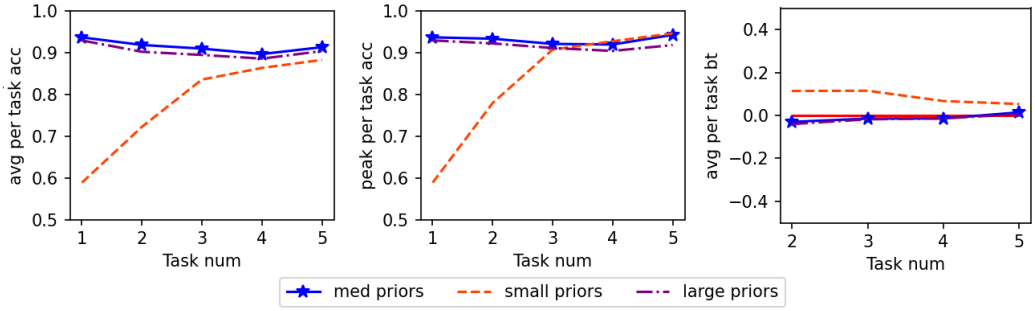


Figure 9: Different prior std sizes on randomly generated preferences over all tasks in 20 News Group.

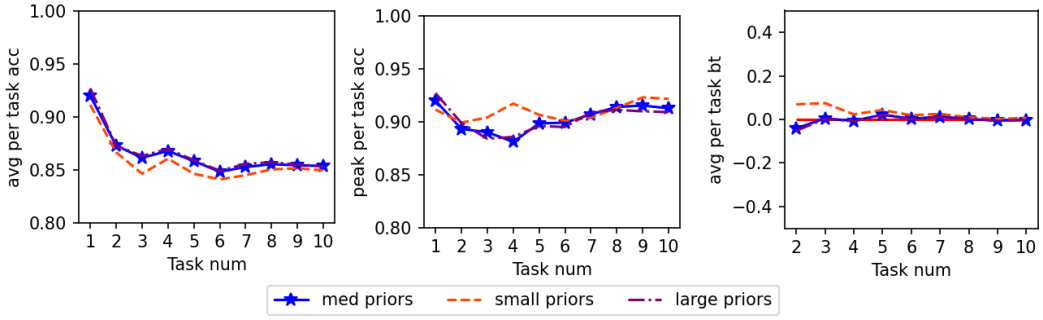


Figure 10: Different prior std sizes on randomly generated preferences over all tasks in Split CIFAR100.

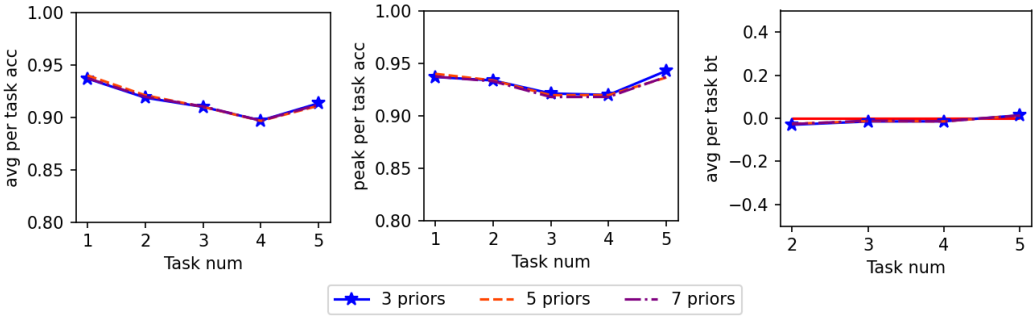


Figure 11: Different numbers of priors on randomly generated preferences over all tasks in 20 News Group.

1. Medium prior stds $= \{2, 2.5, 3\}$, same as the main experiments.
2. Small prior stds $= \{0.2, 0.25, 0.3\}$.
3. Large prior stds $= \{20, 25, 30\}$.

Figure 9 and 10 show the effects of different prior std sizes on the learning performance, on 20 News Group and Split CIFAR100, respectively. We can see that in the beginning of 20 News Group, small prior stds lower the average and peak per task accuracy. However, this decrease in performance is gradually reduced, and eliminated at tasks 4 and 5. This is because the posterior distributions gradually approach a pdf parameterized by the Pareto-optimal model with high probability. The forgetting prevention of the small stds is slightly improved from the other two trials, with a higher backward transfer initially, but also merges with the others at the end due to the same reason. All backward transfers are near zero, meaning there is almost no forgetting. For Split CIFAR100, the same divergence in performance appear in the beginning 3 tasks, most obvious in average accuracy. Then, the same pattern follows.

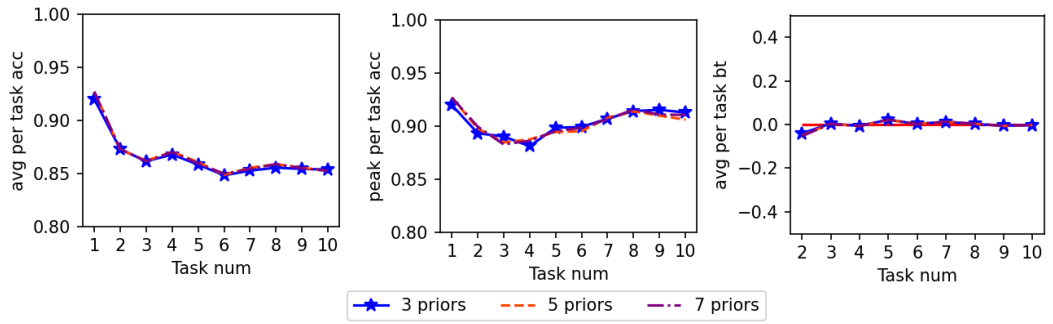


Figure 12: Different numbers of priors on randomly generated preferences over all tasks in Split CIFAR100.

We conclude that different choices of priors may lower the performance in the initial tasks, but the performance will gradually improve and align with each other to achieve Pareto-optimality. Next, we evaluate different numbers of priors.

1. 3 priors, stds $= \{2, 2.5, 3\}$, same as the main experiments.
2. 5 priors, stds $= \{1.5, 2, 2.5, 3, 3.5\}$.
3. 7 priors, stds $= \{1, 1.5, 2, 2.5, 3, 3.5, 4\}$.

As shown in Figure 11 and 12, different numbers of priors (3, 5 and 7) show very similar trends in performance. We therefore conclude that a small number of 3 priors is sufficient.

### 5.3.4 Different $\beta$'s

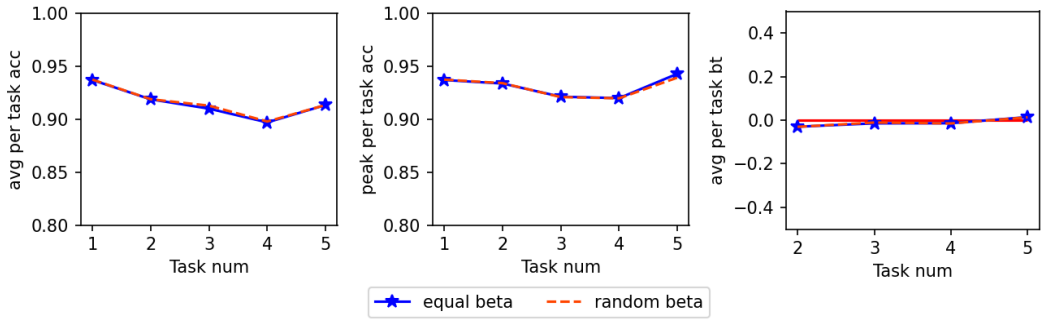


Figure 13: Equal and randomized $\beta$'s on randomly generated preferences over all tasks in 20 News Group.

We also evaluate the effects of different $\beta$'s. On 20 News Group and Split CIFAR100, we have

1. Equal $\beta$'s, same as the main experiments.
2. Randomized $\beta$'s.

As shown in Figure 13 and 14, equal sized $\beta$'s and randomly split sizes of $\beta$'s have almost the same performance trends. This provides evidence for our statement in Section 4.2, that using different choices of $\beta$'s would not affect the overall performance. Therefore, using equal $\beta$'s would be sufficient.

## 6 Conclusion

We propose IBCL to tackle the CLuST problem, where models for an unbounded number of stability-plasticity trade-off preferences can be requested at each task.

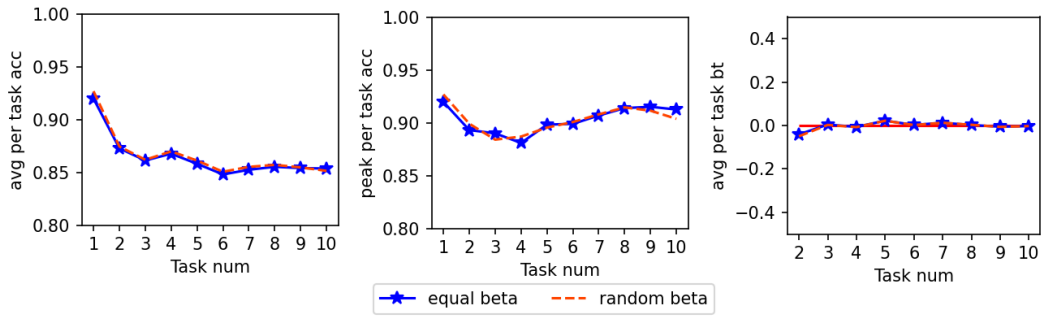


Figure 14: Equal and randomized $\beta$'s on randomly generated preferences over all tasks in Split CIFAR100.

**Advantages of IBCL.** The design of IBCL improves not only learning performance, but also efficiency when solving the CLuST problem, as state-of-the-art methods require retraining per preference, while IBCL only needs convex combinations. This benefit applies to various scales of models. It will be an interesting future direction to find a use case on large-scale models.

**Limitations of IBCL.** Poorly performing models can also be sampled from IBCL's HDRs. However, in practice, we can fine-tune $\alpha$ to reduce HDR to avoid poorly performing ones, as shown in ablation studies. In addition, one future research direction is to derive the preference vector $\bar{w}$ from some inputs. For example, we may learn it from an additional sequence of prompts (Wu et al., 2024). In that case, the preference vector itself might be different according to the design, including loss functions other than cross-entropy, which is currently used.

**Broader Impacts.** IBCL is potentially useful in deriving user-customized models from large multi-task models. These include large language models, recommendation systems, and other applications.

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## Appendix A Reason to adopt a Bayesian continual learning approach

Let $q_0(\theta)$ be our prior probability density / mass function (pdf / pmf) on the parameter $\theta \in \Theta$ at time $t = 0$. At time $t = 1$, we collect data $(\bar{x}_1, \bar{y}_1)$ related to task 1, we elicit likelihood pdf/pmf $\ell_1(\bar{x}_1, \bar{y}_1 \mid \theta)$, and we compute $q_1(\theta \mid \bar{x}_1, \bar{y}_1) \propto q_0(\theta) \times \ell_1(\bar{x}_1, \bar{y}_1 \mid \theta)$. At time $t = 2$, we collect data $(\bar{x}_2, \bar{y}_2)$ related to task 2 and we elicit likelihood pdf/pmf $\ell_2(\bar{x}_2, \bar{y}_2 \mid \theta)$. Now we have two options.

(i) Bayesian Continual Learning (BCL): we let the prior pdf/pmf at time $t = 2$ be the posterior pdf/pmf at time $t = 1$. That is, our prior pdf/pmf is $q_1(\theta \mid \bar{x}_1, \bar{y}_1)$, and we compute $q_2(\theta \mid \bar{x}_1, \bar{y}_1, \bar{x}_2, \bar{y}_2) \propto q_1(\theta \mid \bar{x}_1, \bar{y}_1) \times \ell_2(\bar{x}_2, \bar{y}_2 \mid \theta) \propto q_0(\theta) \times \ell_1(\bar{x}_1, \bar{y}_1 \mid \theta) \times \ell_2(\bar{x}_2, \bar{y}_2 \mid \theta)$;[6]

(ii) Bayesian Isolated Learning (BIL): we let the prior pdf/pmf at time $t = 2$ be a generic prior pdf/pmf $q_0'(\theta)$. We compute $q_2'(\theta \mid \bar{x}_2, \bar{y}_2) \propto q_0'(\theta) \times \ell_2(\bar{x}_2, \bar{y}_2 \mid \theta)$. We can even re-use the original prior, so that $q_0' = q_0$.

As we can see, in option (i) we assume that the data generating process at time $t = 2$ takes into account both tasks, while in option (ii) we posit that it only takes into account task 2. Denote by $\sigma(X)$ the sigma-algebra generated by a generic random variable $X$. Let also $Q_2$ be the probability measure whose pdf/pmf is $q_2$, and $Q_2'$ be the probability measure whose pdf/pmf is $q_2'$. Then, we have the following.

**Proposition A.1.** *Posterior probability measure $Q_2$ can be written as a $\sigma(\bar{X}_1, \bar{Y}_1, \bar{X}_2, \bar{Y}_2)$-measurable random variable taking values in $[0, 1]$, while posterior probability measure $Q_2'$ can be written as a $\sigma(\bar{X}_2, \bar{Y}_2)$-measurable random variable taking values in $[0, 1]$.*

*Proof.* Pick any $A \subset \Theta$. Then, $Q_2[A \mid \sigma(\bar{X}_1, \bar{Y}_1, \bar{X}_2, \bar{Y}_2)] = \mathbb{E}_{Q_2}[\mathbb{1}_A \mid \sigma(\bar{X}_1, \bar{Y}_1, \bar{X}_2, \bar{Y}_2)]$, a $\sigma(\bar{X}_1, \bar{Y}_1, \bar{X}_2, \bar{Y}_2)$-measurable random variable taking values in $[0, 1]$. Notice that $\mathbb{1}_A$ denotes the indicator function for set $A$. Similarly, $Q_2'[A \mid \sigma(\bar{X}_2, \bar{Y}_2)] = \mathbb{E}_{Q_2'}[\mathbb{1}_A \mid \sigma(\bar{X}_2, \bar{Y}_2)]$, a $\sigma(\bar{X}_2, \bar{Y}_2)$-measurable random variable taking values in $[0, 1]$. This is a well-known result in measure theory (Billingsley, 1986). □

Of course Proposition A.1 holds for all $t \geq 2$. Recall that the sigma-algebra $\sigma(X)$ generated by a generic random variable $X$ captures the idea of information encoded in observing $X$. An immediate corollary is the following.

**Corollary A.2.** *Let $t \geq 2$. Then, if we opt for BIL, we lose all the information encoded in $\{(\bar{X}_i, \bar{Y}_i)\}_{i=1}^{t-1}$.*

In turn, if we opt for BIL, we obtain a posterior that is not measurable with respect to $\sigma(\{(\bar{X}_i, \bar{Y}_i)\}_{i=1}^{t}) \setminus \sigma(\bar{X}_t, \bar{Y}_t)$. If the true data generating process $p_t$ is a function of the previous data generating processes $p_{t'}$, $t' \leq t$, this leaves us with a worse approximation of the "true" posterior $Q^{\text{true}} \propto Q_0 \times p_t$.

The phenomenon in Corollary A.2 is commonly referred to as *catastrophic forgetting*. Continual learning literature is unanimous in labeling catastrophic forgetting as undesirable – see e.g. (Farquhar & Gal, 2019; Li et al., 2020). For this reason, in this work we adopt a BCL approach. In practice, we cannot compute the posterior pdf/pmf exactly, and we will resort to variational inference to approximate them – an approach often referred to as Variational Continual Learning (VCL) (Nguyen et al., 2018). As shown in Section 3.2, Assumption 3.2 is needed in VCL to avoid catastrophic forgetting.

### A.1 Relationship between IBCL and other BCL techniques

Like (Farquhar & Gal, 2019; Li et al., 2020), the weights in our Bayesian neural networks (BNNs) have Gaussian distribution with diagonal covariance matrix. Because IBCL is rooted in Bayesian continual learning, we can initialize IBCL with a much smaller number of parameters to solve a complex task as long as it can solve a set of simpler tasks. In addition, IBCL does not need to evaluate the importance of parameters by measures such as computing the Fisher information, which are computationally expensive and intractable in large models.

[6] Here we tacitly assume that the likelihoods are independent.

### A.1.1 Relationship between IBCL and MAML

In this section, we discuss the relationship between IBCL and the Model-Agnostic Meta-Learning (MAML) and Bayesian MAML (BMAML) procedures introduced in (Finn et al., 2017; Yoon et al., 2018), respectively. These are inherently different than IBCL, since the latter is a continual learning procedure, while MAML and BMAML are meta-learning algorithms. Nevertheless, given the popularity of these procedures, we feel that relating IBCL to them would be useful to draw some insights on IBCL itself.

In MAML and BMAML, a task $i$ is specified by a $n_i$-shot dataset $D_i$ that consists of a small number of training examples, e.g. observations $(x_{1_i}, y_{1_i}), \dots, (x_{n_i}, y_{n_i})$. Tasks are sampled from a task distribution $\mathbb{T}$ such that the sampled tasks share the statistical regularity of the task distribution. In IBCL, Assumption 3.2 guarantees that the tasks $p_i$ share the statistical regularity of class $\mathcal{F}$. MAML and BMAML leverage this regularity to improve the learning efficiency of subsequent tasks.

At each meta-iteration $i$,

1. *Task-Sampling*: For both MAML and BMAML, a mini-batch $T_i$ of tasks is sampled from the task distribution $\mathbb{T}$. Each task $\tau_i \in T_i$ provides task-train and task-validation data, $D_{\tau_i}^{\text{trn}}$ and $D_{\tau_i}^{\text{val}}$, respectively.
2. *Inner-Update*: For MAML, the parameter of each task $\tau_i \in T_i$ is updated starting from the current generic initial parameter $\theta_0$, and then performing $n_i$ gradient descent steps on the task-train loss. For BMAML, the posterior $q(\theta_{\tau_i} \mid D_{\tau_i}^{\text{trn}}, \theta_0)$ is computed, for all $\tau_i \in T_i$.
3. *Outer-Update*: For MAML, the generic initial parameter $\theta_0$ is updated by gradient descent. For BMAML, it is updated using the Chaser loss (Yoon et al., 2018, Equation (7)).

Notice how in our work $\bar{w}$ is a probability vector. This implies that if we fix a number of task $k$ and we let $\bar{w}$ be equal to $(w_1, \dots, w_k)^\top$, then $\bar{w} \cdot \bar{p}$ can be seen as a sample from $\mathbb{T}$ such that $\mathbb{T}(p_i) = w_i$, for all $i \in \{1, \dots, k\}$.

Here lies the main difference between IBCL and BMAML. In the latter the information provided by the tasks is used to obtain a refinement of the (parameter of the) distribution $\mathbb{T}$ on the tasks themselves. In IBCL, instead, we are interested in the optimal parameterization of the posterior distribution associated with $\bar{w} \cdot \bar{p}$. Notice also that at time $k+1$, in IBCL the support of $\mathbb{T}$ changes: it is $\{p_1, \dots, p_{k+1}\}$, while for MAML and BMAML it stays the same.

Also, MAML and BMAML can be seen as ensemble methods, since they use different values (MAML) or different distributions (BMAML) to perform the Outer-Update and come up with a single value (MAML) or a single distributions (BMAML). Instead, IBCL keeps distributions separate via FGCS, thus capturing the ambiguity faced by the designer during the analysis.

Furthermore, we want to point out how while for BMAML the tasks $\tau_i$ are all "candidates" for the true data generating process (dgp) $p_i$, in IBCL we approximate the pdf/pmf of $p_i$ with the product $\prod_{h=1}^{i} \ell_h$ of the likelihoods up to task $i$. The idea of different candidates for the true dgp is beneficial for IBCL as well: in the future, we plan to let go of Assumption 3.2 and let each $p_i$ belong to a credal set $\mathcal{P}_i$. This would capture the epistemic uncertainty faced by the agent on the true dgp.

To summarize, IBCL is a continual learning technique whose aim is to find the correct parameterization of the posterior associated with $\bar{w} \cdot \bar{p}$. Here, $\bar{w}$ expresses the developer's preferences on the tasks. MAML and BMAML, instead, are meta-learning algorithms whose main concern is to refine the distribution $\mathbb{T}$ from which the tasks are sampled. While IBCL is able to capture the preferences of, and the ambiguity faced by, the designer, MAML and BMAML are unable to do so. On the contrary, these latter seem better suited to solve meta-learning problems. An interesting future research direction is to come up with imprecise BMAML, or IBMAML, where a credal set $\text{Conv}(\{\mathbb{T}_1, \dots, \mathbb{T}_k\})$ is used to capture the ambiguity faced by the developer in specifying the correct distribution on the possible tasks. The process of selecting one element from such credal set may lead to computational gains.

## Appendix B Proofs of the Propositions

*Proof of Proposition 4.1.* Without loss of generality, suppose we have encountered $i = 2$ tasks so far, so the FGCS is $\mathcal{Q}_2$. Let $\text{ex}[\mathcal{Q}_1] = \{q_1^j\}_{j=1}^{m_1}$ and $\text{ex}[\mathcal{Q}_2] \setminus \text{ex}[\mathcal{Q}_1] = \{q_2^j\}_{j=1}^{m_2}$. Let $\hat{q}$ be any element of $\mathcal{Q}_2$.

Since $\mathcal{Q}_2$ is a convex set, with extreme elements $\{q_1^j\}_{j=1}^{m_1} \cup \{q_2^j\}_{j=1}^{m_2}$, there exists a probability vector $\bar{\beta} = (\beta_1^1, \dots, \beta_1^{m_1}, \beta_2^1, \dots, \beta_2^{m_2})^\top$ such that

$$\hat{q} = \sum_{j=1}^{m_1} \beta_1^j q_1^j + \sum_{j=1}^{m_2} \beta_2^j q_2^j. \tag{9}$$

That is, $\beta_1^j \geq 0$, $\beta_2^j \geq 0$, for all $j$, and $\sum_{j=1}^{m_1} \beta_1^j + \sum_{j=1}^{m_2} \beta_2^j = 1$. Due to the fact that every $q_1^j$ is learned by variational inference (Nguyen et al., 2018) from a prior $q_0^j$ in Algorithm 1, for each $q_1^j$, we have

$$q_1^j(\theta) \approx \frac{\ell_1(\bar{x}_1, \bar{y}_1|\theta) q_0^j(\theta)}{\int_\Theta \ell_1(\bar{x}_1, \bar{y}_1|\theta) q(\theta) d\theta} \propto \ell_1(\bar{x}_1, \bar{y}_1|\theta) q_0^j(\theta) = \hat{p}_1(\bar{x}_1, \bar{y}_1|\theta) q_0^j(\theta) \tag{10}$$

where $\ell_1$ is the likelihood at task 1, and $\hat{p}_1 \equiv \ell_1$ estimates the pdf of task 1's true data generating process $p_1$.

Recall that in Bayesian continual learning, we use the previous task's posterior as the next task's prior. Then, since every $q_2^j$ is learned by variational inference from a prior $q_1^j$, we have that

$$q_2^j(\theta) \propto \ell_2(\bar{x}_2, \bar{y}_2|\theta) \underbrace{q_1^j(\theta)}_{\propto \ell_1(\bar{x}_1, \bar{y}_1|\theta) q_0^j(\theta)} \propto \underbrace{\ell_2(\bar{x}_2, \bar{y}_2|\theta) \ell_1(\bar{x}_1, \bar{y}_1|\theta)}_{=: \hat{p}_2(\bar{x}_1, \bar{y}_1, \bar{x}_2, \bar{y}_2|\theta)} q_0^j(\theta), \tag{11}$$

where $\hat{p}_2 \coloneqq \ell_1 \times \ell_2$ estimates the pdf of task 2's true data generating process $p_2$. In general, $\hat{p}_i = \prod_{k=1}^{i} \ell_k$, and $\ell_k$ is the likelihood at task $k$ (Servia-Rodriguez et al., 2021). Distribution $\hat{p}_k$ estimates the pdf of the true data generating process $p_k$ of task $k$, $k \in \{1, \dots, i\}$. Therefore, we expand on equation 9 as

$$\hat{q} = \sum_{j=1}^{m_1} \beta_1^j q_1^j + \sum_{j=1}^{m_2} \beta_2^j q_2^j \propto \hat{p}_1 \sum_{j=1}^{m_1} \beta_1^j q_0^j + \hat{p}_2 \sum_{j=1}^{m_2} \beta_2^j q_0^j. \tag{12}$$

As a consequence of the proportionality relation in equation 12, we can then find a vector $\bar{w} = (w_1 = \sum_{j=1}^{m_1} \beta_1^j, w_2 = \sum_{j=1}^{m_2} \beta_2^j)^\top$ that expresses the designer's preferences over tasks 1 and 2. In turn, we can write $\hat{q} \equiv \hat{q}_{\bar{w}}$. As we can see, then, the act of selecting a generic distribution $\hat{q} \in \mathcal{Q}_2$ is equivalent to specifying a preference vector $\bar{w}$ over tasks 1 and 2. This concludes the proof. □

*Proof of Proposition 4.2.* For maximum generality, assume $\Theta$ is uncountable. Recall from Definition 2.2 that *$\alpha$-level Highest Density Region* $\Theta_{\bar{w}}^\alpha$ is defined as the subset of the parameter space $\Theta$ such that

$$\int_{\Theta_{\bar{w}}^\alpha} \hat{q}_{\bar{w}}(\theta) \mathrm{d}\theta \geq 1 - \alpha \quad \text{and} \quad \int_{\Theta_{\bar{w}}^\alpha} \mathrm{d}\theta \text{ is a minimum.}$$

We need $\int_{\Theta_{\bar{w}}^\alpha} \mathrm{d}\theta$ to be a minimum because we want $\Theta_{\bar{w}}^\alpha$ to be the smallest possible region that gives us the desired probabilistic coverage. Equivalently, from Definition 2.3 we can write that $\Theta_{\bar{w}}^\alpha = \{\theta \in \Theta : \hat{q}_{\bar{w}}(\theta) \geq \hat{q}_{\bar{w}}^\alpha\}$, where $\hat{q}_{\bar{w}}^\alpha$ is the largest constant such that $\Pr_{\theta \sim \hat{q}_{\bar{w}}}[\theta \in \Theta_{\bar{w}}^\alpha] \geq 1 - \alpha$. Our result $\Pr_{\theta_{\bar{w}}^\star \sim \hat{q}_{\bar{w}}}[\theta_{\bar{w}}^\star \in \Theta_{\bar{w}}^\alpha)] \geq 1 - \alpha$, then, comes from the fact that $\Pr_{\theta_{\bar{w}}^\star \sim \hat{q}_{\bar{w}}}[\theta_{\bar{w}}^\star \in \Theta_{\bar{w}}^\alpha)] = \int_{\Theta_{\bar{w}}^\alpha} \hat{q}_{\bar{w}}(\theta) \mathrm{d}\theta$, a consequence of a well-known equality in probability theory (Billingsley, 1986). □

## Appendix C Details of Experiment Configurations

We select 15 tasks from CelebA. All tasks are binary image classification on celebrity face images. Each task $i$ is to classify whether the face has an attribute such as wearing eyeglasses or having a mustache. The first 15 attributes (out of 40) in the attribute list (Liu et al., 2015) are selected for our tasks. The training, validation and testing sets are already split upon download, with 162,770, 19,867 and 19,962 images, respectively. All images are annotated with binary labels of the 15 attributes in our tasks. We use the same training, validation and testing set for all tasks, with labels being the only difference.

We select 20 classes from CIFAR100 (Krizhevsky et al., 2009) to construct 10 Split-CIFAR100 tasks (Zenke et al., 2017). Each task is a binary image classification between an animal class (label 0) and a non-animal class (label 1). The classes are (in order of tasks):

1. Label 0: aquarium fish, beaver, dolphin, flatfish, otter, ray, seal, shark, trout, whale.
2. Label 1: bicycle, bus, lawn mower, motorcycle, pickup truck, rocket, streetcar, tank, tractor, train.

That is, the first task is to classify between aquarium fish images and bicycle images, and so on. We want to show that the continual learning model incrementally gains knowledge of how to identify animals from non-animals throughout the task sequence. For each class, CIFAR100 has 500 training data points and 100 testing data points. We hold out 100 training data points for validation. Therefore, at each task we have 400 $\times$ 2 = 800 training data, 100 $\times$ 2 = 200 validation data and 100 $\times$ 2 = 200 testing data.

We also select 20 classes from TinyImageNet (Le & Yang, 2015). The setup is similar to Split-CIFAR100, with label 0 being animals and 1 being non-animals.

1. Label 0: goldfish, European fire salamander, bullfrog, tailed frog, American alligator, boa constrictor, goose, koala, king penguin, albatross.
2. Label 1: cliff, espresso, potpie, pizza, meatloaf, banana, orange, water tower, via duct, tractor.

The dataset already splits 500, 50 and 50 images for training, validation and testing per class. Therefore, each task has 1000, 100 and 100 images for training, validation and testing, respectively.

20NewsGroups (Lang, 1995) contains news report texts on 20 topics. We select 10 topics for 5 binary text classification tasks. Each task is to distinguish whether the topic is computer-related (label 0) or not computer-related (label 1), as follows.

1. Label 0: comp.graphics, comp.os.ms-windows.misc, comp.sys.ibm.pc.hardware, comp.sys.mac.hardware, comp.windows.x.
2. Label 1: misc.forsale, rec.autos, rec.motorcycles, rec.sport.baseball, rec.sport.hockey.

Each class has different number of news reports. On average, a class has 565 reports for training and 376 for testing. We then hold out 100 reports from the 565 for validation. Therefore, each binary classification task has 930, 200 and 752 data points for training, validation and testing, on average respectively.

All data points are first preprocessed by a feature extractor. For images, the feature extractor is a pre-trained ResNet18 (He et al., 2016). We input the images into the ResNet18 model and obtain its last hidden layer's activations, which has a dimension of 512. For texts, the extractor is TF-IDF (Aizawa, 2003) succeeded with PCA to reduce the dimension to 512 as well.

Each Bayesian network model is trained with evidence lower bound (ELBO) loss, with a fixed feed-forward architecture (input=512, hidden=64, output=1). The hidden layer is ReLU-activated and the output layer is sigmoid-activated. Therefore, our parameter space $\Theta$ is the set of all values that can be taken by this network's weights and biases.

The three variational inference priors, learning rate, batch size and number of epcohs are tuned on validation sets. The tuning results are as follows. Here, "lr" stands for learning rate.

1. CelebA: priors = $\{\mathcal{N}(0, 0.2^2 I), \mathcal{N}(0, 0.25^2 I), \mathcal{N}(0, 0.3^2 I)\}$, lr = $1e-3$, batch size = 64, epochs = 10.
2. Split-CIFAR100: priors = $\{\mathcal{N}(0, 2^2 I), \mathcal{N}(0, 2.5^2 I), \mathcal{N}(0, 3^2 I)\}$, lr = $5e-4$, batch size = 32, epochs = 50.
3. TinyImageNet: priors = $\{\mathcal{N}(0, 2^2 I), \mathcal{N}(0, 2.5^2 I), \mathcal{N}(0, 3^2 I)\}$, lr = $5e-4$, batch size = 32, epochs = 30.
4. 20NewsGroup: priors = $\{\mathcal{N}(0, 2^2 I), \mathcal{N}(0, 2.5^2 I), \mathcal{N}(0, 3^2 I)\}$, lr = $5e-4$, batch size = 32, epochs = 100.

For the baseline methods, we use exactly the same learning rate, batch sizes and epochs. For probabilistic baseline methods (VCL), we use the prior with the median standard deviation. For example, on CelebA tasks, VCL uses the normal prior $\mathcal{N}(0, 0.25^2 I)$.

For rehearsal-based baselines, the memory size per task for CelebA is 200, and for the rest is 50. Together with the numbers above, we can compute the numerical values in Table 1.