

TSLM: TREE-STRUCTURED LANGUAGE MODELING FOR DIVERGENT THINKING

Anonymous authors

Paper under double-blind review

ABSTRACT

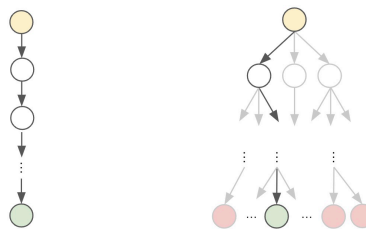
Language models generate reasoning sequentially, preventing them from decoupling irrelevant exploration paths during search. We introduce Tree-Structured Language Modeling (TSLM), which uses special tokens to encode branching structure, enabling models to generate and selectively expand multiple search paths within a single generation process. By training on complete search trees including both successful and failed attempts, TSLM learns to internalize systematic exploration without redundant recomputation of shared prefixes. TSLM achieves robust performance and superior inference efficiency by avoiding the multiple independent forward passes required by external search methods. These results suggest a new paradigm of inference-time scaling for robust reasoning, demonstrating that supervised learning on complete tree-structured traces provides an efficient alternative for developing systematic exploration capabilities in language models.

1 INTRODUCTION

Complex reasoning often requires exploring multiple solution paths before converging on an answer. Consider solving the Game of 24 with numbers [8, 4, 3, 6]: a systematic approach would branch out $8 + 4$, $8 - 4$ and other combinations simultaneously, rather than committing to a single path early. However, current language models generate solutions sequentially, making it difficult to systematically explore alternatives or recover from early mistakes.

Recent reasoning models like o1 OpenAI (2024) and DeepSeek-R1 DeepSeek-AI et al. (2025) have shown impressive capabilities through extended reasoning traces, but still fundamentally operate as sequential generators. While they may internally consider multiple options, they cannot explicitly decouple parallel branches within their generation process. External methods like Tree-of-Thought Yao et al. (2023) address this through post-hoc search, but require multiple independent model calls and external orchestration.

The key challenge is that models cannot coherently construct diverse exploration branches. Sequential models collapse all reasoning into one path. External search methods like Tree-of-Thought sample multiple trajectories independently, but this produces fragmented distribution where search sample is drawn without coordination, leading to redundant overlap in some regions while missing critical branches in others. The model has no mechanism to systematically reconstruct the complete search space.



(a) **Sequential thinking:** A purely sequential approach that traverses one deterministic path. (b) **Tree-structured thinking:** A branching exploration that expands multiple possibilities simultaneously.

Figure 1: Sequential vs. Tree-Structured Reasoning. (a) Sequential approaches commit to single paths, limiting exploration of alternatives. (b) Tree-structured approaches systematically explore multiple possibilities, enabling recovery from mistakes and comprehensive solution space coverage. TSLM bridges this gap by teaching language models to generate tree-structured explorations natively.

TSLM addresses this through *coherent tree generation*: instead of sampling multiple independent trajectories, the model generates the complete branching structure in one forward pass using special tokens ([SEP], [FAIL], [GOAL]) to mark viable paths, dead ends, and goals. During training, models learn to systematically construct diverse branches for each node. During inference, the model generates this coherent tree structure, then selectively “stitches” each branch into isolated contexts for expansion, maintaining the complete search topology rather than hoping parallel samples will cover it.

This approach enables models to learn the complete reasoning process by including both successful paths and failed attempts, leading to more robust problem-solving capabilities. Our experimental results demonstrate that this internalized search strategy not only outperforms sequential modeling baselines but also exhibits remarkable parameter efficiency and emergent capabilities that arise naturally from learning complete exploration patterns rather than just final answers.

1.1 OUR CONTRIBUTIONS

This paper makes the following contributions:

- We introduce *Tree-Structured Language Modeling* (TSLM), a token-based serialization framework that enables sequential language model to learn tree-structured reasoning through supervised learning on cohesive search traces
- TSLM demonstrate consistent performance gains across diverse tasks: 100% vs. 17% on Game of 24, robust extrapolation to unseen, larger environments (91.5% vs. 42.7% for Tree-of-Thought in 20×20 Gridworld environments), and enhanced performance on open-ended reasoning
- We uncover emergent capabilities including systematic identification of unsolvable problems and robust extrapolation beyond training complexity
- We demonstrate a new test-time scaling paradigm: rather than parallel sampling k independent trajectories, TSLM scales by exploring k branches within a single coherent tree, showing better scaling efficiency and higher convergence accuracy

Our findings challenge the granted assumption that we need inference-time scaling (either via reinforcement learning or scaffolding) is necessary for developing robust reasoning capabilities in language models. Instead, we demonstrate that properly structured supervised learning of tree-formatted reasoning traces may provide a more direct and efficient path toward enhanced reasoning and planning. We provide detailed answers to common questions in Appendix §A. We also release our implementation here.

2 BACKGROUND

Contemporary language models generate tokens sequentially, modeling $p(y | x) = \prod_{t=1}^{|y|} p(y_t | x, y_{<t})$ Brown et al. (2020). For problems requiring exploration of multiple solution paths, this sequential approach has limitations: (1) linear commitment to single paths, (2) error propagation, (3) redundant computation when multiple solutions are needed, and (4) inability to systematically explore alternatives in parallel.

Current multi-path reasoning approaches rely on inference-time scaling methods, either for post-hoc sampling or policy gradient update from self rollouts. Tree-of-Thoughts Yao et al. (2023) samples multiple candidates at each step using external search algorithms, but faces exponential computational costs. Reasoning traces generated from autoregressive models like o1 OpenAI (2024) are due to the number of rollouts for gradient update, but remain constrained by linear generation and may produce redundant information Chen et al. (2025). We provide a more extensive related works in Appendix §B.

3 TREE-STRUCTURED LANGUAGE MODELING (TSLM)

We introduce Tree-Structured Language Modeling (TSLM) as a framework to natively incorporate divergent exploration for natural language generation. TSLM differs from standard sequential

language modeling by generating multiple possible next actions or statements and linking them into a coherent tree structure.

3.1 MODELING MULTIPLE NEXT ACTIONS

Let s be the current state or partial solution. In a sequential language model, we predict a single next action a from s and transition to $s' = T(s, a)$. By contrast, in TSLM, we represent multiple possible successors:

$$\pi_{\theta}(s) = \{T(s, a_1), \dots, T(s, a_k)\},$$

where each a_i denotes a distinct branch and k is the branching factor. TSLM learns to expand s into these k successors within a single forward pass, retaining the relationships among them rather than generating them independently. This branching representation is fundamentally different from sampling k independent trajectories. Each a_i is generated conditionally on the previous actions a_1, \dots, a_{i-1} at the same node, ensuring systematic coverage of the action space rather than redundant sampling from a marginalized distribution.

3.2 ENCODING AND DECODING WITH TREE STRUCTURE

To enable standard transformer architectures to learn tree-structured reasoning, we develop a serialization scheme that encodes complete search trees into linear sequences. This approach allows us to train language models on tree data while preserving the branching structure.

Token-Based Tree Serialization. We introduce special tokens that TSLM learns to encode tree structure:

- [SEP]: Indicates a viable action that can be further expanded
- [FAIL]: Indicates a non-viable action (dead end)
- [GOAL]: Marks the desired goal state
- [BOS] and [EOS]: Mark sequence boundaries of child expansions for each node

This serialization captures both successful paths and unsuccessful explorations, teaching the model the complete search process rather than just final answers. A detailed worked example showing the complete serialization format is provided in Appendix §C.

Training Procedure. During training, we apply standard language modeling loss to the entire serialized sequence:

$$\mathcal{L} = - \sum_{t=1}^T \log p(y_t | y_{<t}, x) \quad (1)$$

where y_t includes both reasoning content and structural tokens. This standard cross entropy loss has a non standard training signal. Unlike sequential modeling where each token depends only on its prefix, TSLM tokens depend on tree structure. For example, [SEP] after action a_i is conditioned on whether $T(s, a_i)$ is expandable **and** whether it differs from a_1, \dots, a_{i-1} . This structural conditioning teaches the model to:

1. Generate multiple actions at each decision point
2. Assign appropriate viability markers ([SEP], [FAIL], or [GOAL])
3. Structure the exploration systematically

Inference Procedure. During inference, TSLM reconstructs the tree structure:

1. Generate the next reasoning step with multiple candidate actions
2. Parse structural tokens to identify viable branches ([SEP])
3. Fork different branches into independent sequences for parallel expansion
4. Recursively expand each viable state until finding a solution ([GOAL]) or exhausting options

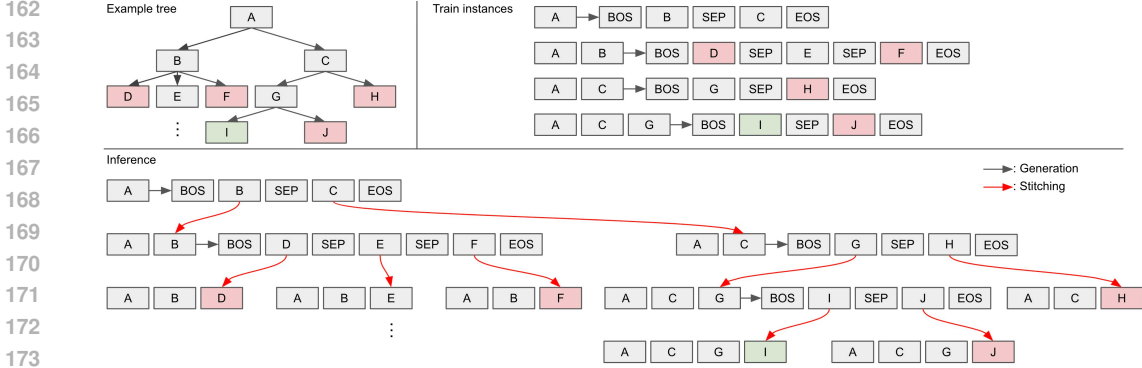


Figure 2: TSLM (Tree-Structured Language Model) is a language model designed for hierarchical exploration in sequence generation tasks. Beginning with an initial state (e.g., “A”) and progressing toward a goal state (e.g., “I”), TSLM constructs a tree structure where nodes represent states and branches signify possible paths. During training, the model serializes the tree into linear sequences using special tokens to separate branches and mark the start/end of a sequence, allowing it to learn structured expansions effectively. During inference, TSLM generates multiple branching actions to explore diverse sequences (e.g., expanding from “C” to “G” and “H”). These branches are independently expanded in parallel using a stitching process, enabling broad exploration toward the goal while efficiently pruning unwanted paths.

This approach enables systematic exploration. While TSLM forks divergent branches into different sequences (involving multiple calls), the construction of the whole search tree is coherent and internalized, avoiding the redundant sampling inherent in Tree-of-Thought.

3.3 FORMAL TRAINING OBJECTIVE

For a search tree \mathcal{T} with nodes \mathcal{N} , let $\pi(s_i)$ denote the path from root to node s_i . Each node s_i generates a token sequence $y_{s_i} = [a_i, m_i]$ where a_i is the action description and $m_i \in \{[\text{SEP}], [\text{FAIL}], [\text{GOAL}]\}$ is the structural marker.

The context for node s_i consists of:

$$\text{ctx}(s_i) = \bigcup_{s_j \in \pi(s_i)} y_{s_j} \cup \bigcup_{\substack{s_k \in \text{siblings}(s_i) \\ k < i}} y_{s_k} \quad (2)$$

where $\text{siblings}(s_i)$ are nodes sharing the same parent, and $k < i$ indicates siblings generated before s_i (enforcing left-to-right ordering within each branching factor).

The training objective becomes:

$$\mathcal{L} = -\frac{1}{|\mathcal{N}|} \sum_{s_i \in \mathcal{N}} \sum_{t=1}^{|y_{s_i}|} \log p(y_{s_i,t} \mid y_{s_i,<t}, \text{ctx}(s_i)) \quad (3)$$

This formulation makes explicit that each node conditions on (1) its ancestral path and (2) previously generated siblings, but **not** on unrelated subtrees. This selective conditioning is what enables context decoupling.

In practice, we serialize the tree using depth-first traversal, converting it into a linear sequence that transformers can process. Crucially, while the input is linear, the conditional dependencies are dictated by tree topology: node s_i depends only on $\text{ctx}(s_i)$ (its ancestors and prior siblings), not on all tokens that precede it in the serialized sequence. This structured conditioning is what distinguishes TSLM from simply concatenating all exploration paths sequentially.

4 SEARCH TREE SUPERVISION

4.1 TRAINING ON STRUCTURED TASKS

For structured tasks with predefined search trees, we can directly train TSLM to learn and reproduce the tree structure. Formally, let $t \in \mathcal{T}$ be a task with a solution of a finite depth, $T(s, a)$ be a transition function that maps state-action pairs to new states, and $A(s)$ be a finite action space that defines valid actions at state s . Since these components are explicitly defined, we can generate the complete search tree and employ TSLM to predict the branching structure ($T(s, a) \mid a \in A(s)$) at each state in the search tree. By directly imitating the predefined tree expansions, TSLM guides the model to faithfully reproduce structured exploration patterns. Examples include board games and planning problems with well-defined rules.

4.2 SEARCH TREE SUPERVISION FOR OPEN-ENDED REASONING TASKS

While structured tasks have predefined search trees that TSLM can directly learn from, most real-world tasks lack explicit tree structures, providing only correct answers or gold trajectories. In this paper, we adopt a simple bootstrapping method to construct synthetic training trees by combining model-generated explorations with known solutions; For each training instance, we deploy the Tree-of-Thoughts sampling Yao et al. (2023) from supervision language model to generate pseudo search tree. The process involves:

1. Sampling a set of candidate actions at each state using beam search
2. Building a tree structure by propagating these actions forward
3. Incorporating known gold trajectories as high-priority branches
4. Ordering remaining branches using a reward function $R(s, a)$
5. Deduplicating redundant paths while preserving the tree structure

Algorithm 1 details this procedure. Our approach ensures each training tree contains at least one valid solution while exploring diverse alternatives. The reward-based ordering helps prioritize promising actions, while deduplication prevents redundant search. We adopt the original RAP Hao et al. (2023) reward function to refine exploration by prioritizing promising branches.¹

5 EXPERIMENTAL RESULTS

5.1 BASELINES FOR COMPARISON

Baseline Model Architecture We compare TSLM to the following baselines:

- **Sequence Cloning (SC)**: A standard sequential modeling that clones a single linear sequence of gold Chain-of-Thought (language modeling similar to GPT-3 (Brown et al., 2020) or Llama 3 (Touvron et al., 2023)).
- **Procedure Cloning (PC)**: A sequential modeling that clones Chain-of-Thought reasonings of the entire search trace in a single linear sequence (o1-like reasoning models, trained with systematic supervision (Kim et al., 2024; Yang et al., 2022)).
- **GRPO**: A reinforcement learning approach to incentivize reasoning trace during post-training (o1-like reasoning models, trained with GRPO (Shao et al., 2024) objectives)
- **Tree-of-Thought (ToT)**: Scaling the number of inference of SC model during test-time using external search algorithms (Yao et al., 2023) with beam search across multiple reasoning paths.

¹Caveat here is that this is *not* the best method to train TSLM. For example, we can use supervision from better reasoning language models using techniques like ReJump (Zeng et al., 2025). Also we can use self-training with external search methods like SoS (Gandhi et al., 2024). We rather focus on the concept of TSLM itself rather than high-level training techniques.

Algorithm 1 Guided Search Tree Bootstrapping

```

270 Input: Task  $\mathcal{T}$ , transition  $T$ , reward  $R$ , branch factor  $k$ , supervision model  $\pi_\theta$ 
271 Data = []
272 for  $t \in \mathcal{T}$  do
273   Initialize queue = [ $s$ ].
274   Gold trajectory  $s_0 = s, s_1, \dots, s_n = g$ 
275   while  $g \notin \text{queue}$  do
276      $tmp = \text{queue.pop}(0)$ 
277     for  $i = 1$  to  $k$  do
278        $tmpqueue = []$ 
279       if  $i = 1$  and  $tmp \in \{s_0, \dots, s_n\}$  then
280          $a_i = a^*(tmp)$  {Add gold action}
281          $tmpqueue.add(T(tmp, a_i))$ 
282       else
283          $a_i \sim \pi_\theta(tmp)$ 
284         if  $a_i \notin \{a_1, \dots, a_{i-1}\}$  then
285            $tmpqueue.add(T(tmp, a_i))$ 
286           {Deduplication}
287         end if
288       end if
289     end for
290      $tmpqueue = \sigma_R(tmpqueue)$ {Sort by reward}
291      $queue += tmpqueue$ 
292   end while
293   Data.append(queue)
294 end for

```

To ensure a rigorous comparison, we evaluate ToT using a **pass@100** metric (success if any of the first 100 terminal nodes is correct), while all other methods (SC, PC, GRPO, TSLM) are evaluated using **pass@1** (single attempt). We chose ToT@100 because it represents the convergence point of the search tree in our experiments, effectively serving as an upper bound for sampling-based scaffolding methods. Thus, comparing TSLM (pass@1) against ToT (pass@100) provides a highly conservative estimate of TSLM’s relative performance.

We test models using greedy decoding for sequential methods, and breadth of 5, temperature of 0.3 across ToT experiments. For TSLM inference, we explore different tree traversal and solution selection strategies. Unless otherwise specified, we use Breadth-First Search (BFS) as the default algorithm to systematically explore the generated tree structure until finding a successful solution. (We analyze implications of BFS versus alternative search strategies such as Depth-First Search (DFS) in Appendix §E.1.) For each expansion during the inference, we select the first $k = 5$ actions generated and deduplicate them with exact matching.

To evaluate Tree-Structured Language Modeling (TSLM), we conduct experiments on both structured and unstructured tasks. Our experiments use Llama-3-8B Grattafiori et al. (2024) as base experiments unless specified, comparing TSLM against sequential language modeling baselines. Also, We aim to compare architectural differences rather than scaling effects, using modest training data (less than 10K instances per task) for post-training. We have two task scenarios: Structured Planning Tasks, which are tasks with predefined search trees, while Open-ended Reasoning Tasks are tasks with undefined solution spaces. Structured planning tasks include Game of 24 and Textualized Gridworld, while Open-ended reasoning tasks include ProntoQA and GSM8K. Refer to Appendix §F for more details regarding each task and supervision examples.

5.2 BASE RESULTS

Table 1 summarizes our experimental findings across all tasks, revealing key insights about different reasoning approaches. TSLM consistently outperforms sequential models, achieving perfect accuracy on structured tasks. Most notably, the Gridworld scaling results reveal a striking limitation of Tree-of-

Task	SC	pass@1			pass@100
		PC	GRPO	TSLM	ToT
§F.2 Game of 24	17.0%	47.0%	15.0%	100%	32.0%
	78.2%	99.7%	24.0%	100%	95.0%
	33.0%	81.1%	6.0%	91.5%	42.7%
§F.3 ProntoQA	99.7%	97.5%	99.8%	100%	100%
	GSM8K	55.8%	55.9%	60.8%	61.6%

Table 1: Success rates across different tasks and methods. Note that ToT results are reported as pass@100 (success if any of 100 attempts is correct), while all other methods are pass@1. Despite this disadvantage, TSLM matches or outperforms ToT on most tasks. The Gridworld results show both in-domain (10×10) and scaling (20×20) performance, highlighting TSLM’s robustness when complexity scales beyond training boundaries.

Thought: while ToT achieves excellent in-domain performance (95.0% on 10×10 grids), it suffers catastrophic degradation when complexity scales (dropping to 42.7% on 20×20 grids).

This ToT scaling failure is particularly surprising given its strong in-domain performance and sophisticated external search mechanisms. In contrast, TSLM maintains robust performance across the complexity boundary (100% → 91.5%), demonstrating that internalized search procedures generalize better than external search algorithms. For open-ended tasks, ToT shows comparable performance on GSM8K (62.3% vs 61.6%) but this advantage disappears when systematic exploration is needed, as evidenced by Game of 24 results where ToT performs no better than basic sequential methods (17.0%).

5.3 FAIR COMPARISON: TEST-TIME SCALING

A critical distinction: TSLM introduces a fundamentally different test-time scaling paradigm. Traditional methods scale by generating k **independent** trajectories through parallel sampling (ToT) or sequential enumeration. Each trajectory recomputes shared prefixes redundantly. TSLM scales by exploring k candidates within a **single, coherent search tree** generated by the model in one forward pass. The model explicitly constructs the branching structure, then we select which branches to verify.

We define test-time scaling for each method:

- **TSLM**: Generates one coherent tree structure, then verifies the first k terminal states (ending with either [FAIL] or [GOAL]) from Breadth-First Search traversal. Crucially, the k candidates come from a single tree construction, not k independent samplings.
- **Procedure Cloning (PC)**: Verifies the first k terminal states from the sequential search trace.

To rigorously compare the effectiveness of internalized search (TSLM) versus external scaffolding (ToT), we evaluate performance under matched candidate budgets. This comparison is crucial because TSLM’s “test-time scaling” differs fundamentally from standard parallel sampling. In ToT or standard scaffolding, scaling k implies generating k independent, redundant traces. In TSLM, scaling k means exploring more branches within a *single, cohesive search tree* structure generated by the model. This internalized scaling allows for more efficient resource allocation, as the model can prioritize promising branches without regenerating common prefixes or exploring completely disjoint paths. Figure 3a presents the results on GSM8K as we scale the number of candidates k .

TSLM consistently outperforms ToT and PC across all budgets. Notably, TSLM with just a single candidate (61.3%) nearly matches ToT’s converged performance (62.3%), and TSLM saturates at a significantly higher accuracy (67.2%). This confirms that TSLM’s training objective successfully internalizes a superior search tree compared to the parallel sampling in ToT.

5.4 COMPUTATIONAL EFFICIENCY ANALYSIS

An immediate followup advantage of TSLM is its inference efficiency. Unlike external scaffolding methods that require multiple independent model calls, TSLM generates the complete search tree in a single forward pass (or a few passes if using beam search decoding), sharing computation for common prefixes.

We compared the average wall-clock time required to reach solution convergence across methods. As shown in Figure 3b, TSLM is significantly faster than ToT. ToT suffers from time explosion due to redundant sampling and lack of computation sharing. TSLM is also faster than Procedure Cloning (PC) because TSLM’s tree structure allows the model to ignore irrelevant subtrees during inference, whereas PC must process the entire linearized sequence of the search trace.

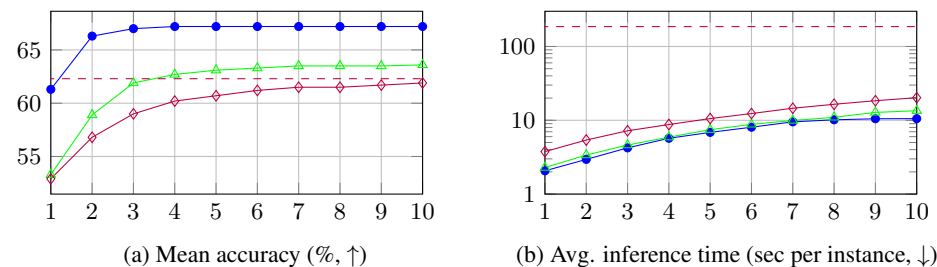


Figure 3: Comparison of (a) mean accuracy and (b) average inference time when scaling the number of candidates across different methods on GSM8K. Methods: TSLM (●), PC (△), and SC (◇), convergence at ToT@100: - - -. All the cache control during the inference time is done by vLLM (Kwon et al., 2023) v.0.6.6.

6 ANALYSIS

Beyond core performance gains, TSLM exhibits several remarkable capabilities that distinguish it from traditional approaches.

6.1 IDENTIFYING UNSOLVABLE CASES

One challenging aspect for language models is avoiding hallucination on problems that have no valid solution. Sequential models trained on SC, PC, or GRPO implicitly learn to generate answers within their training distribution, which can be problematic when faced with unsolvable cases. For example, in the Game of 24 task, the numbers 1, 1, 2, and 3 cannot generate 24 through any sequence of arithmetic operations. Since sequential models have not been trained on examples with no solution, they tend to hallucinate and generate invalid answers.

TSLM, however, demonstrates a unique capability to identify unsolvable cases. Quantitatively, across 100 unsolvable Game of 24 instances, TSLM correctly identified 97 cases by terminating without a solution. In contrast, baseline methods (SC, PC, GRPO) failed to identify any unsolvable cases, instead hallucinating invalid solutions (There are 3 unintentional instances where PC correctly refused, only because the model generated excessively long traces that failed to terminate properly due to skewed search tree generation). This suggests the tree-structured exploration enables TSLM to systematically explore the full solution space and recognize when no valid path exists. Unlike sequential models that are pressured to always generate some answer, TSLM’s cohesive search tree allows it to confidently determine and declare when a problem is unsolvable.

We position this capability as addressing a fundamental limitation of sequential language models. Hallucination remains a major problem in language modeling, with theoretical work suggesting it’s inevitable in the next-token prediction paradigm (Xu et al., 2025) Frontier labs like OpenAI attempt to address this through post-training to modify refusal behavior (Kalai et al., 2025), but the mechanisms remain unclear and heavily rely on RL approaches.

432
433
434
435
436
437
438
439
440
441
442
443
444
445
446
447
448
449
450
451
452
453
454
455
456
457
458
459
460
461
462
463
464
465
466
467
468
469
470
471
472
473
474
475
476
477
478
479
480
481
482
483
484
485

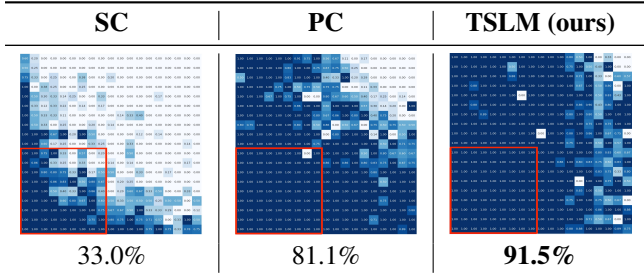


Table 2: Qualitative comparison of model extrapolation capabilities on Gridworlds of varying sizes (maximum 20×20). Each heatmap shows performance at different grid dimensions (x,y), with darker colors indicating better performance. The red box indicates the boundary of training data. Overall accuracy shown above each plot.

6.2 QUALITATIVE ANALYSIS ON EXTRAPOLATION BEYOND TRAINING DATA

We qualitatively show how TSLM demonstrates strong extrapolation capabilities when tested on larger grid sizes than seen during training for each grid size (Table 2). This suggests TSLM’s structured tree representation learns generalizable navigation patterns rather than memorizing specific configurations.

6.3 TEST-TIME SCALING: BEYOND PARALLEL SAMPLING

Beyond the quantitative improvements shown in §5.3, TSLM’s test-time scaling reveals a qualitative shift in how models can control exploration. Traditional scaling methods treat the model as a fixed sampler: increasing k simply draws more independent samples from the same learned distribution. TSLM scaling is fundamentally different: the model explicitly generates branching structure, enabling adaptive control over which regions of the search space to explore. This transforms scaling from “sample more” to “explore strategically.”

The robustness of TSLM’s internalized search extends across different base models. Appendix §E.2 demonstrates that TSLM maintains consistent improvements over baselines regardless of the underlying architecture (Llama-3-8B, Llama-3-8B-Instruct, Qwen-2.5-7B), showing that the benefits stem from structured exploration rather than model-specific quirks.

More importantly, TSLM enables flexible search strategies unavailable to parallel sampling methods. By generating an explicit tree structure, we can choose between breadth-first search (prioritizing solution optimality) and depth-first search (prioritizing model confidence), or implement adaptive strategies that expand high-reward branches more aggressively. As shown in Appendix §E.1, DFS achieves 63.1% top-1 accuracy (vs BFS 61.3%) by immediately following high-confidence actions, while BFS provides better coverage for top-k exploration. This controllability is impossible with independent sampling, where each trajectory commits to a full path without coordinating exploration priorities.

Finally, TSLM’s scaling extends beyond the training branching factor. TSLM’s scaling extends beyond the training branching factor (Appendix §E.3). Figure 6 demonstrates that models trained with k=5 successfully extrapolate to k=10 at inference time, improving convergence from 67.2% to 71.1%. The model generalizes its learned exploration strategy to broader search, unlike parallel sampling which simply draws more independent samples from the same distribution.

7 CONCLUSION

We introduce Tree-Structured Language Modeling (TSLM), which enables language models to generate complete search trees within a single generation process using token-based serialization. TSLM achieves superior performance across structured planning and open-ended reasoning tasks: 100% accuracy on Game of 24 (vs. 17% for baselines), robust extrapolation to larger environments (76.5% vs. 26% for Tree-of-Thought).

Our work challenges the prevailing assumption that reasoning capabilities require reinforcement learning or inference-time search orchestration. By training on complete tree-structured traces rather than single solution paths, supervised learning can internalize systematic exploration strategies. Key advantages over existing approaches include: (1) coherent tree generation vs fragmented parallel sampling, (2) selective context decoupling enabling efficient branch exploration, (3) emergent capabilities like unsolvable problem detection, and (4) a new test-time scaling paradigm that explores within structured trees rather than across independent trajectories.

REFERENCES

- Maciej Besta, Nils Blach, Ales Kubicek, Robert Gerstenberger, Michal Podstawski, Lukas Gianinazzi, Joanna Gajda, Tomasz Lehmann, Hubert Niewiadomski, Piotr Nyczyk, and Torsten Hoefler. Graph of thoughts: Solving elaborate problems with large language models. *Proceedings of the AAAI Conference on Artificial Intelligence*, 38(16):17682–17690, March 2024. ISSN 2159-5399. doi: 10.1609/aaai.v38i16.29720. URL <http://dx.doi.org/10.1609/aaai.v38i16.29720>.
- Tom Brown, Benjamin Mann, Nick Ryder, Melanie Subbiah, Jared D Kaplan, Prafulla Dhariwal, Arvind Neelakantan, Pranav Shyam, Girish Sastry, Amanda Askell, et al. Language models are few-shot learners. *Advances in neural information processing systems*, 33:1877–1901, 2020.
- Xingyu Chen, Jiahao Xu, Tian Liang, Zhiwei He, Jianhui Pang, Dian Yu, Linfeng Song, Qiuzhi Liu, Mengfei Zhou, Zhuosheng Zhang, Rui Wang, Zhaopeng Tu, Haitao Mi, and Dong Yu. Do not think that much for $2+3=?$ on the overthinking of o1-like llms, 2025. URL <https://arxiv.org/abs/2412.21187>.
- Tianzhe Chu, Yuexiang Zhai, Jihan Yang, Shengbang Tong, Saining Xie, Dale Schuurmans, Quoc V. Le, Sergey Levine, and Yi Ma. Sft memorizes, rl generalizes: A comparative study of foundation model post-training, 2025. URL <https://arxiv.org/abs/2501.17161>.
- Karl Cobbe, Vineet Kosaraju, Mohammad Bavarian, Jacob Hilton, Reiichiro Nakano, Christopher Hesse, and John Schulman. Training verifiers to solve math word problems, 2021.
- DeepSeek-AI, Daya Guo, Dejian Yang, Haowei Zhang, Junxiao Song, Ruoyu Zhang, Runxin Xu, Qihao Zhu, Shirong Ma, Peiyi Wang, Xiao Bi, Xiaokang Zhang, Xingkai Yu, Yu Wu, Z. F. Wu, Zhibin Gou, Zhihong Shao, Zhuoshu Li, Ziyi Gao, Aixin Liu, Bing Xue, Bingxuan Wang, Bochao Wu, Bei Feng, Chengda Lu, Chenggang Zhao, Chengqi Deng, Chenyu Zhang, Chong Ruan, Damai Dai, Deli Chen, Dongjie Ji, Erhang Li, Fangyun Lin, Fucong Dai, Fuli Luo, Guangbo Hao, Guanting Chen, Guowei Li, H. Zhang, Han Bao, Hanwei Xu, Haocheng Wang, Honghui Ding, Huajian Xin, Huazuo Gao, Hui Qu, Hui Li, Jianzhong Guo, Jiashi Li, Jiawei Wang, Jingchang Chen, Jingyang Yuan, Junjie Qiu, Junlong Li, J. L. Cai, Jiaqi Ni, Jian Liang, Jin Chen, Kai Dong, Kai Hu, Kaige Gao, Kang Guan, Kexin Huang, Kuai Yu, Lean Wang, Lecong Zhang, Liang Zhao, Litong Wang, Liyue Zhang, Lei Xu, Leyi Xia, Mingchuan Zhang, Minghua Zhang, Minghui Tang, Meng Li, Miaojun Wang, Mingming Li, Ning Tian, Panpan Huang, Peng Zhang, Qiancheng Wang, Qinyu Chen, Qiushi Du, Ruiqi Ge, Ruisong Zhang, Ruizhe Pan, Runji Wang, R. J. Chen, R. L. Jin, Ruyi Chen, Shanghao Lu, Shangyan Zhou, Shanhuang Chen, Shengfeng Ye, Shiyu Wang, Shuiping Yu, Shunfeng Zhou, Shuting Pan, S. S. Li, Shuang Zhou, Shaoqing Wu, Shengfeng Ye, Tao Yun, Tian Pei, Tianyu Sun, T. Wang, Wangding Zeng, Wanbiao Zhao, Wen Liu, Wenfeng Liang, Wenjun Gao, Wenqin Yu, Wentao Zhang, W. L. Xiao, Wei An, Xiaodong Liu, Xiaohan Wang, Xiaokang Chen, Xiaotao Nie, Xin Cheng, Xin Liu, Xin Xie, Xingchao Liu, Xinyu Yang, Xinyuan Li, Xuecheng Su, Xuheng Lin, X. Q. Li, Xiangyue Jin, Xiaojin Shen, Xiaosha Chen, Xiaowen Sun, Xiaoxiang Wang, Xinnan Song, Xinyi Zhou, Xianzu Wang, Xinxia Shan, Y. K. Li, Y. Q. Wang, Y. X. Wei, Yang Zhang, Yanhong Xu, Yao Li, Yao Zhao, Yaofeng Sun, Yaohui Wang, Yi Yu, Yichao Zhang, Yifan Shi, Yiliang Xiong, Ying He, Yishi Piao, Yisong Wang, Yixuan Tan, Yiyang Ma, Yiyuan Liu, Yongqiang Guo, Yuan Ou, Yuduan Wang, Yue Gong, Yuheng Zou, Yujia He, Yunfan Xiong, Yuxiang Luo, Yuxiang You, Yuxuan Liu, Yuyang Zhou, Y. X. Zhu, Yanhong Xu, Yanping Huang, Yaohui Li, Yi Zheng, Yuchen Zhu, Yunxian Ma, Ying Tang, Yukun Zha, Yuting Yan, Z. Z. Ren, Zehui Ren, Zhangli Sha, Zhe Fu, Zhean Xu, Zhenda Xie, Zhengyan Zhang, Zhewen Hao, Zhicheng Ma, Zhigang Yan, Zhiyu Wu, Zihui Gu, Zijia Zhu, Zijun Liu, Zilin Li, Ziwei Xie, Ziyang Song, Zizheng Pan, Zhen Huang, Zhipeng Xu, Zhongyu Zhang, and Zhen Zhang. Deepseek-r1: Incentivizing reasoning capability in llms via reinforcement learning, 2025. URL <https://arxiv.org/abs/2501.12948>.

- 540 Xidong Feng, Ziyu Wan, Muning Wen, Stephen Marcus McAleer, Ying Wen, Weinan Zhang, and Jun
541 Wang. Alphazero-like tree-search can guide large language model decoding and training, 2024.
542 URL <https://arxiv.org/abs/2309.17179>.
- 543
544 Kanishk Gandhi, Denise Lee, Gabriel Grand, Muxin Liu, Winson Cheng, Archit Sharma, and
545 Noah D. Goodman. Stream of search (sos): Learning to search in language, 2024. URL <https://arxiv.org/abs/2404.03683>.
- 546
547 Aaron Grattafiori, Abhimanyu Dubey, Abhinav Jauhri, Abhinav Pandey, Abhishek Kadian, Ahmad
548 Al-Dahle, Aiesha Letman, Akhil Mathur, Alan Schelten, Alex Vaughan, Amy Yang, Angela Fan,
549 Anirudh Goyal, Anthony Hartshorn, Aobo Yang, Archi Mitra, Archie Sravankumar, Artem Korenev,
550 Arthur Hinsvark, Arun Rao, Aston Zhang, Aurelien Rodriguez, Austen Gregerson, Ava Spataru,
551 Baptiste Roziere, Bethany Biron, Binh Tang, Bobbie Chern, Charlotte Caucheteux, Chaya Nayak,
552 Chloe Bi, Chris Marra, Chris McConnell, Christian Keller, Christophe Touret, Chunyang Wu,
553 Corinne Wong, Cristian Canton Ferrer, Cyrus Nikolaidis, Damien Allonsius, Daniel Song, Danielle
554 Pintz, Danny Livshits, Danny Wyatt, David Esiobu, Dhruv Choudhary, Dhruv Mahajan, Diego
555 Garcia-Olano, Diego Perino, Dieuwke Hupkes, Egor Lakomkin, Ehab AlBadawy, Elina Lobanova,
556 Emily Dinan, Eric Michael Smith, Filip Radenovic, Francisco Guzmán, Frank Zhang, Gabriel
557 Synnaeve, Gabrielle Lee, Georgia Lewis Anderson, Govind Thattai, Graeme Nail, Gregoire Mialon,
558 Guan Pang, Guillem Cucurell, Hailey Nguyen, Hannah Korevaar, Hu Xu, Hugo Touvron, Iliyan
559 Zarov, Imanol Arrieta Ibarra, Isabel Kloumann, Ishan Misra, Ivan Evtimov, Jack Zhang, Jade Copet,
560 Jaewon Lee, Jan Geffert, Jana Vranes, Jason Park, Jay Mahadeokar, Jeet Shah, Jelmer van der Linde,
561 Jennifer Billock, Jenny Hong, Jenya Lee, Jeremy Fu, Jianfeng Chi, Jianyu Huang, Jiawen Liu, Jie
562 Wang, Jiecao Yu, Joanna Bitton, Joe Spisak, Jongsoo Park, Joseph Rocca, Joshua Johnstun, Joshua
563 Saxe, Junteng Jia, Kalyan Vasuden Alwala, Karthik Prasad, Kartikeya Upasani, Kate Plawiak,
564 Ke Li, Kenneth Heafield, Kevin Stone, Khalid El-Arini, Krithika Iyer, Kshitiz Malik, Kuenley
565 Chiu, Kunal Bhalla, Kushal Lakhotia, Lauren Rantala-Yearly, Laurens van der Maaten, Lawrence
566 Chen, Liang Tan, Liz Jenkins, Louis Martin, Lovish Madaan, Lubo Malo, Lukas Blecher, Lukas
567 Landzaat, Luke de Oliveira, Madeline Muzzi, Mahesh Pasupuleti, Mannat Singh, Manohar Paluri,
568 Marcin Kardas, Maria Tsimpoukelli, Mathew Oldham, Mathieu Rita, Maya Pavlova, Melanie
569 Kambadur, Mike Lewis, Min Si, Mitesh Kumar Singh, Mona Hassan, Naman Goyal, Narjes
570 Torabi, Nikolay Bashlykov, Nikolay Bogoychev, Niladri Chatterji, Ning Zhang, Olivier Duchenne,
571 Onur Çelebi, Patrick Alrassy, Pengchuan Zhang, Pengwei Li, Petar Vasic, Peter Weng, Prajwal
572 Bhargava, Pratik Dubal, Praveen Krishnan, Punit Singh Koura, Puxin Xu, Qing He, Qingxiao Dong,
573 Ragavan Srinivasan, Raj Ganapathy, Ramon Calderer, Ricardo Silveira Cabral, Robert Stojnic,
574 Roberta Raileanu, Rohan Maheswari, Rohit Girdhar, Rohit Patel, Romain Sauvestre, Ronnie
575 Polidoro, Roshan Sumbaly, Ross Taylor, Ruan Silva, Rui Hou, Rui Wang, Saghar Hosseini, Sahana
576 Chennabasappa, Sanjay Singh, Sean Bell, Seohyun Sonia Kim, Sergey Edunov, Shaoliang Nie,
577 Sharan Narang, Sharath Rapparthi, Sheng Shen, Shengye Wan, Shruti Bhosale, Shun Zhang, Simon
578 Vandenhende, Soumya Batra, Spencer Whitman, Sten Sootla, Stephane Collot, Suchin Gururangan,
579 Sydney Borodinsky, Tamar Herman, Tara Fowler, Tarek Sheasha, Thomas Georgiou, Thomas
580 Scialom, Tobias Speckbacher, Todor Mihaylov, Tong Xiao, Ujjwal Karn, Vedanuj Goswami,
581 Vibhor Gupta, Vignesh Ramanathan, Viktor Kerkez, Vincent Gonguet, Virginie Do, Vish Vogeti,
582 Vitor Albiero, Vladan Petrovic, Weiwei Chu, Wenhan Xiong, Wenyin Fu, Whitney Meers, Xavier
583 Martinet, Xiaodong Wang, Xiaofang Wang, Xiaojing Ellen Tan, Xide Xia, Xinfeng Xie, Xuchao
584 Jia, Xuewei Wang, Yaelle Goldschlag, Yashesh Gaur, Yasmine Babaei, Yi Wen, Yiwen Song,
585 Yuchen Zhang, Yue Li, Yuning Mao, Zacharie Delpierre Coudert, Zheng Yan, Zhengxing Chen, Zoe
586 Papakipos, Aaditya Singh, Aayushi Srivastava, Abha Jain, Adam Kelsey, Adam Shajnfeld, Adithya
587 Gangidi, Adolfo Victoria, Ahuva Goldstand, Ajay Menon, Ajay Sharma, Alex Boesenberg, Alexei
588 Baevski, Allie Feinstein, Amanda Kallet, Amit Sangani, Amos Teo, Anam Yunus, Andrei Lupu,
589 Andres Alvarado, Andrew Caples, Andrew Gu, Andrew Ho, Andrew Poulton, Andrew Ryan, Ankit
590 Ramchandani, Annie Dong, Annie Franco, Anuj Goyal, Aparajita Saraf, Arkabandhu Chowdhury,
591 Ashley Gabriel, Ashwin Bharambe, Assaf Eisenman, Azadeh Yazdan, Beau James, Ben Maurer,
592 Benjamin Leonhardi, Bernie Huang, Beth Loyd, Beto De Paola, Bhargavi Paranjape, Bing Liu,
593 Bo Wu, Boyu Ni, Braden Hancock, Bram Wasti, Brandon Spence, Brani Stojkovic, Brian Gamido,
Dustin Holland, Edward Dowling, Eissa Jamil, Elaine Montgomery, Eleonora Presani, Emily

- 594 Hahn, Emily Wood, Eric-Tuan Le, Erik Brinkman, Esteban Arcaute, Evan Dunbar, Evan Smothers,
595 Fei Sun, Felix Kreuk, Feng Tian, Filippos Kokkinos, Firat Ozgenel, Francesco Caggioni, Frank
596 Kanayet, Frank Seide, Gabriela Medina Florez, Gabriella Schwarz, Gada Badeer, Georgia Sweeney,
597 Gil Halpern, Grant Herman, Grigory Sizov, Guangyi, Zhang, Guna Lakshminarayanan, Hakan Inan,
598 Hamid Shojanazeri, Han Zou, Hannah Wang, Hanwen Zha, Haroun Habeeb, Harrison Rudolph,
599 Helen Suk, Henry Aspegren, Hunter Goldman, Hongyuan Zhan, Ibrahim Damlaj, Igor Molybog,
600 Igor Tufanov, Ilias Leontiadis, Irina-Elena Veliche, Itai Gat, Jake Weissman, James Geboski, James
601 Kohli, Janice Lam, Japhet Asher, Jean-Baptiste Gaya, Jeff Marcus, Jeff Tang, Jennifer Chan, Jenny
602 Zhen, Jeremy Reizenstein, Jeremy Teboul, Jessica Zhong, Jian Jin, Jingyi Yang, Joe Cummings,
603 Jon Carvill, Jon Shepard, Jonathan McPhie, Jonathan Torres, Josh Ginsburg, Junjie Wang, Kai
604 Wu, Kam Hou U, Karan Saxena, Kartikay Khandelwal, Katayoun Zand, Kathy Matosich, Kaushik
605 Veeraraghavan, Kelly Michelena, Keqian Li, Kiran Jagadeesh, Kun Huang, Kunal Chawla, Kyle
606 Huang, Lailin Chen, Lakshya Garg, Lavender A, Leandro Silva, Lee Bell, Lei Zhang, Liangpeng
607 Guo, Licheng Yu, Liron Moshkovich, Luca Wehrstedt, Madian Khabsa, Manav Avalani, Manish
608 Bhatt, Martynas Mankus, Matan Hasson, Matthew Lennie, Matthias Reso, Maxim Groshev, Maxim
609 Naumov, Maya Lathi, Meghan Keneally, Miao Liu, Michael L. Seltzer, Michal Valko, Michelle
610 Restrepo, Mihir Patel, Mik Vyatskov, Mikayel Samvelyan, Mike Clark, Mike Macey, Mike Wang,
611 Miquel Jubert Hermoso, Mo Metanat, Mohammad Rastegari, Munish Bansal, Nandhini Santhanam,
612 Natascha Parks, Natasha White, Navyata Bawa, Nayan Singhal, Nick Egebo, Nicolas Usunier,
613 Nikhil Mehta, Nikolay Pavlovich Laptev, Ning Dong, Norman Cheng, Oleg Chernoguz, Olivia
614 Hart, Omkar Salpekar, Ozlem Kalinli, Parkin Kent, Parth Parekh, Paul Saab, Pavan Balaji, Pedro
615 Rittner, Philip Bontrager, Pierre Roux, Piotr Dollar, Polina Zvyagina, Prashant Ratanchandani,
616 Pritish Yuvraj, Qian Liang, Rachad Alao, Rachel Rodriguez, Rafi Ayub, Raghotham Murthy,
617 Raghu Nayani, Rahul Mitra, Rangaprabhu Parthasarathy, Raymond Li, Rebekkah Hogan, Robin
618 Battey, Rocky Wang, Russ Howes, Ruty Rinott, Sachin Mehta, Sachin Siby, Sai Jayesh Bondu,
619 Samyak Datta, Sara Chugh, Sara Hunt, Sargun Dhillon, Sasha Sidorov, Satadru Pan, Saurabh
620 Mahajan, Saurabh Verma, Seiji Yamamoto, Sharadh Ramaswamy, Shaun Lindsay, Shaun Lindsay,
621 Sheng Feng, Shenghao Lin, Shengxin Cindy Zha, Shishir Patil, Shiva Shankar, Shuqiang Zhang,
622 Shuqiang Zhang, Sinong Wang, Sneha Agarwal, Soji Sajuyigbe, Soumith Chintala, Stephanie
623 Max, Stephen Chen, Steve Kehoe, Steve Satterfield, Sudarshan Govindaprasad, Sumit Gupta,
624 Summer Deng, Sungmin Cho, Sunny Virk, Suraj Subramanian, Sy Choudhury, Sydney Goldman,
625 Tal Remez, Tamar Glaser, Tamara Best, Thilo Koehler, Thomas Robinson, Tianhe Li, Tianjun
626 Zhang, Tim Matthews, Timothy Chou, Tzook Shaked, Varun Vontimitta, Victoria Ajayi, Victoria
627 Montanez, Vijai Mohan, Vinay Satish Kumar, Vishal Mangla, Vlad Ionescu, Vlad Poenaru,
628 Vlad Tiberiu Mihalescu, Vladimir Ivanov, Wei Li, Wenchen Wang, Wenwen Jiang, Wes Bouaziz,
629 Will Constable, Xiaocheng Tang, Xiaojian Wu, Xiaolan Wang, Xilun Wu, Xinbo Gao, Yaniv
630 Kleinman, Yanjun Chen, Ye Hu, Ye Jia, Ye Qi, Yenda Li, Yilin Zhang, Ying Zhang, Yossi Adi,
631 Youngjin Nam, Yu, Wang, Yu Zhao, Yuchen Hao, Yundi Qian, Yunlu Li, Yuzi He, Zach Rait,
632 Zachary DeVito, Zef Rosnbrick, Zhaoduo Wen, Zhenyu Yang, Zhiwei Zhao, and Zhiyu Ma. The
633 llama 3 herd of models, 2024. URL <https://arxiv.org/abs/2407.21783>.
- 634 Shibo Hao, Yi Gu, Haodi Ma, Joshua Jiahua Hong, Zhen Wang, Daisy Zhe Wang, and Zhiting Hu.
635 Reasoning with language model is planning with world model, 2023. URL <https://arxiv.org/abs/2305.14992>.
- 636 Adam Tauman Kalai, Ofir Nachum, Santosh S. Vempala, and Edwin Zhang. Why language models
637 hallucinate, 2025. URL <https://arxiv.org/abs/2509.04664>.
- 638 Doyoung Kim, Jongwon Lee, Jinho Park, and Minjoon Seo. How language models extrapolate
639 outside the training data: A case study in textualized gridworld, 2024. URL <https://arxiv.org/abs/2406.15275>.
- 640 Woosuk Kwon, Zhuohan Li, Siyuan Zhuang, Ying Sheng, Lianmin Zheng, Cody Hao Yu, Joseph E.
641 Gonzalez, Hao Zhang, and Ion Stoica. Efficient memory management for large language model
642 serving with pagedattention, 2023.
- 643 Yujia Li, David Choi, Junyoung Chung, Nate Kushman, Julian Schrittwieser, Rémi Leblond, Tom
644 Eccles, James Keeling, Felix Gimeno, Agustin Dal Lago, Thomas Hubert, Peter Choy, Cyprien
645 de Masson d’Autume, Igor Babuschkin, Xinyun Chen, Po-Sen Huang, Johannes Welbl, Sven
646 Goyal, Alexey Cherepanov, James Molloy, Daniel J. Mankowitz, Esme Sutherland Robson,
647

- 648 Pushmeet Kohli, Nando de Freitas, Koray Kavukcuoglu, and Oriol Vinyals. Competition-level code
649 generation with alphacode. *Science*, 378(6624):1092–1097, December 2022. ISSN 1095-9203. doi:
650 10.1126/science.abq1158. URL <http://dx.doi.org/10.1126/science.abq1158>.
- 651 OpenAI. Openai o1 system card. *preprint*, 2024.
- 652
- 653 Abulhair Saparov and He He. Language models are greedy reasoners: A systematic formal analysis
654 of chain-of-thought, 2023. URL <https://arxiv.org/abs/2210.01240>.
- 655 Zhihong Shao, Peiyi Wang, Qihao Zhu, Runxin Xu, Junxiao Song, Xiao Bi, Haowei Zhang,
656 Mingchuan Zhang, Y. K. Li, Y. Wu, and Daya Guo. Deepseekmath: Pushing the limits of
657 mathematical reasoning in open language models, 2024. URL <https://arxiv.org/abs/2402.03300>.
- 658
- 659 Noah Shinn, Federico Cassano, Edward Berman, Ashwin Gopinath, Karthik Narasimhan, and Shunyu
660 Yao. Reflexion: Language agents with verbal reinforcement learning, 2023.
- 661
- 662 Hugo Touvron, Thibaut Lavril, Gautier Izacard, Xavier Martinet, Marie-Anne Lachaux, Timothée
663 Lacroix, Baptiste Rozière, Naman Goyal, Eric Hambro, Faisal Azhar, et al. Llama: Open and
664 efficient foundation language models. *arXiv preprint arXiv:2302.13971*, 2023.
- 665 Xuezhi Wang, Jason Wei, Dale Schuurmans, Quoc Le, Ed Chi, Sharan Narang, Aakanksha Chowdh-
666 ery, and Denny Zhou. Self-consistency improves chain of thought reasoning in language models,
667 2023. URL <https://arxiv.org/abs/2203.11171>.
- 668
- 669 Ziwei Xu, Sanjay Jain, and Mohan Kankanhalli. Hallucination is inevitable: An innate limitation of
670 large language models, 2025. URL <https://arxiv.org/abs/2401.11817>.
- 671 Mengjiao Yang, Dale Schuurmans, Pieter Abbeel, and Ofir Nachum. Chain of thought imitation with
672 procedure cloning, 2022. URL <https://arxiv.org/abs/2205.10816>.
- 673 Shunyu Yao, Dian Yu, Jeffrey Zhao, Izhak Shafran, Thomas L. Griffiths, Yuan Cao, and Karthik
674 Narasimhan. Tree of thoughts: Deliberate problem solving with large language models, 2023.
- 675
- 676 Yuchen Zeng, Shuibai Zhang, Wonjun Kang, Shutong Wu, Lynnix Zou, Ying Fan, Heeju Kim,
677 Ziqian Lin, Jungtaek Kim, Hyung Il Koo, Dimitris Papailiopoulos, and Kangwook Lee. Rejump:
678 A tree-jump representation for analyzing and improving llm reasoning, 2025. URL <https://arxiv.org/abs/2512.00831>.
- 679
- 680 Andy Zhou, Kai Yan, Michal Shlapentokh-Rothman, Haohan Wang, and Yu-Xiong Wang. Language
681 agent tree search unifies reasoning acting and planning in language models. *arXiv preprint*
682 *arXiv:2310.04406*, 2023.

684 A FREQUENTLY ASKED QUESTIONS

- 685
- 686 **Q1: What do these results mean for reasoning model development more broadly?** Our results
687 challenge three prevailing assumptions. First, TSLM (67.2%) outperforms ToT (Yao et al., 2023)
688 (62.3%) despite ToT using 100 independent samples, revealing that coherent tree generation beats
689 fragmented parallel sampling. The bottleneck is not inference compute but systematic search space
690 construction. Second, ToT’s catastrophic failure on 20×20 Gridworld (42.7%) versus TSLM’s
691 extrapolation (91.5%) shows external search does not generalize when complexity scales; the search
692 procedure must be internalized during training. Third, TSLM achieves perfect Game of 24 accuracy
693 via supervised learning while GRPO (Shao et al., 2024) achieves only 15%, challenging the “SFT
694 Memorizes, RL Generalizes” paradigm (Chu et al., 2025). When supervision includes complete
695 exploration traces rather than just answers, supervised learning can acquire systematic reasoning
696 capabilities typically associated with RL methods.
- 697 **Q2: How does TSLM differ fundamentally from existing approaches (ToT, standard CoT,
698 GRPO)?** TSLM generates coherent tree structures within a single model forward pass, unlike
699 ToT (independent parallel sampling), standard CoT (single linear path), or GRPO (RL-based trace
700 generation). The key is selective context decoupling: TSLM conditions each branch only on relevant
701 ancestors and siblings, not the entire exploration history. This enables both efficient training (learning
from complete trees) and efficient inference (avoiding redundant prefix computation).

Q3: Can TSLM be applied to more complex real-world tasks? While our current experiments focus on relatively constrained tasks, the principles of TSLM can extend to more complex domains. For tasks with well-defined structure (like code generation or game playing), direct application is straightforward. For more open-ended tasks, our bootstrapping approach provides a foundation for constructing synthetic tree-structured training data. Future work should explore applications to more diverse and complex reasoning domains.

Q4: Why compare only with ToT and not advanced methods like RAP, LATS, or Reflexion?

These methods represent online planning (external feedback loops during inference), while TSLM represents offline planning (internalizing search during training). ToT is the representative baseline for external scaffolding without environment interaction. Methods like LATS(Zhou et al., 2023) and Reflexion(Shinn et al., 2023) are orthogonal and could potentially enhance TSLM, but address different problems (online refinement vs native tree generation). Note that our ToT@100 evaluation upper-bounds related sampling methods: mathematically $E[P(\text{RAP}@1)] \leq E[P(\text{ToT}@b^d)]$ and $E[P(\text{Self-consistency}@b^d)] \leq E[P(\text{ToT}@b^d)]$ since tree expansion has the same expectation as parallel sampling

Q5: Does TSLM work on truly open-ended tasks? TSLM relies on the ability to construct training trees, which requires verifiable solutions (even if only for a subset of data). For truly open-ended domains with subjective correctness (e.g., creative writing) or no clear verification, constructing the training signal is challenging. This is a limitation of the current bootstrapping approach, though the core principle of structured exploration may still apply if suitable reward signals can be defined.

B RELATED WORK

External Search-Augmented Language Models. A major line of work augments language models with external search algorithms during inference. Tree-of-Thought Yao et al. (2023) applies breadth-first and depth-first search externally, sampling multiple reasoning paths with external evaluation. Graph-of-Thought Besta et al. (2024) extends this to general graph structures. Monte Carlo Tree Search (MCTS) approaches include RAP (Reasoning via Planning) Hao et al. (2023), which employs MCTS with world models, and LATS (Language Agent Tree Search) Zhou et al. (2023), which combines MCTS with reflection mechanisms. TS-LLM Feng et al. (2024) and AlphaCode-style approaches Li et al. (2022) integrate MCTS with language generation. These methods achieve strong performance but require multiple model invocations and external orchestration, limiting efficiency and integration with model training.

Multi-Path Generation and Reasoning. Various approaches explore multiple reasoning paths without structured search. Self-consistency decoding Wang et al. (2023) generates multiple independent reasoning paths and selects the most consistent answer. Ensemble methods combine predictions from multiple reasoning chains through diverse beam search and nucleus sampling variants. Recent models like o1 OpenAI (2024) and DeepSeek-R1 DeepSeek-AI et al. (2025) generate extended reasoning traces, but remain fundamentally sequential and may include redundant computation Chen et al. (2025). These approaches explore multiple paths but typically generate them independently without capturing structural relationships, unlike TSLM’s coherent tree structures.

Learning-Based Reasoning Enhancement. Reinforcement learning has emerged as a dominant paradigm for improving reasoning capabilities. DeepSeek Math Shao et al. (2024) applies GRPO for mathematical reasoning, while other work uses RL from human feedback for instruction following. Actor-Critic methods and policy gradient approaches have shown promise in multi-step reasoning and mathematical problem solving. Algorithmic reasoning approaches train models to imitate procedures like sorting, graph traversal, and dynamic programming Kim et al. (2024); Yang et al. (2022), learning to execute classical algorithms step-by-step. However, these methods either require complex RL training or focus on single algorithmic traces rather than dynamic exploration strategies. TSLM demonstrates that carefully structured supervised learning can achieve comparable performance without RL’s complexity.

756 C TSLM SERIALIZATION EXAMPLE

757
758 This section provides a detailed worked example showing how TSLM serializes tree structures for
759 the Game of 24 task.

760
761 **Problem Setup.** Consider a Game of 24 problem with numbers [4, 5, 6, 10]. The goal is to find
762 arithmetic operations that result in 24.

763
764 **Complete Tree Serialization.** The tree structure is serialized as:

765
766 Input:
767 $4 + 5 = 9$ [SEP]
768 $6 + 9 = 15$ [SEP]
769
770 Output:
771 [BOS]
772 $9 + 15 = 24$ [GOAL]
773 $9 - 15 = -6$ [FAIL] [EOS]

774 This format captures:

- 775 • Multiple candidate actions at each step
- 776 • Viability markers ([SEP] for expandable, [FAIL] for dead ends)
- 777 • Step boundaries ([BOS], [EOS])
- 778 • Both successful and unsuccessful exploration paths

781 D LIMITATIONS AND FUTURE WORK

782
783 **Computational Overhead.** Training on complete trees requires processing all nodes rather than
784 single paths, increasing computational cost by a factor of the average tree size. Future work should
785 explore efficient training strategies like tree-aware attention caching and sparse gradient updates for
786 selective subtree optimization.

787
788 **Supervision Quality Dependence.** Synthetic tree generation quality depends heavily on the super-
789 vision model’s capabilities. For instance, Llama-3-8B produces poor-quality trees for GSM8K, while
790 Llama-3-8B-Instruct works well. Models must match the target reasoning format (some generate
791 Python code instead of step-by-step reasoning, others include extraneous dialogue). Better automated
792 methods for supervision model selection and tree quality validation are needed.

793
794 **Open-Ended Domains.** TSLM requires verifiable solution paths to construct training trees. In truly
795 open-ended domains with subjective correctness (creative writing, open dialogue), defining branch
796 viability becomes challenging. Adapting TSLM to such domains may require preference learning or
797 soft verification signals rather than binary correctness.

798 E DETAILED INFERENCE-TIME SCALING ANALYSIS

800 E.1 BFS VS DFS: EXPLORING SEARCH STRATEGIES

801
802 Within TSLM’s inference framework, the choice between Breadth-First Search (BFS) and Depth-First
803 Search (DFS) reveals fundamental differences in exploration priorities. We analyze these search
804 strategies when TSLM’s branching is guided by preference-based ordering.

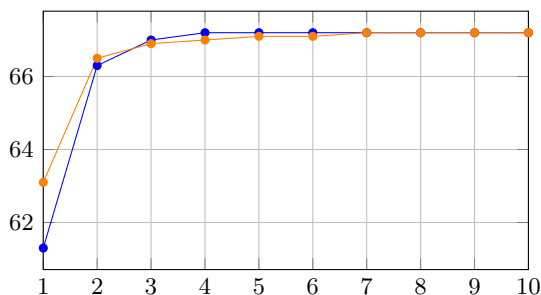
805
806 **Theoretical Analysis.** Consider two distinct solution paths $A = a_1, \dots, a_n$ and $B = b_1, \dots, b_m$ in a
807 preference-ordered search tree generated by TSLM.

808
809 BFS employs a dual prioritization strategy: it first considers path length (traversing A before B if
 $n < m$), then uses preference signals to break ties. When paths have equal length ($n = m$), BFS

810 selects based on the first differing action’s reward. If $R(a_k) > R(b_k)$ at the earliest divergence
 811 point k , BFS favors path A . This means **BFS prioritizes solution optimality first, using learned**
 812 **preferences to resolve ties.**

813 In contrast, DFS operates purely on learned preference ordering, disregarding path length con-
 814 siderations. For the same paths A and B , DFS immediately follows the higher-reward action at
 815 any divergence point k , regardless of whether this leads to longer solution paths. This approach
 816 means **DFS prioritizes learned preferences consistently, potentially sacrificing optimality for**
 817 **high-confidence actions.**

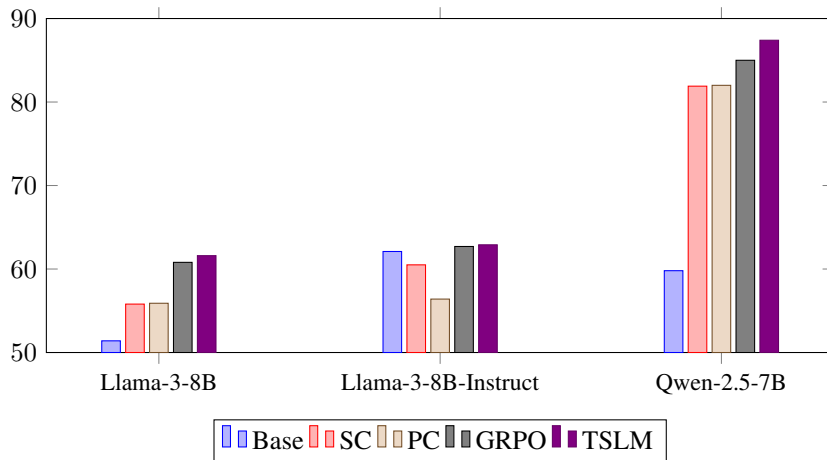
818
 819 **Empirical Results.** Figure 4 shows that DFS achieves better top-1 accuracy by finding preferred
 820 solutions first, but slightly underperforms BFS for top-3 to top-6 candidates since it does not prioritize
 821 optimality. However, both methods converge to the same accuracy as they traverse the complete
 822 tree. This validates our theoretical analysis that DFS provides faster convergence to high-confidence
 823 solutions while BFS offers better overall exploration coverage.



825
826
827
828
829
830
831
832
833
834
835 Figure 4: Comparison of mean accuracy (%) over number of candidates for different search strategies
 836 (BFS: ●, DFS ●)

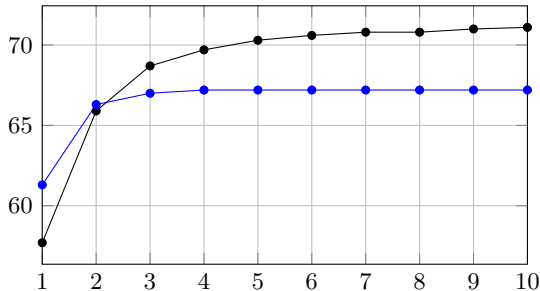
837 838 839 E.2 BASE MODEL PERFORMANCE IMPACT

840 We investigate whether the base model’s performance influences TSLM’s effectiveness by comparing
 841 performance across different models with different GSM8K performance (Llama-3-8B, Llama-3-8B-
 842 Instruct, and Qwen-2.5-7B). TSLM consistently outperforms baseline methods across both model
 843 variants, with improvements remaining robust across architectures. This indicates that TSLM’s
 844 structured exploration provides consistent benefits independent of the base model.



845
846
847
848
849
850
851
852
853
854
855
856
857
858
859
860
861
862
863
Figure 5: GSM8K accuracy comparison across different base models and methods.

864
865
866
867
868
869
870
871
872
873



874 Figure 6: Comparison of TSLM performance for the base branching factors $k = 5$ (●) vs. extrapolated
875 $k = 10$ (●). Extrapolating the branching factor improves accuracy in a scalable manner.
876

877
878 E.3 BRANCHING FACTOR ANALYSIS

879 The branching factor k in search tree generation (Algorithm 1) controls the maximum number of
880 candidates expanded at each node. We initially set $k = 5$ to match the supervision breadth, but a
881 natural question arises: Can TSLM effectively scale to larger branching factors?
882

883 In principle, if we traverse the complete search tree, a larger k should yield better accuracy by
884 exploring more candidates. However, when limited to examining just the first few terminated
885 candidates, increasing the branching factor could potentially add noise that degrades performance.

886 We investigate this empirically by comparing TSLM performance between $k = 5$ and $k = 10$
887 branching factors. As shown in Figure 6, extrapolating to the larger branching factor yields consistent
888 improvements, with convergence rate increasing from 67.2% to 71.1%. This enhanced performance
889 indicates TSLM can effectively leverage broader exploration without being overwhelmed by the
890 expanded search space.

891
892 F TASK DETAILS AND EXAMPLES

893
894

Task	# Depth(Train)	# Depth(Test)	# Train Instance	# Test Instance	Evaluation Metric
Game of 24	3	3	1.2K	100	Equation Validation
Gridworld	10 × 10	10 × 10	10K	1.5K	Exact Matching
Gridworld (o.o.d)	10 × 10	20 × 20	10K	1.5K	Exact Matching
ProntoQA	1-5	1-5	4.5K	450	Exact Matching
GSM8K	2-9	2-11	7.5K	1.3K	Answer Matching

895
896
897
898
899
900

901
902 Table 3: General task settings

903
904
905 F.1 GENERAL TASK SETUP DETAILS

906 Table 3 illustrates the general task settings for structured and open-ended tasks. Depth refers to the
907 number of reasoning steps.
908

909 F.2 TASK SCENARIO 1: STRUCTURED PLANNING

910 We first evaluate TSLM and baselines on two different structured planning tasks (Left in Figure
911 7). These tasks involve predefined search trees, enabling us to evaluate how well models reproduce
912 algorithmic patterns:
913

914
915 **Game of 24** Given four numbers, the task is to obtain 24 with basic arithmetic operations
916 ($+$, $-$, \times , \div). For example, with inputs 8, 4, 3, and 6, a valid solution is $(8 + 4) \times (6 \div 3) = 24$
917 Yao et al. (2023). Each action selects two operands and an operator, while states represent partial
expressions. Since there may be multiple paths that reaches the answer (e.g. $(8 + 4) \times (6 \div 3)$ and

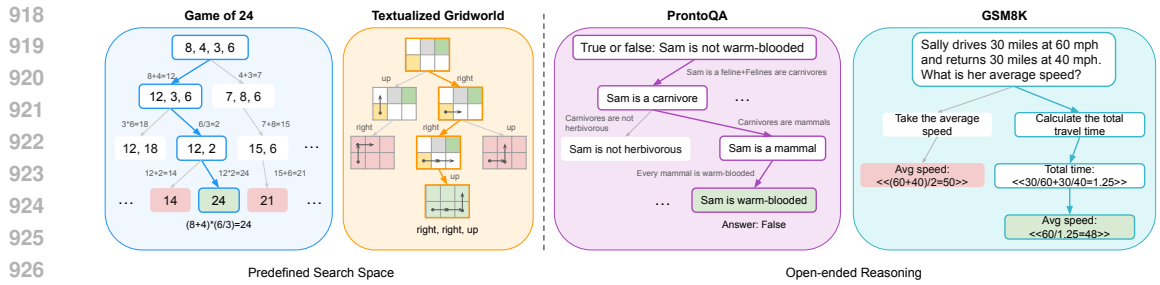


Figure 7: (Left) Structured planning tasks with predefined search spaces (e.g., Game of 24, Textualized Gridworld) where success is measured by the model’s ability to reproduce algorithmic search patterns. (Right) General open-ended reasoning tasks (e.g., ProntoQA, GSM8K) requiring adaptive exploration of undefined search spaces.

$(6 \div 3) \times (4 + 8)$), we check if the final expression equals 24. We train each method on 1.2K instances and test on 100 instances.

Textualized Gridworld A text-based navigation task where an agent navigates a grid using cardinal directions (up, down, left, right) while avoiding obstacles (Kim et al., 2024). Given a start position (typically bottom left) and goal position (typically top right), the agent must find a valid path with shortest length. For instance, in a 3x2 grid with a pit at (1,1), the optimal solution is (right, right, up). Actions are single moves (up/down/left/right) and states are grid coordinates. We ensure all the environments to have unique shortest paths. We use 10K training instances with maximum grid size 10x10 and 1.5K test instances up to the same maximum size, evaluating exact path matching.

F.3 TASK SCENARIO 2: OPEN-ENDED REASONING TASKS

We next evaluate TSLM and baselines on two open-ended reasoning tasks (Right on Figure 7) requiring exploration of undefined solution spaces. We use the synthetic tree generation method from §4.2 for training.

ProntoQA ProntoQA Saporov & He (2023) is a logical reasoning dataset where models verify statement truth given premises. For example, given a premise “Every jompus is not small. Each impus is small. Each jompus is a dumpus. Alex is a jompus.” and a query “True or false: Alex is not small?”, the solution path to verify the query should be “Alex is a jompus. Every jompus is not small. Alex is not small. So the answer is True.”

Actions are logical deduction steps and states are accumulated facts. With unique valid deduction paths, we verify whether the generated path is correct. We use 4.5K training and 450 test instances. For TSLM/PC, we use tree generation of beam search ($k=5$, $temp=0.3$) with Llama-3-8B as a base model for supervision.

GSM8K GSM8K Cobbe et al. (2021) is a mathematical word problem dataset requiring multi-step reasoning. For example, for a given problem “James runs 3 sprints 3 times a week. Each sprint is 60 meters. How many total meters per week?”, one valid solution would be “3 sprints \times 3 times = 9 sprints per week. 9 sprints \times 60 meters = 540 meters. The answer is 540.”

Actions are solution steps and states are partial solutions. Since multiple valid solution paths can reach the correct answer, we just validate whether the answer is correct. We use 7.5K training instances and 1.3K test instances. For TSLM/PC, we use tree generation of beam search ($k=5$, $temp=0.1$) with Llama-3-8B-Instruct as a base model for supervision.

F.4 TASK EXAMPLES

Gridworld SC Instance:

972 **USER:** You are given a rectangular gridworld, where you can move up, down, left, or right
973 as long as each of your x, y coordinate is within 0 to the x, y size of the grid. If you move
974 up, your y coordinate increases by 1. If you move down, your y coordinate decreases by 1.
975 If you move left, your x coordinate decreases by 1. If you move right, your x coordinate
976 increases by 1.
977 You will interact with the gridworld environment to reach the goal state, while avoiding the
978 pit and the wall. You cannot move through the wall or move outside the grid. If you fall
979 into the pit, you lose. If you reach the goal, you win. For each of your turn, you will be
980 given the possible moves.
981 You should respond your move with either one of 'up', 'down', 'left', or 'right'.
982 **ASSISTANT: OK**
983 **USER:** Grid is from (7, 7) to (18, 13). Goal: (18, 13) Current: (7, 7) The pit is at (16, 9),
984 (17, 7), (14, 8), (11, 9), and (16, 11). The wall is at (11, 7), (15, 10), (18, 12), (9, 9), (8, 7),
985 (17, 10), (9, 12), (12, 7), (10, 13), (15, 12), (17, 9), (9, 11), (17, 12), (8, 9), (13, 11), (10, 9),
986 (12, 9), (11, 11), (12, 12), (16, 10), (15, 8), (9, 7), (8, 11), (7, 13), (13, 13), and (13, 10).
987 **ASSISTANT:**
988 Step 1
989 up
990 (7, 8)[SEP]
991 Step 2
992 up
993 (7, 9)[SEP]
994 Step 3
995 up
996 (7, 10)[SEP]
997 Step 4
998 up
999 (7, 11)[SEP]
1000 ...

999 **Gridworld PC Instance:**
1000 **USER:** You are given a rectangular gridworld, where you can move up, down, left, or right
1001 as long as each of your x, y coordinate is within 0 to the x, y size of the grid. If you move
1002 up, your y coordinate increases by 1. If you move down, your y coordinate decreases by 1.
1003 If you move left, your x coordinate decreases by 1. If you move right, your x coordinate
1004 increases by 1.
1005 You will interact with the gridworld environment to reach the goal state, while avoiding the
1006 pit and the wall. You cannot move through the wall or move outside the grid. If you fall
1007 into the pit, you lose. If you reach the goal, you win. For each of your turn, you will be
1008 given the possible moves.
1009 You should respond your move with either one of 'up', 'down', 'left', or 'right'.
1010 **ASSISTANT: OK**
1011 **USER:** Grid is from (7, 7) to (18, 13). Goal: (18, 13) Current: (7, 7) The pit is at (16, 9),
1012 (17, 7), (14, 8), (11, 9), and (16, 11). The wall is at (11, 7), (15, 10), (18, 12), (9, 9), (8, 7),
1013 (17, 10), (9, 12), (12, 7), (10, 13), (15, 12), (17, 9), (9, 11), (17, 12), (8, 9), (13, 11), (10, 9),
1014 (12, 9), (11, 11), (12, 12), (16, 10), (15, 8), (9, 7), (8, 11), (7, 13), (13, 13), and (13, 10).
1015 **ASSISTANT:**
1016 Step 1
1017 up
1018 (7, 8)[SEP]
1019 down
1020 (7, 6)[FAIL]
1021 left
1022 (6, 7)[FAIL]
1023 right
1024 (8, 7)[FAIL]
1025 Step 2
1026 up
1027 (7, 9)[SEP]

1026 down
1027 (7, 7)[FAIL]
1028 left
1029 (6, 8)[FAIL]
1030 right
1031 (8, 8)[SEP]
1032 ...

1033 **Gridworld TSLM Instance:**
1034 **USER:** You are given a rectangular gridworld, where you can move up, down, left, or right
1035 as long as each of your x, y coordinate is within 0 to the x, y size of the grid. If you move
1036 up, your y coordinate increases by 1. If you move down, your y coordinate decreases by 1.
1037 If you move left, your x coordinate decreases by 1. If you move right, your x coordinate
1038 increases by 1.
1039 You will interact with the gridworld environment to reach the goal state, while avoiding the
1040 pit and the wall. You cannot move through the wall or move outside the grid. If you fall
1041 into the pit, you lose. If you reach the goal, you win. For each of your turn, you will be
1042 given the possible moves.
1043 You should respond your move with either one of 'up', 'down', 'left', or 'right'.
1044 **ASSISTANT:** OK
1045 **USER:** Grid is from (7, 7) to (18, 13). Goal: (18, 13) Current: (7, 7) The pit is at (16, 9),
1046 (17, 7), (14, 8), (11, 9), and (16, 11). The wall is at (11, 7), (15, 10), (18, 12), (9, 9), (8, 7),
1047 (17, 10), (9, 12), (12, 7), (10, 13), (15, 12), (17, 9), (9, 11), (17, 12), (8, 9), (13, 11), (10, 9),
1048 (12, 9), (11, 11), (12, 12), (16, 10), (15, 8), (9, 7), (8, 11), (7, 13), (13, 13), and (13, 10).
1049 **ASSISTANT:**
1050 Step 1
1051 up
1052 (7, 8)[SEP]
1053 Step 2
1054 up
1055 (7, 9)[SEP]
1056 Step 3
1057 up
1058 (7, 10)[SEP]
1059 Step 4
1060 up
1061 (7, 11)[SEP]
1062 down
1063 (7, 9)[FAIL]
1064 left
1065 (6, 10)[FAIL]
1066 right
1067 (8, 10)[SEP]

1068 **Game24 SC Instance:**
1069 **USER:** Input: 4, 5, 6, 10
1070 **ASSISTANT:** Step 1
1071 $4 + 5 = 9$ [SEP]
1072 Step 2
1073 $6 + 10 = 16$ [SEP]

1074 **Game24 PC Instance:**
1075 **USER:** Input: 4, 5, 6, 10
1076 **ASSISTANT:** Step 1
1077 $4 + 5 = 9$ [SEP]
1078 $4 - 5 = -1$ [SEP]
1079 $4 * 5 = 20$ [SEP]
 $4 / 5 = 4/5$ [SEP]
 $4 + 6 = 10$ [SEP]
 $4 - 6 = -2$ [SEP]

1080 $4 * 6 = 24$ [SEP]
1081 $4 / 6 = 2/3$ [SEP]
1082 $4 + 10 = 14$ [SEP]
1083 $4 - 10 = -6$ [SEP]
1084 $4 * 10 = 40$ [SEP]
1085 $4 / 10 = 2/5$ [SEP]
1086 $5 + 4 = 9$ [SEP]
1087 $5 - 4 = 1$ [SEP]
1088 $5 * 4 = 20$ [SEP]
1089 $5 / 4 = 5/4$ [SEP]
1090 $5 + 6 = 11$ [SEP]
1091 $5 - 6 = -1$ [SEP]
1092 $5 * 6 = 30$ [SEP]
1093 $5 / 6 = 5/6$ [SEP]
1094 $5 + 10 = 15$ [SEP]
1095 $5 - 10 = -5$ [SEP]
1096 $5 * 10 = 50$ [SEP]
1097 $5 / 10 = 1/2$ [SEP]
1098 $6 + 4 = 10$ [SEP]
1099 $6 - 4 = 2$ [SEP]
1100 $6 * 4 = 24$ [SEP]
1101 $6 / 4 = 3/2$ [SEP]
1102 $6 + 5 = 11$ [SEP]
1103 $6 - 5 = 1$ [SEP]
1104 $6 * 5 = 30$ [SEP]
1105 $6 / 5 = 6/5$ [SEP]
1106 $6 + 10 = 16$ [SEP]
1107 $6 - 10 = -4$ [SEP]
1108 $6 * 10 = 60$ [SEP]
1109 $6 / 10 = 3/5$ [SEP]
1110 $10 + 4 = 14$ [SEP]
1111 $10 - 4 = 6$ [SEP]
1112 $10 * 4 = 40$ [SEP]
1113 $10 / 4 = 5/2$ [SEP]
1114 $10 + 5 = 15$ [SEP]
1115 $10 - 5 = 5$ [SEP]
1116 $10 * 5 = 50$ [SEP]
1117 $10 / 5 = 2$ [SEP]
1118 $10 + 6 = 16$ [SEP]
1119 $10 - 6 = 4$ [SEP]
1120 $10 * 6 = 60$ [SEP]
1121 $10 / 6 = 5/3$ [SEP][FAIL]
1122 Step 2
1123 $6 + 10 = 16$ [SEP]
1124 $6 - 10 = -4$ [SEP]
1125 $6 * 10 = 60$ [SEP]
1126 $6 / 10 = 3/5$ [SEP]
1127 $6 + 9 = 15$ [SEP]
1128 $6 - 9 = -3$ [SEP]
1129 $6 * 9 = 54$ [SEP]
1130 $6 / 9 = 2/3$ [SEP]
1131 $10 + 6 = 16$ [SEP]
1132 $10 - 6 = 4$ [SEP]
1133 $10 * 6 = 60$ [SEP]
 $10 / 6 = 5/3$ [SEP]
 $10 + 9 = 19$ [SEP]
 $10 - 9 = 1$ [SEP]
 $10 * 9 = 90$ [SEP]
 $10 / 9 = 10/9$ [SEP]

1134 $9 + 6 = 15$ [SEP]
1135 $9 - 6 = 3$ [SEP]
1136 $9 * 6 = 54$ [SEP]
1137 $9 / 6 = 3/2$ [SEP]
1138 $9 + 10 = 19$ [SEP]
1139 $9 - 10 = -1$ [SEP]
1140 $9 * 10 = 90$ [SEP]
1141 $9 / 10 = 9/10$ [SEP][FAIL]
1142 $6 + 10 = 16$ [SEP]
1143 $6 - 10 = -4$ [SEP]
1144 $6 * 10 = 60$ [SEP]
1145 $6 / 10 = 3/5$ [SEP]
1146 $6 + -1 = 5$ [SEP]
1147 $6 - -1 = 7$ [SEP]
1148 $6 * -1 = -6$ [SEP]
1149 $6 / -1 = -6$ [SEP]
1150 $10 + 6 = 16$ [SEP]
1151 $10 - 6 = 4$ [SEP]
1152 $10 * 6 = 60$ [SEP]
1153 $10 / 6 = 5/3$ [SEP]
1154 $10 + -1 = 9$ [SEP]
1155 $10 - -1 = 11$ [SEP]
1156 $10 * -1 = -10$ [SEP]
1157 $10 / -1 = -10$ [SEP]
1158 $-1 + 6 = 5$ [SEP]
1159 $-1 - 6 = -7$ [SEP]
1160 $-1 * 6 = -6$ [SEP]
1161 $-1 / 6 = -1/6$ [SEP]
1162 $-1 + 10 = 9$ [SEP]
1163 $-1 - 10 = -11$ [SEP]
1164 $-1 * 10 = -10$ [SEP]
1165 $-1 / 10 = -1/10$ [SEP][FAIL]
1166 $6 + 10 = 16$ [SEP]
1167 $6 - 10 = -4$ [SEP]
1168 $6 * 10 = 60$ [SEP]
1169 $6 / 10 = 3/5$ [SEP]
1170 $6 + 20 = 26$ [SEP]
1171 $6 - 20 = -14$ [SEP]
1172 $6 * 20 = 120$ [SEP]
1173 $6 / 20 = 3/10$ [SEP]
1174 $10 + 6 = 16$ [SEP]
1175 $10 - 6 = 4$ [SEP]
1176 $10 * 6 = 60$ [SEP]
1177 $10 / 6 = 5/3$ [SEP]
1178 $10 + 20 = 30$ [SEP]
1179 $10 - 20 = -10$ [SEP]
1180 $10 * 20 = 200$ [SEP]
1181 $10 / 20 = 1/2$ [SEP]
1182 $20 + 6 = 26$ [SEP]
1183 $20 - 6 = 14$ [SEP]
1184 $20 * 6 = 120$ [SEP]
1185 $20 / 6 = 10/3$ [SEP]
1186 $20 + 10 = 30$ [SEP]
1187 $20 - 10 = 10$ [SEP]
1188 $20 * 10 = 200$ [SEP]
1189 $20 / 10 = 2$ [SEP][FAIL]

Game24 TSLM Instance:

USER: Input: 4, 5, 6, 10

1188 **ASSISTANT:** Step 1
 1189 $4 + 5 = 9$ [SEP]
 1190 Step 2
 1191 $6 + 10 = 16$ [SEP]
 1192 $6 - 10 = -4$ [SEP]
 1193 $6 * 10 = 60$ [SEP]
 1194 $6 / 10 = 3/5$ [SEP]
 1195 $6 + 9 = 15$ [SEP]
 1196 $6 - 9 = -3$ [SEP]
 1197 $6 * 9 = 54$ [SEP]
 1198 $6 / 9 = 2/3$ [SEP]
 1199 $10 + 6 = 16$ [SEP]
 1200 $10 - 6 = 4$ [SEP]
 1201 $10 * 6 = 60$ [SEP]
 1202 $10 / 6 = 5/3$ [SEP]
 1203 $10 + 9 = 19$ [SEP]
 1204 $10 - 9 = 1$ [SEP]
 1205 $10 * 9 = 90$ [SEP]
 1206 $10 / 9 = 10/9$ [SEP]
 1207 $9 + 6 = 15$ [SEP]
 1208 $9 - 6 = 3$ [SEP]
 1209 $9 * 6 = 54$ [SEP]
 1210 $9 / 6 = 3/2$ [SEP]
 1211 $9 + 10 = 19$ [SEP]
 1212 $9 - 10 = -1$ [SEP]
 1213 $9 * 10 = 90$ [SEP]
 1214 $9 / 10 = 9/10$ [SEP][FAIL]

ProntoQA SC Instance:

1214 **USER:** Arthropods are small. Insects are not eight-legged. Lepidopteran are insects.
 1215 Invertebrates are animals. Every spider is eight-legged. Insects are arthropods. Every
 1216 butterfly is a lepidopteran. Animals are multicellular. Arthropods are invertebrates. Sam is
 1217 a butterfly.

1218 True or false: Sam is not eight-legged.

1219 **ASSISTANT:**

1220 Step 1
 1221 Sam is a butterfly.[SEP]
 1222 Step 2
 1223 Every butterfly is a lepidopteran.[SEP]
 1224 Step 3
 1225 Sam is a lepidopteran.[SEP]
 1226 Step 4
 1227 Lepidopteran are insects.[SEP]
 1228 Step 5
 1229 Sam is an insect.[SEP]
 1230 Step 6
 1231 Insects are not eight-legged.[SEP]
 1232 Step 7
 1233 Sam is not eight-legged.[SEP]
 1234 Step 8
 The answer is true.[GOAL]

1235 **ProntoQA PC Instance:** c Arthropods are small. Insects are not eight-legged. Lepi-
 1236 dopterans are insects. Invertebrates are animals. Every spider is eight-legged. Insects are
 1237 arthropods. Every butterfly is a lepidopteran. Animals are multicellular. Arthropods are
 1238 invertebrates. Sam is a butterfly.

1239 True or false: Sam is not eight-legged.

1240 **ASSISTANT:**

1241 Step 1
 Sam is a butterfly.[SEP]

1242 Step 2
 1243 Every butterfly is a lepidopteran.[SEP]
 1244 Step 3
 1245 Sam is a lepidopteran.[SEP]
 1246 Step 4
 1247 Lepidopterans are insects.[SEP]
 1248 Step 5
 1249 Sam is an insect.[SEP]
 1250 Step 6
 1251 Insects are not eight-legged.[SEP]
 1252 Insects are arthropods.[SEP]
 1253 Step 7
 1254 Sam is not eight-legged.[SEP]
 1255 Sam is an arthropod.[GOAL]

ProntoQA TSLM Instance:

1256 **USER:** Arthropods are small. Insects are not eight-legged. Lepidopterans are insects.
 1257 Invertebrates are animals. Every spider is eight-legged. Insects are arthropods. Every
 1258 butterfly is a lepidopteran. Animals are multicellular. Arthropods are invertebrates. Sam is
 1259 a butterfly.

1260 True or false: Sam is not eight-legged.

1261 **ASSISTANT:**

1262 Step 1
 1263 Sam is a butterfly.[SEP]
 1264 Step 2
 1265 Every butterfly is a lepidopteran.[SEP]
 1266 Step 3
 1267 Sam is a lepidopteran.[SEP]
 1268 Step 4
 1269 Lepidopterans are insects.[SEP]
 1270 Step 5
 1271 Sam is an insect.[SEP]
 1272 Step 6
 1273 Insects are not eight-legged.[SEP]
 1274 Step 7
 1275 Sam is not eight-legged.[SEP]
 1276 Sam is an arthropod.[GOAL]

GSM8K SC Instance:

1277 **USER:** Kylar went to the store to buy glasses for his new apartment. One glass costs \$5,
 1278 but every second glass costs only 60% of the price. Kylar wants to buy 16 glasses. How
 1279 much does he need to pay for them?

1280 **ASSISTANT:**

1281 Step 1
 1282 Every second glass costs $5 * 0.6 = \$ \ll 5 * 0.6 = 3 \gg 3$. [SEP]
 1283 Step 2
 1284 So for 16 glasses, Kylar needs to pay $3 * 8 = \$ \ll 16/2 = 8 \gg$ for every second
 1285 glass.[GOAL]

GSM8K PC Instance:

1286 **USER:** A robe takes 2 bolts of blue fiber and half that much white fiber. How many bolts
 1287 in total does it take?

1288 **ASSISTANT:**

1289 Step 1
 1290 It takes $2 * 0.5 = \ll 2 * 0.5 = 1 \gg 1$ bolt of white fiber.[SEP]
 1291 The robe takes 2 bolts of blue fiber.[SEP]
 1292 Step 2
 1293 So it takes $2 + 1 = \ll 2 + 1 = 3 \gg 3$ bolts in total.[SEP]
 1294 It also takes half as much white fiber, which means it takes 1 bolt of white fiber (since half
 1295 of 2 is 1).[SEP]

1296
1297
1298
1299
1300
1301
1302
1303
1304
1305
1306
1307
1308
1309
1310
1311
1312
1313
1314
1315
1316
1317
1318
1319
1320
1321
1322
1323
1324
1325
1326
1327
1328
1329
1330
1331
1332
1333
1334
1335
1336
1337
1338
1339
1340
1341
1342
1343
1344
1345
1346
1347
1348
1349

Step 3
The answer is 3.[GOAL]

GSM8K TSLM Instance:

USER: Kylar went to the store to buy glasses for his new apartment. One glass costs \$5, but every second glass costs only 60% of the price. Kylar wants to buy 16 glasses. How much does he need to pay for them?

ASSISTANT:

Step 1

Every second glass costs $5 * 0.6 = \$ \ll 5 * 0.6 = 3 \gg 3$. [SEP]

Step 2

So for 16 glasses, Kylar needs to pay $3 * 8 = \$ \ll 16/2 = 8 \gg$ for every second glass. [SEP]

So for 8 glasses, it would cost $8 * 3 = 24$ dollars. For the remaining 8 glasses, it would cost $8 * 5 = 40$ dollars. So in total, it would cost $24 + 40 = 64$ dollars. [GOAL]

G USE OF LARGE LANGUAGE MODELS

We acknowledge the use of large language models to assist in the preparation of this manuscript. Specifically:

Writing assistance. Large language models were used to aid in polishing and refining the writing throughout the paper, including improving clarity, grammar, and expression of technical concepts.

Related work discovery. Large language models were employed for retrieval and discovery tasks, particularly in identifying and organizing relevant related work and ensuring comprehensive coverage of the literature.

All technical contributions, experimental design, implementation, analysis, and conclusions presented in this work are the original work of the authors. The use of LLMs was limited to editorial assistance and literature search support, and did not influence the core scientific contributions or findings reported in this paper.