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Stability of disturbance decomposition near traveling wave solutions for the second-order Camassa-Holm equation

Danping Ding[1] · Di Wen[1]

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Abstract
Based on the pseudo-conformal transformation, this paper discusses the stability of the perturbation decomposition near the traveling wave solutions of the second-order Camassa-Holm equation. Through the characterization of the parameter properties in the modulation equation, the solution near the solitons is qualitatively studied, and the decomposition is given on the geometric parameters And the stability results of disturbances, establish a feasible theoretical basis for research on dynamics.

Keywords Second-order Camassa-Holm equation · Stability · Pseudo-conformal transformation · Traveling wave solutions · Modulation equation

1 Introduction

In 2003, Constantin and Kolev [1] obtained a geodesic equation when they studied Hamiltonian vector fields on the regular dual of the Lie algebra of the diffeomorphism group of the circle, the associated flow relative to a modified Lie-Poisson structure being given by (1.1).

$$u_t = B_k(u, u), \;\; k \in \mathbb{N}. \tag{1.1}$$

Where $u = u(t, x)$, $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$,

$$B_k(u, u) \triangleq A_k^{-1} C_k(u) - u\partial_x u$$

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✉ Di Wen
Wendy19970222@outlook.com

1 School of Mathematical Sciences, Jiangsu University, Xuefu Road, Zhenjiang 212013, Jiangsu, China

$$A_k(u) \triangleq \sum_{j=0}^{k}(-1)^j \partial_x^{2j} u$$
$$C_k(u) \triangleq -uA_k(\partial_x u) + A_k(u\partial_x u) - 2\partial_x u A_k(u)$$

Operation A_k^{-1} is given by the equivalent convolution form

$$A_k^{-1}(f)(x) = P_k * f = \int_R P_k(x-y)f(y)dy, \ x \in R$$
$$\widehat{P_k}(\xi) = \frac{1}{1+\xi^2+\cdots+\xi^{2k}}, \ \xi \in R \tag{1.2}$$

For $k=0$, Eq. (1.1) yields the inviscid Burgers equation [2].

$$\partial_t u + u\partial_x u = -\partial_x(u^2) \ or \ u_t + 3uu_x = 0. \tag{1.3}$$

For $k=1$, Eq. (1.1) become the Camassa-Holm equation [3].

$$\partial_t u + u\partial_x u = -\partial_x A_1^{-1}\left[u^2 + \frac{1}{2}\left(\partial_x u^2\right)\right], \tag{1.4}$$

or

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0.$$

When $k=0$ and $k=1$, Eqs. (1.3) (1.4) are integrable [4–6].

For $k=2$, Eq.(1.1) change into

$$\partial_t u + u\partial_x u = -A_2^{-1}\partial_x\left[u^2 + \frac{1}{2}(\partial_x u)^2 - \frac{1}{2}\left(\partial_x^2 u\right)^2 - 3(\partial_x u\partial_x^2 u)_x\right], \tag{1.5}$$

the right side is equivalent to $-P_2 * \partial_x\left[u^2 + \frac{1}{2}(\partial_x u)^2 - \frac{1}{2}\left(\partial_x^2 u\right)^2 - 3\left(\partial_x u \cdot \partial_x^2 u\right)_x\right]$, where

$$P_2(x) = \frac{\sqrt{3}}{3}e^{-\frac{\sqrt{3}}{2}|x|}\sin\left(\frac{|x|}{2} + \frac{\pi}{6}\right), \ x \in \mathbb{R}. \tag{1.6}$$

Then Eq. (1.5) can be written as

$$u_t + uu_x + P_2 * \left[u^2 + \frac{1}{2}(\partial_x u)^2 - \frac{1}{2}\left(\partial_x^2 u\right)^2 - 3\left(\partial_x u \cdot \partial_x^2 u\right)_x\right]_x = 0 \tag{1.7}$$

it is equivalent to

$$u_t - u_{txx} + u_{txxxx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} + 2uu_{xxxx} + uu_{xxxxx} = 0 \tag{1.8}$$

Where P_2 satisfy $u - u_{xx} + u_{xxxx} = \delta$($\delta$ means Dirac measure).This equation is called second-order Camassa-Holm equation(Next referred to as CH equation).When $k \geq 2$, Eq. (1.1) is called as higher-order CH equation. Since this article only involves the study of the properties of the second-order CH equation, for the sake of brevity, denote P_2 in Eq. (1.6) by P.

In 2009, Coclite , Holden and Karlsen discussed the well-posedness of the higher-order CH equation [7], and got :

- If $u_0 \in H_{k,p} = \left\{ f \in H^k(\mathbb{R}) \middle| \partial_x^k f \in L^p(\mathbb{R}) \right\}$, where $2 < p < +\infty$, then there is a global weak solution $u(t,x) \in C\left([0,+\infty]; C^{k-1}(\mathbb{R})\right) \cap L^\infty\left([0,+\infty]; H^k(\mathbb{R})\right)$;
- If $u_0 \in H^{k+1}(\mathbb{R})$, then $u(t,x) \in L^\infty\left([0,T]; H^{k+1}(\mathbb{R})\right), T > 0$;
- If $u_0 \in H_{k,r}(\mathbb{R})$, $2 \leq r < +\infty$, then $u(t,x) \in L^\infty\left([0,T]; H_{k,r}(\mathbb{R})\right), T > 0$.When $k = 2$, if there is a mapping $h \in L^1([0,T]), T > 0$, such that $\left\|\partial_x^2 u\right\|_{L^\infty(\mathbb{R})} \leq h(t)$, and the solution $u(t,x)$ exists and is unique.

In 2010, Ding and Lv [8] discussed the existence of conservative solutions under Cauchy problem for Eq. (1.1):Let $u_0 \in H^k(\mathbb{R})$ and there exists a constant $M > 0$, s.t. $p_{>M} u_o = 0$(where p is smooth Littlewood-Paley operator), then Eq. (1.1) corresponds initial value problem has a unique global conservation solution $u(t,x) \in C\left([0,+\infty]; H^{k-1}(\mathbb{R})\right) \cap L^\infty\left([0,+\infty]; H^k(\mathbb{R})\right)$.

In 2011, Tian,Zhang and Xia got Eq. (1.8) the conclusion of the local well-posedness of the solution:Let $u \in H^s(\mathbb{R})$, $s > 9/2$, then Eq. (1.1) has a unique solution [9] $u(t,x) \in C([0,T); H^s(\mathbb{R})) \cap C^1\left([0,T); H^{s-1}(\mathbb{R})\right)$, where $T = T(\|u_0\|_{H^s}) > 0$.

In 2016, Lin,Lv and Wei [10] discussed the well-posedness and limit behavior of stochastic high-order CH equations.The limit behaviors of the solution are examined as $\varepsilon \to 0$.

In 2019, Ding and Wang studied the attenuation near the traveling wave solutions of the second-order CH equation [11]: Assumed $u_0 \in U_{\alpha_0}, u_0 = Q + \varepsilon_0, \varepsilon_0(x) \in H^5(R)$, satisfy $\forall x \geq 0$, $|\varepsilon_0(x)| + |\varepsilon_{0xx}(x)| \leq a_0 e^{-\frac{\sqrt{3}}{2}x}$, then exist $\theta > 0$, for $t > 0, x \geq 0$, established $|\varepsilon(t,x)| + |\varepsilon_{xx}(t,x)| \leq \theta \left(e^{-\frac{\sqrt{3}}{4}t} + 1\right) e^{-\frac{\sqrt{3}}{2}x}$.

In 2020, Qu and Fu [12] established for a class of higher-order Camassa-Holm equations, the local well-posed properties of the Cauchy problem in Besov space and Sobolev space are :For $s > 1/2$ let $m_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$. Suppose there is a point $x_0 \in \mathbb{R}$ s.t. $m_0(x_0) > 0$ and

$$\frac{|B_1|}{C_2} - \frac{C_1}{C_2} < \frac{1}{m_0(x_0)} < \sqrt{\frac{C_1^2}{C_1^2} + \frac{B_1^2}{C_2}} - \frac{C_1}{C_2},$$

Where $B_1, C_{1,2}$ are obtained by calculationis. Then the solution $u(t,x)$ blows up in T_0, and

$$T_0 \leq t_1 = \frac{1}{\sqrt{C_2}} \ln \left[\frac{\frac{C_1}{C_2} - \sqrt{\frac{C_1^2}{C_1^2} - g^2(0) + \frac{B_1^2}{C_2}}}{g(0) - \frac{B_1}{\sqrt{C_2}}} \right].$$

Where $g(0) = \frac{1}{m_0(x_0)} + \frac{C_1}{C_2}$. In the same year, Shirvani and Nadjafihah [13] used Lie symmetry group method to discuss symmetries and nonclassical sym-metries for the high-order CH equation,and the conserved quantity is the conclusion produced by the multiplier of the homotopy operator. If $u = f(x, t)$ is the solution of the higher-order CH equation, then the set of functions

$$\begin{aligned} G_1 \cdot (s) f(x, t) &= f(x - s, t), \\ G_2 \cdot (s) f(x, t) &= f(x, t - s), \\ G_3 \cdot (s) f(x, t) &= e^{-s} f\left(x, te^{-s}\right). \end{aligned}$$

also the solution of this equation. Where $G_1 : (t, x, u) \mapsto (x + s, t, u), G_2 : (t, x, u) \mapsto (x, t + s, u), G_3 : (t, x, u) \mapsto \left(x, e^s t, e^{-s} u\right)$.

For second-order CH equation, the form of traveling wave solutions is [14]:

$$Q_A(x - ct) = \begin{cases} Ae^{-\frac{\sqrt{3}}{2}(x-ct)} \left(\cos \frac{x-ct}{2} + \sqrt{3} \sin \frac{x-ct}{2}\right), & x \geq ct, \\ Ae^{\frac{\sqrt{3}}{2}(x-ct)} \left(\cos \frac{x-ct}{2} - \sqrt{3} \sin \frac{x-ct}{2}\right), & x < ct. \end{cases} \tag{1.9}$$

Where $A > 0$ is amplitude, and it is related to c (for convenience, let $A = 1$);$c > 0$ representative wave speed, then $P(\xi) = \frac{\sqrt{3}}{6A} Q_A(\xi)$ and $Q_A - (Q_A)'_{\xi\xi} + (Q_A)'_{\xi\xi\xi\xi} = 2\sqrt{3}A\delta$. The mass and energy are conserved by the flow:

$$M(u(t)) = M_0 \; or \int u(t, x) = \int u_0(x) \tag{1.10}$$

$$E(u(t)) = E(u_0) \; or \sum_{j=0}^{k} \int (\partial_x^j u)^2 (t, x) = \sum_{j=0}^{k} \int (\partial_x^j u_0)^2 (t, x). \tag{1.11}$$

This paper is aim to study the stability of the perturbed decomposition near the traveling wave solutions of the second-order CH equation. The study of the decomposition problem can be traced back to the instability of the solutions to the Cauchy problem of the critical generalized KdV (gKdV) equation by Martel and Merle [15], the asymptotic stability of solitons [16] and other series of work. In 2014, Martel, Merle and *Raphaël* added the parameter b to the solitary wave solutions disturbance [17], classify the dynamics near the solitons.

For $p > 1, u_0 \in (\mathbb{R})$ the gKdV equation *Cauchy* problem:

$$\begin{cases} u_t + \left(u_{xx} + u^p\right)_x = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.12}$$

In [15], they use Pseudo-conformal transformation $v(t, y) = \lambda^{\frac{1}{2}} u(t, \lambda(t) y + x(t))$ and decompositio $v(t,x) = \varepsilon(t,x) + Q(x)$, where $\lambda(t) > 0$, $x(t) \in \mathbb{R}$, to proof the orbit instability of solitons $Q(x) = \left(3/\cosh^2 2x\right)^{1/4}$ in H^1 and L^2 for gKdV equation. In [16], they obtained the asymptotic stability of the solitons by utilizing Liouvill theorem [18].

For the convenience of description, let's define some symbols in this paper

1. $a \lesssim b \Leftrightarrow a \le \widetilde{C} b$, where $\widetilde{C}$ is a certain positive number;
2. For a given $\alpha^* > 0$, let $\rho(\alpha^*)$ denote a small quantity dependin on it, $\rho(\alpha^*) \to 0$ as $\alpha^* \to 0$;
3. Use $\int$ instead of $\int_{\mathbb{R}}$, and omit the integral variable;
4. Let $\|\cdot\|_{H^k}$ represents norm $H^k(\mathbb{R})$ in Sobolev space, $\|f\|_{H^k}^2 = \sum_{j=0}^{k} \int (\partial_x^2 f)^2$, $k \in \mathbb{N}$.

Consider the second-order CH equation in this paper. First decompose the solution of Eq. (1.8) as:

$$u(t,x) = \lambda^{-\frac{1}{2}} (Q_b + \varepsilon)(t, x - x(t)). \tag{1.13}$$

Where Q_b is a suitable O(b) deformation of the traveling wave profile. Let

$$T_{\alpha^*} = \left\{ u \in H^2 : \inf_{\substack{\widehat{\lambda} > 0 \\ \widehat{x} \in \mathbb{R}}} \left\| u - \widehat{\lambda}^{-\frac{1}{2}} Q(\cdot - \widehat{x}) \right\|_{H^2} < \alpha^* \right\}. \tag{1.14}$$

Where $\widehat{\lambda} = \widehat{\lambda}(u)$, $\widehat{x} = \widehat{x}(u)$, let $\widehat{\lambda}(t_0) = \widehat{\lambda}_0$, $\widehat{x}(t_0) = \widehat{x}_0$.

2 Decomposition near the traveling wave

Let $\chi \in C^\infty(\mathbb{R})$, s.t. $0 \le \chi \le 1$, $\chi' \ge 0$ in $\mathbb{R}$, and here

$$\chi(x) \equiv \begin{cases} 1 & x \in [-1, \infty) \\ 0 & x \in (-\infty, -2] \end{cases} \tag{2.1}$$

Let $\chi_b(y) = \chi(|b|^\gamma y)$, where $\gamma \in (0,1)$.

Let $\mathcal{B} = \left\{ f \in C^\infty(\mathbb{R}) : \left| f^{(k)}(y) \right| \le (1+|y|)^p e^{-|y|} \right\}$, where $k = 1, 2$, $p > 0$, $y \in \mathbb{R}$; $Q_b(y) = Q(y) + b\chi_b(y) D(y)$. Where $|b| < |b^*| \ll 1$, $D \in \mathcal{B}$ and $(D, Q') \neq 0$, $\lim_{y \to \infty} D(y) = 0$, $|D(y)| \lesssim e^{-\frac{|y|}{2}}$.

Proposition 2.1 *For* $\forall \xi \in \mathbb{R}$,

$$|Q_b(\xi)| \lesssim e^{-|\xi|} + b\left[\chi_{[-2,-1]}(|b|^\gamma \xi) + e^{-\frac{1}{2}|\xi|}\right].$$

Mass and energy meet

$$\left|\int Q_b - \int Q\right| \lesssim |b|, \tag{2.2}$$

$$\left|E(Q_b) - \|Q\|_{H^2}^2\right| \lesssim |b|^{\frac{1}{2}}. \tag{2.3}$$

Proof

(1) From (1.9), $|Q(\xi)| = e^{-\frac{\sqrt{3}}{2}|\xi|}\left|(\cos\frac{\xi}{2} \pm \sqrt{3}\sin\frac{\xi}{2})\right|$, where $\xi = x - ct$. And then

$$\begin{aligned}|Q_b(\xi)| &= |Q(\xi) + b\chi_b(\xi)D(\xi)| \\ &\lesssim e^{-\frac{1}{4}|\xi|} + b\left[\chi_{[-2,1]}(|b|^{\gamma}\xi) + e^{-\frac{1}{2}|\xi|}\right].\end{aligned}$$

(2) According to (1.10),

$$M(Q_b) = \int Q + b\int \chi_b D,$$

By the decay of D, we get (2.2). Similarly,

$$\begin{aligned}E\left(Q_b\right) = &\|Q\|_{H^2}^2 + b^2\left\|\chi_b D\right\|_{H^2}^2 \\ &+ 2b\left[\int Q\chi_b D + \int Q_y(\chi_b D)_y + \int Q_{yy}(\chi_b D)_{yy}\right].\end{aligned}$$

Obviously

$$\left|E\left(Q_b\right) - \|Q\|_{H^2}^2\right| \le b^2\left\|D\right\|_{H^2}^2 + 2b\left\|QD\right\|_{H^2}^2 \lesssim b^2 + 2b.$$

Where $\|Q\|_{H^2}^2 = 2\sqrt{3}$, $\chi_b \le 1$, then (2.3) is true.

□

We use the modulating tube T_{α^*} of the function near the solitons manifold for studing, more specifically, assume that there exist $(\lambda_1(t), x_1(t)) \in \mathbb{R}_+^* \times \mathbb{R}$ and $\varepsilon_1(t)$ s.t. $\forall t \in [0, t_0)$ established

$$u(t,x) = \frac{1}{\lambda_1^{\frac{1}{2}}(t)}\left(Q + \varepsilon_1\right)\left(t, x - x_1(t)\right), \tag{2.4}$$

Where $u_0 = Q + \varepsilon_0$, $\varepsilon_0(x) \in H^2(\mathbb{R})$.

Proposition 2.2 *If $u(x) \in C^r(R)$, let*

$$v(t,y) = \lambda^{\frac{1}{2}}(t)u\left(t, y + x(t)\right). \tag{2.5}$$

Then $v(y) \in C^r(R)$.Where $x = y + x(t)$.

Definition 2.1 There exist continuous functions $(\lambda, b, x) : [0, t_0] \to \mathbb{R}^+ \times \mathbb{R}^2$

$$\varepsilon(t, y) = v(t, y) - Q_{b(t)}(y) = \lambda^{\frac{1}{2}}(t)u\left(t, y + x(t)\right) - Q_{b(t)}(y). \tag{2.6}$$

For all $t \in [0, t_0]$.

Proposition 2.3 *For* $\forall t \geq 0$, $\varepsilon_0\left(x\right) \in H^2\left(\mathbb{R}\right)$, *there is*

$$M(\varepsilon + Q_b) = \lambda^{\frac{1}{2}} M_0,\ \ E(\varepsilon + Q_b) = \lambda E_0 \tag{2.7}$$

Proof The proof of this statement is based on definition, and is therefore omitted. □

Proposition 2.4 (Modulation parameter selection [19, 20]) *In the decomposition of (2.6), there exist* $\lambda(t) > 0, b(t) > 0, x(t) > 0$, *and unique map* $(\lambda, b, x) : T_\alpha \mapsto \mathbb{R}^+ \times \mathbb{R}^2$, *s.t. if* $u \in T_{\alpha^*}(\alpha^* > 0)$, *then* ε *satisfies the orthogonality condition:*

$$(\varepsilon, Q) = (\varepsilon, Q_y) = (\varepsilon, yQ_y) = 0. \tag{2.8}$$

and has

$$|b(t)| + \left|1 - \frac{\lambda(t)}{\lambda_1(t)}\right| \lesssim \rho(\alpha),\ \ \text{and}\ \ \|\varepsilon(t)\|_{H^2} \lesssim \rho(\|\varepsilon(0)\|_{H^2}) \tag{2.9}$$

Proof Let $f_1 = \int \varepsilon Q$, $f_2 = \int \varepsilon Q_y$, $N_0(\lambda, b) = (1, 0)$

$$\because\ \varepsilon_\lambda|_{N_0} = \frac{1}{2}Q, \quad \varepsilon_b|_{N_0} = -D.$$

then there is

$$\left.\frac{\partial f_1}{\partial \lambda}\right|_{N_0} = \frac{1}{2}\int Q \cdot Q \neq 0, \qquad \left.\frac{\partial f_2}{\partial b}\right|_{N_0} = -\int D \cdot Q,$$
$$\left.\frac{\partial f_2}{\partial \lambda}\right|_{N_0} = \frac{1}{2}\int Q \cdot Q_y = 0, \quad \left.\frac{\partial f_2}{\partial b}\right|_{N_0} = -\int D \cdot Q_y \neq 0.$$

The Jacobian matrix

$$J = \left.\frac{\partial(f_1, f_2)}{\partial(\lambda, b)}\right|_{N_0} = \begin{vmatrix} \frac{1}{2}\int QQ & -\int DQ \\ \frac{1}{2}\int QQ_y & -\int DQ_y \end{vmatrix} = -\frac{1}{2}\|Q\|_{L^2} \cdot (D, Q_y) \neq 0.$$

From implicit function existence theorem, $\exists U(1, 0) \subset (\mathbb{R}^+ \times \mathbb{R})$, $\exists \widetilde{\alpha} > 0$ and a unique map $(\lambda, b) : \{u \in H^2(\mathbb{R}) : \left\|u - \lambda^{-\frac{1}{2}}Q\right\|_{H^2} < \widetilde{\alpha}\} \to U(1, 0)$ s.t. (2.8) holds.

Moreover, if $\left\| u - \lambda^{-\frac{1}{2}} Q \right\|_{H^2} < \overline{\alpha} \le \widetilde{\alpha}$, then $|b(t)| + \left| 1 - \frac{\lambda(t)}{\lambda_1(t)} \right| \lesssim \rho(\alpha)$.

$$\begin{aligned}
\because \|\varepsilon\|_{H^2} &= \left\| \lambda^{\frac{1}{2}} u - Q_b \right\|_{H^2} \\
&\lesssim |\lambda^{\frac{1}{2}} - 1| \cdot \|u\|_{H^2} + \left\| u - \lambda^{-\frac{1}{2}} Q \right\|_{H^2} + |\lambda^{-\frac{1}{2}} - 1| \cdot \|Q\|_{H^2} - b\|\chi_b D\|_{H^2} \\
&\lesssim 2\rho(\alpha) + \alpha. \\
\therefore \|\varepsilon\|_{H^2} &\lesssim \rho(\|\varepsilon(0)\|_{H^2})
\end{aligned}$$

There exist $\widetilde{\alpha} < \alpha^*$ and a unique map in C^1,$r : T_{\alpha^*} \to \mathbb{R}$, s.t. if $u \in T_{\alpha^*}$, then

$$\left\| u(\cdot) - \lambda^{-\frac{1}{2}} Q(\cdot - r) \right\|_{H^2} = \inf_{r \in \mathbb{R}} \left\| u(\cdot) - \lambda^{-\frac{1}{2}} Q(\cdot - r) \right\|_{H^2} < \widetilde{\alpha} < \alpha^*,$$

That is to say, there exist parameters depended on the solution $\lambda_1(u) = \lambda_1(u(\cdot + r(u)))$, $x_1(u) = x_1(u(\cdot + r(u))) + r(u)$, $b_1(u) = b_1(u(\cdot + r(u)))$, s.t. (2.8) and (2.9) established. □

Proposition 2.5 *On* $[0, t_0]$,

$$\begin{aligned}
&\|\varepsilon\|_{L^1} \lesssim \left| \lambda^{\frac{1}{2}} \int u_0 - \int Q \right| + |b|, \\
&|\lambda E_0| \lesssim |b| + \|\varepsilon\|_{H^2}^2 + 2\sqrt{3}.
\end{aligned} \tag{2.10}$$

Proof From (2.7) conservation of mass

$$\int \varepsilon = \lambda^{\frac{1}{2}} \int (u - Q - b\chi_b D).$$

then

$$\|\varepsilon\|_{L^1} \lesssim \left| \lambda^{\frac{1}{2}} \int u_0 - \int Q \right| + |b|.$$

Similarly,

$$\lambda E_0 = E(\varepsilon) + E(Q_b) + 2\int \left[\varepsilon Q_b + \varepsilon_y (Q_b)_y + \varepsilon_y (Q_b)_{yy} \right] \tag{2.11}$$

By (2.3) and (2.11), we can get that

$$\lambda E_0 = \|\varepsilon\|_{H^2}^2 + 2\sqrt{3} + b^2 \|\chi_b D\|_{H^2}^2 + 8\sqrt{3} b\varepsilon(0) + 2bL_Q + 2b^2 L_\varepsilon. \tag{2.12}$$

Here

$$L_Q = \int Q\chi_b D + \int Q_y(\chi_b D)_y + \int Q_{yy}(\chi_b D)_{yy},$$
$$L_\varepsilon = \int \chi_b D\varepsilon + \int (\chi_b D)_y \varepsilon_y + \int (\chi_b D)_{yy}\varepsilon_{yy}.$$

According to $H\ddot{o}lder$ inequality, there is

$$\left(L_Q\right)^2 \leq M_Q \left[\int (\chi_b D)^2 + \int (\chi_b D)_y^2 + \int (\chi_b D)_{yy}^2\right] = M_Q \left\|\chi_b D\right\|_{H^2}^2.$$

Where $M_Q = \max\left\{\int Q^2, \int Q_y^2, \int Q_{yy}^2\right\}$. Then $\left|L_Q\right| \leq \sqrt{M_Q}\|\chi_b D\|_{H^2}$. Similarly

$$(L_\varepsilon)^2 \leq \left(\int e^{-2|y|}\right)\cdot\left(\int \varepsilon^2 + \int \varepsilon_y^2 + \int \varepsilon_{yy}^2\right) = \|\varepsilon\|_{H^2}^2$$

Then $|L_\varepsilon| \leq \|\varepsilon\|_{H^2}$.Bring these results of inquality into (2.12), then

$$\lambda E_0 \leq \|\varepsilon\|_{H^2}^2 + 2\sqrt{3} + b^2\left(\|\chi_b D\|_{H^2}^2 + 2\left|\|\varepsilon\|_{H^2}\right|\right)$$
$$+ b\left[8\sqrt{3}\varepsilon\,(0) + 2\sqrt{M_Q}\left|\|\chi_b D\|_{H^2}\right|\right]$$

Because boundedness of $\|\varepsilon\|_{H^2}$ and $\|\chi_b D\|_{H^2}$, (2.10) is proved. □

Introduce the new time variable s, it satisfy

$$s = \int_0^t \frac{dt'}{\lambda^{1/2}(t')},\ or\ equivalently\ \frac{ds}{dt} = \frac{1}{\lambda^{1/2}(t)}. \tag{2.13}$$

Then the t in (2.6) can be rewritten as $s \in [0, s_0]$, where $s_0 = s(t_0)$. According to the decomposition of ε in (2.6), there is :

Lemma 2.1 *For* $\forall s \in [0, s_0]$, $s \mapsto (\lambda(s), x(s), b(s)) \subset C^1$,

$$\begin{aligned}\varepsilon_s + \frac{1}{2}b\varepsilon =& \frac{1}{2}\left(\frac{\lambda_s}{\lambda} + b\right)(Q_b + \varepsilon) + x_s(Q_b + \varepsilon)_y \\ &- Q_b(Q_b)_y - \varepsilon\varepsilon_y - (Q_b\varepsilon)_y + \Phi_b - \frac{1}{2}bQ_b + R(\varepsilon)\end{aligned} \tag{2.14}$$

where

$$\Phi_b = (Q_b)'_s = -b_s \int (\chi_b + \gamma y {\chi'}_b)D,$$

$$\begin{aligned} R(\varepsilon) = & \frac{3}{2}\int P_{zzz}(y-z)(Q_b)'_z - \int P_z(y-z)\left[Q_b^2 + \frac{1}{2}(Q_b)_z^2 - \frac{1}{2}(Q_b)_{zz}^2\right] \\ & + \frac{3}{2}\int P_{zzz}(y-z)\varepsilon_z^2 - \int P_z(y-z)\left(\varepsilon^2 + \frac{1}{2}\varepsilon_z^2 - \frac{1}{2}\varepsilon_{zz}^2\right) \\ & + 3\int P_{zzz}(y-z)(Q_b)'_z\varepsilon_z - \int P_z(y-z)\left[2Q_b\varepsilon + (Q_b)'_z\varepsilon_z - (Q_b)''_{zz}\varepsilon_{zz}\right]. \end{aligned} \tag{2.15}$$

The expression of P is the same as (1.6).

Proof From (2.5),

$$\begin{aligned} v_t &= \frac{1}{2}\lambda^{-\frac{1}{2}}\lambda_t u + \lambda^{\frac{1}{2}}u_t + \lambda^{\frac{1}{2}}x_t u_y, \\ v_y &= \lambda^{\frac{1}{2}}u_y, \quad v_{yy} = \lambda^{\frac{1}{2}}u_{yy}. \end{aligned} \tag{2.16}$$

Substitute in (1.7)

$$\begin{aligned} & \lambda^{\frac{1}{2}}v_t - \frac{1}{2}\lambda^{-\frac{3}{2}}v_t v - \lambda^{-\frac{1}{2}}v_y x_t + v v_y \\ & = \frac{3}{2}\int P_{zzz}(y-z)v_z^2 - \int P_z(y-z)(v^2 + \frac{1}{2}v_z^2 - \frac{1}{2}v_{zz}^2). \end{aligned} \tag{2.17}$$

According to (2.13), (2.17) can be transformed into

$$\begin{aligned} & v_s - \frac{1}{2}\frac{\lambda_s}{\lambda}v - v_y x_t + v v_y \\ & = \frac{3}{2}\int P_{zzz}(y-z)v_z^2 - \int P_z(y-z)\left(v^2 + \frac{1}{2}v_z^2 - \frac{1}{2}v_{zz}^2\right). \end{aligned} \tag{2.18}$$

From (2.6), there is $\varepsilon_s = v_s - (Q_b)_s$, then

$$\begin{aligned} \varepsilon_s + \frac{1}{2}b\varepsilon = & \frac{1}{2}\left(\frac{\lambda_s}{\lambda} + b\right)(Q_b + \varepsilon) + x_s(Q_b + \varepsilon)_y \\ & - Q_b(Q_b)_y - \varepsilon\varepsilon_y - (Q_b\varepsilon)_y + \Phi_b - \frac{1}{2}bQ_b + R(\varepsilon) \end{aligned}$$

Where

$$\begin{aligned} R(\varepsilon) = & \frac{3}{2}\int P_{zzz}(y-z)(Q_b)'_z - \int P_z(y-z)\left[Q_b^2 + \frac{1}{2}(Q_b)'_z - \frac{1}{2}(Q_b)_{zz}^2\right] \\ & + \frac{3}{2}\int P_{zzz}(y-z)\varepsilon_z^2 - \int P_z(y-z)\left(\varepsilon^2 + \frac{1}{2}\varepsilon_z^2 - \frac{1}{2}\varepsilon_{zz}^2\right) \\ & + 3\int P_{zzz}(y-z)(Q_b)'_z\varepsilon_z - \int P_z(y-z)\left[2Q_b\varepsilon + (Q_b)'_z\varepsilon_z - (Q_b)''_{zz}\varepsilon_{zz}\right], \end{aligned}$$

and

$$P_x = \begin{cases} -\frac{\sqrt{3}}{3}e^{-\frac{\sqrt{3}}{2}x}\sin\frac{x}{2} & x \geq 0, \\ \frac{\sqrt{3}}{3}e^{\frac{\sqrt{3}}{2}x}\sin\frac{x}{2} & x \geq 0. \end{cases} \quad P_{xxx} = \begin{cases} \frac{\sqrt{3}}{3}e^{-\frac{\sqrt{3}}{2}x}\sin\left(\frac{x}{2}+\frac{2}{3}\pi\right) & x \geq 0, \\ -\frac{\sqrt{3}}{3}e^{\frac{\sqrt{3}}{2}x}\sin\left(\frac{x}{2}-\frac{2}{3}\pi\right) & x \geq 0. \end{cases} \tag{2.19}$$

□

Lemma 2.2 (Parametric equation) *If there exist* $0 < \widetilde{\alpha} < \alpha^*$, *s.t. for* $\forall t \geq 0$, $u(t) \in T_{\widetilde{\alpha}}$, *then* $\lambda(s), x(x) \in C^1$. *Moreover,*

$$\begin{aligned} &\frac{1}{2}\left(\frac{\lambda_s}{\lambda}+b\right)\int Q_b Q - x_s \int Q_b Q_y \\ &\quad = -\frac{1}{2}\int Q_y(Q_b+\varepsilon)^2 - \int \Phi_b Q + \frac{1}{2}b\int Q_b Q - \int RQ. \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} &\frac{1}{2}\left(\frac{\lambda_s}{\lambda}+b\right)\int Q_b y Q_y - x_s\left(\int Q_b Q_y + \int Q_b y Q_{yy} + \int \varepsilon y Q_{yy}\right) \\ &\quad = -\frac{1}{2}\int (Q_y + yQ_{yy})(Q_b+\varepsilon)^2 - \int \Phi_b Q_y + \frac{1}{2}b\int Q_b y Q_y - \int R y Q_y. \end{aligned} \tag{2.21}$$

Here R *is given by* (2.15).

Proof This calculation relies on (2.8). In order to perform rigorous calculations, regularization parameters are used for u.

Select $\widetilde{Q} \in C^1(\mathbb{R}\backslash\{0\})$ s.t. for $\forall x \in (\mathbb{R}\backslash\{0\})$,

$$\left|\widetilde{Q}(x)\right| + \left|\widetilde{Q}'(x)\right| \lesssim e^{-\frac{|x|}{2}}.$$

Assume $u_0 \in H^5$, therefore $u \in L^\infty\left([0,+\infty], H^5(\mathbb{R})\right)$, where $\left|\widetilde{Q}(y)\right| \lesssim e^{-\frac{1}{2}|y|}$. According to proposition 2.4, $\lambda(t), x(t), b(t) \in C^1(\mathbb{R})$.From (1.7), we calculate $\frac{d}{ds}\int \widetilde{Q}\varepsilon$,

$$\begin{aligned} \frac{d}{ds}\int \widetilde{Q}u &= \lambda^{\frac{1}{2}}\frac{d}{dt}\int \widetilde{Q}u \\ &= \lambda^{\frac{1}{2}}\int \widetilde{Q}\left\{-uu_x - P_2 * \left[u^2 + \frac{1}{2}u_x^2 - \frac{1}{2}u_{xx}^2 - 3(u_x u_{xx})_x\right]_x\right\} \\ &= \frac{1}{2}\lambda^{\frac{1}{2}}\int u^2\widetilde{Q}_x - \lambda^{\frac{1}{2}}\int \widetilde{Q}\left\{P_2 * \left[u^2 + \frac{1}{2}u_x^2 - \frac{1}{2}u_{xx}^2 - 3(u_x u_{xx})_x\right]_x\right\}. \end{aligned}$$

Therefore

$$\begin{aligned}\frac{d}{ds}\int \widetilde{Q}v =& \frac{d}{ds}\left[\lambda^{\frac{1}{2}}\int \widetilde{Q}\left(x-x(s)\right)u(s,x)dx\right]\\ =& \frac{1}{2}\frac{\lambda_s}{\lambda}\int \widetilde{Q}v - x_s\int \widetilde{Q}v + \frac{1}{2}\int \widetilde{Q}_x v^2\\ &-\int \widetilde{Q}\left\{P_2 * \left[u^2+\frac{1}{2}u_x^2-\frac{1}{2}u_{xx}^2-3(u_x u_{xx})_x\right]_x\right\}\end{aligned} \tag{2.22}$$

From definition 2.1 and (2.14), (2.22) can be rewritten as

$$\begin{aligned}&\frac{d}{ds}\int \varepsilon\widetilde{Q}+\frac{1}{2}b\int \varepsilon\widetilde{Q}\\ &=\frac{1}{2}\left(\frac{\lambda_s}{\lambda}+b\right)\left(\int Q_b\widetilde{Q}+\int \varepsilon\widetilde{Q}\right)-x_s\left[\int Q_b\widetilde{Q}_y+\int \varepsilon\widetilde{Q}_y\right]\\ &\quad+\frac{1}{2}\int \widetilde{Q}_y(Q_b+\varepsilon)^2+\int \Phi_b\widetilde{Q}-\frac{1}{2}b\int Q_b\widetilde{Q}+\int R\widetilde{Q}\end{aligned} \tag{2.23}$$

Let $\widetilde{Q}=Q$, $\widetilde{Q}=yQ_y$ in (2.23).
When $\widetilde{Q}=Q$, because $(\varepsilon, Q)=0$, $(\varepsilon, Q_y)=0$, then

$$\begin{aligned}0=\frac{d}{ds}\int \varepsilon Q =& \frac{1}{2}\left(\frac{\lambda_s}{\lambda}+b\right)\int Q_bQ - x_s\int Q_bQ_y\\ &+\frac{1}{2}\int Q_{yy}(Q_b+\varepsilon)^2+\int \Phi_bQ-\frac{1}{2}b\int Q_bQ+\int RQ.\end{aligned}$$

Analogously, when $\widetilde{Q}=yQ_y$, there is

$$\begin{aligned}0&=\frac{d}{ds}\int \varepsilon yQ_y\\ &=\frac{1}{2}\left(\frac{\lambda_s}{\lambda}+b\right)\int Q_byQ_y-x_s\left(\int Q_bQ_y+\int Q_byQ_{yy}+\int \varepsilon yQ_{yy}\right)\\ &\quad+\frac{1}{2}\int (Q_y+yQ_{yy})(Q_b+\varepsilon)^2+\int \Phi_byQ_y-\frac{1}{2}b\int Q_byQ_y+\int RyQ_y\end{aligned}$$

So (2.20) and (2.21) are proved.

Put the parametric equations as a system of equations related to $\left(\frac{\lambda_s}{\lambda}+b\right)$ and x_s, similar to propsition 2.4, the coefficient determinant $g(s)$ is

$$\begin{aligned}g(s)=&-\left(\int Q_bQ\right)\cdot\left(\int Q_bQ_y+\int Q_byQ_{yy}+\int \varepsilon yQ_{yy}\right)\\ &+\left(\int Q_byQ_y\right)\cdot\left(\int Q_bQ_y\right),\end{aligned}$$

Calculate its value at $N_0(1,0)$,

$$g(s)|_{N_0} = -\left(\int Q^2\right)\cdot\left(\int QQ_y + \int QyQ_{yy} + \int \varepsilon y Q_{yy}\right) + \left(\int QyQ_y\right)\cdot\left(\int QQ_y\right).$$

It can be obtained by $\int Q \cdot yQ_{yy} = 0$, $\int QQ_y = 0$, then

$$g(s)|_{N_0} = -\left(\int Q^2\right)\cdot\left(\int \varepsilon y Q_{yy}\right) \tag{2.24}$$

For $\int \varepsilon y Q_{yy}$, by Cauchy inequality:

$$\int \varepsilon y Q_{yy} \leq \left(\int \varepsilon^2\right)^{\frac{1}{2}}\cdot\left(\int y^2 Q_{yy}^2\right)^{\frac{1}{2}} \triangleq c_g. \tag{2.25}$$

Where $c_g > 0$, from (2.24)

$$g(s)|_{N_0} \geq -\frac{1}{2}\left(\int Q^2\right)\cdot(-c_g)$$

It is obviously that $g(s)|_{N_0} > 0$. According to Cramer rule, $\left(\frac{\lambda_s}{\lambda} + b\right)$ and x_s can be written as the scalar product of $(\partial_y^i \varepsilon)^j$ and $(\partial_y^{i_0} \varepsilon)^{j_0}$, $(i, j) \in \{0, 1, 2\} \times \{0, 1, 2\}$.

Assume $u_0 \in H^4(\mathbb{R})$, take a sequence [15] $u_{n_0} \in H^5(\mathbb{R})$, according to the principle of density, $u_{n_0} \to u_0$ as $n \to +\infty$,and u_n is also the solution of (1.8) in H^4. For each $T > 0$, according to [21] Th 2.4, then for $\forall 0 \leq t \leq T$, there is $u_n \in L^\infty\left([0, T]; H^3(\mathbb{R})\right)$. According to the continuous dependence of the solution on the initial value, $u_n \to u$ as $n \to \infty$ in $L^\infty\left([0, T]; H^4(\mathbb{R})\right)$.

$\varepsilon_n, \lambda_n, b_n$ and x_n are defined [15] by the same way in (2.4), from propsition 2.4, in $L^\infty([0, T])$ there is $\lambda_n \to \lambda$, $b_n \to b$ and $x_n \to x$ as $n \to \infty$, then $\int \widetilde{Q}\,(\partial_y^i \varepsilon_n)^j \to \int \widetilde{Q}\,(\partial_y^i \varepsilon)^j$, $(i, j) \in \{0, 1, 2\} \times \{0, 1, 2\}$.Using $i = 0$, $j = 1$ for example, the remaining is proved in the same way. Where $\varepsilon_n(t,y) = \lambda_n^{\frac{1}{2}} u_n(t, y + x_n) - Q_{b(n)}(y)$

$$\begin{aligned}
&\left|\int \widetilde{Q}(y) \cdot [\varepsilon_n(s,y) - \varepsilon(s,y)]\right| \\
&= \left|\int \widetilde{Q}\left(x - x_n(s)\right) \cdot \left[\lambda_n^{\frac{1}{2}} u_n(s,x) - b_n(s)\chi_b(x)D(x)\right]\right. \\
&\quad \left. - \int \widetilde{Q}\left(x - x(s)\right)\left[\lambda^{\frac{1}{2}} u(s,x) - b(s)\chi_b(x)D(x)\right]\right| \\
&\le \lambda_n^{\frac{1}{2}} \left|\int \left\{\widetilde{Q}\left(x - x_n(s)\right) \cdot [u_n(s,x) - u(s,x)]\right\}\right| \\
&\quad + \left|\int \left\{\left[\widetilde{Q}\left(x - x_n(s)\right) \cdot \lambda_n^{\frac{1}{2}} - \widetilde{Q}\left(x - x(s)\right) \cdot \lambda^{\frac{1}{2}}\right] u(s,x)\right\}\right| \\
&\quad + \left|\int [\widetilde{Q}\left(x - x_n(s)\right) \chi_b(x)D(x) \cdot [b(s) - b_n(s)]\right| \\
&\quad + \left|\int \left[\widetilde{Q}\left(x - x(s)\right) - \widetilde{Q}\left(x - x_n(s)\right)\right] \cdot b(s)\chi_b(x)D(x)\right| \\
&\triangleq G_1 + G_2 + G_3 + G_4.
\end{aligned}$$

Next, to estimate $G_1 \sim G_4$.

From Cauchy-Schwarz inquality,

$$\begin{aligned}
G_1 &\le \lambda_n^{\frac{1}{2}} \left\|\widetilde{Q}\left(x - x_n(s)\right)\right\|_{L^2} \cdot \|u_n(s,x) - u(s,x)\|_{L^2}, \\
G_3 &\le \left\|\widetilde{Q}\left(x - x_n(s)\right) \chi_b(x)D(x)\right\|_{L^2} \cdot \|b_n(s) - b(s)\|_{L^2}.
\end{aligned}$$

$b_n(s) \to b(s)$ as $n \to \infty$, and $u_n \to u$ in $L^\infty([0,T]; H^4(\mathbb{R}))$. By the boundedness of λ, then $G_{1,3} \to 0$.

According to $H\ddot{o}lder$ inquality,

$$\begin{aligned}
G_2 &\le \left\|\widetilde{Q}\left(x - x_n(s)\right) \lambda_n^{\frac{1}{2}} - \widetilde{Q}\left(x - x(s)\right) \lambda^{\frac{1}{2}}\right\|_{L^\infty} \cdot \int u(s,x), \\
G_4 &\le \left\|\widetilde{Q}\left(x - x(s)\right) - \widetilde{Q}\left(x - x_n(s)\right)\right\|_{L^\infty} \cdot b(s) \int \chi_b(x)D(x).
\end{aligned}$$

From the conservation of mass, $\|u(t,x)\|_{H^2} \lesssim \|u_0(t,x)\|_{H^2}$, that is to say $G_{2,4} \to 0$.Therefore, $\left|\int \widetilde{Q}(y) \cdot [\varepsilon_n(s,y) - \varepsilon(s,y)]\right| \to 0$ as $n \to \infty$. In $L^\infty([0,T])$, then

$$\int \widetilde{Q}\left(\partial_y^i \varepsilon_n\right)^j \to \int \widetilde{Q}\left(\partial_y^i \varepsilon\right)^j,$$

Where $(i, j) \in \{0, 1, 2\} \times \{0, 1, 2\}$. Hence

$$\lambda_n + b_n = 2\frac{\int_0^s \left[m_1 m_{x_2} - m_2 m_{x_1}\right](s')ds'}{m_1 m_{(\lambda,b)_1} - m_{x_1} m_{(\lambda,b)_2}},$$
$$x_n = \frac{\int_0^s \left[m_1 m_{(\lambda,b)_2} - m_2 m_{(\lambda,b)_1}\right](s')ds'}{m_{(\lambda,b)_1} m_{x_2} - m_{(\lambda,b)_2 m_{x_1}}}.$$

Where

$$\begin{aligned}
m_{(\lambda,b)_1} &= \int Q_b Q, \quad m_{(\lambda,b)_2} = \int Q_b y Q_y.\\
m_{x_1} &= \int Q_b Q_y, \qquad m_{x_2} = \int Q_b Q_y + \int Q_b y Q_{yy} + \int \varepsilon y Q_{yy}.\\
m_1 &= -\frac{1}{2}\int Q_y (Q_b + \varepsilon)^2 - \int \Phi_b Q + \frac{1}{2} b \int Q_b Q - \int RQ,\\
m_2 &= -\frac{1}{2}\int (Q_y + y Q_{yy})(Q_b + \varepsilon)^2 - \int \Phi_b Q_y + \frac{1}{2} b \int Q_b y Q_y - \int R y Q_y.
\end{aligned}$$

In particular, if $\lambda(s), x(s) \in C^1$, and $u_0 \in H^4(\mathbb{R})$, then (2.20) and (2.21) are valid on the $H^4(\mathbb{R})$. ◻

Proposition 2.6 *For $R(\varepsilon)$ in* (2.15), the following inquality holds

$$R(\varepsilon) \lesssim \int e^{-\frac{9}{10}|y|} + \int \varepsilon e^{-\frac{3}{2}|y|} + \|\varepsilon\|_{H^2}^2 . \tag{2.26}$$

Proof First let $R(\varepsilon) = R_1 + R_2 + R_3$, where

$$\begin{aligned}
R_1 &= \frac{3}{2}\int P_{zzz}(y-z)(Q_b)_z' - \int P_z(y-z)\left[Q_b^2 + \frac{1}{2}(Q_b)_z' - \frac{1}{2}(Q_b)_{zz}^2\right],\\
R_2 &= 3\int P_{zzz}(y-z)(Q_b)_z'\varepsilon_z - \int P_z(y-z)\left[2Q_b\varepsilon + (Q_b)_z'\varepsilon_z - (Q_b)_{zz}''\varepsilon_{zz}\right],\\
R_3 &= \frac{3}{2}\int P_{zzz}(y-z)\varepsilon_z^2 - \int P_z(y-z)\left(\varepsilon^2 + \frac{1}{2}\varepsilon_z^2 - \frac{1}{2}\varepsilon_{zz}^2\right).
\end{aligned}$$

To R_1,

$$\begin{aligned}
R_1 =& \frac{\sqrt{3}}{2}\int e^{-\frac{\sqrt{3}}{2}|y-z|}\sin\left(\frac{|y-z|}{2} + \frac{2}{3}\pi\right)(Q_b)_z'\\
&- \frac{\sqrt{3}}{3}\int e^{-\frac{\sqrt{3}}{2}|y-z|}\sin\frac{|y-z|}{2}\left[Q_b^2 + \frac{1}{2}(Q_b)_z^2 - \frac{1}{2}(Q_b)_{zz}^2\right]\\
\lesssim& \int \left|e^{-\frac{\sqrt{3}}{2}|x|}\cdot e^{-\frac{\sqrt{3}}{2}|x|}\right| + \int \left|e^{-\frac{\sqrt{3}}{2}|x|}\left[Q_b^2 + \frac{1}{2}(Q_b)_z^2 - \frac{1}{2}(Q_b)_{zz}^2\right]\right|.
\end{aligned}$$

So $R_1(y) \lesssim \int e^{-\frac{\sqrt{9}}{10}|y|}$. Similarly, $R_3 \lesssim \|\varepsilon\|_{H^2}$. Using the property of ε and Q_b

$$\begin{aligned} R_2 =& \sqrt{3}\int e^{-\frac{\sqrt{3}}{2}|y-z|}\sin\left(\frac{|y-z|}{2}+\frac{2}{3}\pi\right)(Q_b)_z^{'}\varepsilon_z \\ & -\frac{\sqrt{3}}{3}\int e^{-\frac{\sqrt{3}}{2}|y-z|}\sin\frac{|y-z|}{2}\left[2Q_b\varepsilon+(Q_b)_z^{'}\varepsilon_z-(Q_b)_{zz}^{'}\varepsilon_{zz}\right] \\ & \lesssim \int e^{-\frac{\sqrt{3}}{2}|x|}e^{-\frac{\sqrt{3}}{2}|x|}\varepsilon_z+\int e^{-\frac{\sqrt{3}}{2}|x|}\varepsilon, \end{aligned}$$

So $R_2(y) \lesssim \int e^{-\sqrt{3}|y|}\varepsilon \lesssim \int e^{-\frac{3}{2}|y|}\varepsilon$.

Summarily, $R(\varepsilon) = R_1 + R_2 + R_3$ and

$$R(\varepsilon) \lesssim \int e^{-\frac{9}{10}|y|} + \int \varepsilon e^{-\frac{3}{2}|y|} + \|\varepsilon\|_{H^2}^2 .$$

Then (2.26) holds. □

Proposition 2.7 *(The equation of b) Assume for* $\forall t \in [0, t_0)$,

$$\int \varepsilon_y^2(t,y)e^{-\frac{1}{2}|y|}dy \le \aleph \tag{2.27}$$

Where $\aleph$ *is any non-negative small quantity. The following holds*

$$\begin{aligned} & b_s+\widetilde{c_b}b+\widetilde{c_{b^2}}b^2-b\left[\frac{1}{2}(\frac{\lambda_s}{\lambda}+b)+\frac{\int DQ_{yy}\varepsilon}{\int DQ_y}\right] \\ & = \mathrm{O}\left(\int e^{-\frac{7}{5}|y|}\right)+\mathrm{O}\left(\int \varepsilon e^{-|y|}\right)+\mathrm{O}\left(|b|^5\right). \end{aligned} \tag{2.28}$$

Where

$$c_b=\int \chi_b DQ_{yy}Q,\ c_{b^2}=\int \chi_b D\left(\chi_b DQ_{yy}+\frac{1}{2}Q_y\right),\ \widetilde{c_{b^k}}=-\frac{c_{b^k}}{\int DQ_y}.\quad (k=1,2)$$

Proof According to (2.8), there is

$$\begin{aligned} \frac{d}{ds}\int \varepsilon Q_y =& \frac{1}{2}\left(\frac{\lambda_s}{\lambda}+b\right)\int Q_bQ_y+x_s\int (Q_b+\varepsilon)_yQ_y \\ & +\frac{1}{2}\int (Q_b+\varepsilon)^2\cdot Q_{yy}+\int \Phi_bQ_y-\frac{1}{2}b\int Q_bQ_y+\int R\cdot Q_y=0 \end{aligned} \tag{2.29}$$

Here, s is regarded as a variable independent of y. According to the orthogonal condition,

$$\int Q_bQ_y=\int QQ_y+b\int \chi_bDQ_y=b\int \chi_bDQ_y,$$

Simplify the square term

$$\int \left(Q_b^2 + 2Q_b\varepsilon + \varepsilon^2\right) Q_{yy} = 2b\int \chi_b D Q Q_{yy} + 2b\int \chi_b D\varepsilon Q_{yy} + b^2\int \chi_b^2 D^2 Q_{yy} + \mathrm{O}(\int e^{-\frac{3}{2}|y|}) + \mathrm{O}\left(\int \varepsilon e^{-\frac{3}{2}|y|}\right).$$

The simplification here is based on (2.27). According to the properties of Φ_b

$$\int \Phi_b Q_y = -b_s\int (\chi_b + \gamma y \chi'_b) D \cdot Q_y = -b_s\int DQ_y + \mathrm{O}\left(|b|^5\right).$$

Using the propsition of Q,

$$\left|x_s\int Q_b Q_y\right| + \left|\int \varepsilon Q_{yy}\right| \lesssim |b|^5$$

Put these results of simplification into (2.29), we get

$$\frac{1}{2}\left(\frac{\lambda_s}{\lambda} + b\right) b\int \chi_b D Q_y + c_b b + b\int \chi_b D\varepsilon Q_{yy} + c_{b^2} b^2 - b_s\int DQ_y \tag{2.30}$$

Where

$$c_b = \int \chi_b D Q Q_{yy},\ c_{b^2} = \int \chi_b D\left(\chi_b D Q_{yy} + \frac{1}{2}Q_y\right)$$

From the boundness of D and Q, the scalar product of Q and ε is

$$\mathrm{O}\left(|b|^5\right) + \mathrm{O}\left(\int e^{-\frac{3}{2}|y|}\right) + \mathrm{O}\left(\int \varepsilon e^{-|y|}\right) + \mathrm{O}\left(\int e^{-\frac{7}{5}|y|}\right) + \mathrm{O}\left(\int \varepsilon e^{-\frac{3}{2}|y|}\right) \tag{2.31}$$

Among them, the estimate of $\int R$ is given in the proposition 2.26. Combine (2.31) and (2.29), then

$$b_s + \widetilde{c_b} b + \widetilde{c_{b^2}} b^2 - b\left[\frac{1}{2}\left(\frac{\lambda_s}{\lambda} + b\right) + \frac{\int DQ_{yy}\varepsilon}{\int DQ_y}\right] = \mathrm{O}\left(\int e^{-\frac{7}{5}|y|}\right) + \mathrm{O}\left(\int \varepsilon e^{-|y|}\right) + \mathrm{O}\left(|b|^5\right).$$

Where $\widetilde{c_{b^k}} = -c_{b^k}/\int DQ_y$. □

Lemma 2.3 *(Control of modulation parameter [22]) For* $\forall s \in [0, s_0]$,

$$\left|\frac{\lambda_s}{\lambda} + b\right| + |x_s| \lesssim b^2 + \left(\int \varepsilon e^{-\frac{1}{2}|y|}\right)^{\frac{1}{2}}, \tag{2.32}$$

$$|b_s| \lesssim |b| + \int \varepsilon e^{-\frac{1}{2}|y|}. \tag{2.33}$$

Proof By Cauchy-Schwarz inquality,

$$\int \widetilde{Q}_y (Q_b + \varepsilon)^2 \lesssim \left(\int e^{-\frac{2}{3}|y|} \right)^{\frac{1}{2}} \cdot \left(\int \varepsilon^4 e^{-3|y|} \right)^{\frac{1}{2}} \lesssim \int \varepsilon^4 e^{-\frac{1}{3}|y|}.$$

And

$$b \int Q_b \widetilde{Q} = b \int Q \widetilde{Q} + b^2 \int \chi_b D \widetilde{Q} \lesssim b \int e^{-\frac{1}{2}|y|}$$

From lemma 2.2

$$\begin{aligned}\left| \frac{1}{2} \left(\frac{\lambda_s}{\lambda} + b \right) \int Q_b Q \right| &\lesssim |x_s| \cdot \int Q_b Q_y - \frac{1}{2} \int Q_y (Q_b + \varepsilon)^2 - \int \Phi_b Q \\ &\quad + \frac{1}{2} b \int Q_b Q_y - \int R Q_y \\ &\lesssim |x_s| \left[|b| + \left(\int \varepsilon e^{-\frac{1}{2}|y|} \right)^{\frac{1}{2}} \right] + |b_s| + \int \varepsilon^4 e^{-\frac{1}{3}|y|} \\ &\quad + |b| \int e^{-\frac{1}{2}|y|} + \int e^{-\frac{9}{10}|y|} + \int \varepsilon e^{-\frac{1}{2}|y|} + \|\varepsilon\|_{H^2}^2 .\end{aligned}$$

Similarly, from (2.21) we can get

$$\begin{aligned}&\left| x_s \left(\int Q_b Q_y + \int Q_b y Q_{yy} + \int \varepsilon y Q_{yy} \right) \right| \\ &\lesssim \left| \frac{\lambda_s}{\lambda} + b \right| \left[|b| + \left(\int \varepsilon^2 e^{-\frac{1}{3}|y|} \right)^{\frac{1}{2}} \right] + |b_s| + \int \varepsilon^4 e^{-\frac{1}{3}|y|} \\ &\quad + |b| \int e^{-\frac{1}{2}|y|} + \int e^{-\frac{9}{10}|y|} + \int \varepsilon e^{-\frac{1}{2}|y|} + \|\varepsilon\|_{H^2}^2 .\end{aligned}$$

According to $H\ddot{o}lder$ inquality,

$$\begin{aligned}\int \varepsilon^4 e^{-\frac{1}{3}|y|} &\lesssim \left\| \varepsilon^2 e^{-\frac{1}{6}|y|} \right\|_{L^\infty} \cdot \left(\int \varepsilon^2 e^{-\frac{1}{6}|y|} \right), \\ \int \varepsilon e^{-\frac{1}{2}|y|} &\lesssim \left\| \varepsilon e^{-\frac{1}{6}|y|} \right\|_{L^\infty} \cdot \int e^{-\frac{1}{3}|y|}.\end{aligned}$$

Hence,

$$\left|\frac{\lambda_s}{\lambda}+b\right|+|x_s|\lesssim\left(\left|\frac{\lambda_s}{\lambda}+b\right|+|x_s|+|b|\right)\cdot\left[|b|+\left(\int\varepsilon e^{-\frac{1}{2}|y|}\right)^{\frac{1}{2}}\right]$$
$$+|b_s|+\int\varepsilon^4e^{-\frac{1}{3}|y|}+\int e^{-\frac{9}{10}|y|}+\int\varepsilon e^{-\frac{3}{2}|y|}+\|\varepsilon\|^2_{H^2}.$$

That is

$$\left|\frac{\lambda_s}{\lambda}+b\right|+|x_s|\lesssim b^2+|b_s|+\left(\int\varepsilon e^{-\frac{1}{2}|y|}\right)^{\frac{1}{2}}$$

Then according to propsition 2.7,

$$b_s\lesssim|b|+b\left(\int\varepsilon e^{-\frac{1}{3}|y|}\right)^{\frac{1}{2}}+\int e^{-\frac{7}{5}|y|}+\int\varepsilon e^{-\frac{1}{2}|y|}.$$

We can also get

$$|b_s+cb|\lesssim|b|^2+b\left(\int\varepsilon e^{-\frac{1}{2}|y|}\right)+\int\varepsilon e^{-|y|}+\int e^{-\frac{7}{5}|y|}.$$

□

Proposition 2.8 *Assume the uniform L^1 control on the right*

$$\int_{y>0}|\varepsilon(t)|\lesssim\rho(\aleph_0)\ for\ \forall t\in[0,t_0),\tag{2.34}$$

where $\aleph_0$ is a small enough universal constant, for $y\to+\infty$, the following holds :

(i) Let

$$\eta_1(y)=\frac{2}{(Q,\int_{-\infty}^{y}Q_y)}\int_{-\infty}^{y}Q_y,\quad J_1(s)=\int\varepsilon(s)\cdot\eta_1;\tag{2.35}$$

then

$$\left|\left(\frac{\lambda_s}{\lambda}+b\right)-b^2-2\left[(J_1)_s-\frac{1}{2}\frac{\lambda_s}{\lambda}J_1\right]\right|$$
$$\lesssim\int\varepsilon e^{-\frac{3}{2}|y|}+|b|\left(\int\varepsilon e^{-\frac{3}{2}|y|}\right)+\int e^{-\frac{9}{10}|y|}+|b|^5.\tag{2.36}$$

(ii) Let

$$\eta_2(y)=\frac{\int_{-\infty}^{y}(yQ_{yy}+Q_y)}{(D,\int_{-\infty}^{y}(yQ_{yy}+Q_y))},\quad J_2(s)=\int\varepsilon(s)\cdot\eta_2;\tag{2.37}$$

and for $k = -\frac{1}{2}\left(\chi_b D, \int_{-\infty}^{y}\left(yQ_{yy} + Q_y\right)\right)$, *there is*

$$\left|b_s + kb^2 + \left[(J_2)_s - \frac{1}{2}\frac{\lambda_s}{\lambda}J_2\right]\right| \lesssim \int e^{-\frac{9}{10}|y|} + \int \varepsilon e^{-\frac{3}{2}|y|} + \int \varepsilon^2 e^{-\frac{1}{2}|y|} + |b|\,. \tag{2.38}$$

(iii) Let $\eta = (2\eta_1 + \eta_2) \in \mathcal{B}$, $J(s) = (\varepsilon(s), \eta)$, α *is positive constant, then*

$$\left|\frac{d}{ds}\left(\frac{b}{\lambda^\alpha}\right) + \frac{b}{\lambda^\alpha}\left[J_s - \frac{1}{2}\frac{\lambda_s}{\lambda}J\right]\right| \lesssim \frac{1}{\lambda^\alpha}\left(\int \varepsilon^2 e^{-\frac{1}{2}|y|} + |b|\right). \tag{2.39}$$

Proof According to (2.34), due to the L^1 bound, from (2.14),

$$\begin{aligned}\frac{d}{ds}\left(\varepsilon, \int_{-\infty}^{y} f\right) =& \frac{1}{2}\left(\frac{\lambda_s}{\lambda} + b\right)\left(Q_b, \int_{-\infty}^{y} f\right) + \frac{1}{2}\frac{\lambda_s}{\lambda}\left(\varepsilon, \int_{-\infty}^{y} f\right) \\ & - x_s\left(Q_b + \varepsilon, f\right) + \frac{1}{2}\left((Q_b + \varepsilon)^2, f\right) \\ & + \left(\Phi_b, \int_{-\infty}^{y} f\right) - \frac{1}{2}b\left(Q_b, \int_{-\infty}^{y} f\right) + \left(R, \int_{-\infty}^{y} f\right).\end{aligned} \tag{2.40}$$

Consider (2.32), (2.33) and (2.28), (2.40) is equivalent to

$$\begin{aligned}2\frac{d}{ds}\left(\varepsilon, \int_{-\infty}^{y} f\right) =& \left(\frac{\lambda_s}{\lambda} + b\right)\left(Q, \int_{-\infty}^{y} f\right) + \frac{\lambda_s}{\lambda}\left(\varepsilon, \int_{-\infty}^{y} f\right) \\ & - 2x_s\left(Q, f\right) + \left((Q_b + \varepsilon)^2, f\right) \\ & + 2\left(\Phi_b, \int_{-\infty}^{y} f\right) - b\left(Q_b, \int_{-\infty}^{y} f\right) + 2\left(R, \int_{-\infty}^{y} f\right).\end{aligned} \tag{2.41}$$

In the calculation of $J_1$and J_2, take f meets $(f, \varepsilon) = 0$.

(i) The boundedness of J_1
We apply (2.41) to $f_1 = Q_y$, then

$$\begin{aligned}2\frac{d}{ds}\left(\varepsilon, \int_{-\infty}^{y} Q_y\right) =& \left(\frac{\lambda_s}{\lambda} + b\right)\left(Q, \int_{-\infty}^{y} Q_y\right) + \frac{\lambda_s}{\lambda}\left(\varepsilon, \int_{-\infty}^{y} Q_y\right) \\ & + \left((Q_b + \varepsilon)^2, Q_y\right) + 2\left(\Phi_b, \int_{-\infty}^{y} Q_y\right) \\ & - b\left(Q_b, \int_{-\infty}^{y} Q_y\right) + 2\left(R, \int_{-\infty}^{y} Q_y\right).\end{aligned} \tag{2.42}$$

Where

$$\begin{aligned}&\int (Q_b+\varepsilon)^2 Q_y\\&=\int Q^2Q_y+2b\int \chi_b DQ_y+b^2\int \chi_b^2D^2Q_y\\&\quad+2\left(\int Q\varepsilon Q_y+b\int \chi_b D\varepsilon Q_y\right)+\int \varepsilon^2 Q_y\\&=\mathrm{O}\left(|b|\right)+\mathrm{O}\left(\int \varepsilon e^{-\frac{3}{2}|y|}\right)+\mathrm{O}\left(|b|\int \varepsilon e^{-\frac{3}{2}|y|}\right)+\mathrm{O}\left(\int \varepsilon^2 e^{-\frac{1}{2}|y|}\right).\end{aligned}\tag{2.43}$$

From (2.33), we can get

$$\begin{aligned}\left(\Phi_b,\int_{-\infty}^{y}Q_y\right)&=-\left(b_s(\chi_b+\gamma y{\chi'}_b)D,\int_{-\infty}^{y}Q_y\right)\\&=-\,b_s(D,Q_y)+\mathrm{O}\left(|b|^5\right)\\&=-\,\mathrm{O}\left(|b|\right)-\mathrm{O}\left(\int \varepsilon e^{-\frac{1}{2}|y|}\right)+\mathrm{O}\left(|b|^5\right).\end{aligned}$$

As the definition of Q_b,

$$b\left(Q_b,\int_{-\infty}^{y}Q_y\right)=b\left(Q,\int_{-\infty}^{y}Q_y\right)+b^2\left(\chi_b D,\int_{-\infty}^{y}Q_y\right)=\mathrm{O}\left(|b|\right)\tag{2.44}$$

According to (2.15), we have

$$\left(R,\int_{-\infty}^{y}Q_y\right)=\mathrm{O}\left(\int e^{-\frac{9}{10}|y|}\right)+\mathrm{O}\left(\int \varepsilon e^{-\frac{3}{2}|y|}\right)+\rho\left(\|\varepsilon\|_{H^2}^2\right).\tag{2.45}$$

Substituting the (2.43)$\sim$(2.45) into (2.42), then

$$\begin{aligned}2\frac{\left(\varepsilon_s,\int_{-\infty}^{y}Q_y\right)}{\left(Q,\int_{-\infty}^{y}Q_y\right)}&=\left(\frac{\lambda_s}{\lambda}+b\right)+\frac{\lambda_s}{\lambda}\frac{\left(\varepsilon,\int_{-\infty}^{y}Q_y\right)}{\left(Q,\int_{-\infty}^{y}Q_y\right)}+\mathrm{O}\left(\int \varepsilon e^{-\frac{3}{2}|y|}\right)\\&\quad+\mathrm{O}\left[|b|\left(\int \varepsilon e^{-\frac{3}{2}|y|}\right)\right]-\mathrm{O}\left(\int \varepsilon e^{-\frac{1}{2}|y|}\right)+\mathrm{O}\left(|b|^5\right)\\&\quad-\mathrm{O}\left(|b|^2\right)+\mathrm{O}\left(\int e^{-\frac{9}{10}|y|}\right)+\rho\left(\|\varepsilon\|_{H^2}^2\right).\end{aligned}\tag{2.46}$$

From (2.35), then $(J_1)_s=\left(\varepsilon_s,\int_{-\infty}^{y}Q_y\right)/\left(Q,\int_{-\infty}^{y}Q_y\right)$. Hence

$$\left|\left(\frac{\lambda_s}{\lambda}+b\right)-b^2-2\left[(J_1)_s-\frac{1}{2}\frac{\lambda_s}{\lambda}J_1\right]\right|$$

$$\lesssim \int \varepsilon e^{-\frac{3}{2}|y|} + |b| \left(\int \varepsilon e^{-\frac{3}{2}|y|} \right) + \int e^{-\frac{9}{10}|y|} + |b|^5.$$

(ii) Boundedness of J_2
Similar to the structure of J_1, we now apply (2.41) to $f_2 = yQ_{yy} + Q_y$, using the algebraic relations,

$$\left(Q, \int_{-\infty}^{y} yQ_{yy} + \int_{-\infty}^{y} Q_y \right) = -\left(Q, \int_{-\infty}^{y} Q_y \right) + \left(Q, \int_{-\infty}^{y} Q_y \right) = 0.$$

we can obtain that

$$\begin{aligned}
&2\frac{d}{ds}\left(\varepsilon, \int_{-\infty}^{y} \left(yQ_{yy} + Q_y\right)\right) \\
&\quad = \frac{\lambda_s}{\lambda}\left(\varepsilon, \int_{-\infty}^{y} \left(yQ_{yy} + Q_y\right)\right) \\
&\qquad + \left((Q_b + \varepsilon)^2, yQ_{yy} + Q_y\right) + 2\left(\Phi_b, \int_{-\infty}^{y} \left(yQ_{yy} + Q_y\right)\right) \\
&\qquad - b\left(Q_b, \int_{-\infty}^{y} \left(yQ_{yy} + Q_y\right)\right) + 2\left(R, \int_{-\infty}^{y} \left(yQ_{yy} + Q_y\right)\right).
\end{aligned} \tag{2.47}$$

and similarly

$$\begin{aligned}
&\int (Q_b + \varepsilon)^2 \cdot \left(yQ_{yy} + Q_y\right) \\
&\quad = 2b \int \chi_b D \left(yQ_{yy} + Q_y\right)(Q + \varepsilon) + b^2 \int \chi_b^2 D^2 \left(yQ_{yy} + Q_y\right) \\
&\qquad + 2\int Q\varepsilon \left(yQ_{yy} + Q_y\right) + \int \varepsilon^2 \left(yQ_{yy} + Q_y\right) \\
&\quad = \mathrm{O}(|b|) + \mathrm{O}\left(\int \varepsilon e^{-\frac{3}{2}|y|}\right) \\
&\qquad + \mathrm{O}\left(|b| \int \varepsilon e^{-\frac{3}{2}|y|}\right) + \mathrm{O}\left(\int \varepsilon^2 e^{-\frac{1}{2}|y|}\right).
\end{aligned}$$

Estimate the scalar product term of Φ_b,

$$\begin{aligned}
&\left(\Phi_b, \int_{-\infty}^{y} yQ_{yy} + \int_{-\infty}^{y} Q_y\right) \\
&\quad = -b_s\left(\left(\chi_b + \gamma y {\chi'}_b\right) D, \int_{-\infty}^{y} yQ_{yy} + \int_{-\infty}^{y} Q_y\right) \\
&\quad = -2b_s\left(D, \int_{-\infty}^{y} yQ_{yy} + \int_{-\infty}^{y} Q_y\right) + \mathrm{O}\left(|b|^5\right).
\end{aligned}$$

Substituting these results into (2.47), here

$$
\begin{aligned}
&\frac{\left(\varepsilon_s, \int_{-\infty}^{y} yQ_{yy}+Q_y\right)}{\left(D, \int_{-\infty}^{y} yQ_{yy}+Q_y\right)} \\
&\quad = \frac{\lambda_s}{\lambda}\frac{\left(\varepsilon, \int_{-\infty}^{y} yQ_{yy}+Q_y\right)}{\left(D, \int_{-\infty}^{y} yQ_{yy}+Q_y\right)} - b_s - \mathrm{O}\left(|b|^2\right) \\
&\qquad + \mathrm{O}\left(|b|\right) + \mathrm{O}\left(\int \varepsilon e^{-\frac{3}{2}|y|}\right) + \mathrm{O}\left[|b|\left(\int \varepsilon e^{-\frac{3}{2}|y|}\right)\right] \\
&\qquad + \mathrm{O}\left(\int \varepsilon^2 e^{-\frac{1}{2}|y|}\right) + \mathrm{O}\left(|b|^5\right) + \mathrm{O}\left(\int e^{-\frac{9}{10}|y|}\right) + \rho\left(\|\varepsilon\|_{H^2}^2\right).
\end{aligned} \tag{2.48}
$$

According to (2.25), we find that $\int \varepsilon y Q_{yy} \not\equiv 0$. From (2.37),

$$
\left|b_s + kb^2 + \left[(J_2)_s - \frac{1}{2}\frac{\lambda_s}{\lambda}J_2\right]\right| \lesssim \int e^{-\frac{9}{10}|y|} + \int \varepsilon e^{-\frac{3}{2}|y|} + \int \varepsilon^2 e^{-\frac{1}{2}|y|} + |b|\,.
$$

Where $k = -\frac{1}{2}\left(\chi_b D, \int_{-\infty}^{y}\left(yQ_{yy}+Q_y\right)\right)$.

(iii) Boundedness of J

It can be computed form (2.36) and (2.38):

$$
\begin{aligned}
&\frac{d}{ds}\left(\frac{b}{\lambda^\alpha}\right) = \frac{b_s}{\lambda^\alpha} - \alpha\frac{\lambda_s}{\lambda}\frac{b}{\lambda^\alpha} = \frac{b_s+\alpha b^2}{\lambda^\alpha} - \frac{\alpha b}{\lambda^\alpha}\left(\frac{\lambda_s}{\lambda}+b\right) \\
&\quad = \frac{1}{\lambda^\alpha}\left[\mathrm{O}\left(\int e^{-\frac{9}{10}|y|}\right) + \mathrm{O}\left(\int \varepsilon e^{-\frac{3}{2}|y|}\right) + \mathrm{O}\left(\int \varepsilon^2 e^{-\frac{1}{2}|y|}\right)\right. \\
&\qquad \left. + \mathrm{O}\left(|b|\right) - (J_2)_s + \frac{1}{2}\frac{\lambda_s}{\lambda}J_2\right] \\
&\qquad - \frac{\alpha}{\lambda^\alpha}\left[\mathrm{O}\left(b\int \varepsilon e^{-\frac{3}{2}|y|}\right) + \mathrm{O}\left(b^2\int \varepsilon e^{-\frac{3}{2}|y|}\right) + \mathrm{O}\left(b\int e^{-\frac{9}{10}|y|}\right)\right. \\
&\qquad \left. + \mathrm{O}\left(b^6\right) + \mathrm{O}\left(b^2\right) + 2b(J_1)_s - b\frac{\lambda_s}{\lambda}J_1\right]
\end{aligned} \tag{2.49}
$$

Where α positive constant, for $|b| < 1$, then (2.49) becomes

$$
\begin{aligned}
\frac{d}{ds}\left(\frac{b}{\lambda^\alpha}\right) &= -\frac{1}{\lambda^\alpha}\left[(J_2)_s - \frac{1}{2}\frac{\lambda_s}{\lambda}J_2\right] \\
&\quad - \frac{b}{\lambda^\alpha}\left[2(J_1)_s - \frac{\lambda_s}{\lambda}J_1\right] + \frac{1}{\lambda^\alpha}\mathrm{O}\left(\int \varepsilon^2 e^{-\frac{1}{2}|y|} + |b|\right)
\end{aligned} \tag{2.50}
$$

According to the definition of η, there is $J = J_1 + J_2$, then

$$
\left|\frac{d}{ds}\left(\frac{b}{\lambda^\alpha}\right) + \frac{b}{\lambda^\alpha}\left[J_s - \frac{1}{2}\frac{\lambda_s}{\lambda}J\right]\right| \lesssim \frac{1}{\lambda^\alpha}\left(\int \varepsilon^2 e^{-\frac{1}{2}|y|} + |b|\right).
$$

Which proves (2.39) .

□

3 The stability of ε

Consider the following Cauchy problem:

$$\begin{cases} \varepsilon_s + \frac{1}{2}b\varepsilon = \frac{1}{2}\left(\frac{\lambda_s}{\lambda} + b\right)(Q_b + \varepsilon) + x_s(Q_b + \varepsilon)_y \\ \qquad - Q_b(Q_b)_y - \varepsilon\varepsilon_y - (Q_b\varepsilon)_y + \Phi_b \\ \qquad - \frac{1}{2}bQ_b + R(\varepsilon), \ (s, y) \in \mathbb{R}^+ \times \mathbb{R}, \\ \varepsilon(0, y) = \varepsilon_0(y), \ y \in \mathbb{R}. \end{cases} \tag{3.1}$$

Lemma 3.1 *The energy relationship of ε holds as following*

$$\begin{aligned} \|\varepsilon\|_{H^2}^2 =& \lambda\|Q\|_{H^2}^2 - \|Q_b\|_{H^2}^2 + \lambda\|\varepsilon_0\|_{H^2}^2 \\ &+ 4\sqrt{3}\left[\lambda\varepsilon_0(0) - \varepsilon(s, 0)\right] \\ &- 2b\left[\int \chi_b D\varepsilon + \int (\chi_b D)_y\varepsilon_y + \int (\chi_b D)_{yy}\varepsilon_{yy}\right]. \end{aligned} \tag{3.2}$$

Proof From proposition 2.3,

$$E(\varepsilon + Q_b) = \lambda E_0 = \lambda E(\varepsilon_0 + Q). \tag{3.3}$$

According to the definition of energy ,

$$\begin{aligned} E(Q_b + \varepsilon) &= \|Q_b\|_{H^2}^2 + \|\varepsilon\|_{H^2}^2 + 4\sqrt{3}\varepsilon(s, 0) \\ &\quad + 2b\left[\int \chi_b D\varepsilon + \int (\chi_b D)_y\varepsilon_y + \int (\chi_b D)_{yy}\varepsilon_{yy}\right]. \\ E(Q + \varepsilon_0) &= \|Q\|_{H^2}^2 + \|\varepsilon_0\|_{H^2}^2 + 4\sqrt{3}\varepsilon_0(0). \end{aligned} \tag{3.4}$$

Substitute (3.4) into (3.3), we have

$$\begin{aligned} \|\varepsilon\|_{H^2}^2 =& \lambda\|Q\|_{H^2}^2 - \|Q_b\|_{H^2}^2 + \lambda\|\varepsilon_0\|_{H^2}^2 \\ &+ 4\sqrt{3}\left[\lambda\varepsilon_0(0) - \varepsilon(s, 0)\right] \\ &- 2b\left[\int \chi_b D\varepsilon + \int (\chi_b D)_y\varepsilon_y + \int (\chi_b D)_{yy}\varepsilon_{yy}\right]. \end{aligned}$$

□

Lemma 3.2 *Suppose $u(t) \in T_{\alpha^*}$ is the solution of the problem (1.8) responding to u_0, and $\varepsilon \in H^2(\mathbb{R})$ is the solution of (2.14) responding to $\varepsilon_0 = u_0 - Q$, then*

$$\|\varepsilon(s)\|_{H^2} \lesssim \|\varepsilon_0(s)\|_{H^2}, \ s \geq 0. \tag{3.5}$$

Proof From proposition 2.3,

$$\|\varepsilon(s)\|^2_{H^2} = \lambda \|Q\|^2_{H^2} - \|Q_b\|^2_{H^2} + \lambda \|\varepsilon_0\|^2_{H^2} + 4\sqrt{3}\left[\lambda\varepsilon_0(0) - \varepsilon(s,0)\right] - 2b \cdot I \tag{3.6}$$

Where $I = \int \chi_b D\varepsilon + \int (\chi_b D)_y \varepsilon_y + \int (\chi_b D)_{yy}\varepsilon_{yy}$.

Obviously $\|Q\|_{H^2} \leq \|Q_b\|_{H^2}$. From propsition 2.4, then $|\lambda - 1| \lesssim \rho(\alpha)$, $\|\varepsilon(s)\|_{H^2} \lesssim \rho(\alpha)$, so $|\lambda - 1| \lesssim \|\varepsilon(s)\|_{H^2}$, then (3.6) is equivalent to

$$\begin{aligned} \|\varepsilon(s)\|^2_{H^2} \leq & 2\sqrt{3}C_Q\|\varepsilon(s)\|_{H^2} + [\rho(\alpha) + 1]\|\varepsilon_0\|^2_{H^2} \\ & + 4\sqrt{3}[\rho(\alpha) + 1]\|\varepsilon_0\|_{H^2} + 2|b| \cdot I \end{aligned} \tag{3.7}$$

Using $H\ddot{o}lder$ inequality,

$$\begin{aligned} I &\leq \left(\int D^2\right)^{\frac{1}{2}} \cdot \left(\int \varepsilon^2\right)^{\frac{1}{2}} + \left(\int D_y^2\right)^{\frac{1}{2}} \cdot \left(\int \varepsilon_y^2\right)^{\frac{1}{2}} + \left(\int D_{yy}^2\right)^{\frac{1}{2}} \cdot \left(\int \varepsilon_{yy}^2\right)^{\frac{1}{2}} \\ &\leq \|\varepsilon\|_{L^2} + \left\|\varepsilon_y\right\|_{L^2} + \left\|\varepsilon_{yy}\right\|_{L^2} \lesssim \|\varepsilon\|_{H^2} \end{aligned}$$

Then (3.7) can be rewritten as

$$\|\varepsilon(s)\|^2_{H^2} \leq C_\varepsilon\|\varepsilon(s)\|_{H^2} + C_{\varepsilon_0}\|\varepsilon_0\|_{H^2}. \tag{3.8}$$

Where $C_\varepsilon = 2\sqrt{3}C_Q + 2|b|$, $C_{\varepsilon_0} = [\rho(\alpha) + 1]\left(\|\varepsilon_0\|_{H^2} + 4\sqrt{3}\right)$.

Argue by contradiction. Suppose $\exists s_0 > 0$, s.t. $\|\varepsilon(s_0)\|_{H^2} \neq 0$, and for $\forall C \in (0, +\infty)$, $\|\varepsilon(s_0)\|_{H^2} > C\|\varepsilon_0\|^{\frac{1}{2}}_{H^2}$. From (3.8),

$$\|\varepsilon(s)\|_{H^2} \leq \frac{1}{2}\sqrt{4C_{\varepsilon_0}\|\varepsilon_0\|_{H^2} + C_\varepsilon^2} + \frac{C_\varepsilon}{2} \triangleq \widetilde{C_\varepsilon} \tag{3.9}$$

Let

$$C = \left(\widetilde{C_\varepsilon}/\|\varepsilon_0\|^{\frac{1}{2}}_{H^2}\right) + \widetilde{C_{\varepsilon_0}}$$

Where $\widetilde{C_{\varepsilon_0}} \in \mathbb{R}$. According to the assumptions and (3.9), we infer that

$$\widetilde{C_\varepsilon} + \widetilde{C_{\varepsilon_0}} \cdot \|\varepsilon_0\|^{\frac{1}{2}}_{H^2} < \|\varepsilon(s_0)\|_{H^2} < \widetilde{C_\varepsilon}$$

This leads to a contradiction that $\widetilde{C_{\varepsilon_0}} \cdot \|\varepsilon_0\|_{H^2} > 0$. Since the supposed $\|\varepsilon(s_0)\|_{H^2} > C\|\varepsilon_0\|_{H^2}$ is wrong. So (3.5) is established. □

Corollary 3.1 *If $u_0 \in T_{\alpha_0}$, then $\exists \alpha^* = \alpha^*(\alpha_0)$, s.t. $u \in T_{\alpha^*}$.*

Theorem 3.1 *Let $\rho > 0$, $\|\widetilde{\varepsilon_0} - \varepsilon_0\|_{H^2} < \rho$, ε and $\widetilde{\varepsilon}$ are the solutions of* (2.14) responding to $\varepsilon_0(y)$ and $\widetilde{\varepsilon_0}(y)$ respectively. Then $\|\widetilde{\varepsilon} - \varepsilon\|_{H^2} \lesssim \rho$.

Proof Let $\varepsilon = \varepsilon$, $\varepsilon = \widetilde{\varepsilon}$ in (3.1), here

$$\varepsilon^*_s + \frac{1}{2}b\varepsilon^* = \frac{1}{2}\left(\frac{\lambda_s}{\lambda} + b\right)\varepsilon^* + x_s\varepsilon^*_y - \varepsilon^*\widetilde{\varepsilon} - \varepsilon^*\varepsilon_y - (Q_b\varepsilon^*)_y + R^*(\varepsilon^*). \tag{3.10}$$

Where $\varepsilon^* = \widetilde{\varepsilon} - \varepsilon$, $R^*(\varepsilon^*) = \sum_{k=1}^4 R^*_k$, and

$$\begin{aligned} R^*_1 &= \frac{3}{2}\int P_{zzz}(y-z)\left[\left(\varepsilon^*_z\right)^2 - \varepsilon_z^2\right],\ R^*_2 = 3\int P_{zzz}(y-z)(Q_b)'_z\varepsilon^*_z, \\ R^*_3 &= -\int P_z(y-z)\left[2Q_b\varepsilon^* + (Q_b)'_z\varepsilon^*_z + (Q_b)''_{zz}\varepsilon^*_{zz}\right], \\ R^*_4 &= -\int P_z(y-z)\left\{\left[\left(\varepsilon^*\right)^2 - \varepsilon^2\right] + \frac{1}{2}\left[\left(\varepsilon^*_z\right)^2 - \varepsilon_z^2\right] - \frac{1}{2}\left[\left(\varepsilon^*_{zz}\right)^2 - \varepsilon_{zz}^2\right]\right\}. \end{aligned}$$

Multiply the equality (3.10) by $\partial_y^{2k}\varepsilon^*$ $(k = 0, 1, 2)$, and perform integral over R. Here only give the calculation detail of $k = 2$, the discussion on the other situation is analogous.

$$x_s\int \varepsilon^*_y\varepsilon^*_{yyyy} = -x_s\int \varepsilon^*_{yyy}\varepsilon^*_{yy} = \frac{1}{2}x_s\left(\varepsilon^*_{yy}\right)^2\Big|_{\mathbb{R}} = 0$$

By integraling

$$\int \varepsilon^*\widetilde{\varepsilon}\varepsilon^*_{yyyy} = \int \left(\varepsilon^*_{yy}\right)^2\widetilde{\varepsilon} + 2\int \varepsilon^*_{yy}\varepsilon^*_y\widetilde{\varepsilon}_y + \int \varepsilon^*_{yy}\varepsilon^*\widetilde{\varepsilon}_{yyy}$$

Homoplastically, we can calculate $\int \varepsilon^*\varepsilon_y\varepsilon^*_{yyyy}$ and $\int (Q_b\varepsilon^*)_y\varepsilon^*_{yyyy}$. Hence

$$\begin{aligned} \frac{1}{2}\frac{d}{ds}\int\left(\varepsilon^*_{yy}\right)^2 = \frac{1}{2}\left(\frac{\lambda_s}{\lambda} + b\right)\int\left(\varepsilon^*_{yy}\right)^2 + \int R^*(\varepsilon^*)\varepsilon^*_{yyyy} \\ -\frac{1}{2}b\int\left(\varepsilon^*_{yy}\right)^2 - \sum_{k=1}^3 I_k - I_Q. \end{aligned}$$

Where

$$I_1 = \int\left(\varepsilon^*_{yy}\right)^2\cdot\left(\widetilde{\varepsilon} + \varepsilon_y\right),\ I_2 = 2\int \varepsilon^*_{yy}\varepsilon^*_y\cdot\left(\widetilde{\varepsilon}_y + \varepsilon_{yy}\right),$$

$$
\begin{aligned}
I_3 &= \int \varepsilon_{yy}^* \varepsilon^* \cdot \left(\widetilde{\varepsilon}_{yy} + \varepsilon_{yyy}\right), \\
I_Q &= \int \varepsilon_{yy}^* (Q_b)_{yyy} \varepsilon^* + 3 \int \varepsilon_{yy}^* (Q_b)_{yy} \varepsilon_y^* + 3 \int \left(\varepsilon_{yy}^*\right)^2 (Q_b)_y + \int \varepsilon_{yy}^* Q_b \varepsilon_{yyy}^*.
\end{aligned}
$$

According to $\varepsilon^* = \widetilde{\varepsilon} - \varepsilon$,

$$
\begin{aligned}
I_1 &\le \int \varepsilon_{yy}^* \left(\left|\varepsilon_{yy}^* \widetilde{\varepsilon}\right| + \left|\varepsilon_{yy}^* \varepsilon_y\right|\right). \\
&\le \left\|\varepsilon_{yy}^*\right\|_{L^\infty} \cdot \left(\int \left|\widetilde{\varepsilon}_{yy} \widetilde{\varepsilon}\right| - \int \left|\varepsilon_{yy} \widetilde{\varepsilon}\right| + \int \left|\widetilde{\varepsilon}_{yy} \varepsilon_y\right| - \int \left|\varepsilon_{yy} \varepsilon_y\right|\right) \\
&\lesssim \left\|\varepsilon_{yy}^*\right\|_{L^\infty} \cdot \left(\|\widetilde{\varepsilon}\|_{H^2}^2 + \|\varepsilon\|_{H^2}^2\right) \triangleq \widetilde{I}
\end{aligned}
$$

We can also get $I_2 \lesssim \widetilde{I}$. For I_3,

$$
I_3 \le \left\|\varepsilon_{yy}^*\right\|_{L^\infty} \cdot \left(\int \left|\varepsilon^* \widetilde{\varepsilon}_{yy}\right| + \int \left|\varepsilon_y^* \varepsilon_{yy}\right|\right) \lesssim \widetilde{I}
$$

Similarly,

$$
I_Q \lesssim M_{L^\infty} \left(\|Q_b\|_{H^3}^2 + \|\varepsilon\|_{H^2}^2 + \|\widetilde{\varepsilon}\|_{H^2}^2\right).
$$

Where $M_{L^\infty} = \max\left\{\left\|\varepsilon_{yy}^*\right\|_{L^\infty}, \left\|\varepsilon_{yy}^*\right\|_{L^\infty}^2\right\}$.

Finally to estimate $\int R^* \varepsilon_{yyyy}^*$. According to the feature of $R^*\left(\varepsilon^*\right)$, we only give the process of R_1^* and R_2^*.

$$
\begin{aligned}
\int R_1^* \varepsilon_{yyyy}^* &= \int \left(R_1^*\right)_{yy} \varepsilon_{yy}^* \\
&= \int \left[\int \frac{\sqrt{3}}{3} e^{-\frac{\sqrt{3}}{2}|y-z|} \sin\left(\frac{|y-z|}{2} + \frac{2}{3}\pi\right) \left[\left(\varepsilon_z^*\right)^2 - \varepsilon_z^2\right] dz\right]_{yy} \varepsilon_{yy}^* dy \\
&\le \frac{\sqrt{3}}{4} \int \left(\left\|\varepsilon_z^*\right\|_{L^2}^2 + \|\varepsilon_z\|_{L^2}^2\right) \varepsilon_{yy}^* dy \\
&\le \left\|\varepsilon^*\right\|_{H^2} \cdot \|\widetilde{\varepsilon}\|_{H^2}^2.
\end{aligned}
$$

Analogously,

$$
\begin{aligned}
\int R_2^* \varepsilon_{yyyy}^* &= \int \left[\int \frac{\sqrt{3}}{3} e^{-\frac{\sqrt{3}}{2}|y-z|} \sin\left(\frac{|y-z|}{2} + \frac{2}{3}\pi\right) (Q_b)'_z \varepsilon_z^* dz\right]_{yy} \varepsilon_{yy}^* dy \\
&\le \left\|\varepsilon^*\right\|_{H^2} \left(\|Q_b\|_{H^2}^2 + \|\widetilde{\varepsilon}\|_{H^2}^2 + \|\varepsilon\|_{H^2}^2\right).
\end{aligned}
$$

Combining the control relationship of I_k, I_Q, R^*, $\int\left(\varepsilon^*_{yy}\right)^2$ and propsition 2.4,

$$\frac{d}{ds}\left\|\varepsilon^*_{yy}\right\|^2_{L^2} \lesssim \left(\|\varepsilon\|^2_{H^2} + \|\widetilde{\varepsilon}\|^2_{H^2} + \|Q_b\|^2_{H^3}\right) \cdot \left\|\varepsilon^*_{yy}\right\|^2_{L^2} \tag{3.11}$$

When $k = 0, 1$, the $\left\|\varepsilon^*_{yy}\right\|^2_{L^2}$ in (3.11) become $\|\varepsilon^*\|^2_{L^2}$ and $\left\|\varepsilon^*_{y}\right\|^2_{L^2}$. As for $H^2 \hookrightarrow L^2$, then

$$\frac{d}{ds}\left\|\varepsilon^*\right\|^2_{H^2} \lesssim \left(\|\varepsilon\|^2_{H^2} + \|\widetilde{\varepsilon}\|^2_{H^2} + \|Q_b\|^2_{H^3}\right) \cdot \left\|\varepsilon^*\right\|^2_{H^2}$$

Because $\widetilde{\varepsilon}_0 \in H^2(\mathbb{R})$, from [21] Th 4, $\widetilde{\varepsilon} \in L^\infty\left([0, T]; H^2(\mathbb{R})\right)$ as $T > 0$ established. Since using lemma 3.2 and *Gronwall* inquality, integral over $[0, T]$

$$\left\|\varepsilon^*\right\|^2_{H^2} \lesssim \left[\exp\left(\|\varepsilon_0\|^2_{H^2} + \|\widetilde{\varepsilon}_0\|^2_{H^2} + \|Q_b\|^2_{H^3}\right)\right] \cdot \left\|\varepsilon^*_0\right\|^2_{H^2}, \quad s > 0 \tag{3.12}$$

Where $\|\widetilde{\varepsilon}_0\| < \rho$, from (3.12), when $T < \infty$, $s \in [0, T]$,

$$\|\widetilde{\varepsilon} - \varepsilon\|^2_{H^2} \lesssim \|\widetilde{\varepsilon}_0 - \varepsilon_0\|^2_{H^2} < \rho$$

this theorem is proved. □

Let $b = b + \Delta b$ in (3.1). Where $\varepsilon_{\Delta b} = \varepsilon_{b+\Delta b} - \varepsilon_b$, $Q_{b+\Delta b} = Q_b + \Delta b \chi_b D$,

$$\begin{aligned} R(b + \Delta b) = &\frac{3}{2}\int P_{zzz}(y - z)\left(Q_{b+\Delta b}\right)'_z + 3\int P_{zzz}(y - z)\left(Q_{b+\Delta b}\right)'_z \varepsilon_z \\ &+ \frac{3}{2}\int P_{zzz}(y - z)\,\varepsilon_z^2 - \int P_z(y - z)\left(\varepsilon^2 + \frac{1}{2}\varepsilon_z^2 - \frac{1}{2}\varepsilon_{zz}^2\right) \\ &- \int P_z(y - z)\left[Q^2_{b+\Delta b} + \frac{1}{2}\left(Q_{b+\Delta b}\right)^2_z - \frac{1}{2}\left(Q_{b+\Delta b}\right)^2_{zz}\right] \\ &- \int P_z(y - z)\left[2Q_{b+\Delta b}\varepsilon + \left(Q_{b+\Delta b}\right)'_z \varepsilon_z + \left(Q_{b+\Delta b}\right)''_{zz}\varepsilon_{zz}\right]. \end{aligned}$$

Theorem 3.2 *Suppose* $\varepsilon_{b+\Delta b}, \varepsilon_b \in L^\infty\left([0, T]\, H^2(\mathbb{R})\right)$ *are the solutions of* (2.14) *responding to* $\varepsilon_{\Delta b}(0, y) = \varepsilon_{\Delta b_0}(y)$, $\varepsilon_b(0, y) = \varepsilon_{b_0}(y)$ *respectively.For* $\forall \rho_b > 0$, $\exists \delta > 0$, *s.t. when* $|b| < \delta_b$ *there is* $\|\varepsilon_{b+\Delta b} - \varepsilon_b\|_{H^1} \lesssim \rho_b$.

Proof By direct calculation,

$$
\begin{aligned}
\frac{d}{ds}\left(\varepsilon_{b+\Delta b}-\varepsilon_b\right)=&\frac{1}{2}\left(\frac{\lambda_s}{\lambda}+b\right)\Delta b\chi_b D+x_s\Delta b(\chi_b D)_y\\
&+\Delta b\left\{\frac{1}{2}\left(Q_b+\varepsilon\right)-\left[(\chi_b D)\cdot(Q_b+\varepsilon)\right]_y\right\}\\
&+(\Delta b)^2\left[\frac{1}{2}\chi_b D-\chi_b D\cdot(\chi_b D)_y\right]\\
&-\frac{1}{2}\Delta b\cdot b\chi_b D+(\Delta b)_s\cdot\chi_b D+R\left(\Delta b\right).
\end{aligned}
\tag{3.13}
$$

Where

$$
\begin{aligned}
R\left(\Delta b\right)=&\frac{3}{2}\int P_{zzz}\left(y-z\right)\left(\Delta b\chi_b D\right)'_z+3\int P_{zzz}\left(y-z\right)\left(\Delta b\chi_b D\right)'_z\varepsilon_z\\
&-\int P_z\left(y-z\right)\left[\left(\Delta b\chi_b D\right)^2+\frac{1}{2}\left(\Delta b\chi_b D\right)_z^2-\frac{1}{2}\left(\Delta b\chi_b D\right)_{zz}^2\right]\\
&-\int P_z\left(y-z\right)\left[2\left(\Delta b\chi_b D\right)\varepsilon+\left(\Delta b\chi_b D\right)'_z\varepsilon_z+\left(\Delta b\chi_b D\right)''_{zz}\varepsilon_{zz}\right].
\end{aligned}
$$

Next to estimate each parts in (3.13).

Let $K_p=\frac{1}{2}\left(\frac{\lambda_s}{\lambda}+b\right)\Delta b\chi_b D+x_s\Delta b(\chi_b D)_y$, then

$$
\left\|K_p\right\|_{H^1}\le\frac{1}{2}\left(\frac{\lambda_s}{\lambda}+b\right)\Delta b\left\|\chi_b D\right\|_{H^1}+x_s\Delta b\left\|(\chi_b D)_y\right\|_{H^1}
\tag{3.14}
$$

From $D\in\mathcal{B}$, then $\|\chi_b D\|_{H^1}\le C_D\sqrt{2}$, $\left\|(\chi_b D)_y\right\|_{H^1}\le C_{D'}\sqrt{2}$. Let $M_{D_H}=\sqrt{2}\max\{C_D,C_{D'}\}$. According to lemma 2.3,

$$
\begin{aligned}
\left\|K_p\right\|_{H^1}&\le\Delta b\cdot M_{D_H}\left[\frac{1}{2}\left(\frac{\lambda_s}{\lambda}+b\right)+x_s\right]\\
&\le\Delta b\cdot M_{D_H}\left[b^2+\left(\int\varepsilon e^{-\frac{1}{2}|y|}\right)^{\frac{1}{2}}\right]\\
&\triangleq\Delta b\cdot B_p.
\end{aligned}
\tag{3.15}
$$

Let $K_1=\Delta b\left\{\frac{1}{2}\left(Q_b+\varepsilon\right)-\left[(\chi_b D)\cdot(Q_b+\varepsilon)\right]_y\right\}\triangleq\Delta b\left(K_{1,1}-K_{1,2}\right)$, using *Minkowski* inquality,

$$
\begin{aligned}
\left\|K_{1,2}\right\|_{H^1}=&\left\|(\chi_b D)_y\cdot(Q_b+\varepsilon)+\chi_b D\cdot(Q_b+\varepsilon)_y\right\|_{H^1}\\
\lesssim&\left\|(\chi_b D)_y\right\|_{H^1}\cdot\left\|Q_b+\varepsilon\right\|_{H^1}+\left\|\chi_b D\right\|_{H^1}\cdot\left\|(Q_b+\varepsilon)_y\right\|_{H^1}.
\end{aligned}
\tag{3.16}
$$

That is

$$\|Q_b+\varepsilon\|_{H^1}^2=\|Q_b\|_{H^1}^2+\|\varepsilon\|_{H^1}^2+2b\left[\int\chi_b D\varepsilon+\int(\chi_b D)_y\varepsilon_y\right],$$

Apparently $\left\|(Q_b+\varepsilon)_y\right\|_{H^1}^2\leq\|Q_b+\varepsilon\|_{H^2}^2=E\left(Q_b+\varepsilon\right)$, according to (3.4)

$$\begin{aligned}\left\|(Q_b+\varepsilon)_y\right\|_{H^1}^2\leq&\|Q_b\|_{H^2}^2+\|\varepsilon\|_{H^2}^2+4\sqrt{3}\varepsilon\left(s,0\right)\\&+2b\left[\int\chi_b D\varepsilon+\int(\chi_b D)_y\varepsilon_y+\int(\chi_b D)_{yy}\varepsilon_{yy}\right]\end{aligned}$$

Substitute these results of inquality into (3.16),

$$\begin{aligned}\left\|K_{1,2}\right\|_{H^1}\leq&\|Q_b\|_{H^2}^2+\|\varepsilon\|_{H^2}^2+\|\chi_b D\|_{H^2}^2+2\sqrt{3}\varepsilon\left(s,0\right)\\&+2b\left[\int\chi_b D\varepsilon+\int(\chi_b D)_y\varepsilon_y\right]+b\left[\int(\chi_b D)_{yy}\varepsilon_{yy}\right]\\\leq&\|Q_b\|_{H^2}^2+\|\varepsilon\|_{H^2}^2+\|\chi_b D\|_{H^2}^2+2\sqrt{3}\varepsilon\left(s,0\right)+2M_{D_L}b\,\|\varepsilon\|_{H^2}^2.\end{aligned}$$

Where $M_{D_L}=\max\left\{\|\chi_b D\|_{L^2},\left\|(\chi_b D)_y\right\|_{L^2},\left\|(\chi_b D)_{yy}\right\|_{L^2}\right\}$. Then

$$\begin{aligned}\|K_1\|_{H^1}=&\Delta b\left\|\frac{1}{2}\left(Q_b+\varepsilon\right)-[(\chi_b D)\cdot(Q_b+\varepsilon)]_y\right\|_{H^1}\\\leq&\frac{1}{2}\Delta b\|Q_b+\varepsilon\|_{H^1}+\Delta b\left\|[(\chi_b D)\cdot(Q_b+\varepsilon)]_y\right\|_{H^1}\\\lesssim&\Delta b\left[M_{Q_b}+M_\varepsilon+\|\chi_b D\|_{H^2}^2+2\sqrt{3}\varepsilon\left(s,0\right)\right].\\\triangleq&\Delta b\cdot B_1\end{aligned}\tag{3.17}$$

Where $M_{Q_b}=\max\left\{\|Q_b\|_{H^1},\|Q_b\|_{H^1}^2\right\}$, $M_\varepsilon=\max\left\{\|\varepsilon\|_{H^1},\|\varepsilon\|_{H^1}^2\right\}$.

Let $K_2=(\Delta b)^2\left[\frac{1}{2}\chi_b D-\chi_b D(\chi_b D)_y\right]$, then

$$\begin{aligned}\|K_2\|_{H^1}\leq&(\Delta b)^2\left[\frac{1}{2}\|\chi_b D\|_{H^1}+\left\|\chi_b D(\chi_b D)_y\right\|_{H^1}\right]\\\leq&(\Delta b)^2\left(M_{D_H}+M_{D_H}^2\right)\triangleq(\Delta b)^2\cdot B_2\end{aligned}\tag{3.18}$$

From $|b|<|b^*|\ll 1$, then

$$\left\|\widehat{K}\right\|_{H^1}\triangleq\left\|-\frac{1}{2}\left(\Delta b\right)b\chi_b D\right\|_{H^1}\leq\Delta b\cdot M_{D_H}\tag{3.19}$$

Let $K_s = (\Delta b)_s \chi_b D$.From property 2.7, we can get the equation of $(b + \Delta b)$

$$(b+\Delta b)_s + \widehat{c_b}\,(b+\Delta b) + \widehat{c_{b^2}}(b+\Delta b)^2 - (b+\Delta b)\left\{\frac{1}{2}\left[\frac{\lambda_s}{\lambda} + (b+\Delta b)\right]\right.$$
$$\left. + \frac{\int DQ_{yy}\varepsilon}{\int DQ_y}\right\} = \mathrm{O}\left(\int e^{-\frac{7}{5}|x|}\right) + \mathrm{O}\left(\int \varepsilon e^{-|y|}\right) + \mathrm{O}\left(|b|^5\right).$$

Where

$$c_b = \int \chi_b D Q_{yy} Q,\ \ c_{b^2} = \int \chi_b D\left(\chi_b D Q_{yy} + \frac{1}{2} Q_y\right),$$
$$\widehat{c_{b^k}} = -\frac{c_{b^k}}{\int DQ_y}.\quad (k = 1, 2)$$

The fact that

$$\left|(b+\Delta b)_s\right| \lesssim |b+\Delta b| + \int \varepsilon e^{-\frac{1}{2}|y|}.$$

So

$$\begin{aligned}\|K_s\|_{H^1} &\leq \left[(b+\Delta b)_s - (b)_s\right]\cdot\|\chi_b D\|_{H^1}\\ &\lesssim [|b+\Delta b| - |b|]\cdot M_{D_H}\\ &\leq \Delta b\cdot M_{D_H}.\end{aligned} \tag{3.20}$$

Combine (3.15),(3.17),(3.18),(3.19) and (3.20), using Minkowski inquality

$$\begin{aligned}\left\|K_p + K_1 + K_2 + \widehat{K} + K_s\right\|_{H^1} &\leq \left\|K_p\right\|_{H^1} + \|K_1\|_{H^1}\\ &\quad + \|K_2\|_{H^1} + \left\|\widehat{K}\right\|_{H^1} + \|K_s\|_{H^1}\\ &\lesssim \Delta b\left(B_p + B_1 + 2M_{D_H}\right) + (\Delta b)^2 B_2\\ &\triangleq \Delta b\widetilde{B}_1 + (\Delta b)^2 B_2.\end{aligned} \tag{3.21}$$

Finally to estimate $\|R(\Delta b)\|_{H^1}$. Let $R(\Delta b) = R_{b_1} + R_{b_2} + R_{b_3} + R_{b_4}$, where

$$R_{b_1} = \frac{3}{2}\int P_{zzz}(y-z)\,(\Delta b\chi_b D)'_z,\quad R_{b_2} = 3\int P_{zzz}(y-z)\,(\Delta b\chi_b D)'_z\varepsilon_z,$$
$$R_{b_3} = -\int P_z(y-z)\left[(\Delta b\chi_b D)^2 + \frac{1}{2}(\Delta b\chi_b D)_z^2 - \frac{1}{2}(\Delta b\chi_b D)_{zz}^2\right],$$
$$R_{b_4} = -\int P_z(y-z)\left[2(\Delta b\chi_b D)\,\varepsilon + (\Delta b\chi_b D)'_z\varepsilon_z + (\Delta b\chi_b D)''_{zz}\varepsilon_{zz}\right].$$

From (2.19),

$$
\begin{aligned}
\left\| R_{b_1} \right\|_{H^1} &= \frac{\sqrt{3}}{2}\Delta b \left\| \int e^{-\frac{\sqrt{3}}{2}|y-z|} \sin\left(\frac{|y-z|}{2} + \frac{2}{3}\pi\right) \cdot (\chi_b D)'_z \right\|_{H^1} \\
&\leq \Delta b \left\{ \int \left[\int e^{-\frac{|y-z|}{2}} \cdot (\chi_b D)'_z dz \right]^2 dy \right. \\
&\quad \left. + \int \left[\int e^{-\frac{|y-z|}{2}} \cdot (\chi_b D)'_z dz \right]_y^2 dy \right\}^{\frac{1}{2}} \\
&\triangleq \Delta b \left(\int R_{b_{1,1}} dy + \int R_{b_{1,2}} dy \right)^{\frac{1}{2}}.
\end{aligned}
\tag{3.22}
$$

For $R_{b_{1,1}}$,

$$
R_{b_{1,1}} \leq \left(\int e^{-|y-z|} dz \right) \cdot \left[\int (\chi_b D)_z^2 dz \right] = 2 \left\| (\chi_b D)_z \right\|_{L^2}^2 .
$$

By calculating

$$
R_{b_{1,2}} = \left[\frac{1}{4} (\chi_b D)(z) \right]^2 \cdot \left(\int e^{-\frac{|y-z|}{2}} dz \right)^2 = \left(\chi_b^2 D^2 \right)(z) .
$$

Bring into (3.22)

$$
\begin{aligned}
\left\| R_{b_1} \right\|_{H^1} &\leq 2\Delta b \left[\int \left\| (\chi_b D)_z \right\|_{L^2}^2 dy + \int \left(\chi_b^2 D^2 \right)(z) dy \right]^{\frac{1}{2}} \\
&\leq 2\Delta b \left(M_{D_L}^2 + M_{D_L}^2 \right)^{\frac{1}{2}} \lesssim \Delta b \cdot M_{D_L} .
\end{aligned}
\tag{3.23}
$$

Similarly,

$$
\begin{aligned}
\left\| R_{b_2} \right\|_{H^1} &\leq \sqrt{3}\Delta b \left\| \int e^{-\frac{|y-z|}{2}} \cdot (\chi_b D)'_z \varepsilon_z dz \right\|_{H^1} \\
&\triangleq \sqrt{3}\Delta b \left(\int R_{b_{2,1}} dy + \int R_{b_{2,2}} dy \right)^{\frac{1}{2}}.
\end{aligned}
\tag{3.24}
$$

Where

$$
R_{b_{2,1}} = \left[\int e^{-\frac{|y-z|}{2}} \cdot (\chi_b D)'_z \varepsilon_z dz \right]^2 \lesssim \left[\int (\chi_b D)_z^2 + \int \varepsilon_z^2 \right]^2 \leq \left(M_{D_L}^2 + M_\varepsilon^2 \right)^2.
$$

And

$$R_{b_{2,2}} = \left[\int e^{-\frac{|y-z|}{2}} \cdot (\chi_b D)'_z \varepsilon_z dz\right]_y^2 = (\chi_b D)_z^2 \varepsilon_z^2.$$

Then

$$\begin{aligned}\left\|R_{b_2}\right\|_{H^1} &\leq \sqrt{3}\Delta b\left[\int \left(M_{D_L}^2 + M_\varepsilon^2\right)^2 dy + \int (\chi_b D)_z^2 \varepsilon_z^2 dy\right]^{\frac{1}{2}} \\ &\lesssim \Delta b\left[\left(M_{D_L}^2 + M_\varepsilon^2\right)^2 + M_\varepsilon\right]^{\frac{1}{2}} \triangleq \Delta b \cdot B_{b_2}.\end{aligned}$$

Similar to R_{b_1}, for R_{b_3}

$$R_{b_3} \leq \frac{\sqrt{3}}{3}(\Delta b)^2\left[\int (\chi_b D)^2 + \int (\chi_b D)_z^2 + \int (\chi_b D)_{zz}^2\right] \leq \sqrt{3}(\Delta b)^2 M_{D_L}^2.$$

That is to say $\left\|R_{b_3}\right\|_{H^1} \lesssim (\Delta b)^2 \cdot M_{D_L}^2$. We can also get $R_{b_4} \lesssim 2\sqrt{3}\Delta b \cdot M_{D_L} M_\varepsilon$, then $\left\|R_{b_4}\right\|_{H^1} \lesssim \Delta b \cdot M_{D_L} M_\varepsilon$.

Combine $\left\|R_{b_k}\right\|_{H^1}$, $(k = 1, \cdots, 4)$,

$$\begin{aligned}\|R(\Delta b)\|_{H^1} &= \left\|R_{b_1} + R_{b_2} + R_{b_3} + R_{b_4}\right\|_{H^1} \\ &\lesssim \Delta b\left(M_{D_L} + B_{b_2} + M_{D_L} M_\varepsilon\right) + (\Delta b)^2 M_{D_L}^2 \\ &\triangleq \Delta b \widetilde{B_{R_1}} + (\Delta b)^2 M_{D_L}^2.\end{aligned} \tag{3.25}$$

To sum up, from (3.21) and (3.25)

$$\begin{aligned}\frac{d}{ds}\|\varepsilon_{\Delta b}\|_{H^1} &= \frac{d}{ds}\|\varepsilon_{b+\Delta b} - \varepsilon_b\|_{H^1} \\ &\leq \left\|K_p\right\|_{H^1} + \|K_1\|_{H^1} + \|K_2\|_{H^1} + \left\|\widehat{K}\right\|_{H^1} + \|K_s\|_{H^1} + \|R(\Delta b)\|_{H^1} \\ &\lesssim \Delta b \cdot \left(\widetilde{B_1} + \widetilde{B_{R_1}}\right) + (\Delta b)^2\left(B_2 + M_{D_L}\right) \triangleq \Delta b \widehat{B_1} + (\Delta b)^2 \widehat{B_2}.\end{aligned}$$

If $\Delta b < (\Delta b)^2$, let $\delta_1 = \sqrt{\rho_b/(\widehat{B_1} + \widehat{B_2})}$, then

$$\left\|(\varepsilon_{\Delta b})_s\right\|_{H^1} \lesssim (\Delta b)^2\left(\widehat{B_1} + \widehat{B_2}\right) < \rho_b.$$

For the same reason, if $\Delta b > (\Delta b)^2$, let $\delta_2 = \rho_b/(\widehat{B_1} + \widehat{B_2})$, then

$$\left\|(\varepsilon_{\Delta b})_s\right\|_{H^1} \lesssim |b|\left(\widehat{B_1} + \widehat{B_2}\right) < \rho_b.$$

Let $\delta_b = \min\{\delta_1, \delta_2\}$. For $\forall \rho_b > 0, \exists \delta > 0$, s.t. when $|b| < \delta_b$, there is $\left\|(\varepsilon_{\Delta b})_s\right\|_{H^1} < \rho_b$, theorem 3.2 is completed. □

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