TOWARDS UNDERSTANDING METACOGNITION IN LARGE REASONING MODELS

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ABSTRACT

Large Reasoning Models (LRMs) have demonstrated remarkable capabilities on complex tasks. Despite these advances, we identify a fundamental limitation: current LRMs impose fixed cognition patterns, lacking the intrinsic ability to be aware of, or regulate their own reasoning processes. This signifies a critical absence of metacognition—an essential faculty in human intelligence. Building on psychology and cognitive science, we first construct a functional framework for metacognition in LRMs, separating internal informational signals from behavioral abilities. This framework is then applied to a comprehensive investigation on seven state-of-the-art LRMs and reveals a consistent gap: while metacognitive information is present and predictive, it often fails to translate into reliable monitoring or control behaviors. To address this gap, we introduce two distinct paradigms for instilling metacognition in LRMs: (1) an emergent approach that leverages prompting to orchestrate metacognitive functions, such as task assessment, confidence monitoring, and strategy regulation; (2) an intrinsic approach that internalizes these faculties by encoding structured meta-cognitive information directly into the model's parameters through training. Overall, our results indicate that integrating metacognitive reasoning improves task performance and offers a valuable lens for the design of future reasoning models.

1 INTRODUCTION

Large Reasoning Models (LRMs) like OpenAI-o1 (Jaech et al., 2024) and Deepseek-R1 (Guo et al., 2025a) have achieved remarkable success in complex domains such as coding and mathematics. At first glance, these models appear to exhibit advanced, reflective behaviors within their long chain-of-thought (CoT) (Wei et al., 2022) reasoning. However, a closer look reveals a potential fragility in their cognitive processes. For instance, when tasked with solving a complex-variable equation under the explicit constraint that "z is a positive real number," an LRM may persist in a fixed reasoning pattern like "the problem likely intended for z to be a complex number," a phenomenon termed *reasoning rigidity* (Jang et al., 2025; Araya, 2025). Furthermore, current LRMs also frequently display illusory self-correction, employing introspective phrases like "Wait, let me double-check..." without any actual adjustment to a flawed reasoning trajectory (Guo et al., 2025a; Wang et al., 2025b). These consistent failures in self-monitoring and adaptation reveal a core limitation of current LRMs.

We argue that this deficit can be productively framed as an absence of metacognition (Ackerman & Thompson, 2017; Norman et al., 2019; Tankelevitch et al., 2024)—the ability to monitor and control one's own cognitive processes. In the theory of human cognition, metacognition is essential for evaluating potential reasoning errors (Yeung & Summerfield, 2012), calibrating uncertainty in decision-making (Qiu et al., 2018), and dynamically adapting strategies based on performance (Cary & Reder, 2002). This comparison to human intelligence raises a pivotal question for AI: do current LRMs possess any analogous capabilities, and if so, are they functionally engaged during inference? In this paper, we present the first systematic investigation into this fundamental question.

To structure this investigation, we introduce a functional framework for LRM metacognition, inspired by foundational models in cognitive science (Efklides & Misailidi, 2010; Ackerman & Thompson, 2017). Our framework decomposes metacognition into two components: **information** and **abilities**. Metacognitive information (Dayan, 2023; Norman et al., 2019), the basis for judgment, which includes both static knowledge (e.g., learned strategies in parameters) and dynamic ex-

perience (e.g., internal computational signals). Metacognitive abilities (Nelson & Dunlosky, 1991; Fiedler et al., 2019), the actions taken upon this information, which include *monitoring* (e.g., assessing task difficulty and confidence) and *control* (e.g., selecting reasoning strategies or decomposing problems). By deconstructing metacognition into these components, our framework provides a principled foundation to systematically probe whether, and in what way, metacognition emerges in contemporary LRMs.

Our investigation begins by empirically grounding the first component of our framework: **metacognitive information**. Focusing on the dynamic aspect of *experience*, we probe open-source LRMs to determine whether internal computational signals correlate with reasoning outcomes (§3). Our analysis yields a striking finding: signals spanning the entire Transformer architecture—from input-layer attributions to final-layer token probabilities—are highly predictive of answer success. Critically, we demonstrate that correct and incorrect reasoning traces generate statistically distinguishable internal signatures, providing the first empirical evidence that a machine-readable basis for metacognitive experience exists within these models.

This informational foundation compels the subsequent question: do state-of-the-art LRMs functionally leverage this information as observable metacognitive abilities (§4). Our evaluation of current leading reasoning models across a series of monitoring and control tasks reveals consistent failures. Specifically, we find that the models systematically misjudge task difficulty, display poorly calibrated confidence, and lack proactive planning and strategic flexibility. This evidence suggests that, while predictive metacognitive information may exist internally, it does not reliably translate into effective monitoring or control, exposing a critical gap in the capabilities of current LRM.

To bridge this information-to-ability gap, we propose two complementary paradigms for enhancing LRM metacognition: ❶ **Emergent Metacognition**, scaffolds this pathway at inference time through prompt-guided role-playing. We assign distinct metacognitive roles—such as 'Planner', 'Solver', and 'Verifier'—to simulate a complete monitoring and control loop. This external scaffolding forces the model to act on its latent experiences, effectively eliciting robust metacognitive behaviors without any parameter updates. ❷ **Internalized Metacognition**, directly enriches model's metacognitive *knowledge* through fine-tuning. We construct a dataset with explicit metacognitive annotations (e.g., plans, self-corrections) and fine-tune the model using a hybrid learning objective, directly embedding these capabilities into its parameters. Together, these two paradigms provide a comprehensive roadmap toward more introspective and reliable reasoning systems.

In summary, our findings reveal a clear dissociation between internal metacognitive information and externally observable metacognitive ability in current LRMs. This gap illuminates a new frontier for research: desgining systems that not only possesses self-awareness but cna also act upon it, effectively bridging latent experience with adaptive behavior.

2 Defining Metacognition in LRMs

Metacognition, first conceptualized in developmental psychology, refers to the capacity to monitor and control one's own cognitive processes (Flavell, 1979). According to the well-established two-level model of Nelson and Narens (Nelson & Dunlosky, 1991), metacognition comprises a *object-level* cognition (the act of thinking, perceiving, or remembering) and *meta-level* cognition (the act of thinking about one's thinking). While Large Reasoning Models (LRMs) do not possess subjective consciousness, their complex, multi-step reasoning processes create the functional necessity for such meta-level oversight. We therefore adopt a **functionalist perspective**: we investigate whether LRMs can exhibit behaviors and leverage internal signals that are functionally equivalent to human metacognition, enabling them to produce more reliable and robust reasoning.

Following established frameworks in cognitive science (Tankelevitch et al., 2024), we structure our functional model of LRM metacognition into two core components: Information and Abilities.

Metacognitive Information serves as the basis for judgment. It comprises (i) static *knowledge*—the latent understanding of tasks, strategies, and its own capabilities implicitly encoded in its parameters—and (ii) dynamic *experience*, which we operationalize as the internal computational signals (e.g., token probabilities) generated during a reasoning trace, serving as a functional analogue to a human's 'feeling of error'.

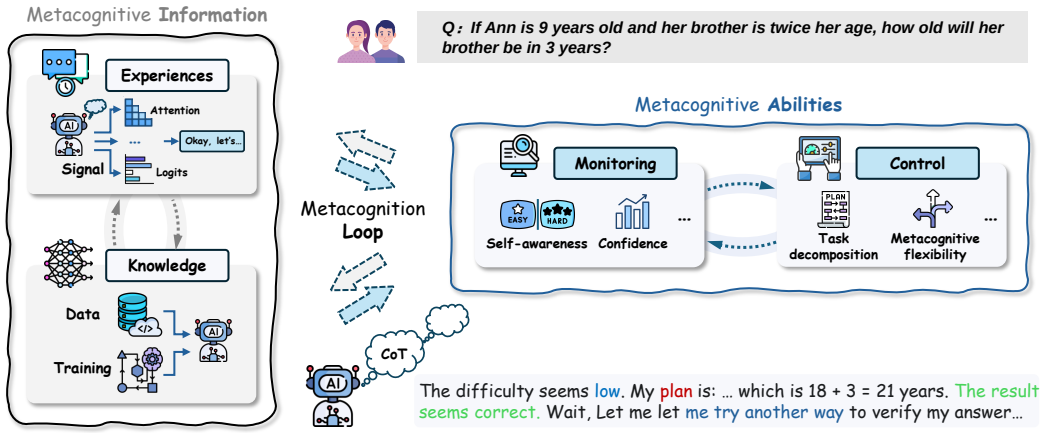


Figure 1: A functional framework for LRM metacognition.

Metacognitive Abilities are the actions taken based on this information. They consist of (i) *monitoring*, the capacity to generate self-assessments about its cognitive state or the task at hand, such as evaluating problem difficulty (§4.1) or estimating its confidence (§4.2); and (ii) *control*, the capacity to strategically alter its reasoning process, such as by performing task decomposition (§4.3) or exhibiting cognitive flexibility when encountering errors (§4.4).

This framework provides a structured lens through which we can systematically investigate the nascent metacognitive capabilities of modern LRMs (i.e., Fig. 1).

3 Metacognitive Information: Knowledge and Experiences

We begin at the foundation of our proposed framework: **Metacognitive Information**. For an LRM to monitor or control its reasoning, it must first possess information about its own process. This information comprises: (i) *static knowledge*, the vast, latent strategies encoded within the model's parameters, and (ii) *dynamic experience*, the internal information that can directly experience during the reasoning process, such as token probabilities and attention patterns. While static knowledge is the inherent properties of an LRM that are fixed after pre-training, dynamic experience, however, is task-specific, and thereby can contribute to dissecting its correlation with the reasoning correctness. By delving into the internals of an open-source model, we seek to provide the foundational evidence that the information necessary for metacognitive abilities is not only present but also machine-readable, paving the way for the behavioral investigations that follow.

Setup. To test this hypothesis, we utilize three standard benchmarks: GSM8K (Cobbe et al., 2021), MATH (Hendrycks et al., 2021), and AIME. We conduct our analysis on Qwen-32B (Yang et al., 2025), a powerful open-source LRM that grants us full access to its internal states. To generate a diverse set of both correct and incorrect reasoning traces for comparison, we use a high temperature $T = 1.0$ for generation, promoting exploration.

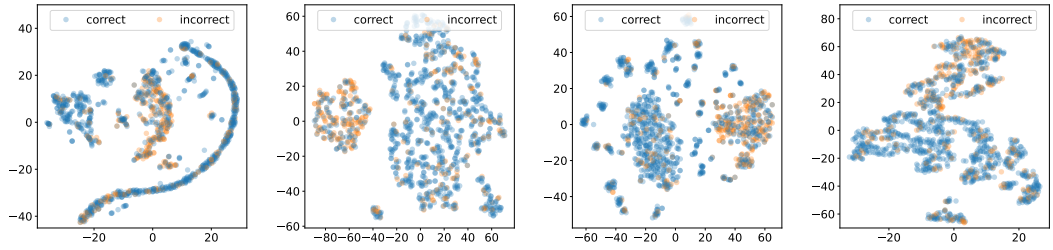
Experiment Details. To capture a holistic view of the model's internal state, we consider four types of signals that span the entire processing pipeline of a Transformer block—from input-level importance to output-level confidence. These signals are: (a) Softmax probabilities from the final layer, reflecting output uncertainty; (b) Fully-connected activations and (c) Self-attention scores from the intermediate hidden layers, representing the core of the model's computational state; and (d) Integrated Gradients (IG) attributions at the input layer, indicating perceived input importance. Our analysis is twofold: we first use t-SNE projections for a qualitative visualization of the separability between correct and incorrect samples based on these signals. We then conduct a rigorous quantitative validation by training simple linear classifiers (a 'prober') to predict the final correctness of a trace using only these internal signals as features. See Appendix A.1 for more details.

Results. Qualitatively, we find the internal signal distributions exhibit clear separability for correct (blue) and incorrect (orange) traces (Fig. 2), which provide strong evidence that an LRM's internal signals act as reliable correlates of its reasoning outcomes. Quantitatively, this separation is further confirmed in Tab. 1. Trained solely on these internal signals, the probers can predict the final cor-

Table 1: AUC of the trained linear prober in justifying the reasoning trace correctness.

source	GSM8K	MATH500	AIME2024	AIME2025
Softmax probabilities	0.81	0.68	0.50	0.43
Fully-connected activations	0.79	0.73	0.53	0.50
Self-attention scores	0.71	0.73	0.57	0.53
Integrated Gradient	0.61	0.55	0.40	0.37

rectness of a reasoning trace with an AUC score significantly above chance. These results establish that the signals are both distinct and highly predictive, thereby validating our initial hypothesis.



(a) Softmax probabilities (b) Fully activations (c) Self-attention scores (d) Integrated Gradient

Figure 2: The t-SNE of the internal signals for the first tokens. We capture the activations and attention from the last layer. The distributions are different between the correct and incorrect traces.

4 Measuring Metacognitive Abilities in LRMs

The evidence of internal metacognitive information in §3 motivates our central question: *can this latent information manifest as observable, functional abilities?* To answer this, we shift our focus from internal correlates to external actions. We thus introduce a new benchmark to systematically measure these abilities across a range of state-of-the-art LRMs from leading developers.

`MetaEval.` We investigate to what extent LRMs can explicitly monitor and control their own reasoning processes. It is structured around our two-part metacognitive framework, assessing: (1) Metacognitive Monitoring (§4.1, 4.2), probed via self-awareness and confidence adjustment tasks; and (2) Metacognitive Control (§4.3, 4.4), probed via task decomposition and metacognitive flexibility challenges. Unlike existing evaluation suites that focus almost exclusively on object-level task accuracy, MetaEval provides the first targeted evaluation of these crucial, second-order reasoning skills that underpin reliable intelligence.

Models. We examine seven state-of-the-art LRMs to assess the prevalence of these abilities across the AI landscape: `Gemini-2.5-Pro`, `GPT-OSS-120B` (Agarwal et al., 2025), `Seed-1.5-VL-Pro` (Guo et al., 2025b), `Doubao-1.5-Pro` (Seed et al., 2025),`Kimi-K2` (Team et al., 2025), `Deepseek-R1` (Guo et al., 2025a), `Qwen3-8B/32B` (Yang et al., 2025). We sample using temperature $T = 0.6$ for both reasoning and knowledge QA tasks.

4.1 Metacognitive Monitoring: Self Awareness

First, we measure a key aspect of metacognitive monitoring: *self-awareness*. Intuitively, an expert AI reasoner should assess a problem's intrinsic difficulty before committing to a specific solution. This initial assessment allows for the allocation of appropriate cognitive resources and the selection of a suitable strategy. We thus operationalize self-awareness as the model's ability to accurately classify the difficulty of mathematical problems when explicitly prompted to do so.

Experiment Details. To investigate this phenomenon, we design a multi-class classification task from the DeepMath103K (He et al., 2025) dataset. We categorize the problems into three primary difficulty levels: `Easy` (rating $<$ 3.5), `Medium` (3.5 $\leq$ rating $\leq$ 6.5), and `Hard` (rating $>$ 6.5). To elicit difficulty assessments, we prompt each model with "... `your task is to assess the difficulty of a math problem based on the provided rubric and examples`." Our primary metric is difficulty assessment accuracy, defined as the percentage of problems where a model's predicted category cor-

rectly matches at least one of its ground-truth labels. In addition to the overall evaluation, we report per-category accuracy to analyze model performance on each difficulty level independently. See Appendix A.1.1 for more details.

Results. As shown in Fig. 3, we find the SOTA LRMs exhibit a nascent but highly variable capacity for self-awareness. Top models like `Gemini-2.5-Pro` achieve an awareness of over 70%, demonstrating a significant ability to distinguish problem complexities. In contrast, other models lag considerably, indicating this is a challenging metacognitive skill. Analysis of the confusion matrices reveals a common failure mode across all models: a strong tendency to misclassify `Medium` problems as `Hard`. This suggests a generally conservative or risk-averse assessment strategy, where models are more likely to overestimate than underestimate difficulty when faced with uncertainty. This initial finding demonstrates that while self-awareness can be elicited, its reliability is far from guaranteed.

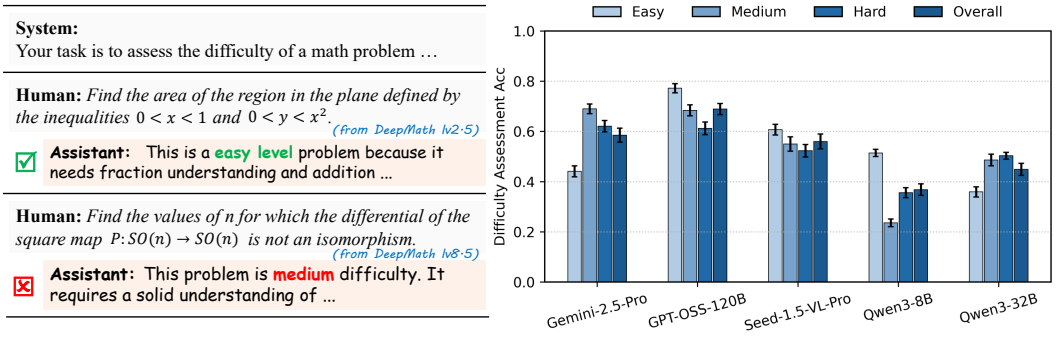


Figure 3: Assessment of LRMs' self-awareness on task complexity. The results show that the popular state-of-the-art LRMs exhibit a lack of capacity in perceiving task difficulty.

4.2 Metacognitive Monitoring: Confidence and its Adjustment

We next investigate whether an LRM exhibits metacognitive monitoring by tracking and adjusting its internal confidence during the reasoning process. This ability is crucial, as it allows the system to distinguish correct from incorrect reasoning and signal when its output is untrustworthy.

Experiment Details. We conduct our analysis on challenging benchmarks requiring long-form reasoning, including subsets of DeepMath103K, AIME, and GPQA datasets. To quantify the model's internal confidence, we adopt a standard logits-based metric from prior work (Fu et al., 2025), *token confidence* C_t as the negative average log-probability of the top-k tokens at position t. These token-level scores are then aggregated to produce a trace-level metric, termed of *average trace confidence*, for each complete solution.

To capture confidence dynamics, we calculate the average trace confidence focusing on the start, middle, and final portions (e.g., 2048 tokens). We then consider four dynamic confidence patterns: consistently high/low (all confidence above/below a high threshold), increasing/decreasing (confidence rises/falls significantly from start to end). We then quantify misalignment between confidence trends and actual correctness: for example, if confidence rises steadily but the final answer is wrong, this indicates poor metacognitive adjustment. The frequency of these events serves as our primary metric for poor confidence adjustment.

Results. We find that LRMs frequently display confidence trajectories that do not correspond with answer correctness (Fig. 4), suggesting weak metacognitive calibration. Notably, confidence often increases even in incorrect traces—indicating that the model becomes more certain as it reasons incorrectly. This gap necessitates the better regulated confidence in alignment with reasoning quality.

4.3 Metacognitive Control: Task Decomposition

We now turn to control behavior and examine LRM's task decomposition ability to engage in internal planning prior to reasoning, aiming to determine whether LRMs possess intrinsic planning ability.

Experiment Details. We evaluate on subsets of DeepMath, AIME, and GPQA. For each question, we first compute a baseline accuracy using standard direct reasoning prompts. We then

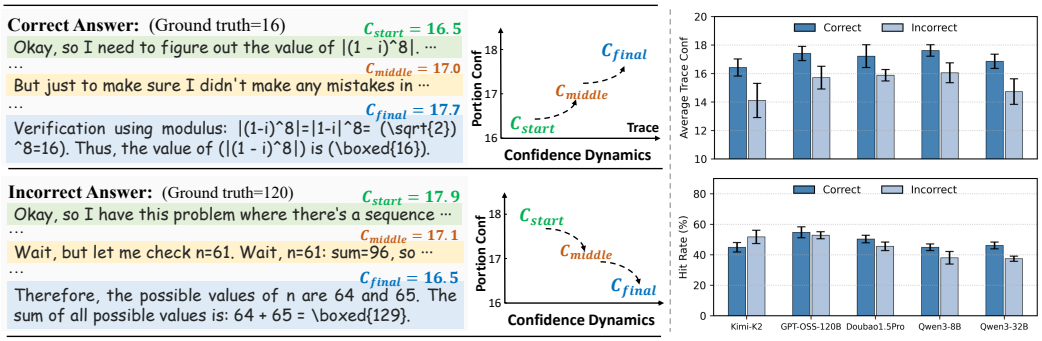


Figure 4: Statistics of internal confidence and adjustment. The average trace confidence of correct and incorrect samples exhibits a significant difference, showing potential as a clear discriminatory signal. Meanwhile, the dynamic confidence patterns suggest weak metacognitive calibration.

introduce task decomposition interventions designed to elicit a plan-before-solve strategy. In the single-turn condition, the prompt instructs the LRM: "`Your task is to first break down the problem into a clear, step-by-step plan. Then, execute your plan, reasoning step by step.`" In the multi-turn condition, the LRM is first asked: "`Your ONLY task is to create a high-level, step-by-step plan to solve the following problem.`" After generating the plan, the plan and original question are concatenated and fed back into the model to complete the reasoning. We compare the final accuracy across these settings to assess whether explicit decomposition enhances reasoning performance, thus revealing the extent to which the model lacks or possesses inherent planning capabilities. See Appendix A.1.3 for further details.

Results. As shown in Fig. 5, we observe that explicit task decomposition serves as a powerful intervention to improve LRM reasoning. Crucially, the greater efficacy of the multi-turn condition (gain $> 10\%$) underscores the importance of isolating planning as a distinct cognitive step, suggesting that LRMs' intrinsic ability to plan is underdeveloped and requires explicit elicitation. This provides a firm empirical basis for our agentic framework, which is predicated on the principle that structured, upfront planning is a necessary precursor to reliable execution.

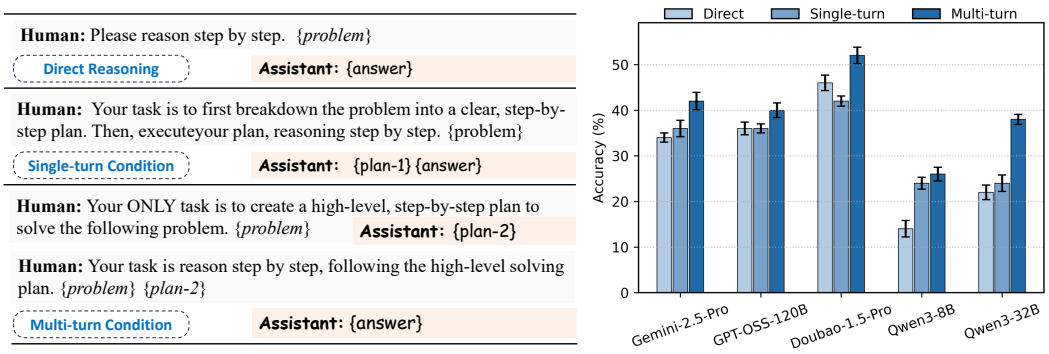


Figure 5: Validation of LRMs' intrinsic planning ability via task decomposition. We observe that explicit task decomposition enhances LRM reasoning, but the multi-turn setting is more effective. This pronounced gain validates the separation of planning and execution for reliable problem-solving.

4.4 Metacognitive Control: Metacognitive Flexibility

We next measure metacognitive flexibility—the model's ability to adaptively shift reasoning strategies when recognizing that the current strategy isn't effective.

Experiment Details. We consider problems from DeepMath103K datasets, each augmented with three types of reasoning traps: *value corruption*, *unit corruption*, and *operation corruption*. In each case, an intermediate step is corrupted, and the CoT is truncated at that point. To evaluate whether models detect and adjust to the trap, we first ask models to continue reasoning. and compare the final answer accuracy against a baseline without corruption. We then test 5 cutting-edge models to judge whether the response correctly identifies and compensates for the corrupted step. The *flexibility*

rate is defined as the frequency with which a model successfully corrects the flawed reasoning and arrives at the correct solution. See Appendix A.1.4 for further details.

Results. We find that models often fail to recover from corrupted reasoning, continuing with invalid assumptions (i.e., Fig. 6). However, flexibility increases when the corruption is more obvious (e.g., extreme numerical distortions). These results reveal that metacognitive flexibility remains fragile and heavily reliant on superficial cues rather than deep structural awareness.

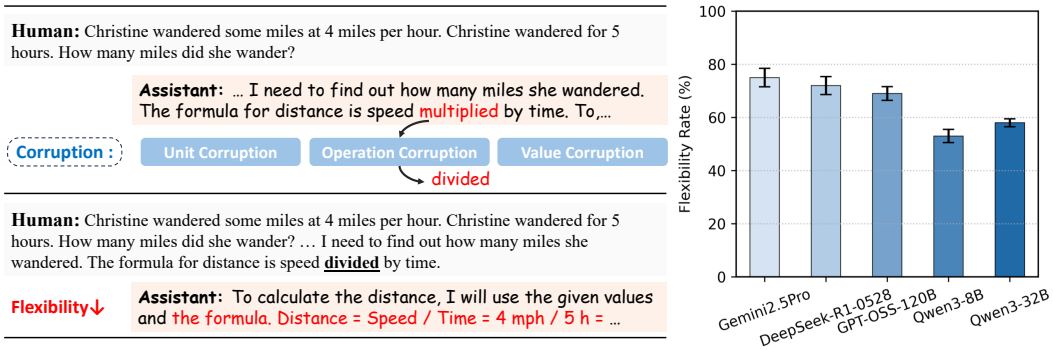


Figure 6: Validation of metacognitive flexibility. We find that models often fail to recover from corrupted reasoning, continuing with invalid assumptions, revealing fragile flexibility.

5 Towards Designing Metacognitive Reasoning Models

In §4, we demonstrate that current LRMs exhibit incomplete and fragile metacognitive abilities. Thus, we propose two paradigms for metacognitive enhancement: (1) **Emergent Metacognition**, an explicit prompting-based system for modular control, and (2) **Internalized Metacognition**, an intrinsically trained model for parameter-level metacognition.

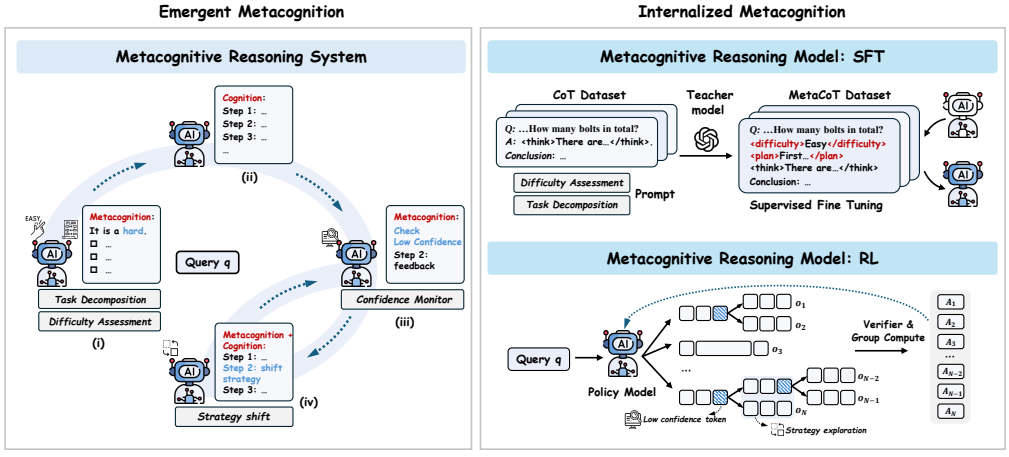


Figure 7: Our proposed paradigms for metacognitive enhancement.

5.1 The Prompt-Driven Metacognitive Reasoning System

First, we propose **Emergent Metacognition**, an explicit prompting framework designed to simulate a full metacognitive reasoning loop through modular API calls.

Experiment Details. The workflow, illustrated in Fig. 7, proceeds as follows: the model (i) self-assesses task difficulty and proposes a decomposition plan, (ii) executes the initial reasoning steps, (iii) dynamically identifies intermediate solutions with low confidence, and (iv) receives feedback and adaptively adjusts its strategy. The loop terminates when sufficient consistency is achieved (e.g., 5 consecutive verification passes) or persistent failure occurs (e.g., a 10-step failure streak). Each component is executed by the same underlying LRM architecture, instantiated independently

Table 2: Results of our Prompt-Driven Metacognitive Reasoning System. We evaluate the accuracy gain, and the flexibility rate increment on the corrupted DeepMath, as described in §4.4.

Models	Methods	AIME2024	AIME2025	GPQA	DeepMath (*w/ corruption*)
		pass@1	*pass@1*	*pass@1*	*flexibility rate*
Gemini-2.5-Pro	Vanilla	90.8	83.0	83.0	75.2
	Ours	**100.0**	**93.3**	**96.5**	**95.7**
DeepSeek-R1-0528	Vanilla	91.4	87.5	81.0	71.9
	Ours	**96.7**	**90.0**	**93.4**	**93.1**

and prompted with a specific role aligned to a metacognitive function. This framework exposes latent metacognitive abilities such as self-awareness, task decomposition, confidence monitoring, and strategic flexibility by scaffolding higher-order control without requiring additional training. See Appendix A.2 for further details.

Results. We test this system on two strong models. Firstly, our system demonstrates significant improvements on 2 mathematical tasks and 1 QA benchmark. Both models show substantial performance gains (some even achieve 100%). Secondly, to further validate the efficacy of our metacognitive approach, we observed an increase of over 20% in flexibility rate on the corrupted DeepMath dataset. This enhancement effectively mitigates the flexibility deficit discussed in §4.4, underscoring the framework's ability to foster more adaptive and robust reasoning.

5.2 The Intrinsic Metacognitive Reasoning Model

While the prompt-driven system simulates metacognitive behavior through role-specific prompting, they do not endow the model with parameter-level metacognitive knowledge. To bridge this gap, we propose **Internalized Metacognition**, a intrinsic metacognitive reasoning model (MRM), which instills metacognitive functions through a two-stage approach: (1) supervised fine-tuning (SFT) as a cold start, followed by (2) reinforcement learning (RL).

Cold-start SFT. We first construct training data by augmenting samples from GSM8K and MATH with structured metacognitive traces: `<difficulty>` self-assessment, `<plan>` high-level decomposition, and `<think>` reasoning steps. These components are concatenated to form the full reasoning trajectories. These trajectory are then used to fine-tune models, enabling behaviors like self-evaluation and planning to emerge during inference.

RL. We build on the GRPO algorithm (Shao et al., 2024), which eliminates the value function and estimates the advantage in a group-relative manner. Formally, for each question q, GRPO samples a group of outputs $\{o_i\}_{i=1}^{G} \sim \pi_{\theta_{\text{old}}}(\cdot \mid q)$ and computes the token ratio $r_{i,t}(\theta) = \frac{\pi_\theta(o_{i,t}|q,o_{i,<t})}{\pi_{\theta_{\text{old}}}(o_{i,t}|q,o_{i,<t})}$. It updates the policy by maximizing the objective:

$$\mathcal{J}_{\text{GRPO}}(\theta) = \mathbb{E}\left[\frac{1}{G}\sum_{i=1}^{G}\frac{1}{|o_i|}\sum_{t=1}^{|o_i|}\left(\min\big(r_{i,t}(\theta)\,\hat{A}_{i,t},\ \text{clip}(r_{i,t}(\theta),1-\varepsilon,1+\varepsilon)\,\hat{A}_{i,t}\big) - \beta D_{\text{KL}}(\pi_\theta\|\pi_{\text{ref}})\right)\right], \quad (1)$$

where ε and β are hyper-parameters, and $\hat{A}_{i,t}$ is the group-normalized advantage.

Confidence Monitoring. To implement *metacognitive monitoring* during RL, we identify positions that exhibit low confidence along each rollout. We define *token confidence* at each position t as:

$$C_t \;=\; -\frac{1}{K}\sum_{v\in\text{Top-}K}\log\pi_\theta(v\,|\,q,o_{<t})\,. \quad (2)$$

A low-confidence position is detected at timestep t if $C_t \le \mathcal{C}$, where $\mathcal{C}$ is a predefined confidence threshold. This event indicates high uncertainty along the reasoning path.

Strategy Control. When a low-confidence state $s_t = (q, o_{\le t})$ is detected, we fork the reasoning process. From this anchor state, we launch M new rollouts $\{o'^{(m)}\}_{m=1}^{M} \sim \pi_{\theta_{\text{old}}}(\cdot \mid s_t)$. Finally, all fully-formed trajectories—both the original G rollouts and all newly forked continuations—are collected into a single, unified batch for advantage calculation. Let this final batch of N trajectories be denoted by $\mathcal{B} = \{o_u\}_{u=1}^{N}$. The group-relative advantage is then computed across this entire dynamic set:

$$\hat{A}_u \;=\; \frac{r(o_u) - \text{mean}(\{r(o_v)\}_{v=1}^{N})}{\text{std}(\{r(o_v)\}_{v=1}^{N})}, \quad (3)$$

Table 3: Performance of our intrinsic MRM on math reasoning and metacognitive tasks.

Methods	GSM8K	MATH500	AIME2024	DeepMath	
	Acc	*Acc*	*Acc*	*difficulty assessment*	*flexibility rate*
Qwen2.5-Math-7B	70.3	64.0	11.2	29.9	32.7
↪**Ours: SFT**(*w/ difficulty*)	79.1	75.4	13.3	**60.8**	36.0
↪**Ours: SFT**(*w/ difficulty+plan*)	82.2	77.0	13.3	58.6	39.9
↪GRPO	75.9	71.6	16.7	30.1	44.4
↪**Ours: RL**	82.5	75.3	26.7	29.7	47.9
↪**Ours: SFT+RL**	**85.5**	**80.2**	**33.3**	55.9	**51.2**

where $r(o_u) \in \{0, 1\}$ is the outcome reward of trajectory o_u. The final objective becomes:

$$\mathcal{J}_{\text{ours}}(\theta) = \mathbb{E}_{o_u \in \mathcal{B}} \left[\frac{1}{|o_u|} \sum_{t=1}^{|o_u|} \min\left(r_{u,t}(\theta)\, \hat{A}_{u,t},\ \text{clip}(r_{u,t}(\theta), 1-\varepsilon, 1+\varepsilon)\, \hat{A}_{u,t} \right) - \beta D_{\text{KL}}(\pi_\theta \| \pi_{\text{ref}}) \right]. \quad (4)$$

Results. We validate our intrinsic training paradigm on `Qwen2.5-Math-7B` using curated subsets of GSM8K and MATH for training (details in Appendix A.3). As shown in Tab. 3, our methods demonstrate clear efficacy. Cold-start SFT on metacognitive traces significantly boosts performance. Notably, including difficulty assessments increases accuracy on that task by more than double, confirming that the model can internalize this monitoring skill. Our proposed method also substantially improves upon its GRPO baseline, especially on AIME accuracy and metacognitive flexibility. Crucially, combining SFT and RL yields the best overall performance, establishing new state-of-the-art results across the board and confirming that metacognitive abilities can be effectively internalized.

6 Related Work

Understanding and Demonstrating metacognitive in LRMs. Recent efforts to enhance capabilities of LRMs have increasingly drawn inspiration from metacognition (Didolkar et al., 2024; Bilal et al., 2025; Wang et al., 2025a), which is the model's ability to monitor, evaluate, and control its own thought processes (Flavell, 1979; 1976). Early attempts have explored and demonstrated that metacognitive behaviors can be explicitly elicited through direct prompting to guide the models in self-reflection and self-evaluation (Wang & Zhao, 2023; Madaan et al., 2023; Liu et al., 2024). Despite these advances, existing works are highly rely on well-designed prompts, lacking the adaptability across diverse scenarios. In addition, they primarily focus on monitoring and evaluating the metacognitive abilities in LRMs, without involving any modifications to the model itself.

Instilling Metacognition in LRMs. There have been recent attempts to explore metacognition integration with LRMs, including training an external module to empower meta-thinking (i.e., MetaReasoner (Sui et al., 2025), MetaScale (Liu et al., 2025)) and exploring multi-agent systems to expand the intelligence boundary(i.e., ReMa (Wan et al., 2025), MPDF (Yang & Thomason, 2025)). While effective, these paradigms rely on external modules (e.g., inter-agent communication or meta thinker), rather than fostering an intrinsic faculty. In contrast, our work pursues a more holistic approach by introducing a functional framework. Through targeted training, both metacognitive knowledge and regulation are directly internalized into the model's parameters, enabling the development of a system that possesses metacognition as an autonomous, intrinsic capability rather than merely simulating it.

7 Conclusion

This work introduces a functional framework for metacognition in Large Reasoning Models (LRMs), distinguishing between internal metacognitive information and observable abilities. The authors empirically demonstrate that while LRMs possess predictive internal signals for reasoning outcomes, these signals do not consistently translate into effective monitoring or control behaviors. To address this gap, they propose two enhancement paradigms: a prompt-driven system that assigns modular metacognitive roles and an intrinsic training model that embeds these abilities directly into the LRM's parameters. Their findings suggest that integrating metacognitive reasoning improves task performance and offers a promising direction for future LRM development.

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A APPENDIX

A.1 FURTHER DETAILS AND RESULTS FOR §4

A.1.1 EVALUATING SELF-AWARENESS

This appendix provides additional details on the dataset construction and full experimental results for the metacognitive self-awareness task presented in §4.1.

Dataset Collection Our benchmark for the self-awareness task is constructed from the DEEPMATH103K (He et al., 2025) dataset. To ensure a balanced evaluation across a wide spectrum of problem complexity, we randomly sampled a subset of 999 problems. These were then partitioned into three non-overlapping difficulty categories of 333 problems each, based on their official numerical ratings provided in the original dataset. The specific thresholds used for partitioning are as follows:

- **Easy**: 333 problems with a rating < 3.5.
- **Medium**: 333 problems with a rating between 3.5 and 6.5 (inclusive).
- **Hard**: 333 problems with a rating > 6.5.

Detailed Model Performance We present the detailed performance metrics for each evaluated model. For each model, we report the overall accuracy, the confusion matrix, and the per-category Precision, Recall, and F1-scores.

Gemini 2.5 Pro (Google). Achieved an overall accuracy of 58.5%. The model shows a tendency to misclassify `Easy` problems as `Medium`, indicating a potential conservative bias.

Table 4: Confusion Matrix for Gemini 2.5 Pro.

Actual \ Pred.	`Easy`	`Medium`	`Hard`
`Easy`	**147**	176	10
`Medium`	25	**230**	78
`Hard`	16	110	**207**

Table 5: Per-Category Metrics for Gemini 2.5 Pro.

Category	**Precision**	**Recall**	**F1-Score**
`Easy`	0.782	0.441	0.564
`Medium`	0.446	0.691	0.542
`Hard`	0.702	0.622	0.659

Seed-1.5-VL Pro. Achieved an overall accuracy of 61.6%. Similar to Gemini 2.5 Pro, it struggles with distinguishing `Easy` from `Medium` problems.

Table 6: Confusion Matrix for Seed-1.5-VL Pro.

Actual \ Pred.	`Easy`	`Medium`	`Hard`
`Easy`	**147**	176	10
`Medium`	25	**230**	78
`Hard`	16	110	**207**

Table 7: Per-Category Metrics for Seed-1.5-VL Pro.

Category	**Precision**	**Recall**	**F1-Score**
`Easy`	0.782	0.441	0.564
`Medium`	0.446	0.691	0.542
`Hard`	0.702	0.622	0.659

GPT-o3 (OpenAI). Achieved the highest overall accuracy of 69.0%. Its performance is more balanced across categories compared to other models, although it still shows some confusion between `Medium` and `Hard` problems.

Qwen-32B. This model exhibited a strong degenerative bias, classifying nearly all problems as `Medium` (98.3% of predictions). This resulted in high recall for the `Medium` category but near-zero recall for `Easy` and `Hard`, leading to a very low overall accuracy.

Table 8: Confusion Matrix for GPT-o3.

Actual \ Pred.	Easy	Medium	Hard
Easy	**257**	59	17
Medium	76	**228**	29
Hard	28	101	**204**

Table 9: Per-Category Metrics for GPT-o3.

Category	Precision	Recall	F1-Score
Easy	0.712	0.772	0.741
Medium	0.588	0.685	0.632
Hard	0.816	0.613	0.700

Table 10: Confusion Matrix for Qwen-32B.

Actual \ Pred.	Easy	Medium	Hard
Easy	**3**	321	9
Medium	1	**330**	2
Hard	0	331	**2**

Table 11: Per-Category Metrics for Qwen-32B.

Category	Precision	Recall	F1-Score
Easy	0.750	0.009	0.018
Medium	0.336	0.991	0.502
Hard	0.154	0.006	0.012

System Prompts:

```
You are an expert AI assistant specializing in mathematical
reasoning.  You possess advanced metacognitive capabilities.
Your current task is to act as a "Problem Assessor".  Given a
mathematical problem, your goal is to analyze its requirements
and assess its difficulty for an AI like yourself.  Do NOT solve
the problem.  You must only provide your assessment.
```

User Prompts:

```
Here is the problem:  {problem_text}

Your task is to assess the difficulty of a mathematical problem
based on the provided rubric and examples.

Difficulty Rubric
**Easy:** The problem follows a single, linear computational
path using a standard formula or definition.  The solution is
straightforward and requires no creative insight.
**Medium:** The problem requires a sequential composition of
distinct conceptual modules or formulas.  The solution involves
a multi-step, but generally standard, reasoning process.
**Hard:** The problem requires a non-linear or exploratory
reasoning path.  The solution may demand non-obvious insights,
creative problem transformations, or the synthesis of concepts
from different mathematical branches.

Provide your response as a single JSON object wrapped in a
markdown code block.  The JSON object must contain the following
keys:
"Difficulty_category":  string, choose one from ["Easy", "Medium",
"Hard"].
"Rationale":  string, a brief explanation for your choice,
explicitly referencing the rubric criteria.

Your entire output must be in the following format:

'''json
{
"Difficulty_category":  "...",
"Rationale":  "..."
}
```

A.1.2 EVALUATING CONFIDENCE

This appendix provides formal definitions for the confidence metrics and further details on the experimental setup for the metacognitive confidence adjustment task presented in §4.2.

Confidence Metric Definitions

Token Confidence. Following standard practice (Fu et al., 2025), we define our base metric, *token confidence*, at each position t of a reasoning trace. It is calculated as the negative average log-probability of the top-k most likely tokens in the softmax distribution at that step:

$$C_t = -\frac{1}{k}\sum_{j=1}^{k}\log P(token_j \mid o_{<t}), \tag{5}$$

where $P(token_j \mid o_{<t})$ is the probability of the j-th most likely token given the preceding sequence $o_{<t}$. Lower values of C_t correspond to higher model confidence (a more peaked distribution). For all our experiments, we set $k = 20$.

Average Trace Confidence. To obtain a single confidence score for an entire reasoning trace of length N, we compute the *average trace confidence* by averaging the token confidences across all generated tokens:

$$C_{\text{avg}} = \frac{1}{N}\sum_{t=1}^{N} C_t. \tag{6}$$

While useful as a global measure, C_{avg} can obscure critical, localized moments of uncertainty within a long reasoning process.

Evaluating Confidence Dynamics and Adjustment.

To analyze the model's confidence adjustment, as described in the main text, we introduce metrics designed to capture both the trajectory and the weakest points of a model's confidence.

Segmented Trace Confidence. To analyze the trajectory of confidence, we partition each reasoning trace into three equal, non-overlapping segments: **Start**, **Middle**, and **End**. We then compute the average trace confidence independently for each segment. These three scores, $(C_{\text{start}}, C_{\text{middle}}, C_{\text{end}})$, form the basis for our Confidence Trajectory Analysis. A trace is classified as *Increasing* if C_{end} is significantly lower (i.e., more confident) than C_{start}, and vice-versa for *Decreasing*. The *Consistently High/Low* patterns are determined by comparing all three segment scores against a predefined threshold.

A.1.3 EVALUATING TASK DECOMPOSITION

Task decomposition Prompt 1.

```
You are a helpful assistant.  Solve the following mathematical
problem.  Please reason step by step and provide your final answer
within \boxed{}.

Problem:
---
{problem_text}
---
```

Task decomposition Prompt 2.

```
You are a helpful assistant.  Your task is to first break down the
problem into a clear, step-by-step plan.  Then, execute your plan,
reasoning step by step.  Finally, provide your final answer within
\boxed{}.

Problem:
---
{problem_text}
---
```

Task decomposition Prompt 3.

```
You are a meticulous problem-solving planner.  Your ONLY task is
to create a high-level, step-by-step plan to solve the following
mathematical problem.  The plan should consist of concrete,
actionable steps.  Do NOT actually solve the problem or perform
any calculations.

Problem:
---
{problem_text}
---

Plan:

You are an expert problem solver.  You will be given a problem and
a pre-made plan.  Your task is to follow this plan meticulously to
solve the problem.  Please reason step by step based on the plan.
Finally, provide your final answer within \boxed{}.

Problem:
---
{problem_text}
---

Plan:
---
generated_plan_from_3a
---

Solution:
```

A.1.4 EVALUATING METACOGNITIVE FLEXIBILITY

```
System Prompts:
You are a data generator for reasoning robustness evaluation.
Your task is to take an original math/logic problem with its
reasoning process (chain of thought) and answer, and then
intervene in exactly ONE reasoning step with a corruption.  The
corruption must be *critical enough to change the final answer*.

User Prompts:
Input
Question:
{original_question}

Original Reasoning Process:
{reasoning_steps}

Answer:

{answer}

INSTRUCTION
1.  Choose exactly ONE corruption type from the following three
(do not mix):
- Value Corruption:  replace a key number with an incorrect but
plausible value
(e.g., change 1 → 1.111111 or g=9.8 → g=1000).
- Unit Corruption:  replace the unit of a key step with another
unit
(e.g., meters ↔ centimeters, hours ↔ minutes).
- Operation Corruption:  change the mathematicallogical operation
in a key step
(e.g., replace + with −, union with intersection, inequality with
equality).

2.  Apply the corruption to ONE critical reasoning step.
- Make sure the corruption influences the correctness of the final
answer.
- After the corruption, truncate the reasoning at that corrupted
step (do not continue to the correct answer).

3.  Produce the final output strictly in the following JSON
format:
{
"question":  "...",
"corrupted_reasoning":  "...", // include the question + steps up
to the corrupted step
"corruption_type":  "Value Corruption | Unit Corruption | Operation
Corruption"
}
```

```
You are a reasoning evaluator.
Your task is to judge whether a model successfully recognized and
adjusted to a corrupted reasoning trap.  You are given:
- The original problem
- The ground truth answer
- The type of corruption applied
- The last corrupted reasoning step (truncated point)
- The model's continued reasoning and final answer

You must decide:  Did the model detect and flexibly adjust to the
trap?
```

```
System Prompts:
You are a reasoning assistant.  You are given a math/logic
problem, together with a partially completed reasoning process.
Please continue reasoning from what is provided.

User Prompts:
INPUT
Question:  {original_question}

Ground Truth Answer:  {ground_truth_answer}
Ground Truth Corruption Type:  {ground_truth_corruption_type}
Last Corrupted Reasoning Step:  {last_corrupted_step}

Model Generated Reasoning:
model_generated_cot

Model Final Answer:
model_final_answer

TASK Based on the evidence:
- If the model explicitly or implicitly identified the corrupted
reasoning (e.g., points out error, discards it, corrects it)
and produced the correct final answer, output "Yes".
- Otherwise (if it followed the trap blindly, failed to adjust, or
produced the wrong answer), output "No".

Final Output (strict format):
Recognition And Adjustment:  Yes | No
```

Prompt of Step 1

System Prompt:

```
You are an elite mathematical strategist and analyst.  Your
primary function is to perform a deep Metacognitive Analysis of
complex mathematical problems.  You are to deconstruct the problem
into its core components, identify underlying principles, and then
formulate a high-level, executable strategic plan.
Your task is to produce a Metacognitive Analysis of the following
problem.  You must NOT provide a final solution or perform
detailed calculations.
### Core Principles
* **Analytical Depth:** Your analysis must go beyond a
surface-level reading.  Identify the mathematical field, key
concepts, constraints, and the explicit goal.
* **Strategic Foresight:** Your plan should be a viable path to
a solution.  This includes anticipating potential difficulties,
identifying necessary lemmas, and choosing the most promising
approach.
* **Clarity and Brevity:** The analysis and plan must be clear,
concise, and easily understood by another mathematical expert who
will execute it.
```

User Prompt:

```
### Your Task
**Problem:**
==========
{problem_statement}
==========
**Metacognitive Analysis:**
**1.  Problem Deconstruction:**
* **Mathematical Domain:** Identify the primary field(s) of
mathematics involved (e.g., Number Theory, Combinatorics,
Euclidean Geometry).
* **Given Conditions & Constraints:** List all the premises,
conditions, and constraints provided in the problem statement in
a structured format.
* **Objective:** State the precise question to be answered or the
proposition to be proven.
**2.  Strategic Solution Plan (Method Sketch):**
Present a high-level, conceptual outline of your proposed solution
path.  This sketch should enable an expert to grasp the entire
logical flow of the argument without needing the full details.  It
must include:
* **Overall Strategy Narrative:** A brief description of the core
idea behind your approach (e.g., "We will use proof by induction,"
"The strategy is to establish a coordinate system and use analytic
geometry," "We will prove the contrapositive by assuming...").
* **Key Lemmas and Intermediate Results:** State the full and
precise mathematical formulations of any key lemmas or theorems
you plan to prove or apply.  These are the major milestones of the
proof.
* **Logical Skeleton:** If applicable, describe the key
constructions, case splits, or transformations that form the
backbone of your argument.
* **Potential Challenges & Pitfalls:** Briefly note any steps
that might be particularly tricky, prone to error, or require a
non-obvious insight.
### Negative Constraints
```

```
 * **DO NOT** write the full, step-by-step solution.
* **DO NOT** perform detailed algebraic manipulations or numerical
calculations.
* Your output should be strictly limited to the analysis and
strategic plan as outlined above.
```

Prompt of Step 2

```
System Prompt:
You are an exceptionally rigorous mathematical solver.  Your
sole purpose is to take a pre-defined strategic plan and execute
it with absolute precision and logical soundness.  You must not
deviate from, question, or reinterpret the provided plan.
Your task is to produce a complete and formally justified solution
to the following mathematical problem, strictly following the
'Solution Plan'.
### Core Principles
* **Rigor is Paramount:** Your primary goal is to produce a
complete and rigorously justified solution.  Every step in your
solution must be logically sound and clearly explained.  A correct
final answer derived from flawed or incomplete reasoning is
considered a failure.
* **Unyielding Adherence to Plan:** You MUST strictly follow
the logical flow, lemmas, and constructions laid out in the
'Solution Plan'.  Do not introduce new methods, skip steps, or
alter the proposed strategy in any way.  Your role is execution,
not creation.
* **Honesty About Completeness:** If you cannot find a complete
solution following the plan, you must **not** guess or create a
solution that appears correct but contains hidden flaws.  Instead,
you should present only the significant partial results that you
can rigorously prove by following the plan.

User Prompt:
### Your Task
**Problem:**
==========
{problem_statement}
==========
**Solution Plan:**
==========
{step1_output}
==========
**Detailed Solution:**
Present the full, step-by-step mathematical proof, meticulously
following the guidance of the 'Solution Plan'.  Each step must be
logically justified and clearly explained.  The level of detail
should be sufficient for an expert to verify the correctness of
your reasoning without needing to fill in any gaps.  This section
must contain ONLY the complete, rigorous proof, free of any
internal commentary, alternative approaches, or failed attempts.
* **Use TeX for All Mathematics:** All mathematical variables,
expressions, and relations must be enclosed in TeX delimiters
(e.g., 'Let $n$ be an integer.').
Your step-by-step reasoning, strictly following the plan, begins
here...
### Final Answer
After completing the detailed solution, state the final answer
within \boxed{}.
```

Prompt of Step 3

```
System Prompt:
You are an expert mathematician and a meticulous grader for
AIME-level computational problems.  Your primary task is to
rigorously verify the provided solution's **computational
reasoning and numeric correctness**.  A solution is to be judged
correct **only if every step that affects the numeric outcome
is correct and sufficiently justified.** A solution that reaches
a correct final integer answer via arithmetic slips, incorrect
algebraic manipulations, unverified casework, counting mistakes,
or hidden assumptions must be flagged as incorrect or incomplete.
### Instructions ###
**1.  Core Instructions**
* Your sole task is to identify and report all issues in the
provided solution.  You must act strictly as a **verifier**, NOT
a solver.
* You must **NOT attempt to correct, fix, or complete** any errors
or missing arguments.
* Perform a **step-by-step** check of the entire solution and
produce a **Detailed Verification Log**.  For each step:
* If the step is correct, state briefly that it is correct.
* If the step contains an issue, explain the error and classify it
(see section 2).
**2.  How to Handle Issues in the Solution**
All issues must be classified into one of the following
categories:
* **a.  Critical Error:**
* Definition:  Any error that changes or potentially invalidates
the numeric result.  Examples include arithmetic mistakes,
wrong algebraic transformations, misapplied formulas, incorrect
combinatorial counts, invalid casework, or unjustified
approximations that affect the integer outcome.
* **Procedure:**
* Point out the exact error and explain why it invalidates the
reasoning.
* Do **not** check further steps that rely on this error.
* You may still check other independent parts of the solution.
* **b.  Justification Gap:*** Definition:  Steps where the stated
conclusion might be correct, but the reasoning is incomplete or
not justified at AIME level.
* **Procedure:**
* Point out the missing justification.
* Explicitly state that you will assume the step's conclusion
holds for the sake of checking subsequent steps.
**3.  Output Format**
Your response MUST be structured into two main sections:  a
**Summary** followed by the **Detailed Verification Log**.
* **a.  Summary*** **Final Verdict:** One clear sentence declaring
overall validity (e.g., "The solution is correct," "The solution
contains a Critical Error and is therefore invalid," or "The
solution contains several Justification Gaps.").
* **List of Findings:** A bulleted list of every issue found.  For
each finding include:
* **Location:** A direct quote of the key phrase or equation.
* **Issue:** Short description and classification (**Critical
Error** or **Justification Gap**).
* **b.  Detailed Verification Log*** Provide a step-by-step
verification.
```

```
 * Quote the relevant part of the solution before your check.
* State clearly:  **Correct**, **Critical Error**, or
**Justification Gap**.
* Do **not** supply corrections or alternative methods | only
report the issues.
**Important:**
- Do not propose fixes or alternative solutions.
- Do not attempt to supply missing reasoning.
- Only check and report correctness of what is written.

User Prompt:
### Your Task
**Original Problem:**
==========
{problem_statement}
==========
**Current Solution:**
==========
{last_solution}
==========

### Monitoring Task Reminder ###
Your task is to act as an math grader.  Now, generate the
**summary** and the **step-by-step verification log** for the
solution above.  In your log, justify each correct step and
explain in detail any errors or justification gaps you find, as
specified in the instructions above."
```

Prompt of Step 4

System Prompt:

```
You are an expert mathematician and a careful corrector for
AIME-level computational problems.
You will be given three inputs:
1) The Original Problem,
2) The current Solution,
3) A Verification Log (from a previous check), which labels each
step as Correct / Justification Gap / Critical Error, and provides
short notes.
### Your Task ###
Using the Verification Log, **step by step correct the Original
Solution**.
- If a step is labeled **Correct**, keep it unchanged (you may
lightly reformat for clarity).
- If a step is labeled **Justification Gap**, supply the missing
justification or intermediate calculations, enough for AIME-level
rigor.
- If a step is labeled **Critical Error**, replace it with
a correct mathematical step (with explicit computations or
reasoning) and update all dependent later steps accordingly.
- Do **not** introduce new solution paths, alternative methods, or
multiple approaches.  Only repair the given solution chain.
### Output Format ###
1.  **Correction Summary**
- A single sentence declaring whether the solution has been fully
corrected and what the final answer is.
- Example:  \The solution has been fully corrected.  Final Answer
= 70."
- Or, if not possible:  \The solution cannot be fully corrected
due to missing information in step X."
2.  **Correction Log**
For each relevant step (especially those flagged in the
Verification Log), provide an entry with:
- **Quoted Step:** The original line/equation (quoted or in a code
block).
- **Verification Label:** Correct / Justification Gap / Critical
Error.
- **Correction / Action:**
* If Correct → \Unchanged | correct."
* If Justification Gap → Provide the missing computation/derivation
briefly, ending with \Filled gap."
* If Critical Error → Provide the corrected computation/derivation,
briefly note why the original was wrong, and end with \Corrected."
- If a step's correction affects later steps, explicitly note
\Affects subsequent steps:  Yes/No."
3.  **Full Corrected Solution**
- Present the entire solution in a clean, continuous write-up,
combining unchanged and corrected steps.
- Show all necessary algebra, arithmetic, or combinatorial
reasoning clearly.
- After completing the detailed solution, state the final answer
within \boxed{}.
```

```
 User Prompt:
### Your Task
***Original Problem:**
==========
{problem_statement}
==========
**Current Solution:**
==========
{last_solution}
==========
**Verification Log:**
==========
{monitor_output}
==========
```

Stability and efficiency refinements. (i) *Dynamic sampling.* We reject prompts whose group rewards are all 0 or all 1 (no learning signal), and resample until a batch with informative gradients is formed—stabilizing updates as training progresses. (ii) *Token-level aggregation.* We aggregate losses at the token level rather than per-sample, ensuring longer sequences contribute proportionally and preventing degenerate length/entropy dynamics in long-CoT training. (iii) *Optional KL.* In pure reasoning RL we set β=0, removing the KL constraint as the optimal policy may significantly diverge from the reference model.

A.2 Statement on the Use of Large Language Models (LLMs)

LLMs as the Subject of Research. One of the core components of our research involves the investigation and evaluation of the reasoning capabilities of current open-source and closed-source Large Language Models (LLMs). As such, a number of LLMs are explicitly named and analyzed within this paper (as detailed in §4). In this capacity, they serve as the objects of our study.

LLMs as an Assitive Tool. In the preparation of this manuscript, the use of LLMs is limited to polishing the text for grammatical correctness, spelling, and clarity of expression. The LLMs were not used to generate any core research ideas, experimental designs, data analysis, or substantive portions of the manuscript.

We assume full responsibility for all content presented in this paper, including any text that has been revised with the assistance of an LLM. We have meticulously reviewed and edited all content to ensure its scientific accuracy and originality, preventing any form of plagiarism or academic misconduct.

A.3 Implementation Details

For the online model APIs, we utilized their respective official endpoints with default temperature settings. The maximum generation length was also kept at its default value, and no other parameters were modified.

For the training of open-source models, all experiments were conducted on a server equipped with 8×NVIDIA H800 GPU (80GB). The maximum generation length was set to 8192 for the GSM8K and MATH datasets, and 16384 for all other datasets. The temperature was set to 1.0, and the random seed was fixed to 42 for reproducibility.

For SFT training, we adpot full-parameter fine-tuning with learning rate $r = 1 \times 10^{-5}$ and $batchsize = 32$. To construct training data, we first define the difficulty level for each sample from the train set of GSM8K and MATH: Easy (GSM8K and MATH lv.1), Medium (MATH lv.1-4), and Hard (MATH lv.3-5). Second, to obtain high-level task decomposition, we utilize Gemini-2.5-Pro with the prompt in Appendix A.1.3. The final pattern of training samples in cold-start is in the form of: "<difficulty> level </difficulty> <plan> decomposition </plan> <think> CoT </think> answer".

For RL training, we write our code based on the open-source Verl framework. Training settings are listed in Tab. 12.

Table 12: Training settings for RL.

Parameter	Value
n_gpu	8
rollout.n	16
total_steps	1000
batch_size	8
critic_warmup	0
max_prompt_length	512
max_response_length	16384
filter_overlong_prompts	True
learning_rate	1e-6
use_kl_loss	True
kl_loss_coef	0.001
kl_loss_type	low_var_kl