Large-Scale Contextual Market Equilibrium Computation through Deep Learning

Anonymous Author(s)
Affiliation
Address
`email`

Abstract

Market equilibrium is one of the most fundamental solution concepts in economics and social optimization analysis. Existing works on market equilibrium computation primarily focus on settings with a relatively small number of buyers. Motivated by this, our paper investigates the computation of market equilibrium in scenarios with a large-scale buyer population, where buyers and goods are represented by their contexts. Building on this realistic and generalized contextual market model, we introduce MarketFCNet, a deep learning-based method for approximating market equilibrium. We start by parameterizing the allocation of each good to each buyer using a neural network, which depends solely on the context of the buyer and the good. Next, we propose an efficient method to estimate the loss function of the training algorithm unbiasedly, enabling us to optimize the network parameters through gradient descent. To evaluate the approximated solution, we introduce a metric called Nash Gap, which quantifies the deviation of the given allocation and price pair from the market equilibrium. Experimental results indicate that MarketFCNet delivers competitive performance and significantly lower running times compared to existing methods as the market scale expands, demonstrating the potential of deep learning-based methods to accelerate the approximation of large-scale contextual market equilibrium.

1 Introduction

Market equilibrium is a solution concept in microeconomics theory, which studies how *individuals* amongst groups will exchange their *goods* to get each one better off [51]. The importance of market equilibrium is evidenced by the 1972 Nobel Prize awarded to John R. Hicks and Kenneth J. Arrow "for their pioneering contributions to general economic equilibrium theory and welfare theory" [58]. Market equilibrium has wide application in fair allocation [32], as a few examples, fairly assigning course seats to students [11] or dividing estates, rent, fares, and others [35]. Besides, market equilibrium are also considered for ad auctions with budget constraints where money has real value [15, 16].

Existing works often use traditional optimization method or online learning technique to solve market equilibrium, which can tackle one market with around 400 buyers and goods in experiments [30, 52]. However, in realistic scenarios, there might be millions of buyers in one market (*e.g.* job market, online shopping market). In these scenarios, the description complexity for the market is $O(nm)$ and it needs at least $O(nm)$ cost to do one optimization step for the market, if there are n buyers and m goods in the market, which is unacceptable when n is extremely large and potentially infinite. In this case, and traditional optimization methods do not work anymore.

However, contextual models come to the rescue. The success of contextual auctions[21, 5] demonstrate the power of contextual models, in which each bidder and item are represented as context and the value (or the distribution) of item to bidder is determined by the contexts. In this way, auctions as well as other economic problems can be described in a more memory-efficient way, making it possible to accelerate the computation on these problems. Inspired by the models of contextual auctions, we propose the concept of contextual markets in a similar way. We verify that contextual markets can be useful to model large-scale markets aforementioned, since the real market can be assumed to be within some low dimension space, and the values of goods to buyers are often not hard to speculate given the knowledge of goods and buyers [46, 45]. Besides, contextual models never lose expressive power compared with raw models[7], giving contextual markets capabilities to generalize over traditional markets.

This paper initiates the study of *deep learning* for *contextual* market equilibrium computation with a large number of buyers. The description complexity of contextual markets is $O(n + m)$, if there are n buyers and m items in the market, making them memory-efficient and helpful for follow-up equilibrium computation while holding the market structure. Following the framework of differentiable economics [18, 26, 62], we propose a deep-learning based approach, MarketFCNet, in which one optimization step costs only $O(m)$ rather than $O(nm)$ in traditional methods, greatly accelerating the computation of market equilibrium. MarketFCNet takes the representations of one buyer and one good as input, and outputs the allocation of the good to the buyer. The training on MarketFCNet targets at an unbiased estimator of the objective function of EG-convex program, which can be formed by independent samples of buyers. By this way, we optimize the allocation function on "buyer space" implicitly, rather than optimizing the allocation to each buyer directly. Therefore, MarketFCNet can reduce the algorithm complexity such that it becomes independent of n, *i.e.*, the number of buyers.

The effectiveness of MarketFCNet is demonstrated by our experimental results. As the market scale expands, MarketFCNet delivers competitive performance and significantly lower running times compared to existing methods in different experimental settings, demonstrating the potential of deep learning-based methods to accelerate the approximation of large-scale contextual market equilibrium.

The contributions of this paper consist of three parts,

- We proposes a method, MarketFCNet, to approximate the contextual market equilibrium in which the number of buyers is large.
- We proposes Nash Gap to quantify the deviation of the given allocation and price pair from the market equilibrium.
- We conduct extensive experiments, demonstrating promising performance on the approximation measure and running time compared with existing methods.

2 Related Works

The history of market equilibrium arises from microeconomics theory, where the concept of competitive equilibrium [51, §10] was proposed, and the existence of market equilibrium is guaranteed in a general setting [3, 61]. Eisenberg and Gale [28] first considered the linear market case, and proved that the solution of EG-convex program constitutes a market equilibrium, which lays the polynomial-time algorithmic foundations for market equilibrium computation. Eisenberg [27] later showed that EG program also works for a class of CCNH utility functions. Shmyrev program later is also proposed to solve market equilibrium with linear utility with a perspective shift from allocation to price [57], while Cole et al. [14] later found that Shmyrev program is the dual problem of EG program with a change of variables. There are also a branch of literature that consider computational perspective in more general settings such as indivisible goods [54, 19, 20] and piece-wise linear utility [60, 33, 34].

There are abundant of works that present algorithms to solve the market equilibrium and shows the convergence results theoretically [13]. Gao and Kroer [30] discusses the convergence rates of first-order algorithms for EG convex program under linear, quasi-linear and Leontief utilities. Nan et al. [52] later designs stochastic optimization algorithms for EG convex program and Shmyrev program with convergence guarantee and show some economic insight. Jalota et al. [42] proposes an ADMM algorithm for CCNH utilities and shows linear convergence results. Besides, researchers are more engaged in designing dynamics that possess more economic insight. For example, PACE dynamic [32, 48, 65] and proportional response dynamic [63, 66, 12], though the original idea of PACE arise from auction design [16, 15].

With the fast growth of machine learning and neural network, many existing works aim at resolving economic problem by deep learning approach, which falls into the differentiate economy framework [26]. A mainstream is to approximate the optimal auction with differentiable models by neural networks [25, 29, 36, 55]. The problem of Nash equilibrium computation in normal form games [22, 50, 23] and optimal contract design [62] through deep learning also attracts researchers' attentions. Among these methodologies, transformer architecture [50, 21, 47] is widely used in solving economic problems.

To the best of our knowledge, no existing works try to approximate market equilibrium through deep learning. Besides, although some literature focuses on low-rank markets and representative markets [46, 45], our works firstly propose the concept of contextual market. We believe that our approach will pioneer a promising direction for large-scale contextual market equilibrium computation.

3 Contextual Market Modelling

In this section, we focus on the model of contextual market equilibrium in which goods are assumed to be divisible. Let the market consist of n buyers, denoted as $1, ..., n$, and m goods, denoted as $1, ..., m$. We denote $[k]$ as the abbreviation of the set $\{1, 2, \dots, k\}$. Each buyer $i \in [n]$ has a representation b_i, and each good $j \in [m]$ has a representation g_j. We assume that b_i belongs to the buyer representation space $\mathcal{B}$, and g_j belongs to the good representation space $\mathcal{G}$. For a buyer with representation $b \in \mathcal{B}$, she has budget $B(b) > 0$. Denote $Y(g) > 0$ as the supply of good with representation g. Although many existing works [30] assume that each good j has *unit* supply (i.e. $Y(g) \equiv 1$ for all $g \in \mathcal{G}$) without loss of generality, their results can be easily generalized to our settings.

An *allocation* is a matrix $\boldsymbol{x} = (x_{ij})_{i\in[n],j\in[m]} \in \mathbb{R}_+^{n\times m}$, where x_{ij} is the amount of good j allocated to buyer i. We denote $\boldsymbol{x}_i = (x_{i1}, \dots, x_{im})$ as the vector of bundle of goods that is allocated to buyer i. The buyers' utility function is denoted as $u : \mathcal{B} \times \mathbb{R}_+^m \to \mathbb{R}_+$, here $u(b_i; \boldsymbol{x}_i)$ denotes the utility of buyer i with representation b_i when she chooses to buy $\boldsymbol{x}_i$. We denote $u_i(\boldsymbol{x}_i)$ as an equivalent form of $u(b_i; \boldsymbol{x}_i)$ and often refer them as the same thing. Similarly, $B(b_i), Y(g_j)$ and B_i, Y_j are often referred to as the same thing, respectively.

Let $\boldsymbol{p} = (p_1, \dots, p_m) \in \mathbb{R}_+^m$ be the prices of the goods, the *demand set* of buyer with representation b_i is defined as the set of utility-maximizing allocations within budget constraint.

$$D(b_i; \boldsymbol{p}) := \arg\max_{\boldsymbol{x}_i} \left\{ u(b_i; \boldsymbol{x}_i) \mid \boldsymbol{x}_i \in \mathbb{R}_+^m, \langle \boldsymbol{p}, \boldsymbol{x}_i \rangle \leq B(b_i) \right\}. \tag{1}$$

A *contextual market* is a 4-tuple: $\mathcal{M} = \langle n, m, (b_i)_{i\in[n]}, (g_j)_{j\in[m]} \rangle$, where buyer utility $u(b_i; \boldsymbol{x}_i)$ is known given the information of the market. We also assume budget function $B : \mathcal{B} \to \mathbb{R}_+$ represents the budget of buyers and capacity function $Y : \mathcal{G} \to \mathbb{R}_+$ represents the supply of goods. All of u, B and Y are assumed to be public knowledge and excluded from a market representation. This assumption mainly comes from two aspects: (1) these functions can be learned from historical data and (2) budgets and supplies can be either encoded in b and g in some way.

The *market equilibrium* is represented as a pair $(\boldsymbol{x}, \boldsymbol{p})$, $\boldsymbol{x} \in \mathbb{R}_+^{n\times m}$, $\boldsymbol{p} \in \mathbb{R}_+^m$, which satisfies the following conditions.

- *Buyer optimality*: $\boldsymbol{x}_i \in D(b_i, \boldsymbol{p})$ for all $i \in [n]$,
- *Market clearance*: $\sum_{i=1}^n x_{ij} \leq Y(g_j)$ for all $j \in [m]$, and equality must hold if $p_j > 0$.

We say that u_i is *homogeneous* (with degree 1) if it satisfies $u_i(\alpha \boldsymbol{x}_i) = \alpha u_i(\boldsymbol{x}_i)$ for any $\boldsymbol{x}_i \geq 0$ and $\alpha > 0$ [53, §6.2]. Following existing works, we assume that u_is are CCNH utilities, where CCNH represents for concave, continuous, non-negative, and homogeneous functions[30]. For CCNH utilities, a market equilibrium can be computed using the following *Eisenberg-Gale convex program* (EG):

$$\max \sum_{i=1}^n B_i \log u_i(\boldsymbol{x}_i) \quad \text{s.t.} \sum_{i=1}^n x_{ij} \leq Y_j, \ \boldsymbol{x} \geq 0. \tag{EG}$$

Theorem 3.1 shows that the market equilibrium can be represented as the optimal solution of (EG).

Theorem 3.1 (Gao and Kroer [30]). *Let u_i be concave, continuous, non-negative and homogeneous (CCNH). Assume $u_i(\mathbf{1}) > 0$ for all i. Then, (i) (EG) has an optimal solution and (ii) any optimal solution $\boldsymbol{x}$ to (EG) together with its optimal Lagrangian multipliers $\boldsymbol{p}^* \in \mathbb{R}_+^m$ constitute a market equilibrium, up to arbitrary assignment of zero-price items. Furthermore, $\langle \boldsymbol{p}^*, \boldsymbol{x}_i^* \rangle = B_i$ for all i.*

Based on Theorem 3.1, it's easy to find that we can always assume $\sum_{i\in[n]} x_{ij} = Y_j$ while preserving the existence of market equilibrium, which states as follows.

Proposition 3.2. *Following the assumptions in Theorem 3.1. For the following* EG convex program with equality constraints,

$$\max \sum_{i=1}^{n} B_i \log u_i(\boldsymbol{x}_i) \quad \text{s.t.} \sum_{i=1}^{n} x_{ij} = Y_j,\ \boldsymbol{x} \geq 0. \tag{2}$$

Then, an optimal solution $\boldsymbol{x}^*$ together with its Lagrangian multipliers $\boldsymbol{p}^* \in \mathbb{R}_+^m$ constitute a market equilibrium. Moreover, assume more that for each good j, there is some buyer i such that $\frac{\partial u_i}{\partial x_{ij}} > 0$ always hold whenever $u_i(\boldsymbol{x}_i) > 0$, then all prices are strictly positive in market equilibrium. As a consequence, Equation (EG) and Equation (2) derive the same solution.

We leave all proofs to Appendix B. Since the additional assumption in Proposition 3.2 is fairly weak, without further clarification, we always assume the conditions in Proposition 3.2 hold and the market clearance condition becomes $\sum_{i\in[n]} x_{ij} = Y(g_j),\ \forall j \in [m]$.

4 MarketFCNet

In this section, we introduce the MarketFCNet (denoted as Market Fully-Connected Network) approach to solve the market equilibrium when the number of buyers is large and potentially infinite. MarketFCNet is a sampling-based methodology, and the key point is to design an unbiased estimator of an objective function whose solution coincides with the market equilibrium. The main advantage is that it has the potential to fit the infinite-buyer case without scaling the computational complexity. Therefore, MarketFCNet is scalable with the number of buyers varies.

4.1 Problem Reformulation

Following the idea of differentiable economics [26], we consider parameterized models to represent the allocation of good j to buyer i, denoted as $x_\theta(b_i, g_j)$, and call it allocation network, where θ is the network parameter. Given buyer i and good j, the network can automatically compute the allocation $x_{ij} = x_\theta(b_i, g_j)$. The allocation to buyer i is represented as $\boldsymbol{x}_i = \boldsymbol{x}_\theta(b_i, \boldsymbol{g})$ and the allocation matrix is represented as $\boldsymbol{x} = \boldsymbol{x}_\theta(\boldsymbol{b}, \boldsymbol{g})$. Then the market clearance constraint can be reformulated as $\sum_{i\in[n]} x_\theta(b_i, g_j) = Y(g_j), \forall j \in [m]$ and the price constraint can be reformulated as $\boldsymbol{x}_\theta(\boldsymbol{b}, \boldsymbol{g}) \geq 0$. Let b be uniformly distributed from $\mathcal{B} = \{b_i : i \in [n]\}$, then the EG program (EG) becomes,

$$\begin{aligned} \max_{x_\theta} \quad & \text{OBJ}(x_\theta) = \mathbb{E}_b[B(b) \log u(b; \boldsymbol{x}_\theta(b, \boldsymbol{g}))] \\ \text{s.t.} \quad & \mathbb{E}_b[x_\theta(b, g_j)] = Y(g_j)/n, \forall j \in [m] \\ & \boldsymbol{x}_\theta(\boldsymbol{b}, \boldsymbol{g}) \geq 0 \end{aligned} \tag{EG-FC}$$

For simplicity, we take $Y(g_j)/n \equiv 1$ for all g_j.

4.2 Optimization

The second constraint in (EG-FC) can be easily handled by the network architecture (for example, network with a softplus layer $\sigma(x) = \log(1 + \exp(x))$. As for the first constraint, from Theorem 3.1, we know the prices of goods are simply the Lagrangian multipliers for the first constraint in (EG-FC). Therefore, we employ the Augmented Lagrange Multiplier Method (ALMM) to solve the problem (EG-FC). We define $\mathcal{L}_\rho(x_\theta, \lambda)$ as the Lagrangian, which has the form:

$$\mathcal{L}_\rho(x_\theta; \boldsymbol{\lambda}) = -\,\text{OBJ}(x_\theta) + \sum_{j=1}^{m} \lambda_j \left(\mathbb{E}_b[x_\theta(b, g_j)] - 1\right) + \frac{\rho}{2} \sum_{j=1}^{m} \left(\mathbb{E}_b[x_\theta(b, g_j)] - 1\right)^2 \tag{3}$$

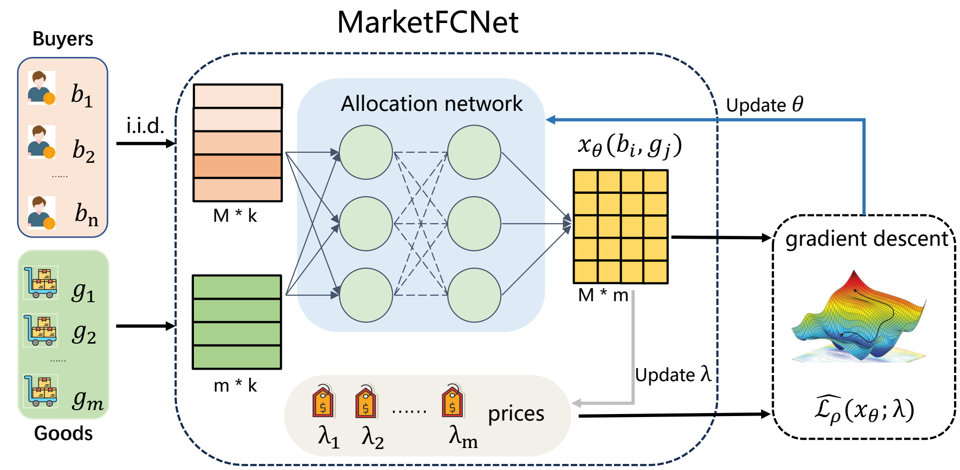


Figure 1: Training process of MarketFCNet. On each iteration, the batch of M independent buyers are drawn. each buyer and each good are represented as k-dimension context. The (i, j)'th element in the allocation matrix represents the allocation computed from i'th buyer and j'th good. MarketFCNet training process alternates between the training of allocation network and prices. The training of allocation network need to achieve an unbiased estimator $\widehat{\mathcal{L}}_\rho(x_\theta; \lambda)$ of the loss function $\mathcal{L}_\rho(x_\theta; \lambda)$, followed by gradient descent. The training of prices need to get an unbiased estimator $\widehat{\Delta}\lambda_j$ of $\Delta\lambda_j$, followed by ALMM updating rule $\lambda_j \leftarrow \lambda_j + \beta_t \widehat{\Delta}\lambda_j$.

Directly computing the objective function seems intractable due to the potentially infinite data size. Therefore, we follow the framework in learning theory culture that we only guarantee to achieve an unbiased gradient of the objective function [1, 8]. The training process of MarketFCNet is presented in Figure 1.

To finish the ALMM algorithm, we need to obtain unbiased estimators of following two expressions.

- An unbiased estimator of $\mathcal{L}_\rho(x_\theta; \boldsymbol{\lambda})$.
- An unbiased estimator of $\Delta\lambda_j$, where $\Delta\lambda_j$ is given by $\Delta\lambda_j = \rho\left(\mathbb{E}_b[x_\theta(b, g_j)] - 1\right)$.

Unbiased estimator of $\Delta\lambda_j$ We aim to obtain an unbiased estimator of $\mathbb{E}_b[x_\theta(b, g_j)]$. By applying Monte Carlo method, we can choose batch size M and sample $b_1, b_2, ..., b_M \sim U(\mathcal{B})$, then $\frac{1}{M}\sum_{i=1}^{M} x_\theta(b_i, g_j)$ forms an unbiased estimator.

Unbiased estimator of $\mathcal{L}_p(x_\theta; \boldsymbol{\lambda})$ For $\mathrm{OBJ}(x_\theta)$ and the second term, the technique to achieve an unbiased estimator is similar. $u(b; x_\theta(b, \boldsymbol{g}))$ in $\mathrm{OBJ}(x_\theta)$ can be calculated directly by summing over all goods. For the last term, notice that

$$\left(\mathbb{E}_b\left[x_\theta(b, g_j)\right] - 1\right)^2 = \left(\mathbb{E}_b\left[x_\theta(b, g_j)\right] - 1\right) \cdot \left(\mathbb{E}_{b'}\left[x_\theta(b', g_j)\right] - 1\right) \tag{4}$$

Therefore, we can sample $b_1, ..., b_M, b'_1, ..., b'_M \sim U(\mathcal{B})$ and compute

$$\frac{\rho}{2} \cdot \frac{1}{M}\sum_{i=1}^{M}\sum_{j=1}^{m}\left(x_\theta(b_i, g_j) - 1\right) \cdot \left(x_\theta(b'_i, g_j) - 1\right) \tag{5}$$

which provides an unbiased estimator for the last term, capturing the squared deviation of output allocations from the constraint.

5 Performance Measures of Market Equilibrium

In this section, we propose *Nash Gap* to measure the performance of an approximated market equilibrium and show that Nash Gap preserves the economic interpretation for market equilibrium. To introduce Nash Gap, we first introduce two types of welfare, Log Nash Welfare and Log Fixed-price Welfare in Definition 5.1 and Definition 5.2, respectively.

Definition 5.1 (Log Nash Welfare)**.** The Log Nash Welfare (abbreviated as LNW) is defined as

$$\mathrm{LNW}(\boldsymbol{x}) = \frac{1}{B_{\text{total}}} \sum_{i \in [n]} B_i \log u_i(\boldsymbol{x}_i), \tag{6}$$

where $B_{\text{total}} = \sum_{i \in [n]} B_i$ is the total budgets for buyers.

Notice that $\mathrm{LNW}(\boldsymbol{x})$ is identical to the objective function in Equation (EG), differing only in the constant term coefficient.

Definition 5.2 (Fixed-price and Log Fixed-price Welfare)**.** We define the fixed-price utility for buyer i as,

$$\tilde{u}(b_i; \boldsymbol{p}) = \max_{\boldsymbol{x}_i} \{ u(b_i; \boldsymbol{x}_i) \mid \boldsymbol{x}_i \in \mathbb{R}_+^m, \langle \boldsymbol{p}, \boldsymbol{x}_i \rangle \leq B(b_i) \} \tag{7}$$

which represents the optimal utility that buyer i can obtain at the price level $\boldsymbol{p}$, regardless of the market clearance constraints. The Log Fixed-price Welfare (abbreviated as LFW) is defined as the logarithm of Fixed-price Welfare,

$$\mathrm{LFW}(\boldsymbol{p}) = \frac{1}{B_{\text{total}}} \sum_{i \in [n]} B_i \log \tilde{u}_i(\boldsymbol{p}) \tag{8}$$

Based on these definitions, we present the definition of Nash Gap.

Definition 5.3 (Nash Gap)**.** We define Nash Gap (abbreviated as NG) as the difference of Log Nash Welfare and Log Fixed-price Welfare, *i.e.*

$$\mathrm{NG}(\boldsymbol{x}, \boldsymbol{p}) = \mathrm{LFW}(\boldsymbol{p}) - \mathrm{LNW}(\boldsymbol{x}) \tag{9}$$

5.1 Properties of Nash Gap

To show why NG is useful in the measure of market equilibrium, we first observe that,

Proposition 5.4 (Price constraints)**.** *If $(\boldsymbol{x}, \boldsymbol{p})$ constitute a market equilibrium, the following identity always hold,*

$$\sum_{j \in [m]} p_j Y_j = \sum_{i \in [n]} B_i \tag{10}$$

Below, we state the most important theorem in this paper.

Theorem 5.5. *Let $(\boldsymbol{x}, \boldsymbol{p})$ be a pair of allocation and price. Assuming the allocation satisfies market clearance and the price meets price constraint, then we have $\mathrm{NG}(\boldsymbol{x}, \boldsymbol{p}) \geq 0$.*

Moreover, $\mathrm{NG}(\boldsymbol{x}, \boldsymbol{p}) = 0$ if and only if $(\boldsymbol{x}, \boldsymbol{p})$ is a market equilibrium.

Theorem 5.5 show that Nash Gap is an ideal measure of the solution concept of market equilibrium, since it holds following properties,

- $\mathrm{NG}(\boldsymbol{x}, \boldsymbol{p})$ is continuous on the inputs $(\boldsymbol{x}, \boldsymbol{p})$.
- $\mathrm{NG}(\boldsymbol{x}, \boldsymbol{p}) \geq 0$ always hold. (under conditions in Theorem 5.5)
- $\mathrm{NG}(\boldsymbol{x}, \boldsymbol{p}) = 0$ if and only if $(\boldsymbol{x}, \boldsymbol{p})$ meets the solution concept.
- The computation of NG does not require the knowledge of an equilibrium point $(\boldsymbol{x}^*, \boldsymbol{p}^*)$

Since some may argue that $\mathrm{NG}(\boldsymbol{x}, \boldsymbol{p})$ is not intuitive to understand, we consider some more intuitive measures, the Euclidean distance to the market equilibrium, *i.e.*, $||\boldsymbol{x} - \boldsymbol{x}^*||$ and $||\boldsymbol{p} - \boldsymbol{p}^*||$, as well as the difference on Weighted Social Welfare, $|\mathrm{WSW}(\boldsymbol{x}) - \mathrm{WSW}(\boldsymbol{x}^*)|$, where $\mathrm{WSW}(\boldsymbol{x}) := \sum_{i \in [n]} \frac{B_i}{B_{\text{total}}} u_i(\boldsymbol{x}_i)$, and show the connection between NG and these intuitive measures.

Proposition 5.6. *Under some technical assumptions (which is presented in Appendix B.4), if $\mathrm{NG}(\boldsymbol{x}, \boldsymbol{p}) = \varepsilon$, we have:*

- $||\boldsymbol{p} - \boldsymbol{p}^*|| = O(\sqrt{\varepsilon})$.

- $||\boldsymbol{x}_i - \boldsymbol{x}_i^*|| = O(\sqrt{\varepsilon})$ *for all* i.
- $|\mathrm{WSW}(\boldsymbol{x}) - \mathrm{WSW}(\boldsymbol{x}^*)| = O(\varepsilon)$.

Finally, we give a saddle-point explaination for Nash Gap.

Corollary 5.7. *Within market clearance and price constraint, we have*

$$\min_{\boldsymbol{p}} \mathrm{LFW}(\boldsymbol{p}) = \max_{\boldsymbol{x}} \mathrm{LNW}(\boldsymbol{x}) \tag{11}$$

Corollary 5.7 provides an economic interpretation for GAP. Market equilibrium can be seen as the saddle point over social welfare, and the social welfare for $\boldsymbol{x}$ can be actually implemented while the social welfare for $\boldsymbol{p}$ is virtual and desired by buyers. Nash Gap measures the gap between the "desired welfare" and the "implemented welfare" for buyers.

5.2 Measures in General Cases

Since NG only works for $(\boldsymbol{x}, \boldsymbol{p})$ that satisfies market clearance and price constraints, we generalize the measure of NG to a more general case, which need to give a measure for all positive $(\boldsymbol{x}, \boldsymbol{p})$.

We first notice that any equilibrium must satisfy the conditions of *market clearance* and *price constraint*, we first make a projection on arbitrary positive $(\boldsymbol{x}, \boldsymbol{p})$ to the space where these constraints hold. Specifically, if we let

$$\alpha_j = \frac{V_j}{\sum_i x_{ij}}, \quad \tilde{x}_{ij} = x_{ij} \cdot \alpha_j \qquad \beta = \frac{\sum_i B_i}{\sum_j V_j p_j}, \quad \tilde{p}_j = \beta \cdot p_j \tag{12}$$

then $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{p}})$ satisfies these constraints and we consider $\mathrm{NG}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{p}})$ as the equilibrium measure.

Besides, we also need to measure how far is the point $(\boldsymbol{x}, \boldsymbol{p})$ to the space within the conditions of *market clearance* and *price constraint*. we propose following two measurement, called Violation of Allocation (abbreviated as VoA) and Violation of Price (abbreviated as VoP), respectively.

$$\mathrm{VoA}(\boldsymbol{x}) := \frac{1}{m} \sum_j |\log \alpha_j|, \qquad \mathrm{VoP}(\boldsymbol{p}) := |\log \beta| \tag{13}$$

From the expressions of VoA and VoP, we know that these two constraints hold if and only if $\mathrm{VoA}(\boldsymbol{x}) = 0$ and $\mathrm{VoP}(\boldsymbol{p}) = 0$.

We argue that this projection is of economic meaning. If $(\boldsymbol{x}, \boldsymbol{p})$ constitute a market equilibrium and we scale budget with a factor of β, then $(\boldsymbol{x}, \beta\boldsymbol{p})$ constitute a market equilibrium in the new market. Similarly, if we scale the value for each buyer with factor $1/\boldsymbol{\alpha}$ (here $\boldsymbol{\alpha}$ can be a vector in $\mathbb{R}_+^m$) and capacity with factor α, then, $(\boldsymbol{\alpha x}, \frac{1}{\boldsymbol{\alpha}}\boldsymbol{p})$ constitute a market equilibrium in the new market. These instances are evidence that market equilibrium holds a linear structure over market parameters. Therefore, a linear projection can eliminate the effect from linear scaling, while preserving the effect from orthogonal errors.

Notice that $\boldsymbol{x} = \tilde{\boldsymbol{x}}$ and $\boldsymbol{p} = \tilde{\boldsymbol{p}}$ if and only if $\mathrm{VoA}(\boldsymbol{x}) = 0$ and $\mathrm{VoP}(\boldsymbol{p}) = 0$, respectively. From Theorem 5.5 We can easy derive following statements:

Proposition 5.8. *For arbitrary* $\boldsymbol{x} \in \mathbb{R}_+^{n\times m}, \boldsymbol{p} \in \mathbb{R}_+^m$, *we have* $\mathrm{VoA}(\boldsymbol{x}) \geq 0, \mathrm{VoP}(\boldsymbol{p}) \geq 0, \mathrm{NG}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{p}}) \geq 0$ *always hold. Moreover,* $(\boldsymbol{x}, \boldsymbol{p})$ *is a market equilibrium if and only if* $\mathrm{VoA}(\boldsymbol{x}) = \mathrm{VoP}(\boldsymbol{p}) = \mathrm{NG}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{p}}) = 0$.

Proposition 5.8 is a certificate that $\mathrm{VoA}(\boldsymbol{x}), \mathrm{VoP}(\boldsymbol{p}), \mathrm{NG}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{p}})$ together form a good measure for market equilibrium. Therefore, in our experiments we compute these measures of solutions and prefer a lower measure without further clarification.

6 Experiments

In this section, we present empirical experiments that evaluate the effectiveness of MarketFCNet. Though briefly mentioned in this section, we leave the details of baselines, implementations, hyper-parameters and experimental environments to Appendix C.

Table 1: Comparison of MarketFCNet with baselines: $n = 1,048,576$ buyers and $m = 10$ goods. The GPU time for MarketFCNet represents the training time and testing time, respectively.

Methods	NG	VoA	VoP	GPU Time
Naïve	3.65e-1	0	0	3.57e-3
EG	2.17e-2	2.620e-1	7.031e-2	197
EG-m	**2.49e-4**	6.01e-2	9.77e-2	100
FC	1.63e-3	**1.416e-2**	**6.750e-3**	**43.6**; **9.63e-2**

6.1 Experimental Settings

In our experiments, all utilities are chosen as CES utilities, which captures a wide utility class including linear utilities, Cobb-Douglas utilities and Leontief utilities and has been widely studied in literature [59, 4]. CES utilities have the form,

$$u_i(x_i) = \left(\sum_{j \in [m]} v_{ij}^{\alpha} x_{ij}^{\alpha} \right)^{1/\alpha}$$

with $\alpha \leq 1$. The fixed-price utilities for CES utility is derived in Appendix A.

In order to evaluate the performance of MarketFCNet, we compare them mainly with a baseline that directly maximizes the objective in EG convex program with gradient ascent algorithm (abbreviated as *EG*), which is widely used in the field of market equilibrium computation. Besides, we also consider a momentum version of *EG* algorithm with momentum $\beta = 0.9$ (abbreviated as *EG-m*). We move the details of all baselines, experimental environments and implementations of algorithms to Appendix C.1 and Appendix C.2.

We also consider a naïve allocation and pricing rule (abbreviated as *Naïve*), which can be regarded as the benchmark of the experiments:

$$x_{ij} = 1, \quad p_j = \frac{\sum_{i \in [n]} B_i}{m V_j}, \quad \text{for all } i, j \tag{14}$$

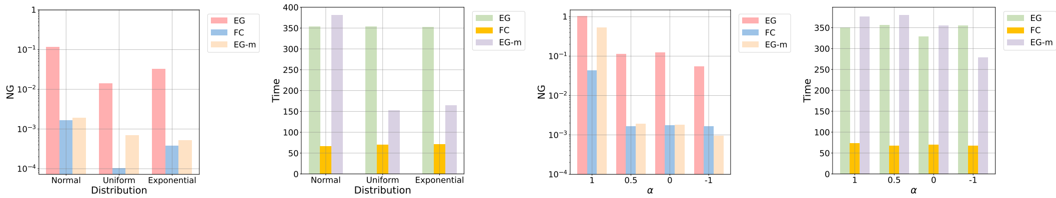
In the following experiments, MarketFCNet is abbreviated as *FC*. Notice that *Naïve* always gives an allocation that satisfies market clearance and price constraints, while *EG*, *EG-m* and *FC* do not.

6.2 Experiment Results

Comparing with Baselines We choose number of buyers $n = 1,048,576 = 2^{20}$, number of items $m = 10$, CES utilities parameter $\alpha = 0.5$ and representation with standard normal distribution as the basic experimental environment of MarketFCNet; We consider $\mathrm{NG}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{p}}), \mathrm{VoA}(\boldsymbol{x}), \mathrm{VoP}(\boldsymbol{p})$ and the running time of algorithms as the measures. Without special specification, these parameters are default settings among other experiments. Results are presented in Table 1. From these results we can see that the approximations of MarketFCNet are competitive with *EG* and *EG-m* and far better than Naïve, which means that the solution of MarketFCNet are very close to market equilibrium. MarketFCNet also achieve a much lower running time compared with *EG* and *EG-m*, which indicates that these methods are more suitable to large-scale market equilibrium computation. In following experiments, VoA and VoP measures are omitted and we only report NG and running time.

Experiments in different parameters settings In this experiments, the market scale is chosen as $n = 4,194,304$ and $m = 10$. We consider experiments on different distribution of representation, including normal distribution, uniform distribution and exponential distribution. See (a) and (b) in Figure 2 for results. We also consider different α in our experimental settings. Specifically, our settings consist of: 1) $\alpha = 1$, the utility functions are linear; 2) $\alpha = 0.5$, where goods are substitutes; 3) $\alpha = 0$, where goods are neither substitutes or complements; 4) $\alpha = -1$, where goods are complements. More detailed results are shown in (c) and (d) Figure 2. The performance of MarketFCNet is robust in both settings.

Figure 2: The Nash Gap and GPU running time for different algorithms: MarketFCNet, EG and EG-m. Different colors represent for different algorithm. Market size is chosen as $n = 4,194,304$ buyers and $m = 10$ goods.



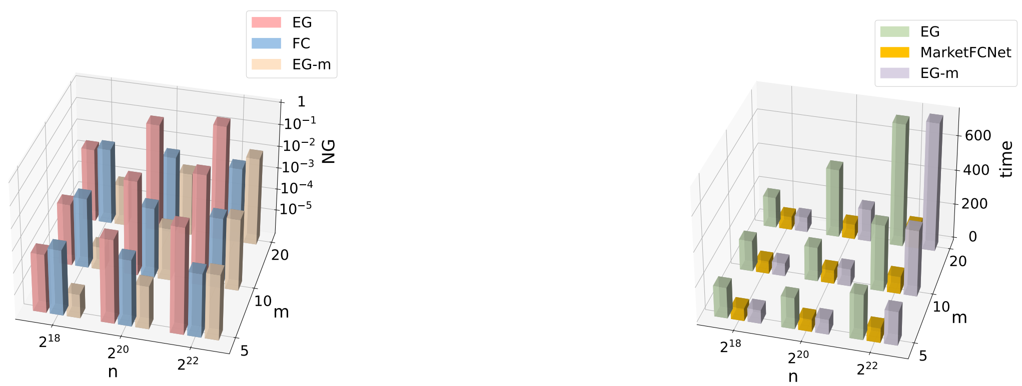
(a) Nash Gap on different context distributions.

(b) GPU running time on different context distributions.

(c) Nash Gap on different CES utilities parameter α.

(d) GPU running time on different CES utilities parameter α.

Figure 3: The Nash Gap and GPU running time for different algorithms: MarketFCNet, EG and EG-m. Different colors represent for different algorithm. Market size is chosen as $n = 2^{18}, 2^{20}, 2^{22}$ buyers and $m = 5, 10, 20$ goods.



(a) Nash Gap on different market size, $n = 2^{18}, 2^{20}, 2^{22}$ buyers and $m = 5, 10, 20$ goods.

(b) GPU running time on different market size, $n = 2^{18}, 2^{20}, 2^{22}$ buyers and $m = 5, 10, 20$ goods.

Different market scale for MarketFCNet In this section we ask that how market size (here n and m) will have impact on the efficiency of MarketFCNet. We set $m = 5, 10, 20$ and $n = 2^{18} = 262,114, 2^{20} = 1,048,576, 2^{22} = 4,194,304$ as the experimental settings. For each combination of n and m, we train MarketFCNet and compared with EG and EG-m, see results in Figure 3. As the market size varies, MarketFCNet has almost the same Nash Gap and running time, which shows the robustness of MarketFCNet method over different market sizes. However, as the market size increases, both EG and EG-m have larger Nash Gaps and longer running times, demonstrating that MarketFCNet is more suitable to large-scale contextual market equilibrium computation.

7 Conclusions and Future Work

This paper initiates the problem of large-scale contextual market equilibrium computation from a deep learning perspective. We believe that our approach will pioneer a promising direction for large-scale contextual market equilibrium computation.

For future works, it would be promising to extend these methods to the case when only the number of goods is large, or both the numbers of goods and buyers are large, which stays a blank throughout our works. Since many existing works proposed dynamics for online market equilibrium computation, it's also promising to extend our approaches to tackle the online market setting with large buyers. Besides, both existing works and ours consider sure budgets and values for buyers, and it would be interesting to extend the fisher market and equilibrium concept when the budgets or values of buyers are stochastic or uncertain.

References

[1] Shun-ichi Amari. Backpropagation and stochastic gradient descent method. *Neurocomputing*, 5 (4-5):185–196, 1993.

[2] Kenneth J Arrow. An extension of the basic theorems of classical welfare economics. In *Proceedings of the second Berkeley symposium on mathematical statistics and probability*, volume 2, pages 507–533. University of California Press, 1951.

[3] Kenneth J Arrow and Gerard Debreu. Existence of an equilibrium for a competitive economy. *Econometrica: Journal of the Econometric Society*, pages 265–290, 1954.

[4] Kenneth J Arrow, Hollis B Chenery, Bagicha S Minhas, and Robert M Solow. Capital-labor substitution and economic efficiency. *The review of Economics and Statistics*, pages 225–250, 1961.

[5] Santiago Balseiro, Christian Kroer, and Rachitesh Kumar. Contextual standard auctions with budgets: Revenue equivalence and efficiency guarantees. *Management Science*, 69(11):6837–6854, 2023.

[6] Siddhartha Banerjee, Vasilis Gkatzelis, Artur Gorokh, and Billy Jin. Online nash social welfare maximization with predictions. In *Proceedings of the 2022 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 1–19. SIAM, 2022.

[7] Yoshua Bengio, Jérôme Louradour, Ronan Collobert, and Jason Weston. Curriculum learning. In *Proceedings of the 26th annual international conference on machine learning*, pages 41–48, 2009.

[8] Léon Bottou. Large-scale machine learning with stochastic gradient descent. In *Proceedings of COMPSTAT'2010: 19th International Conference on Computational StatisticsParis France, August 22-27, 2010 Keynote, Invited and Contributed Papers*, pages 177–186. Springer, 2010.

[9] Simina Brânzei, Yiling Chen, Xiaotie Deng, Aris Filos-Ratsikas, Søren Frederiksen, and Jie Zhang. The fisher market game: Equilibrium and welfare. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 28, 2014.

[10] Jonathan Brogaard, Terrence Hendershott, and Ryan Riordan. High-frequency trading and price discovery. *The Review of Financial Studies*, 27(8):2267–2306, 2014.

[11] Eric Budish. The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. *Journal of Political Economy*, 119(6):1061–1103, 2011.

[12] Yun Kuen Cheung, Richard Cole, and Yixin Tao. Dynamics of distributed updating in fisher markets. In *Proceedings of the 2018 ACM Conference on Economics and Computation*, pages 351–368, 2018.

[13] Richard Cole and Lisa Fleischer. Fast-converging tatonnement algorithms for one-time and ongoing market problems. In *Proceedings of the Fortieth Annual ACM Symposium on Theory of Computing*, pages 315–324, 2008.

[14] Richard Cole, Nikhil Devanur, Vasilis Gkatzelis, Kamal Jain, Tung Mai, Vijay V Vazirani, and Sadra Yazdanbod. Convex program duality, fisher markets, and nash social welfare. In *Proceedings of the 2017 ACM Conference on Economics and Computation*, pages 459–460, 2017.

[15] Vincent Conitzer, Christian Kroer, Debmalya Panigrahi, Okke Schrijvers, Nicolas E Stier-Moses, Eric Sodomka, and Christopher A Wilkens. Pacing equilibrium in first price auction markets. *Management Science*, 68(12):8515–8535, 2022.

[16] Vincent Conitzer, Christian Kroer, Eric Sodomka, and Nicolas E Stier-Moses. Multiplicative pacing equilibria in auction markets. *Operations Research*, 70(2):963–989, 2022.

[17] Michael Curry, Alexander R Trott, Soham Phade, Yu Bai, and Stephan Zheng. Finding general equilibria in many-agent economic simulations using deep reinforcement learning. 2021.

[18] Michael Curry, Tuomas Sandholm, and John Dickerson. Differentiable economics for randomized affine maximizer auctions. *arXiv preprint arXiv:2202.02872*, 2022.

[19] Xiaotie Deng, Christos Papadimitriou, and Shmuel Safra. On the complexity of equilibria. In *Proceedings of the Thiry-fourth Annual ACM Symposium on Theory of Computing*, pages 67–71, 2002.

[20] Xiaotie Deng, Christos Papadimitriou, and Shmuel Safra. On the complexity of price equilibria. *Journal of Computer and System Sciences*, 67(2):311–324, 2003.

[21] Zhijian Duan, Jingwu Tang, Yutong Yin, Zhe Feng, Xiang Yan, Manzil Zaheer, and Xiaotie Deng. A context-integrated transformer-based neural network for auction design. In *International Conference on Machine Learning*, pages 5609–5626. PMLR, 2022.

[22] Zhijian Duan, Wenhan Huang, Dinghuai Zhang, Yali Du, Jun Wang, Yaodong Yang, and Xiaotie Deng. Is nash equilibrium approximator learnable? In *Proceedings of the 2023 International Conference on Autonomous Agents and Multiagent Systems*, pages 233–241, 2023.

[23] Zhijian Duan, Yunxuan Ma, and Xiaotie Deng. Are equivariant equilibrium approximators beneficial? In *Proceedings of the 40th International Conference on Machine Learning*, ICML'23. JMLR.org, 2023.

[24] Zhijian Duan, Haoran Sun, Yurong Chen, and Xiaotie Deng. A scalable neural network for DSIC affine maximizer auction design. 2023. URL https://openreview.net/forum?id=cNb5hkTfGC.

[25] Paul Dütting, Zhe Feng, Harikrishna Narasimhan, David Parkes, and Sai Srivatsa Ravindranath. Optimal auctions through deep learning. In *International Conference on Machine Learning*, pages 1706–1715. PMLR, 2019.

[26] Paul Dütting, Zhe Feng, Harikrishna Narasimhan, David C Parkes, and Sai Srivatsa Ravindranath. Optimal auctions through deep learning: Advances in differentiable economics. *Journal of the ACM (JACM)*, 2023.

[27] Edmund Eisenberg. Aggregation of utility functions. *Management Science*, 7(4):337–350, 1961.

[28] Edmund Eisenberg and David Gale. Consensus of subjective probabilities: The pari-mutuel method. *The Annals of Mathematical Statistics*, 30(1):165–168, 1959.

[29] Zhe Feng, Harikrishna Narasimhan, and David C Parkes. Deep learning for revenue-optimal auctions with budgets. In *Proceedings of the 17th International Conference on Autonomous Agents and Multiagent Systems*, pages 354–362, 2018.

[30] Yuan Gao and Christian Kroer. First-order methods for large-scale market equilibrium computation. *Advances in Neural Information Processing Systems*, 33:21738–21750, 2020.

[31] Yuan Gao and Christian Kroer. Infinite-dimensional fisher markets and tractable fair division. *Operations Research*, 71(2):688–707, 2023.

[32] Yuan Gao, Alex Peysakhovich, and Christian Kroer. Online market equilibrium with application to fair division. *Advances in Neural Information Processing Systems*, 34:27305–27318, 2021.

[33] Jugal Garg, Ruta Mehta, Vijay V Vazirani, and Sadra Yazdanbod. Settling the complexity of leontief and plc exchange markets under exact and approximate equilibria. In *Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing*, pages 890–901, 2017.

[34] Jugal Garg, Yixin Tao, and László A Végh. Approximating equilibrium under constrained piecewise linear concave utilities with applications to matching markets. In *Proceedings of the 2022 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 2269–2284. SIAM, 2022.

[35] Jonathan Goldman and Ariel D Procaccia. Spliddit: Unleashing fair division algorithms. *ACM SIGecom Exchanges*, 13(2):41–46, 2015.

[36] Noah Golowich, Harikrishna Narasimhan, and David C Parkes. Deep learning for multi-facility location mechanism design. In *International Joint Conferences on Artificial Intelligence*, pages 261–267, 2018.

[37] Xue-Zhong He and Shen Lin. Reinforcement learning equilibrium in limit order markets. *Journal of Economic Dynamics and Control*, 144:104497, 2022.

[38] Howard Heaton, Daniel McKenzie, Qiuwei Li, Samy Wu Fung, Stanley Osher, and Wotao Yin. Learn to predict equilibria via fixed point networks. *arXiv preprint arXiv:2106.00906*, 2021.

[39] Edward Hill, Marco Bardoscia, and Arthur Turrell. Solving heterogeneous general equilibrium economic models with deep reinforcement learning. *arXiv preprint arXiv:2103.16977*, 2021.

[40] Zhiyi Huang, Minming Li, Xinkai Shu, and Tianze Wei. Online nash welfare maximization without predictions. In *International Conference on Web and Internet Economics*, pages 402–419. Springer, 2023.

[41] Devansh Jalota and Yinyu Ye. Stochastic online fisher markets: Static pricing limits and adaptive enhancements. *arXiv preprinted arXiv:2205.00825*, 2023.

[42] Devansh Jalota, Marco Pavone, Qi Qi, and Yinyu Ye. Fisher markets with linear constraints: Equilibrium properties and efficient distributed algorithms. *Games and Economic Behavior*, 141:223–260, 2023.

[43] Nils Kohring, Fabian Raoul Pieroth, and Martin Bichler. Enabling first-order gradient-based learning for equilibrium computation in markets. In *International Conference on Machine Learning*, pages 17327–17342. PMLR, 2023.

[44] Christian Kroer. Ai, games, and markets. 2023. https://www.columbia.edu/~ck2945/files/main_ai_games_markets.pdf.

[45] Christian Kroer and Alexander Peysakhovich. Scalable fair division for'at most one'preferences. *arXiv preprint arXiv:1909.10925*, 2019.

[46] Christian Kroer, Alexander Peysakhovich, Eric Sodomka, and Nicolas E Stier-Moses. Computing large market equilibria using abstractions. In *Proceedings of the 2019 ACM Conference on Economics and Computation*, pages 745–746, 2019.

[47] Ningyuan Li, Yunxuan Ma, Yang Zhao, Zhijian Duan, Yurong Chen, Zhilin Zhang, Jian Xu, Bo Zheng, and Xiaotie Deng. Learning-based ad auction design with externalities: The framework and a matching-based approach. In *Proceedings of the 29th ACM SIGKDD Conference on Knowledge Discovery and Data Mining*, pages 1291–1302, 2023.

[48] Luofeng Liao, Yuan Gao, and Christian Kroer. Nonstationary dual averaging and online fair allocation. *Advances in Neural Information Processing Systems*, 35:37159–37172, 2022.

[49] Yuxuan Lu, Qian Qi, and Xi Chen. A framework of transaction packaging in high-throughput blockchains. *arXiv preprint arXiv:2301.10944*, 2023.

[50] Luke Marris, Ian Gemp, Thomas Anthony, Andrea Tacchetti, Siqi Liu, and Karl Tuyls. Turbocharging solution concepts: Solving nes, ces and cces with neural equilibrium solvers. *Advances in Neural Information Processing Systems*, 35:5586–5600, 2022.

[51] Andreu Mas-Colell, Michael Dennis Whinston, Jerry R Green, et al. *Microeconomic theory*, volume 1. Oxford University Press New York, 1995.

[52] Tianlong Nan, Yuan Gao, and Christian Kroer. Fast and interpretable dynamics for fisher markets via block-coordinate updates. *arXiv preprint arXiv:2303.00506*, 2023.

[53] Noam Nisan, Tim Roughgarden, Eva Tardos, and Vijay V Vazirani. Algorithmic game theory, 2007. *Book available for free online*, 2007.

[54] Christos Papadimitriou. Algorithms, games, and the internet. In *Proceedings of the Thirty-third Annual ACM Symposium on Theory of Computing*, pages 749–753, 2001.

[55] Jad Rahme, Samy Jelassi, Joan Bruna, and S Matthew Weinberg. A permutation-equivariant neural network architecture for auction design. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 35, pages 5664–5672, 2021.

[56] Weiran Shen, Sébastien Lahaie, and Renato Paes Leme. Learning to clear the market. In *International Conference on Machine Learning*, pages 5710–5718. PMLR, 2019.

[57] Vadim I Shmyrev. An algorithm for finding equilibrium in the linear exchange model with fixed budgets. *Journal of Applied and Industrial Mathematics*, 3:505–518, 2009.

[58] The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1972. Nobelprize.org. Nobel Prize Outreach AB 2024, Sun. 28 Jan 2024. https://www.nobelprize.org/prizes/economic-sciences/1972/summary/.

[59] Hal R Varian and Hal R Varian. *Microeconomic analysis*, volume 3. Norton New York, 1992.

[60] Vijay V Vazirani and Mihalis Yannakakis. Market equilibrium under separable, piecewise-linear, concave utilities. *Journal of the ACM (JACM)*, 58(3):1–25, 2011.

[61] Leon Walras. *Elements of pure economics*. Routledge, 2013.

[62] Tonghan Wang, Paul Dütting, Dmitry Ivanov, Inbal Talgam-Cohen, and David C Parkes. Deep contract design via discontinuous piecewise affine neural networks. *arXiv preprint arXiv:2307.02318*, 2023.

[63] Fang Wu and Li Zhang. Proportional response dynamics leads to market equilibrium. In *Proceedings of the Thirty-ninth Annual ACM Symposium on Theory of Computing*, pages 354–363, 2007.

[64] Ruitu Xu, Yifei Min, Tianhao Wang, Michael I Jordan, Zhaoran Wang, and Zhuoran Yang. Finding regularized competitive equilibria of heterogeneous agent macroeconomic models via reinforcement learning. In *International Conference on Artificial Intelligence and Statistics*, pages 375–407. PMLR, 2023.

[65] Zongjun Yang, Luofeng Liao, and Christian Kroer. Greedy-based online fair allocation with adversarial input: Enabling best-of-many-worlds guarantees. *arXiv preprint arXiv:2308.09277*, 2023.

[66] Li Zhang. Proportional response dynamics in the fisher market. *Theoretical Computer Science*, 412(24):2691–2698, 2011.

Appendix

A Derivation of Fixed-price Utility for CES Utility Functions

In this section we show the explicit expressions of Fixed-price Utility for CES utility functions.

We first consider the case $\alpha \neq 0, 1, -\infty$. The optimization problem for consumer i is:

$$\max_{x_{ij}, j\in[m]} \quad u_i(\boldsymbol{x}_i) = \left[\sum_{j\in[m]} v_{ij}^{\alpha} x_{ij}^{\alpha}\right]^{1/\alpha} \tag{15}$$

$$s.t. \quad \sum_{j\in[m]} x_{ij} p_j = B_i \tag{Budget Constraint}$$

$$x_{ij} \geq 0 \tag{16}$$

Not hard to verify that in an optimal solution with Equation (Budget Constraint), Equation (16) always holds, therefore we omit this constraint in our derivation.

We write the Lagrangian $L(\boldsymbol{x}_i, \lambda)$

$$L(\boldsymbol{x}_i, \lambda) = u_i(\boldsymbol{x}_i) + \lambda(B_i - \sum_{j\in[m]} x_{ij} p_j) \tag{17}$$

By $\frac{\partial L}{\partial x_{ij}} = 0$, we have

$$\frac{\partial u_i}{\partial x_{ij}^*}(\boldsymbol{x}_i) = \lambda p_j \tag{18}$$

We derive that

$$\frac{\partial u_i}{\partial x_{ij}}(\boldsymbol{x}_i) = \frac{1}{\alpha}\left[\sum_{j\in[m]} v_{ij}^{\alpha} x_{ij}^{\alpha}\right]^{1/\alpha - 1} \cdot \alpha v_{ij}^{\alpha} x_{ij}^{\alpha-1} \tag{19}$$

$$v_{ij}^{\alpha} x_{ij}^{\alpha-1} = c p_j \qquad \cdots \text{let } c = \lambda \cdot \left[\sum_{j\in[m]} v_{ij}^{\alpha} x_{ij}^{\alpha}\right]^{1/\alpha-1} \tag{20}$$

$$x_{ij}^* = \frac{v_{ij}^{\frac{\alpha}{1-\alpha}}}{c^{\frac{1}{1-\alpha}} \cdot p_j^{\frac{1}{1-\alpha}}} \tag{21}$$

Taking (21) into (Budget Constraint), we get

$$B_i = \sum_{j\in[m]} \frac{v_{ij}^{\frac{\alpha}{1-\alpha}}}{c^{\frac{1}{1-\alpha}}} \cdot p_j^{-\frac{\alpha}{1-\alpha}} \tag{22}$$

$$c^{\frac{1}{1-\alpha}} = \frac{1}{B_i} \sum_{j\in[m]} \left(\frac{v_{ij}}{p_j}\right)^{\frac{\alpha}{1-\alpha}} \tag{23}$$

Taking Equation (23) into Equation (21), we get

$$x_{ij}^* = \frac{v_{ij}^{\frac{\alpha}{1-\alpha}}}{p_j^{\frac{1}{1-\alpha}}} \cdot \frac{B_i}{c_0} \tag{24}$$

where $c_0 = \sum_{j\in[m]} \left(\frac{v_{ij}}{p_j}\right)^{\frac{\alpha}{1-\alpha}}$

Taking Equation (24) into Equation (15), we finally have

$$\begin{aligned} u_i(\boldsymbol{x}_i^*) &= \left[v_{ij}^{\alpha} x_{ij}^{\alpha}\right]^{\frac{1}{\alpha}} \\ &= \left[\sum_{j\in[m]} v_{ij}^{\alpha} \frac{v_{ij}^{\frac{\alpha^2}{1-\alpha}}}{p_j^{\frac{\alpha}{1-\alpha}}} c_0^{\alpha}\right] \\ &= \left[\sum_{j\in[m]} \left(\frac{v_{ij}}{p_j}\right)^{\frac{\alpha}{1-\alpha}} c_0^{\alpha}\right] \\ &= B_i c_0^{\frac{1-\alpha}{\alpha}} \\ \log \tilde{u}_i(\boldsymbol{p}) &= \log u_i(\boldsymbol{x}_i^*) = \log B_i + \frac{1-\alpha}{\alpha}\log c_0 \end{aligned} \tag{25}$$

For $\alpha = 1$, by simple arguments we know that consumer will only buy the good that with largest value-per-cost, *i.e.*, v_{ij}/p_j. Therefore, we have

$$\log \tilde{u}_i(\boldsymbol{p}) = \log B_i + \log \max_j \frac{v_{ij}}{p_j} \tag{26}$$

For $\alpha = 0$, we have $\log u_i(\boldsymbol{x}_i) = \frac{1}{v_t}\sum_{j\in[m]} v_{ij} \log x_{ij}$ where $v_t = \sum_{j\in[m]} v_{ij}$.

Similarly, we have

$$cp_j = \frac{\partial \log u_i}{\partial x_{ij}} = \frac{v_{ij}}{x_{ij}} \tag{27}$$

$$x_{ij}^* = \frac{v_{ij}}{cp_j} \tag{28}$$

By solving budget constraints we have $c = \frac{v_t}{B_i}$, and therefore, $x_{ij}^* = \frac{v_{ij}B_i}{p_j v_t}$ and

$$\log u_i(\boldsymbol{x}_i^*) = \frac{1}{v_t}\sum_{j\in[m]} (v_{ij} \log \frac{v_{ij}B_i}{p_j v_t}) \tag{29}$$

$$= \log B_i + \sum_{j\in[m]} \frac{v_{ij}}{v_t} \log \frac{v_{ij}}{p_j v_t} \tag{30}$$

For $\alpha = -\infty$, we can easily know that $v_{ij}x_{ij}^* \equiv c$ for some c. By solving budget constraint we have

$$\sum_{j\in[m]} \frac{cp_j}{v_{ij}} = B_i \tag{31}$$

$$c = B_i \left(\sum_{j\in[m]} \frac{p_j}{v_{ij}}\right)^{-1} \tag{32}$$

$$\log \tilde{u}_i(\boldsymbol{p}) = \log c = \log B_i - \log \sum_{j\in[m]} \frac{p_j}{v_{ij}} \tag{33}$$

Above all, the log Fixed-price Utility for CES functions is

$$\log \tilde{u}_i(\boldsymbol{p}) = \begin{cases} \log B_i + \max_j \log \frac{v_{ij}}{p_j} & \text{for } \alpha = 1 \\ \log B_i + \sum_{j\in[m]} \frac{v_{ij}}{v_t} \log \frac{v_{ij}}{p_j v_t} & \text{for } \alpha = 0 \\ \log B_i - \log \sum_{j\in[m]} \frac{p_j}{v_{ij}} & \text{for } \alpha = -\infty \\ \log B_i + \frac{1-\alpha}{\alpha} \log c_0 & \text{others} \end{cases} \tag{34}$$

B Omitted Proofs

B.1 Proof of Proposition 3.2

We consider Lagrangian multipliers $\boldsymbol{p}$ and use the KKT condition. The Lagrangian becomes

$$L(\boldsymbol{p}, \boldsymbol{x}) = \sum_i B_i \log u_i(\boldsymbol{x}_i) - \sum_j p_j (\sum_i x_{ij} - Y_j) \tag{35}$$

and the partial derivative of x_{ij} is

$$\frac{\partial L(\boldsymbol{p}, \boldsymbol{x}_i)}{\partial x_{ij}} = \frac{B_i}{u_i(\boldsymbol{x}_i)} \frac{\partial u_i}{\partial x_{ij}} - p_j \tag{36}$$

By complementary slackness of $x_{ij} \geq 0$, we have

$$\frac{B_i}{u_i(\boldsymbol{x}_i)} \frac{\partial u_i}{\partial x_{ij}} \leq p_j \text{ for all } i \tag{37}$$

By theorem 3.1, we know that if $(\boldsymbol{x}, \boldsymbol{p})$ is a market equilibrium, we must have $u_i(\boldsymbol{x}_i) > 0$ for all i, and by condition in Proposition 3.2, we can always select buyer i such that $\frac{\partial u_i}{\partial x_{ij}} > 0$. Therefore, we have $p_j > 0$.

As a consequence, $p_j > 0$ indicates that $\sum_j x_{ij} = V_j$ by market clearance condition.

B.2 Proof of Proposition 5.4

Consider the market equilibrium condition $\langle \boldsymbol{p}^*, \boldsymbol{x}_i^* \rangle = B_i$, we have $\sum_j p_j x_{ij} = B_i$. sum over this expression, we have $\sum_i \sum_j p_j x_{ij} = \sum_i B_i$. Then, $\sum_j p_j \sum_i x_{ij} = \sum_i B_i$. Notice that we have $\sum_{i=1}^n x_{ij} = Y_j$ in market equilibrium, so $\sum_j p_j Y_j = \sum_i B_i$, that completes the proof.

B.3 Proof of Theorem 5.5

Proof of Theorem 5.5. Denote $(\boldsymbol{x}, \boldsymbol{p})$ as the market equilibrium, $\boldsymbol{p}$ as the price for goods and $\boldsymbol{x}_i^*(\boldsymbol{p})$ as the optimal consumption set of buyer i when the price is $\boldsymbol{p}$.

We have following equation:

$$\sum_j x_{ij} p_j = B_i \tag{38}$$

$$\boldsymbol{x}_i \in \boldsymbol{x}_i^*(\boldsymbol{p}) \tag{39}$$

$$\sum_{i\in[n]} x_{ij} = Y_j \tag{40}$$

$$u_i(\boldsymbol{p}) = u_i(\boldsymbol{x}_i),\ \forall \boldsymbol{p} \in \mathbb{R}_+^m,\ \forall \boldsymbol{x}_i \in \boldsymbol{x}_i^*(\boldsymbol{p}) \tag{41}$$

From Proposition 5.4 we know $\sum_{i\in[n]} B_i = \sum_{j\in[m]} Y_j p_j$.

Let $\boldsymbol{p}'$ be some price for items such that $\sum_{j\in[m]} Y_j \boldsymbol{p}'_j = \sum_{i\in[n]} B_i$. Let $\boldsymbol{x}'_i \in \boldsymbol{x}_i^*(\boldsymbol{p}')$ and $B'_i = \langle \boldsymbol{p}', \boldsymbol{x}_i \rangle$. We know that

$$\sum_{i\in[n]} B'_i = \langle \boldsymbol{p}', \sum_{i\in[n]} \boldsymbol{x}_i \rangle = \langle \boldsymbol{p}', \boldsymbol{Y} \rangle = \sum_{i\in[n]} B_i \tag{42}$$

For consumer i, $\boldsymbol{x}_i$ costs B'_i at price $\boldsymbol{p}'$, thus $\frac{B_i}{B'_i}\boldsymbol{x}_i$ costs B_i at price $\boldsymbol{p}'$. Besides, $\boldsymbol{x}'_i$ also costs B_i for price $\boldsymbol{p}'$, and $\boldsymbol{x}'$ is the optimal consumption for buyer i. Then we have

$$u_i(\boldsymbol{p}') = u_i(\boldsymbol{x}'_i) \geq u_i(\frac{B_i}{B'_i}\boldsymbol{x}_i) = \frac{B_i}{B'_i}u_i(\boldsymbol{x}_i) \tag{43}$$

where the last equation is from the homogeneity(with degree 1) of utility function.

Taking logarithm and weighted sum with B_i, we have

$$\sum_{i\in[n]} B_i \log u_i(\boldsymbol{p}') \geq \sum_{i\in[n]} B_i \log \frac{B_i}{B'_i} + \sum_{i\in[n]} B_i \log u_i(\boldsymbol{x}_i) \tag{44}$$

Take $B_{\text{total}} = \sum_{i\in[n]} B_i$, the first term in RHS becomes

$$\sum_{i\in[n]} B_i \log \frac{B_i}{B'_i} \tag{45}$$

$$= B_{\text{total}} \sum_{i\in[n]} \left(\frac{B_i}{B_{\text{total}}} \log \frac{B_i / B_{\text{total}}}{B'_i / B_{\text{total}}} \right) \tag{46}$$

$$= B_{\text{total}} \cdot \text{KL}(\frac{\boldsymbol{B}}{B_{\text{total}}} || \frac{\boldsymbol{B}'}{B_{\text{total}}}) \tag{47}$$

$$\geq 0 \tag{48}$$

Therefore,

$$\sum_{i\in[n]} B_i \log u_i(\boldsymbol{p}') \geq \sum_{i\in[n]} B_i \log u_i(\boldsymbol{x}_i) \tag{49}$$

For $\boldsymbol{x}'$ that satisfies market clearance, by optimality of EG program(EG), we have

$$\sum_{i\in[n]} B_i \log u_i(\boldsymbol{x}_i) \geq \sum_{i\in[n]} B_i \log u_i(\boldsymbol{x}'_i) \tag{50}$$

Equation (49) and Equation (50) together complete the proof of the first part.

If $(\boldsymbol{x}, \boldsymbol{p})$ constitutes a market equilibrium, it's obvious that $\text{LFW}(\boldsymbol{p})$ and $\text{LNW}(\boldsymbol{x})$ are identical, therefore $\text{NG}(\boldsymbol{x}, \boldsymbol{p}) = 0$.

On the other hand, if $(\boldsymbol{x}, \boldsymbol{p})$ is not a market equilibrium, but $\text{NG}(\boldsymbol{x}, \boldsymbol{p}) = 0$, it means that the KL convergence term must equal to 0, and $B_i = B'_i$ for all i, which means that $\boldsymbol{x}_i$ costs buyer i with money B_i and $\boldsymbol{x}_i$ are in the consumption set of buyer i. Since $(\boldsymbol{x}, \boldsymbol{p})$ is not a market equilibrium, there is at least one buyer that can choose a better allocation $\boldsymbol{x}'_i$ to improve her utility, therefore improve $\text{LFW}(\boldsymbol{p})$, and it cannot be the case that $\text{LFW}(\boldsymbol{p}) = \text{LNW}(\boldsymbol{x})$, which makes a contradiction.

□

B.4 Proof of Proposition 5.6

We leave the formal presentation of Proposition 5.6 and proofs to three theorems below.

Lemma B.1. *Assume that $u_i(\boldsymbol{x}_i)$ is twice differentiable and denote $H(\boldsymbol{x}_i)$ as the Hessian matrix of $u_i(\boldsymbol{x}_i)$. If following hold:*

- $H(\boldsymbol{x}_i^*)$ *has rank* $m - 1$
- $||\boldsymbol{x}_i - \boldsymbol{x}_i^*|| = \varepsilon$ *for some* i
- $\boldsymbol{x}_i^* > \boldsymbol{0}$

then we have $\text{OPT} - \text{LNW}(\boldsymbol{x}) = \Omega(\varepsilon^2)$.

Lemma B.2. *Denote $\tilde{u}_i(\boldsymbol{p}, B_i)$ and $\boldsymbol{x}_i^*(\boldsymbol{p}, B_i)$ as the maximum utility buyer i can get and the corresponding consumption for buyer i when her budget is B_i and prices are $\boldsymbol{p}$. If following hold:*

- $||\boldsymbol{p} - \boldsymbol{p}^*|| = \varepsilon$
- *$\boldsymbol{x}_i^*(\boldsymbol{p}, B_i)$ is differentiable with $\boldsymbol{p}$.*
- *$H_X \coloneqq (\sum_{i\in[n]} \frac{\partial x_{ij}^*}{\partial p_k}(\boldsymbol{p}^*, B_i))_{j,k\in[m]}$ has full rank.*

then we have $LFW(\boldsymbol{p}) - OPT = \Omega(\varepsilon^2)$.

Remark B.3. It's worth notice that $H(\boldsymbol{x}_i^*)$ can not has full rank m, since $u_i(\boldsymbol{x})$ is assumed to be homogeneous and thus linear in the direction $\boldsymbol{x}$. Therefore, we have $H(\boldsymbol{x}_i)\boldsymbol{x}_i = \boldsymbol{0}$ for all $\boldsymbol{x}_i$.

Let $C_i = \{\boldsymbol{x}_i \in \mathbb{R}_+^m : \langle \boldsymbol{p}, \boldsymbol{x}_i \rangle = B_i\}$ be the consumption set of buyer i, since $\boldsymbol{x}_i$ can not be parallel with C_i, the condition that $H(\boldsymbol{x}_i^*)$ has rank $m-1$ means that, $H(\boldsymbol{x}_i)$ is strongly concave at point $\boldsymbol{x}_i^*$ on the consumption set C_i.

Besides, we emphasize that the conditions in Lemma B.1 and Lemma B.2 are satisfied for CES utility with $\alpha < 1$.

Corollary B.4. *Under the assumptions in Lemma B.1 and Lemma B.2, if $\mathrm{NG}(\boldsymbol{x}, \boldsymbol{p}) = \varepsilon$, we have:*

- $||\boldsymbol{p} - \boldsymbol{p}^*|| = O(\sqrt{\varepsilon})$
- *$||\boldsymbol{x}_i - \boldsymbol{x}_i^*|| = O(\sqrt{\varepsilon})$ for all i*

Proof of Corollary B.4. A direct inference from Lemma B.1 and Lemma B.2, notice that $\mathrm{NG} = \varepsilon$ indicates that $\mathrm{OPT} - \mathrm{LNW}(\boldsymbol{x}) \le \varepsilon$ and $\mathrm{LFW}(\boldsymbol{p}) - \mathrm{OPT} \le \varepsilon$. □

Corollary B.4 states that, for a pair of $(\boldsymbol{x}, \boldsymbol{p})$ that satisfy market clearance and price constraints, a small Nash Gap indicates that the point $(\boldsymbol{x}, \boldsymbol{p})$ is close to the equilibrium point $(\boldsymbol{x}^*, \boldsymbol{p}^*)$, in the sense of Euclidean distance.

Lemma B.5. *Assume following hold:*

- *buyers have same utilities at $\boldsymbol{x}^*$, i.e. $u_i(\boldsymbol{x}_i^*) = u_j(\boldsymbol{x}_j^*) \equiv c$ for all $i,\ j$*
- *$||\boldsymbol{x}_i - \boldsymbol{x}_i^*|| \le \varepsilon$ for all i*

then, we have $|\mathrm{WSW}(\boldsymbol{x}) - \mathrm{WSW}(\boldsymbol{x}^*)| = O(\varepsilon^2)$.

Remark B.6. These conditions can be held when buyers are homogeneous, *i.e.*, $B_i = B_j$ and $u_i(\boldsymbol{x}) = u_j(\boldsymbol{x})$ for all $i, j, \boldsymbol{x} \in \mathbb{R}_+^m$. Besides, consider buyers with same budgets, these conditions can also be held if the market has some "equivariance property", *e.g.*, there is a n-cycle permutation of buyers ρ and permutation of goods τ, such that $u_i(\boldsymbol{x}_i) = u_{\rho(i)}(\tau(\boldsymbol{x}_{\rho(i)}))$ for all i and $\tau(Y_1, ..., Y_m) = (Y_1, ..., Y_m)$.

Corollary B.7. *Under the assumptions in Lemma B.1 and Lemma B.5, if $\mathrm{NG}(\boldsymbol{x}, \boldsymbol{p}) = \varepsilon$, we have*

- $|\mathrm{WSW}(\boldsymbol{x}) - \mathrm{WSW}(\boldsymbol{x}^*)| = O(\varepsilon)$.

Proof. A direct inference from Lemma B.1 and Lemma B.5. □

B.4.1 Proof of Lemma B.1

Proof of Lemma B.1. We observe that

$$\mathrm{OPT} - \mathrm{LNW}(\boldsymbol{x}) = \sum_{i\in[n]} B_i \left[\log u_i(\boldsymbol{x}_i^*) - \log u_i(\boldsymbol{x}_i)\right]$$

Consider the Taylor expansion of $\log u_i(\boldsymbol{x}_i)$ and $u_i(\boldsymbol{x}_i)$:

$$\begin{aligned}\log u_i(\boldsymbol{x}_i) =& \log u_i(\boldsymbol{x}_i^*) + \frac{1}{u_i(\boldsymbol{x}_i^*)}(u_i(\boldsymbol{x}_i) - u_i(\boldsymbol{x}_i^*)) \\ &- \frac{1}{2u_i(\boldsymbol{x}_i^*)^2}(u_i(\boldsymbol{x}_i) - u_i(\boldsymbol{x}_i^*))^2 \\ &+ O((u_i(\boldsymbol{x}_i) - u_i(\boldsymbol{x}_i^*))^3) \\ u_i(\boldsymbol{x}_i) =& u_i(\boldsymbol{x}_i^*) + \frac{\partial u_i}{\partial \boldsymbol{x}_i}(\boldsymbol{x}_i^*)(\boldsymbol{x}_i - \boldsymbol{x}_i^*) \\ &+ \frac{1}{2}(\boldsymbol{x}_i - \boldsymbol{x}_i^*)^T H(\boldsymbol{x}_i^*)(\boldsymbol{x}_i - \boldsymbol{x}_i^*) + O(||\boldsymbol{x}_i - \boldsymbol{x}_i^*||^3)\end{aligned}$$

Notice that $||\boldsymbol{x}_i - \boldsymbol{x}_i^*|| = \varepsilon$, we have

$$\begin{aligned}\log u_i(\boldsymbol{x}_i) =& \log u_i(\boldsymbol{x}_i^*) \\ &+ \frac{1}{u_i(\boldsymbol{x}_i^*)}[\frac{\partial u_i}{\partial x_i}(\boldsymbol{x}_i^*)(\boldsymbol{x}_i - \boldsymbol{x}_i^*) \cdots \varepsilon \text{ term} && (51) \\ &+ \frac{1}{2}(\boldsymbol{x}_i - \boldsymbol{x}_i^*)^T H(\boldsymbol{x}_i^*)(\boldsymbol{x}_i - \boldsymbol{x}_i^*)] \cdots \varepsilon^2 \text{ term} && (52) \\ &- \frac{1}{2u_i(\boldsymbol{x}_i^*)^2}\left(\frac{\partial u_i}{\partial \boldsymbol{x}_i}(\boldsymbol{x}_i^*)(\boldsymbol{x}_i - \boldsymbol{x}_i^*)\right)^2 \cdots \varepsilon^2 \text{ term} && (53) \\ &+ O(\varepsilon^3)\end{aligned}$$

We next deal with Equation (51) to Equation (53) separately.

Derivation of Equation (51) Since $\boldsymbol{x}_i^*$ solves the buyer i's problem, we must have

$$\frac{\partial u_i}{\partial x_i}(\boldsymbol{x}_i^*) = \lambda_i \boldsymbol{p}^* \tag{54}$$

where λ_i is the Lagrangian Multipliers for buyer i.

We also know that $u_i(\boldsymbol{x}_i)$ is homogeneous with degree 1, by Euler formula, we derive

$$\langle \frac{\partial u_i}{\partial x_i}(\boldsymbol{x}_i), \boldsymbol{x}_i \rangle = u_i(\boldsymbol{x}_i) \tag{55}$$

Combine Equation (54) and Equation (55) and take $\boldsymbol{x}_i = \boldsymbol{x}_i^*$, we derive

$$\begin{aligned}\lambda_i \langle \boldsymbol{p}^*, \boldsymbol{x}_i^* \rangle =& u_i(\boldsymbol{x}_i^*) \\ \lambda_i =& \frac{u_i(\boldsymbol{x}_i^*)}{B_i} \\ \frac{\partial u_i}{\partial x_i}(\boldsymbol{x}_i^*) =& \frac{u_i(\boldsymbol{x}_i^*)}{B_i}\boldsymbol{p}^*\end{aligned}$$

Sum up over i for Equation (51), we have

$$\begin{aligned}&\sum_{i \in [n]} B_i \frac{1}{u_i(\boldsymbol{x}_i^*)} \frac{\partial u_i}{\partial x_i}(\boldsymbol{x}_i^*)(\boldsymbol{x}_i - \boldsymbol{x}_i^*) \\ =& \boldsymbol{p} \sum_{i \in [n]} (\boldsymbol{x}_i - \boldsymbol{x}_i^*) && (56) \\ =& 0 \cdots \text{by market clearance}\end{aligned}$$

Derivation of Equation (52) and Equation (53) Combining Equation (52) and Equation (53), we have

$$\begin{aligned}&\frac{B_i}{2u_i(\boldsymbol{x}_i^*)}(\boldsymbol{x}_i - \boldsymbol{x}_i^*)^T H(\boldsymbol{x}_i^*)(\boldsymbol{x}_i - \boldsymbol{x}_i^*) - \frac{1}{2B_i}(\boldsymbol{x}_i - \boldsymbol{x}_i^*)^T(\boldsymbol{p}^* \boldsymbol{p}^{*T})(\boldsymbol{x}_i - \boldsymbol{x}_i^*) \\ =& \frac{1}{2B_i}(\boldsymbol{x}_i - \boldsymbol{x}_i^*)^T(\frac{B_i^2}{u_i(\boldsymbol{x}_i^*)} H(\boldsymbol{x}_i^*) - \boldsymbol{p}^* \boldsymbol{p}^{*T})(\boldsymbol{x}_i - \boldsymbol{x}_i^*)\end{aligned}$$

Denote $H_0(\boldsymbol{x}_i^*) = \frac{B_i^2}{u_i(\boldsymbol{x}_i^*)} H(\boldsymbol{x}_i^*) - \boldsymbol{p}^* \boldsymbol{p}^{*T}$, next we assert that $H_0(\boldsymbol{x}_i^*)$ is negative definite.

Since $H(\boldsymbol{x}_i^*)$ and $-\boldsymbol{p}^* \boldsymbol{p}^{*T}$ are negative semi-definite, $H_0(\boldsymbol{x}_i^*)$ must be negative semi-definite with $\mathrm{rank}(H_0(\boldsymbol{x}_i^*)) \geq m-1$.

Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{m-1} < \lambda_m = 0$ be eigenvalues and $v_1, ..., v_n = \boldsymbol{x}_i^*$ be eigenvectors of $H(\boldsymbol{x}_i^*)$. If $\mathrm{rank}(H_0(\boldsymbol{x}_i^*)) = m-1$, it means that $\boldsymbol{x}_i^*$ has to be eigenvectors of $-\boldsymbol{p}^* \boldsymbol{p}^{*T}$ with eigenvalue 0. However, we have $-\boldsymbol{p}^* \boldsymbol{p}^{*T} \boldsymbol{x}_i^* = -B_i \boldsymbol{p}^* \neq 0$, which leads to a contradiction.

Therefore, we have $\mathrm{rank}(H_0(\boldsymbol{x}_i^*)) = m$ and $H_0(\boldsymbol{x}_i^*)$ is negative definite, we denote $\lambda_1^i \leq ..., \leq \lambda_n^i < 0$ as the eigenvalues of $H_0(\boldsymbol{x}_i^*)$, and k as the universal lower bound for $|\lambda_n^i|$, then we have that,

$$\frac{1}{2}(\boldsymbol{x}_i - \boldsymbol{x}_i^*)^T H_0(\boldsymbol{x}_i^*)(\boldsymbol{x}_i - \boldsymbol{x}_i^*) \leq -\frac{k}{2}\varepsilon^2 \tag{57}$$

By combining Equation (56) and Equation (57), we have

$$\begin{aligned} \mathrm{OPT} - \mathrm{LNW}(\boldsymbol{x}) = & -\sum_{i \in [n]} B_i \left[\frac{1}{2B_i} (\boldsymbol{x}_i - \boldsymbol{x}_i^*)^T H_0(\boldsymbol{x}_i^*)(\boldsymbol{x}_i - \boldsymbol{x}_i^*) \right] + O(\varepsilon^3) \\ \geq & \frac{k}{2}\varepsilon^2 + O(\varepsilon^3) \\ = & \Omega(\varepsilon^2) \end{aligned} \tag{58}$$

□

B.4.2 Proof of Lemma B.2

Proof of Lemma B.2. The proof is similar with Appendix B.4.1 by using Taylor expansion technique. Before that, we first derive some identities.

By Roy's identity, we have

$$\frac{\partial \tilde{u}_i}{\partial p_j}(\boldsymbol{p}, B_i) = -x_{ij}^*(\boldsymbol{p}, B_i) \frac{\partial \tilde{u}_i}{\partial B_i}(\boldsymbol{p}, B_i)$$

Since $u(\boldsymbol{x}_i)$ is homogeneous with $\boldsymbol{x}_i$, it's easy to derive that

$$\frac{\partial \tilde{u}_i}{\partial B_i}(\boldsymbol{p}, B_i) = \frac{\tilde{u}_i(\boldsymbol{p}, B_i)}{B_i}$$

Above all,

$$\frac{\partial \tilde{u}_i}{\partial p_j}(\boldsymbol{p}, B_i) = -\frac{1}{B_i} x_{ij}^*(\boldsymbol{p}, B_i) \tilde{u}_i(\boldsymbol{p}, B_i)$$

Besides,

$$\begin{aligned} \frac{\partial^2 \tilde{u}_i}{\partial p_j \partial p_k}(\boldsymbol{p}, B_i) = & \frac{1}{B_i^2} x_{ij}^*(\boldsymbol{p}, B_i) x_{ik}^*(\boldsymbol{p}, B_i) \tilde{u}_i(\boldsymbol{p}, B_i) \\ & - \frac{1}{B_i} \frac{x_{ij}^*(\boldsymbol{p}, B_i)}{\partial p_k} \tilde{u}_i(\boldsymbol{p}, B_i) \end{aligned}$$

Next we consider the Taylor expansion,

$$\begin{aligned} \log \tilde{u}_i(\boldsymbol{p}) = & \log \tilde{u}_i(\boldsymbol{p}^*) \\ & + \frac{1}{\tilde{u}_i(\boldsymbol{p}^*)} [\frac{\partial \tilde{u}_i}{\partial \boldsymbol{p}}(\boldsymbol{p}^*)(\boldsymbol{p} - \boldsymbol{p}^*) \cdots \varepsilon \text{ term} & (59) \\ & + \frac{1}{2}(\boldsymbol{p} - \boldsymbol{p}^*)^T H_p (\boldsymbol{p} - \boldsymbol{p}^*)] \cdots \varepsilon^2 \text{ term} & (60) \\ & - \frac{1}{2\tilde{u}_i(\boldsymbol{p}^*)^2} \left[\frac{\partial \tilde{u}_i}{\partial \boldsymbol{p}}(\boldsymbol{p}^*)(\boldsymbol{p} - \boldsymbol{p}^*) \right]^2 \cdots \varepsilon^2 \text{ term} & (61) \\ & + O(\varepsilon^3) \end{aligned}$$

where H_p is the Hessian matrix for $\tilde{u}_i(\boldsymbol{p})$.

Derivation of Equation (59) We have

$$\begin{aligned}
&\sum_{i\in[n]} B_i \frac{1}{\tilde{u}_i(\boldsymbol{p}^*)}\langle \frac{\partial \tilde{u}_i}{\partial \boldsymbol{p}}(\boldsymbol{p}^*), (\boldsymbol{p}-\boldsymbol{p}^*)\rangle \\
=&\sum_{i\in[n]} \langle \boldsymbol{x}_i^*, (\boldsymbol{p}-\boldsymbol{p}^*)\rangle \\
=&\langle \boldsymbol{1}, (\boldsymbol{p}-\boldsymbol{p}^*)\rangle \cdots \text{by market clearance} \\
=&0 \cdots \text{by price constraints}
\end{aligned}$$

Derivation of Equation (60) and Equation (61) These expressions become

$$\begin{aligned}
&\frac{1}{2\tilde{u}_i(\boldsymbol{p}^*)}[\frac{1}{B_i^2}\tilde{u}_i(\boldsymbol{p}^*)\langle \boldsymbol{x}_i^*, \boldsymbol{p}-\boldsymbol{p}^*\rangle^2 - \frac{1}{B_i}\tilde{u}_i(\boldsymbol{p}^*)(\boldsymbol{p}-\boldsymbol{p}^*)^T(\frac{\partial x_{ij}^*}{\partial p_k}(\boldsymbol{p}^*, B_i))_{j,k\in[m]}(\boldsymbol{p}-\boldsymbol{p}^*)] \\
-&\frac{1}{2\tilde{u}_i(\boldsymbol{p}^*)^2}\frac{\tilde{u}_i(\boldsymbol{p}^*)^2}{B_i^2}\langle \boldsymbol{x}_i^*, \boldsymbol{p}-\boldsymbol{p}^*\rangle^2 \\
=&\frac{1}{2B_i}(\boldsymbol{p}-\boldsymbol{p}^*)^T(\frac{\partial x_{ij}^*}{\partial p_k}(\boldsymbol{p}^*, B_i))_{j,k\in[m]}(\boldsymbol{p}-\boldsymbol{p}^*)
\end{aligned}$$

Summing up over i, we derive that

$$\begin{aligned}
\mathrm{LFW}(\boldsymbol{p}) - \mathrm{OPT} =& \sum_{i\in[n]} B_i \frac{1}{2B_i}(\boldsymbol{p}-\boldsymbol{p}^*)^T(\frac{\partial x_{ij}^*}{\partial p_k}(\boldsymbol{p}^*, B_i))_{j,k\in[m]}(\boldsymbol{p}-\boldsymbol{p}^*) + O(\varepsilon^3) \\
=&\frac{1}{2}(\boldsymbol{p}-\boldsymbol{p}^*)^T H_X (\boldsymbol{p}-\boldsymbol{p}^*) + O(\varepsilon^3)
\end{aligned}$$

Since $\boldsymbol{p}^*$ gets the minimum of $\mathrm{LFW}(\boldsymbol{p})$, we must have that H_X is positive semi-definite. Together with H_X has full rank, we know that H_X is positive definite. Denote λ_m as the minimum eigenvalues of H_X, we have

$$\begin{aligned}
\mathrm{LFW}(\boldsymbol{p}) - \mathrm{OPT} \geq& \frac{\varepsilon^2 \lambda_m}{2} + O(\varepsilon^3) \\
=&\Omega(\varepsilon^2)
\end{aligned}$$

□

B.4.3 Proof of Lemma B.5

Proof of Lemma B.5. Notice that

$$\mathrm{WSW}(\boldsymbol{x}) = \mathrm{WSW}(\boldsymbol{x}^*) + \sum_{i\in[n]} \langle \frac{\partial \mathrm{WSW}}{\partial \boldsymbol{x}_i}(\boldsymbol{x}_i^*), (\boldsymbol{x}_i - \boldsymbol{x}_i^*)\rangle + O(\varepsilon^2)$$

We have

$$\begin{aligned}
&\frac{\partial \mathrm{WSW}}{\partial \boldsymbol{x}_i}(\boldsymbol{x}_i^*) \\
=&B_i \frac{\partial u_i}{\partial \boldsymbol{x}_i}(\boldsymbol{x}_i^*) \\
=&B_i \frac{u_i(\boldsymbol{x}_i^*)}{B_i}\boldsymbol{p}^* \\
=&c\boldsymbol{p}^*
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{WSW}(\boldsymbol{x}) =& \mathrm{WSW}(\boldsymbol{x}^*) + \sum_{i\in[n]} c\langle \boldsymbol{p}^*, \boldsymbol{x}_i - \boldsymbol{x}_i^*\rangle + O(\varepsilon^2) \\
=&\mathrm{WSW}(\boldsymbol{x}^*) + O(\varepsilon^2) \cdots \text{market clearance}
\end{aligned}$$

which completes the proof.

□

C Additional Experiments Details

C.1 More about baselines

EG program solver (abbreviated as EG) We propose the first baseline algorithm EG. Recall the Eisenberg-Gale convex program(EG):

$$\max \quad \frac{1}{n}\sum_{i=1}^{n} B_i \log u_i(\boldsymbol{x}_i) \quad \text{s.t. } \frac{1}{n}\sum_{i=1}^{n} x_{ij} = 1,\ x \geq 0. \tag{62}$$

We use the network module in pytorch to represent the parameters $\boldsymbol{x} \in \mathbb{R}_+^{n\times m}$, and softplus activation function to satisfy $x \geq 0$ automatedly. We use gradient ascent algorithm to optimize the parameters $\boldsymbol{x}$. For constraint $\frac{1}{n}\sum_{i\in[n]} x_{ij} = 1$, we introduce Lagrangian multipliers λ_j and minimize the Lagrangian:

$$\mathcal{L}_\rho(\boldsymbol{x};\boldsymbol{\lambda}) = -\frac{1}{n}\sum_{i\in[n]} B_i \log u_i(\boldsymbol{x}_i) + \sum_{j\in[m]} \lambda_j \left(\frac{1}{n}\sum_{i\in[n]} x_{ij} - 1\right) \tag{63}$$

$$+\frac{\rho}{2}\sum_{j\in[m]}\left(\frac{1}{n}\sum_{i\in[n]} x_{ij} - 1\right)^2 \tag{64}$$

The updates of $\boldsymbol{\lambda}$ is $\lambda_j \leftarrow \lambda_j + \beta_t\rho\left(\frac{1}{n}\sum_{i\in[n]} x_{ij} - 1\right)$, here β_t is step size, which is identical with that in MarketFCNet. The algorithm returns the final $(\boldsymbol{x}, \boldsymbol{\lambda})$ as the approximated market equilibrium.

EG program solver with momentum (abbreviated as EG-m) The program to solve is exactly same with that in EG. The only difference is that we use gradient ascent with momentum to optimize the parameters $\boldsymbol{x}$.

C.2 More Experimental Details

Without special specification, we use the experiment settings as follows. All experiments are conducted in one RTX 4090 graphics cards using 16 CPUs or 1 GPU. We set dimension of representations of buyers and goods to be $d = 5$. Each elements in representation is i.i.d from $\mathcal{N}(0, 1)$ for normal distribution (default) contexts, $U[0, 1]$ for uniform distribution contexts and $Exp(1)$ for exponential distribution contexts. Budget is generated with $B(b) = ||b||_2$, and valuation in utility function is generated with $v(b, g) = \text{softplus}(\langle b, g\rangle)$, where $\text{softplus}(x) = \log(1 + \exp(x))$ is a smoothing function that maps each real number to be positive. α in CES utility are chosen to be 0.5 by default. MarketFCNet is designed as a fully connected network with depth 5 and width 256 per layer. ρ is chosen to be 0.2 in Augmented Lagrange Multiplier Method and the step size β_t is chosen to be $\frac{1}{\sqrt{t}}$. We choose $K = 100$ as inner iteration for each epoch, and training for 30 epochs in MarketFCNet. For *EG* and *EG-m* baselines, we choose the inner iteration $K = 1000$ when $n > 1000$ and $K = 100$ when $n \leq 1000$ for each epoch. Baselines are enssembled with early stopping as long as NG is lower than 10^{-3}. Both baselines are optimized for 30 epochs in total.

We use Adam optimizer and learning rate $1e-4$ to optimize the allocation network in MarketFCNet. When computing $\Delta\lambda_j$ in MarketFCNet, we directly compute $\Delta\lambda_j$ rather than generate an unbiased estimator, since it does not cost too much to consider all buyers for one time. For those baselines, we use gradient descent to optimize the parameters following existing works, and the step size is fine-tuned to be $1e+2$ for $\alpha = 1, n > 1000$; $1e+3$ for $\alpha < 1, n > 1000$ and 1 for $\alpha < 1,\ n \leq 1000$ and 0.1 for $\alpha = 1,\ n \leq 1000$ for better performances of the baselines. Since that Lagrangian multipliers $\lambda \leq 0$ will indicate an illegal Nash Gap measure, therefore, we hard code EG algorithm such that it will only return a result when it satisfies that the price $\lambda_j > 0$ for all good j. All baselines are run in GPU when $n > 1000$ and CPU when $n \leq 1000$.[1]

[1]We find in the experiments when market size is pretty large, baselines run slower on CPU than on GPU and this phenomenon reverses when market size is small. Therefore, the hardware on which baselines run depend on the market size and we always choose the faster one in experiments.

NeurIPS Paper Checklist

1. **Claims**

   Question: Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope?

   Answer: **[TODO]**[Yes]

   Justification: **[TODO]**

   Guidelines:

   - The answer NA means that the abstract and introduction do not include the claims made in the paper.
   - The abstract and/or introduction should clearly state the claims made, including the contributions made in the paper and important assumptions and limitations. A No or NA answer to this question will not be perceived well by the reviewers.
   - The claims made should match theoretical and experimental results, and reflect how much the results can be expected to generalize to other settings.
   - It is fine to include aspirational goals as motivation as long as it is clear that these goals are not attained by the paper.

2. **Limitations**

   Question: Does the paper discuss the limitations of the work performed by the authors?

   Answer: **[TODO]**[Yes]

   Justification: **[TODO]**We discuss the limitations in Section 7.

   Guidelines:

   - The answer NA means that the paper has no limitation while the answer No means that the paper has limitations, but those are not discussed in the paper.
   - The authors are encouraged to create a separate "Limitations" section in their paper.
   - The paper should point out any strong assumptions and how robust the results are to violations of these assumptions (e.g., independence assumptions, noiseless settings, model well-specification, asymptotic approximations only holding locally). The authors should reflect on how these assumptions might be violated in practice and what the implications would be.
   - The authors should reflect on the scope of the claims made, e.g., if the approach was only tested on a few datasets or with a few runs. In general, empirical results often depend on implicit assumptions, which should be articulated.
   - The authors should reflect on the factors that influence the performance of the approach. For example, a facial recognition algorithm may perform poorly when image resolution is low or images are taken in low lighting. Or a speech-to-text system might not be used reliably to provide closed captions for online lectures because it fails to handle technical jargon.
   - The authors should discuss the computational efficiency of the proposed algorithms and how they scale with dataset size.
   - If applicable, the authors should discuss possible limitations of their approach to address problems of privacy and fairness.
   - While the authors might fear that complete honesty about limitations might be used by reviewers as grounds for rejection, a worse outcome might be that reviewers discover limitations that aren't acknowledged in the paper. The authors should use their best judgment and recognize that individual actions in favor of transparency play an important role in developing norms that preserve the integrity of the community. Reviewers will be specifically instructed to not penalize honesty concerning limitations.

3. **Theory Assumptions and Proofs**

   Question: For each theoretical result, does the paper provide the full set of assumptions and a complete (and correct) proof?

   Answer: **[TODO]**[No]

Justification: **[TODO]**The answer is [Yes] except for Theorem 3.1. Theorem 3.1 is a restated theorem of Gao and Kroer [30] and we do not cover that proof in this paper.

Guidelines:

- The answer NA means that the paper does not include theoretical results.
- All the theorems, formulas, and proofs in the paper should be numbered and cross-referenced.
- All assumptions should be clearly stated or referenced in the statement of any theorems.
- The proofs can either appear in the main paper or the supplemental material, but if they appear in the supplemental material, the authors are encouraged to provide a short proof sketch to provide intuition.
- Inversely, any informal proof provided in the core of the paper should be complemented by formal proofs provided in appendix or supplemental material.
- Theorems and Lemmas that the proof relies upon should be properly referenced.

4. **Experimental Result Reproducibility**

   Question: Does the paper fully disclose all the information needed to reproduce the main experimental results of the paper to the extent that it affects the main claims and/or conclusions of the paper (regardless of whether the code and data are provided or not)?

   Answer: **[TODO]**[Yes]

   Justification: **[TODO]**We present the experimental details in Appendix C.

   Guidelines:

   - The answer NA means that the paper does not include experiments.
   - If the paper includes experiments, a No answer to this question will not be perceived well by the reviewers: Making the paper reproducible is important, regardless of whether the code and data are provided or not.
   - If the contribution is a dataset and/or model, the authors should describe the steps taken to make their results reproducible or verifiable.
   - Depending on the contribution, reproducibility can be accomplished in various ways. For example, if the contribution is a novel architecture, describing the architecture fully might suffice, or if the contribution is a specific model and empirical evaluation, it may be necessary to either make it possible for others to replicate the model with the same dataset, or provide access to the model. In general. releasing code and data is often one good way to accomplish this, but reproducibility can also be provided via detailed instructions for how to replicate the results, access to a hosted model (e.g., in the case of a large language model), releasing of a model checkpoint, or other means that are appropriate to the research performed.
   - While NeurIPS does not require releasing code, the conference does require all submissions to provide some reasonable avenue for reproducibility, which may depend on the nature of the contribution. For example
     (a) If the contribution is primarily a new algorithm, the paper should make it clear how to reproduce that algorithm.
     (b) If the contribution is primarily a new model architecture, the paper should describe the architecture clearly and fully.
     (c) If the contribution is a new model (e.g., a large language model), then there should either be a way to access this model for reproducing the results or a way to reproduce the model (e.g., with an open-source dataset or instructions for how to construct the dataset).
     (d) We recognize that reproducibility may be tricky in some cases, in which case authors are welcome to describe the particular way they provide for reproducibility. In the case of closed-source models, it may be that access to the model is limited in some way (e.g., to registered users), but it should be possible for other researchers to have some path to reproducing or verifying the results.

5. **Open access to data and code**

   Question: Does the paper provide open access to the data and code, with sufficient instructions to faithfully reproduce the main experimental results, as described in supplemental material?

Answer: **[TODO]**[No]

Justification: **[TODO]**The code need to be more finely organized before it goes public.

Guidelines:

- The answer NA means that paper does not include experiments requiring code.
- Please see the NeurIPS code and data submission guidelines (`https://nips.cc/public/guides/CodeSubmissionPolicy`) for more details.
- While we encourage the release of code and data, we understand that this might not be possible, so "No" is an acceptable answer. Papers cannot be rejected simply for not including code, unless this is central to the contribution (e.g., for a new open-source benchmark).
- The instructions should contain the exact command and environment needed to run to reproduce the results. See the NeurIPS code and data submission guidelines (`https://nips.cc/public/guides/CodeSubmissionPolicy`) for more details.
- The authors should provide instructions on data access and preparation, including how to access the raw data, preprocessed data, intermediate data, and generated data, etc.
- The authors should provide scripts to reproduce all experimental results for the new proposed method and baselines. If only a subset of experiments are reproducible, they should state which ones are omitted from the script and why.
- At submission time, to preserve anonymity, the authors should release anonymized versions (if applicable).
- Providing as much information as possible in supplemental material (appended to the paper) is recommended, but including URLs to data and code is permitted.

6. **Experimental Setting/Details**

Question: Does the paper specify all the training and test details (e.g., data splits, hyper-parameters, how they were chosen, type of optimizer, etc.) necessary to understand the results?

Answer: **[TODO]**[Yes]

Justification: **[TODO]**These are presented in Appendix C

Guidelines:

- The answer NA means that the paper does not include experiments.
- The experimental setting should be presented in the core of the paper to a level of detail that is necessary to appreciate the results and make sense of them.
- The full details can be provided either with the code, in appendix, or as supplemental material.

7. **Experiment Statistical Significance**

Question: Does the paper report error bars suitably and correctly defined or other appropriate information about the statistical significance of the experiments?

Answer: **[TODO]**[No]

Justification: **[TODO]**Since the difference between baselines and our method is prominent, we believe that one experiment on each setting is an enough certificate to show the effectiveness of our method.

Guidelines:

- The answer NA means that the paper does not include experiments.
- The authors should answer "Yes" if the results are accompanied by error bars, confidence intervals, or statistical significance tests, at least for the experiments that support the main claims of the paper.
- The factors of variability that the error bars are capturing should be clearly stated (for example, train/test split, initialization, random drawing of some parameter, or overall run with given experimental conditions).
- The method for calculating the error bars should be explained (closed form formula, call to a library function, bootstrap, etc.)
- The assumptions made should be given (e.g., Normally distributed errors).

- It should be clear whether the error bar is the standard deviation or the standard error of the mean.
- It is OK to report 1-sigma error bars, but one should state it. The authors should preferably report a 2-sigma error bar than state that they have a 96% CI, if the hypothesis of Normality of errors is not verified.
- For asymmetric distributions, the authors should be careful not to show in tables or figures symmetric error bars that would yield results that are out of range (e.g. negative error rates).
- If error bars are reported in tables or plots, The authors should explain in the text how they were calculated and reference the corresponding figures or tables in the text.

8. **Experiments Compute Resources**

   Question: For each experiment, does the paper provide sufficient information on the computer resources (type of compute workers, memory, time of execution) needed to reproduce the experiments?

   Answer: **[TODO]**[Yes]

   Justification: **[TODO]**See Appendix C.

   Guidelines:

   - The answer NA means that the paper does not include experiments.
   - The paper should indicate the type of compute workers CPU or GPU, internal cluster, or cloud provider, including relevant memory and storage.
   - The paper should provide the amount of compute required for each of the individual experimental runs as well as estimate the total compute.
   - The paper should disclose whether the full research project required more compute than the experiments reported in the paper (e.g., preliminary or failed experiments that didn't make it into the paper).

9. **Code Of Ethics**

   Question: Does the research conducted in the paper conform, in every respect, with the NeurIPS Code of Ethics `https://neurips.cc/public/EthicsGuidelines`?

   Answer: **[TODO]**[Yes]

   Justification: **[TODO]**

   Guidelines:

   - The answer NA means that the authors have not reviewed the NeurIPS Code of Ethics.
   - If the authors answer No, they should explain the special circumstances that require a deviation from the Code of Ethics.
   - The authors should make sure to preserve anonymity (e.g., if there is a special consideration due to laws or regulations in their jurisdiction).

10. **Broader Impacts**

    Question: Does the paper discuss both potential positive societal impacts and negative societal impacts of the work performed?

    Answer: **[TODO]**[Yes]

    Justification: **[TODO]**The accelaration of market equilibrium computation is a positive social impact.

    Guidelines:

    - The answer NA means that there is no societal impact of the work performed.
    - If the authors answer NA or No, they should explain why their work has no societal impact or why the paper does not address societal impact.
    - Examples of negative societal impacts include potential malicious or unintended uses (e.g., disinformation, generating fake profiles, surveillance), fairness considerations (e.g., deployment of technologies that could make decisions that unfairly impact specific groups), privacy considerations, and security considerations.

- The conference expects that many papers will be foundational research and not tied to particular applications, let alone deployments. However, if there is a direct path to any negative applications, the authors should point it out. For example, it is legitimate to point out that an improvement in the quality of generative models could be used to generate deepfakes for disinformation. On the other hand, it is not needed to point out that a generic algorithm for optimizing neural networks could enable people to train models that generate Deepfakes faster.
- The authors should consider possible harms that could arise when the technology is being used as intended and functioning correctly, harms that could arise when the technology is being used as intended but gives incorrect results, and harms following from (intentional or unintentional) misuse of the technology.
- If there are negative societal impacts, the authors could also discuss possible mitigation strategies (e.g., gated release of models, providing defenses in addition to attacks, mechanisms for monitoring misuse, mechanisms to monitor how a system learns from feedback over time, improving the efficiency and accessibility of ML).

11. **Safeguards**

    Question: Does the paper describe safeguards that have been put in place for responsible release of data or models that have a high risk for misuse (e.g., pretrained language models, image generators, or scraped datasets)?

    Answer: **[TODO]**[NA]

    Justification: **[TODO]**

    Guidelines:

    - The answer NA means that the paper poses no such risks.
    - Released models that have a high risk for misuse or dual-use should be released with necessary safeguards to allow for controlled use of the model, for example by requiring that users adhere to usage guidelines or restrictions to access the model or implementing safety filters.
    - Datasets that have been scraped from the Internet could pose safety risks. The authors should describe how they avoided releasing unsafe images.
    - We recognize that providing effective safeguards is challenging, and many papers do not require this, but we encourage authors to take this into account and make a best faith effort.

12. **Licenses for existing assets**

    Question: Are the creators or original owners of assets (e.g., code, data, models), used in the paper, properly credited and are the license and terms of use explicitly mentioned and properly respected?

    Answer: **[TODO]**[NA]

    Justification: **[TODO]**

    Guidelines:

    - The answer NA means that the paper does not use existing assets.
    - The authors should cite the original paper that produced the code package or dataset.
    - The authors should state which version of the asset is used and, if possible, include a URL.
    - The name of the license (e.g., CC-BY 4.0) should be included for each asset.
    - For scraped data from a particular source (e.g., website), the copyright and terms of service of that source should be provided.
    - If assets are released, the license, copyright information, and terms of use in the package should be provided. For popular datasets, `paperswithcode.com/datasets` has curated licenses for some datasets. Their licensing guide can help determine the license of a dataset.
    - For existing datasets that are re-packaged, both the original license and the license of the derived asset (if it has changed) should be provided.

- If this information is not available online, the authors are encouraged to reach out to the asset's creators.

13. **New Assets**

    Question: Are new assets introduced in the paper well documented and is the documentation provided alongside the assets?

    Answer: **[TODO]**[NA]

    Justification: **[TODO]**

    Guidelines:

    - The answer NA means that the paper does not release new assets.
    - Researchers should communicate the details of the dataset/code/model as part of their submissions via structured templates. This includes details about training, license, limitations, etc.
    - The paper should discuss whether and how consent was obtained from people whose asset is used.
    - At submission time, remember to anonymize your assets (if applicable). You can either create an anonymized URL or include an anonymized zip file.

14. **Crowdsourcing and Research with Human Subjects**

    Question: For crowdsourcing experiments and research with human subjects, does the paper include the full text of instructions given to participants and screenshots, if applicable, as well as details about compensation (if any)?

    Answer: **[TODO]**[NA]

    Justification: **[TODO]**

    Guidelines:

    - The answer NA means that the paper does not involve crowdsourcing nor research with human subjects.
    - Including this information in the supplemental material is fine, but if the main contribution of the paper involves human subjects, then as much detail as possible should be included in the main paper.
    - According to the NeurIPS Code of Ethics, workers involved in data collection, curation, or other labor should be paid at least the minimum wage in the country of the data collector.

15. **Institutional Review Board (IRB) Approvals or Equivalent for Research with Human Subjects**

    Question: Does the paper describe potential risks incurred by study participants, whether such risks were disclosed to the subjects, and whether Institutional Review Board (IRB) approvals (or an equivalent approval/review based on the requirements of your country or institution) were obtained?

    Answer: **[TODO]**[NA]

    Justification: **[TODO]**

    Guidelines:

    - The answer NA means that the paper does not involve crowdsourcing nor research with human subjects.
    - Depending on the country in which research is conducted, IRB approval (or equivalent) may be required for any human subjects research. If you obtained IRB approval, you should clearly state this in the paper.
    - We recognize that the procedures for this may vary significantly between institutions and locations, and we expect authors to adhere to the NeurIPS Code of Ethics and the guidelines for their institution.
    - For initial submissions, do not include any information that would break anonymity (if applicable), such as the institution conducting the review.