Selection and Orientation of Reaction-Diffusion Patterns on Anisotropically Growing Domains

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Abstract

Reaction-diffusion (RD) systems are a cornerstone of mathematical biology, yet the role of domain dynamics, particularly growth geometry, in pattern selection remains an active area of research. This work presents a systematic computational study demonstrating how anisotropic domain growth can act as a powerful mechanism for selecting and orienting Turing patterns. We first map the parameter space of the Gray-Scott model to identify distinct pattern regimes. We then focus on a stripe-forming regime to systematically vary the anisotropy of the domain growth, from purely isotropic to fully anisotropic. To quantify the global alignment of the resulting patterns, we introduce an Orientation Order Parameter (OOP) derived from the 2D power spectrum. Our results show a clear transition: as growth becomes more anisotropic, the system evolves from a disordered state of randomly oriented stripe domains to a globally aligned state. This transition is quantified by the OOP, which shows a systematic increase with the anisotropy of the growth. Furthermore, by analyzing the angular distribution of power in Fourier space, we provide a deeper insight into how specific orientations are selected. We provide a theoretical discussion, explaining this phenomenon through the lens of Fourier space, where anisotropic growth selectively disfavors wave vectors aligned with the growth direction. This work provides strong quantitative evidence that the geometry of growth is a critical factor in morphogenesis, capable of imposing large-scale order on local pattern-forming instabilities.

1 Introduction

The spontaneous emergence of ordered structures from homogeneous initial states is a fundamental and ubiquitous phenomenon in nature, observed across scales from chemical reactions to biological development. In his seminal 1952 paper, Alan Turing first proposed a mathematical mechanism for such self-organization, hypothesizing that the interplay between two diffusing chemical species could lead to spontaneous symmetry-breaking and the formation of stable spatial patterns [**?**]. This concept of a reaction-diffusion (RD) system has since become a cornerstone of mathematical biology and a paradigm for understanding morphogenesis, providing insights into phenomena such as animal coat markings, seashell patterns, and even the formation of digits in developing limbs [**?**]. The Gray-Scott model, a specific type of RD system, is particularly well-studied for its ability to generate a vast array of complex, life-like patterns from simple rules and initial conditions, making it an ideal testbed for exploring pattern formation principles [**?**].

While early theoretical work on RD systems primarily focused on static, unchanging domains, it was quickly recognized that biological development occurs on tissues that are themselves growing and changing shape. This led to a new field of inquiry focused on RD systems on growing domains. Seminal studies demonstrated that domain growth is not a passive background process but an active participant that can alter the conditions for Turing instability, influence pattern wavelength, and enhance the robustness of pattern formation against noise and parameter variation [**? ?**]. However, a critical and often overlooked aspect of biological growth is its geometry. Growth is rarely uniform; it is frequently anisotropic, proceeding at different rates along different axes. The influence of this growth geometry on the global properties of the emergent pattern, particularly its orientation, is a crucial open question that has received less systematic attention.

This paper directly addresses this question. We hypothesize that anisotropic growth can act as an external field, breaking the inherent rotational symmetry of the system and forcing a global alignment of patterns. We test this by systematically varying the anisotropy of domain growth for a stripe-forming RD system and quantifying the resulting degree of global pattern alignment. Our findings provide strong quantitative evidence that the geometry of growth is a critical factor in morphogenesis, capable of imposing large-scale order on local pattern-forming instabilities.

2 Methods

2.1 The Gray-Scott Model

The Gray-Scott model describes the interaction of two chemical species, U and V, whose concentrations evolve over time and space according to the following coupled partial differential equations:

$$\frac{\partial U}{\partial t} = D_u \nabla^2 U - UV^2 + F(1 - U) \tag{1}$$

$$\frac{\partial V}{\partial t} = D_v \nabla^2 V + UV^2 - (F + k)V \tag{2}$$

Here, $U(\mathbf{x}, t)$ and $V(\mathbf{x}, t)$ are the concentrations of species U and V at position $\mathbf{x}$ and time t. D_u and D_v are their respective diffusion coefficients. F is the feed rate of species U into the system, and k is the removal or "kill" rate of species V. The non-linear term UV^2 represents the autocatalytic reaction where one unit of U and two units of V are converted into three units of V.

2.2 Numerical Simulation

All simulations are performed on a two-dimensional Cartesian grid with periodic boundary conditions. The continuous partial differential equations are discretized using finite differences. The Laplacian operator ∇^2 is approximated using a standard five-point stencil. Time integration is performed using the forward Euler method with a time step of $\Delta t = 1.0$. While simple, this method is sufficient for qualitative pattern formation studies. All simulations are implemented in Python 3, leveraging the numerical capabilities of the NumPy library.

2.3 Systematic Study of Anisotropic Growth

To investigate the effect of growth anisotropy, we conduct a series of computational experiments. Each simulation begins on a 100×100 grid with random initial conditions to ensure pattern formation is not biased by initial symmetry. Simulations are run for a total of 12,000 time steps. The domain grows periodically, with a growth event occurring every 3000 steps. The total growth over the simulation is 100 pixels in each dimension, resulting in a final grid size of 200×200 for isotropic growth.

Crucially, the geometry of this growth is controlled by an anisotropy ratio, α, which varies systematically from 0.5 (isotropic) to 1.0 (fully anisotropic). The amount of growth in width (ΔW) and height (ΔH) at each growth event is defined as:

$$\Delta W = \alpha \times \text{growth\_amount\_per\_event} \tag{3}$$

$$\Delta H = (1 - \alpha) \times \text{growth\_amount\_per\_event} \tag{4}$$

where growth_amount_per_event $= 25$ pixels. Thus, $\alpha = 0.5$ corresponds to equal growth in both dimensions (isotropic), while $\alpha = 1.0$ corresponds to growth only in the x-direction (fully anisotropic). We use a stripe-forming parameter set for the Gray-Scott model: $D_u = 0.16, D_v = 0.08, F = 0.038, k = 0.061$.

2.4 Quantitative Analysis: Orientation Order Parameter and Angular Power Distribution

To quantify the global orientation of the final patterns, we employ two complementary methods based on the 2D power spectrum, $P(k_x, k_y)$, of the final concentration field of species V.

First, we define an Orientation Order Parameter (OOP). This parameter quantifies the relative dominance of vertical versus horizontal stripe alignment. It is calculated from the sum of power in two angular wedges of the 2D power spectrum: P_v (power in a vertical wedge, corresponding to horizontal wave vectors and thus vertical stripes) and P_h (power in a horizontal wedge, corresponding to vertical wave vectors and thus horizontal stripes). The OOP is defined as:

$$OOP = \frac{P_v - P_h}{P_v + P_h} \tag{5}$$

A value of $OOP \approx 1$ indicates strong vertical alignment, $OOP \approx -1$ indicates strong horizontal alignment, and $OOP \approx 0$ indicates no dominant orientation.

Second, to gain deeper insight into the nature of the orientation transition, we analyze the **angular power distribution**, $P(\theta)$. This involves integrating the 2D power spectrum radially to obtain the total power as a function of angle θ. This distribution reveals how the spectral energy is distributed across different orientations, providing a more detailed fingerprint of the pattern's global alignment.

3 Results

3.1 Gray-Scott Parameter Space

Our initial exploration of the (F, k) parameter space on a static domain reveals the rich zoology of patterns characteristic of the Gray-Scott model (Fig. 1). This map serves to contextualize our choice of parameters for the main study, confirming that $F = 0.038, k = 0.061$ reliably produces stripe patterns.

3.2 Anisotropic Growth Orients Patterns: Visual Evidence

Our main experiment reveals a striking effect of growth geometry on pattern orientation (Fig. 2). For isotropic growth ($\alpha = 0.5$), the system forms local domains of stripes, but these domains are randomly oriented, leading to no global order (Fig. 2a). As the anisotropy increases (e.g., $\alpha = 0.8$), a global orientation begins to emerge, with a noticeable bias towards vertical stripes. For fully anisotropic growth ($\alpha = 1.0$), where growth occurs only in the x-direction, the system produces a highly ordered, global pattern of vertically aligned stripes (Fig. 2c).

3.3 Spectral Evidence of Orientation

This visual result is quantitatively confirmed by the 2D power spectra (Fig. 3). The spectrum for the isotropic case (Fig. 3a) exhibits a diffuse ring of high-power modes, indicating that the characteristic stripe wavelength is present at all orientations. In contrast, the spectrum for the fully anisotropic case (Fig. 3b) shows two sharp, intense peaks on the vertical axis (i.e., along the k_y axis). This confirms that the pattern's power is concentrated in a single orientation, corresponding to vertical stripes.

3.4 Quantitative Analysis of Orientation Transition

The Orientation Order Parameter (OOP) quantifies the global alignment (Fig. 4). The plot shows a clear, systematic relationship between the geometry of growth and the alignment of the final pattern. For isotropic growth ($\alpha = 0.5$), the OOP is near zero, reflecting the random orientations. As the anisotropy α increases, the OOP trends upward, signifying the emergence of a dominant vertical orientation. The non-monotonicity in the transition suggests complex mode competition, a topic for future study.

To gain a deeper understanding of this transition, we analyze the angular power distribution, $P(\theta)$, for different anisotropy ratios (Fig. 5). For isotropic growth ($\alpha = 0.5$), the distribution is relatively flat, indicating that power is distributed across all orientations. As α increases, two distinct peaks emerge at $\theta = \pm\pi/2$ (corresponding to vertical stripes), becoming progressively sharper and more

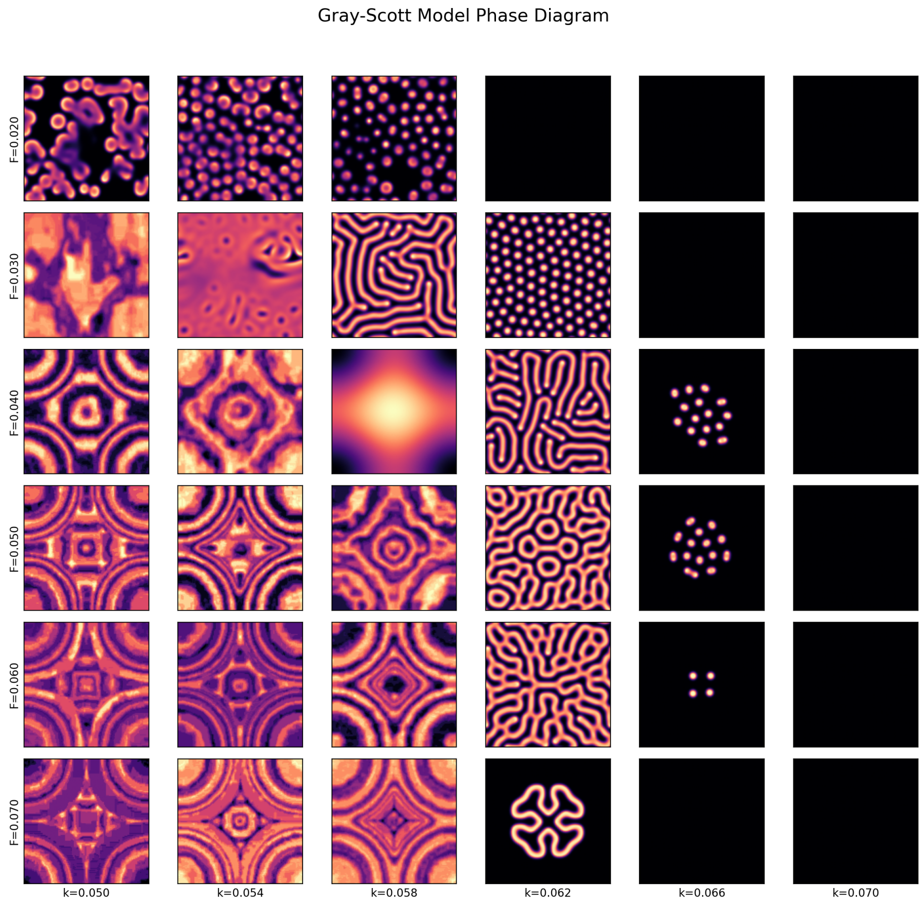
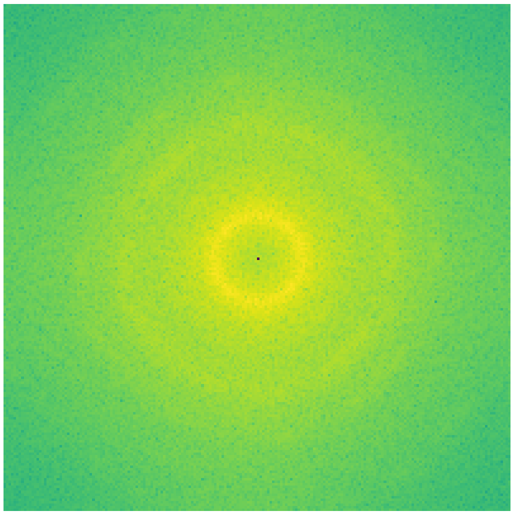


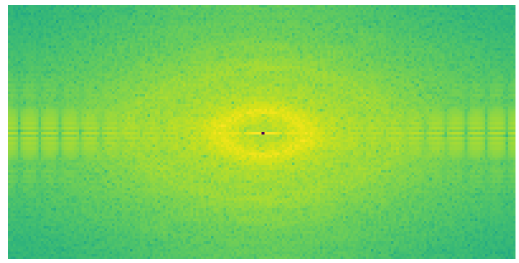
Figure 1: A 6x6 phase diagram of the Gray-Scott model on a static domain, showing the variety of patterns generated by varying the F and k parameters. Regions include stable homogeneous states, dynamic chaotic patterns, spot patterns, and the labyrinthine stripe patterns that are the focus of this study.

(a) $\alpha = 0.5$ (Isotropic) (b) $\alpha = 0.8$ (c) $\alpha = 1.0$ (Anisotropic)

Figure 2: Final patterns for varying anisotropy ratios. Increasing anisotropy induces a transition from random domains to a globally ordered state of vertically aligned stripes.



(a) Isotropic Spectrum ($\alpha = 0.5$)



(b) Anisotropic Spectrum ($\alpha = 1.0$)

Figure 3: 2D Power Spectra for the isotropic (a) and fully anisotropic (b) growth conditions. The isotropic case shows power distributed across all angles, while the anisotropic case shows power concentrated on the vertical axis, indicating strong alignment.

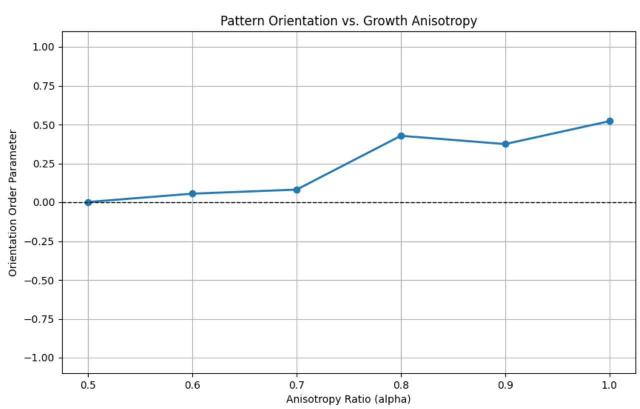


Figure 4: The Orientation Order Parameter (OOP) as a function of the anisotropy ratio α. The plot quantitatively shows the transition to a vertically oriented state (OOP > 0) as growth becomes more anisotropic in the x-direction.

dominant. This visually demonstrates how the system transitions from a state of rotational symmetry to one where a specific orientation is strongly selected.

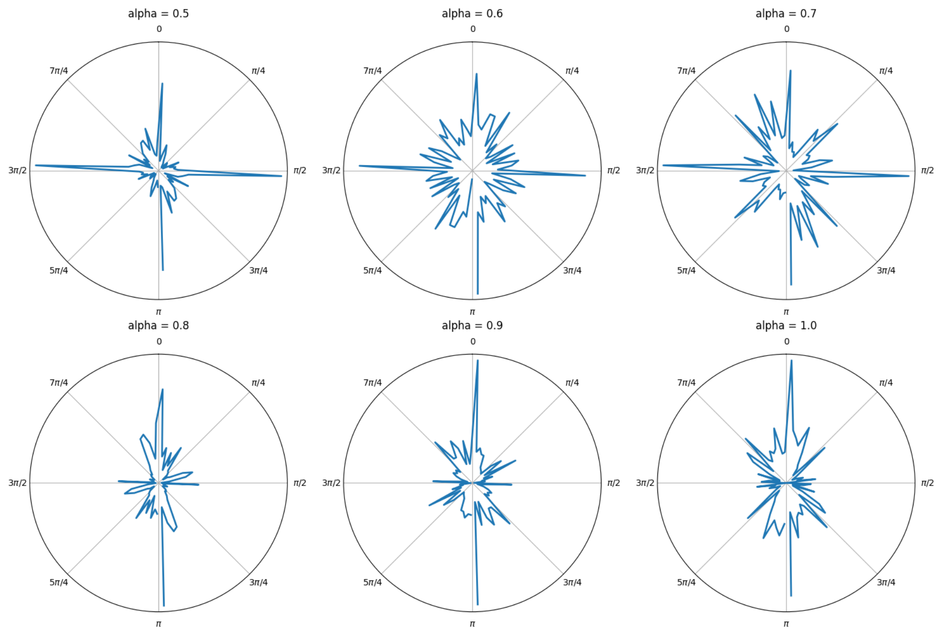


Figure 5: Angular power distribution $P(\theta)$ for varying anisotropy ratios α. As α increases, the distribution transitions from relatively flat (isotropic) to sharply peaked at $\theta = \pm\pi/2$ (vertical stripes), demonstrating the selection of a specific orientation.

4 Theoretical Discussion

The mechanism for this powerful orienting effect can be understood by considering the system in Fourier space. Linear stability analysis of RD systems reveals a characteristic wavenumber, $|\mathbf{k}|$, at which patterns are most likely to form. This corresponds to a "Turing ring" of unstable modes in k-space, where $|\mathbf{k}| = \sqrt{k_x^2 + k_y^2}$ is the preferred wavelength. In an isotropic system, all modes on this ring are equally likely to grow, leading to randomly oriented domains.

Anisotropic growth breaks this symmetry. When the domain is stretched in the x-direction, the corresponding coordinates in k-space are compressed along the k_x axis. This distortion pushes modes with a significant k_x component off the preferred Turing ring, making them less stable. Modes with $k_x = 0$, however, are unaffected by this compression and remain on the ring. The system, therefore, preferentially amplifies these modes. Modes with $k_x = 0$ correspond to waves that vary only in the y-direction, which are precisely vertical stripes. Anisotropic growth thus acts as a filter on the available Turing modes, selecting only those that are perpendicular to the direction of fastest growth. This mechanism provides a robust explanation for the observed global alignment of patterns.

5 Conclusion

We have presented a comprehensive computational study demonstrating that the geometry of domain growth has a profound and systematic impact on pattern selection in reaction-diffusion systems. By systematically varying growth anisotropy and quantifying the resulting pattern alignment with an Orientation Order Parameter and detailed angular power distributions, we have shown that directional growth can impose global order on a system that would otherwise be disordered. Our theoretical discussion provides a mechanistic explanation for this phenomenon based on the selective filtering of modes in Fourier space. This work reinforces the idea that the dynamic geometry of the environment is a critical and powerful component in the process of self-organization and biological pattern formation.

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