## NO LLM SOLVED YU TSUMURA'S 554TH PROBLEM

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#### ABSTRACT

We show, contrary to the optimism about LLM's problem-solving abilities, fueled by the recent gold medals at the International Math Olympiad (IMO) that LLMs attained, that a problem exists—Yu Tsumura's 554th problem—that a) is within the scope of an IMO problem in terms of proof sophistication, b) is not a combinatorics problem, which have caused issues for LLMs, c) requires fewer proof techniques than typical hard IMO problems, d) has a publicly available solution (likely in the training data of LLMs), and e) that cannot be readily solved by *any* existing off-the-shelf LLM (commercial or open-source). We include an analysis of the output traces of 16 SOTA LLMs. Additionally, we compare the generic LLM output to a new proof by a former IMO participant, carried out in a small study, which is significantly better motivated than the original, publicly-available proof, and elaborate on the differences in LLM and human *proof quality*.

## 1 Introduction

The results achieved by several commercial companies at the International Mathematical Olympiad<sup>1</sup> in 2025 (IMO25) have been hailed as a milestone in AI in press releases.<sup>2</sup>, as well as "awfully impressive" by some researchers <sup>3</sup>. Lending credence to these statements is a recent replication of some of these results in a scientific setting (Huang & Yang, 2025) by using a more complex verification scheme, as well as an the *OpenAI* × *AIMO* evaluation from March, released in August, on a version of OpenAI o3's model that solved 47/50 Olympiad-level hidden math problems there were used for the AIMO2 competition.<sup>4</sup> Due to the difficulty of the involved problems, these results paint a very optimistic future for the reasoning abilities of state-of-the-art LLMs.

In this paper, we present a counterclaim that offers a more nuanced perspective on the current state of affairs. The previous results show that there exist LLMs that solve math problems that require intricate reasoning abilities, for which likely no solution was in the training data ahead of time, due to the recency of the problems. We show the converse: There exist mathematical problems that require intricate reasoning abilities that no current off-the-shelf LLMs have, for which a full solution exists online, posted at a time that predates the advent of LLMs.<sup>5</sup>

For IMO25 problems, both AI systems solving the problems formalized in Lean4 were evaluated, as well as system processing the problem formulated in natural-language. We focus our analysis solely on the natural-language problems.

Specifically, we show that there exists a problem that none of the current set of widely used LLMs, whether proprietary or open-weight, can solve. This problem is publicly available and is Yu Tsumura's 554th problem.<sup>6</sup> It is a group theory problem, but we emphasize that no specialized knowledge of group theory is needed. We reproduce it below:

<sup>1</sup>imo-official.org

<sup>2</sup>https://blog.google/products/gemini/gemini-2-5-deep-think/

<sup>3</sup>https://www.nature.com/articles/d41586-025-02343-x

<sup>4</sup>https://aimoprize.com/updates/2025-09-05-the-gap-is-shrinking

<sup>&</sup>lt;sup>5</sup>On this archive.org link, the year 2017 is the first time the problem is listed online. This link contains a copy of the statement and the proof.

<sup>&</sup>lt;sup>6</sup>Yu Tsumura's 554th problem.

Let x, y be generators of a group G with relations

$$xy^2 = y^3x,$$

$$yx^2 = x^3y.$$

*Prove that G is the trivial group.* 

Its proof, which is also provided in the link above, is short and requires nothing more than clever symbolic manipulation—a task that LLMs solving Olympiad-level math problems need to possess. In fact, this problem is similar to an IMO problem, such as a functional equation problem or inequality, where there are established proof techniques. This should be the easiest case for an LLM in terms of problem difficulty. While the proof, provided in the link above, utilizes the concepts of conjugacy and the order of a group element, these are only mentioned once and can be unpacked. This makes the proof independent of any specific knowledge about group theory.

We speculate that the problem is difficult for LLMs because solving the problem involves a deep search through identities that can be derived from the original relations. There are two potential reasons why this poses a problem:

- The probability of the LLM hallucinating/making an algebraic error before finding the required identities is very high.
- The LLM is not trained to search to a high enough expression depth.

Lastly, we note that 60 members of the public have self-reported on Yu Tsumura's website that they solved the problem. In addition to this, we conducted a n=1 study to highlight the differences between LLM's mathematical reasoning abilities and human mathematical reasoning abilities: Even though LLMs are trained on vastly more data, they were not able to solve the problem, as shown in Section 2; on the other hand, we asked a former IMO participant, who did not have any exposure to group theory, to learn about the basic definitions necessary to understand Yu Tsumura's 554th problem and to attempt to solve the problem. He succeeded, and devised a new proof strategy to solve Yu Tsumura's problem. We comment on his approach in Section 3 which shows that whereas the LLMs seem to need to spend a lot of their time just trying random algebraic manipulations with little clear direction, the IMO participant clearly motivated different proof strategies, which highlights a completely approach to problem-solving, that LLMs lack.

#### 2 RESULTS

All our evaluations were performed one-shot, i.e., a single attempt was made to obtain the answer. Our assessment is made from the point of view *of an end user at the present point in time*. Thus, we are assessing whether the model can answer Yu Tsumura's 554th problem *robustly*, which means that the model has to produce the correct answer most of the time, making a one-shot evaluation should be sufficient. Repeated evaluations might produce correct proofs, but if it takes a best-of-*n* approach, majority voting, or other techniques to elicit them, from the perspective of the end user, this would be a different model that is evaluated (namely, one where the tested LLM incorporates an output refinement strategy on top, that mirrors the repeating-evaluation framework).

The list of models that we queried is given in Table 2. These models arguably represent the state of the art among publicly available options. Although this list is not exhaustive, these models likely outperform most others and are the most highly rated ones on website such as lmarena.ai (except GPT-4.5). Therefore, we reason that if these models are unable to solve the problem, it is unlikely that other comparable or less capable models will succeed either.

The failures in each case are fatal to the proof. In all cases, the model relied on the error we listed to complete its output. None of the models makes really significant progress before such an error derails the model, or in the case of the "argument incomplete" annotation, the model appears to give up and declares success before much meaningful progress is made.

The fact that our result transcends the various types of LLMs indicates:

LLM	Access	<b>Eval Date</b>	Failure
o3-pro (B.1)	OpenRouter	28 Jul	D
o3 (B.2)	online GUI	1 Aug	C, T
o4-mini-high (B.3)	online GUI	1 Aug	T
GPT-4o (B.4)	online GUI	1 Aug	I
Gemini 2.5 Pro (B.5)	online GUI	1 Aug	A
DeepSeek R1 (B.6)	online GUI	1 Aug	U
Claude Sonnet 4 (Ext. Thinking) (B.7)	online GUI	2 Aug	U
Claude Opus 4 (Ext. Thinking) (B.8)	OpenRouter	2 Aug	T
Grok 4 0709 (B.9)	LMArena	2 Aug	U
Kimi K2 (B.10)	OpenRouter	2 Aug	A, I
Qwen3 235B A22B Thinking 2507 (B.11)	OpenRouter	2 Aug	U
GLM-4.5 (B.12)	OpenRouter	2 Aug	A
Gemini 2.5 Deep Think (B.13)	online GUI	3 Aug	A
Llama 4 Maverick (B.14)	LMArena	3 Aug	U
DeepSeek v3 0324 (B.15)	OpenRouter	3 Aug	I
QwQ 32B (B.16)	LMArena	3 Aug	A, U
GPT-OSS-120B (B.17)	OpenRouter	14 Aug	U
GPT-5 Thinking (B.18)	online GUI	16 Aug	D

Table 1: A table of all 16 evaluated LLMs on Yu Tsumura's 554th problem, together with the dates at which the models were prompted, and links to the full outputs and detailed failure mode descriptions. For both Claude models the "Extended Thinking" option was turned on. Some models are missing size specifications, e.g., DeepSeek R1 as the GUI, that was used to access the model, did not reveal this information about the underlying model, see Appendix A for more information. None of the listed models were able to solve it flawlessly, as outlined by the failure modes (see key below). We refer to Section B for full output traces, and detailed explanation about the (potentially multiple) critical failure modes, and on which lines of the proof they occur.

Key: A = algebra error, C = missed case, D = incompatible definition, I = argument incomplete, T = inapplicable theorem, U = unwarranted assumption/claim

- Lack of high-quality scientific evaluation. In contrast to final-answer benchmarks and evaluations, such as OlympiadBench (He et al., 2024), for which automatic assessment is possible, there are few benchmarks for assessing proof-based reasoning, due to the high human effort involved. Exceptions are matharena.ai, (Petrov et al., 2025) and the earlier GHOSTS benchmark and evaluation (Frieder et al., 2023) problems, which contains a preliminary assessment on older LLMs on 100 problems from the book by A. Engel, "Problem-Solving Strategies" (Engel, 1998). Benchmarks comprising just six problems, such as the evaluation on IMO25 problems, are too small to make an informed assessment about the (mathematical) reasoning abilities of LLMs. The current results emphasize this, contradicting the optimism that the IMO25 inspires.
- Outcome misalignment. The goal is to increase the reasoning abilities of LLMs, which can be measured by the number of problems an LLM can solve. Relying on final-answer benchmarks can skew this. Hence, problems where proof assessment is performed are necessary to establish the baseline of reasoning abilities, and the current failure shows that some gaps may still exist towards final-answer benchmarks.

## 3 HUMAN COMPARISON AND A NEW PROOF

Yu Tsumura's 554th problem belongs to the domain of group theory, which is not a domain that is present at problems from the IMO.

Inspection of the original proof reveals that no specific group theory knowledge or group theoretic proof strategies are needed to solve Yu Tsumura's problem, beyond the definition of a group and a generator. Nonetheless, it is unclear whether the difficulty of the problem is within the reach of an IMO-level competitor.

To clarify this, we carried out an n=1 study with a former IMO25 participant, who was not yet exposed to any group theory.

Precise instructions were provided to him that he was not to look up information about group theory, and to receive any information he needed to understand the foundational group theoretic definitions solely form interactions with ChatGPT. We shared the full, unredacted transcript of interactions with ChatGPT, as well as his write-up of the proof.<sup>7</sup>

The fact that an IMO participant was able to solve this problem demonstrated, perhaps unsurprisingly, that Yu Tsumura's 554th problem is well within the reach of IMO-level students.

More interesting is to observe that the student engaged, without being explicitly prompted to so in any way, to devise a proof that comes close to what is known as a "motivated proof" (Pólya, 1949; Morris, 2020). These are proofs where each step made in the proof is made more transparent by providing clear motivation. It was observed that LLMs struggle with devising motivated proofs (Frieder et al., 2024), and the current paper highlights more strongly the distinction in human and LLM proof quality. This both pertains to the thoughtful outline he shared that represents his thinking process, as well as the final proof that resulted from this process.

What is noticeable about the solution by the IMO participant is that after spotting the identity  $xy^{2n}x^{-1} = x^{3n}$ , which some LLMs are also able to devise, the participant exploits it by picking special values of n.

In particular, the participants notices that we can replace a factor of 2 with a factor of 3, at the expense of wrapping things in another x and  $x^{-1}$ . But then by just focusing on the power of 3 dividing n, specifically by assuming n is divisible by a sufficiently high power of 3, the participant can control the use of the identity. The main thing that is different here is that he keeps focusing on powers, whereas both Yu Tsumura's original solution and others known correct proofs prove some identity involving  $y^{27}$  and then go back to using the original identities to knock out the rest of the solution (or, in case of LLMs, just "fiddling" until the whole thing collapses).

The IMO participant's solution is much nicer because it shows why 27 is important in the proof and where it comes from - a motivated proof step.

<sup>&</sup>lt;sup>7</sup>https://anonymous.4open.science/r/yutsumura\_solution-OBCD/

## 4 LIMITATIONS

Goodhart's law, which states that "when a measure becomes a target, it ceases to be a good measure" is pervasive in machine learning: This principle highlights that model creators often optimize the models to score highly on a given benchmark, rather than equipping the models with the skills that are partially captured by that benchmark and that are needed to succeed on that benchmark. In this regard, we expect that, having emphasized the problems commercial LLMs face on Yu Tsumura's 554th problem, models will soon be adapted to solve this issue (we hypothesize that for some state-of-the-art models, techniques as straightforward as improved test-time training will lead to the problem being solved). Yet, we conjecture that even in this case, other problems be found on which LLMs will struggle across the board.

Our evaluation pertained exclusively to models that reasoned and did not use a RAG pipeline – since the solution is publicly available, such an approach would not have assessed the reasoning skills of the evaluated model. In the case of o3-Pro, it was necessary to explicitly prompt the model not to look up the solution online.

Our protocol was to give each model a single attempt at a solution. It is reasonable to assume that multiple attempts, especially with the more expensive models, may result in a more complete solution. Of course, commercial models may already do this internally, using techniques such as majority voting, or more sophisticated variants thereof. We did not follow this approach, because our analysis pertains solely to see whether the experience of an *end user* interacting with these language models can live up to the expectation genereted by the strong performance on IMO25.

We have focused on publicly released, widely deployed models, especially flagship models. We cannot exclude that there are boutique models or models that are not yet publicly deployed that can reliably solve the problem.

The difference in model capabilities might also be explained by differences in how much training on the test task was performed (Dominguez-Olmedo et al., 2024).

Lastly, a mathematical problem with a proof that relies mainly on symbolic manipulation will pose few issues for a symbolic solver tailored to this type of reasoning. In this regard we expect that an LLM that has access to a tool, such as Vampire <sup>8</sup> or some other solver, and can translate the problem into the necessary formalism, will be able to solve it.

## 5 CONCLUSION

We have demonstrated that there exists at least one problem drawn from a similar distribution in terms of human difficulty and solution strategies as IMO problems, on which LLMs have demonstrated very strong performance to date, on which LLMs nonetheless systematically fail. Thus, subject to the constraints mentioned in Section 4, reasoning in LLMS remains brittle.

The fact that LLMs attained gold medals in the IMO and that, further, an unreleased variant of OpenAI's o3 solved 50 Olympiad-level problems in the OpenAI  $\times$  AIMO eval, would imply that LLMs should be able to solve Yu Tsumura's 554th problem, too, which we showed is accessible to a human with IMO-level preparation.

Yet, the fact that none of the LLMs solved it highlights that LLMs' "thinking" is different from the thinking of a human, and, in particular, that their reasoning ability is not *transitive*: Solving problems of a similar level of difficulty does not guarantee that another, similar such problem can be solved.

We are cognizant that Yu Tsumura's 554th problem will soon be solved by LLMs, in particular, once attention has been drawn to the fact that it is not solvable. Our analysis was carried out over a time-frame of less than three weeks between July and August 2025, and meant to capture a noteworthy snapshot at the time.

<sup>8</sup>https://github.com/vprover/vampire

Nonetheless, this doesn't change our overall message, as in that case other problems that should be accessible to LLMs based on their performance on comprehensive benchmarks, will turn out to be not solvable by a wide set of LLMs.

We conclude with concerns we have going forward: In announcements on strong LLM performance, it is not always clear what score was used. Several common options exist to score problems from a benchmark, such as a binary score per problem (correct/incorrect), or a score that takes into account repeated sampling (pass@n, first introduced in (Kulal et al., 2019)). Not being fully explicit in how exactly the methodology was set up can make the numbers hard to interpret.

For very long running commercial models, it will become difficult to rule out human intervention behind the scenes as benchmarks are being carried out, especially if the models are only deployed to a very small number of individuals or can only be afforded by very few researchers. This will potentially skew results of evaluations and make scientific evaluation difficult, if not impossible. We note that better evaluation standards are needed to address this issue. Pre-registered evaluation, akin to pre-registered studies, where time constraints are discussed and fixed in advance, are needed to exclude issues like these, by making sure (among other things) that the time alloted for the evaluation is not longer than the average expected runtime of the model.

#### REPRODUCIBILITY STATEMENT

Due to the stochastic nature of LLMs, re-generating the outputs is not possible. However, we have included the full output traces of all LLMs in Appendix B.

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#### A ADDITIONAL LLM INFORMATION

In order to emulate the end user experience, we used various web GUIs to access the models. In some cases, only these were available.

We accessed DeepSeek R1 via the DeepSeek website. We presume this is DeepSeek R1 0528. This is a 685B model, comprising 671B main model weights. It is accessed by turning on the DeepThink option in the DeepSeek web interface.

The Claude Sonnet 4 model was run with Extended Thinking turned on in the web interface.

The Claude Opus 4 model was accessed via OpenRouter and it spent 146.5s reasoning, indicating that the Extended Thinking mode was switched on.

On release day, GPT-5 suffered technical issues, such as returning responses unrelated to the prompt, or crashes. An output for GPT-5 was obtained, however OpenAI changed the available options quickly over the coming days and we decided not to use the release day response and wait until the models stabilised.

A later attempt on Aug 14 returned an empty answer after 14m 54s.

The included output for GPT-5 was obtained on Aug 16 with GPT-5 Thinking. The mathematical error was essentially the same as the one produced by GPT-5 on release day. However, the Aug 16 generation only thought for 2min 7s vs 4min 43s on release day. We don't know if this was due to server load or a change in the model.

#### B OUTPUT TRACES

This appendix contains the unmodified solutions produced by the model, along with the lines at which critical errors are made. We also consider a substantially incomplete proof to be a critical error.

The prompt for all LLMs was identical: Let G be a group with generators x and y and relations xy2 = y3x and yx2 = x3y. Can you prove that G is the trivial group. For models that do a web search, we add: Do not perform a web search to solve this problem to incentivize the model to reason rather than to regurgitate the web-based proof. For each model below, we first list the errors, and then reproduce the generated proof. At least one error was always critical, and we indicate which.

## B.1 03-PRO

Lines 3, 8 and 12: The definition of the commutator used is  $[x,y] = xyx^{-1}y^{-1}$ . The definition of conjugation used is  $x^y = y^{-1}xy$ . But the identity  $[x,yz] = [x,y][x,z]^y$  is not valid with these definitions. (Critical)

```
362
      Write the two relations in commutator form.
363
364
                           \iff x y^2 x^{-1} = y^3 \setminus [x, y^2] = y^3
            xy^2 = y^3x
365
                          \iff y x^2 y^{-1} = x^3 \setminus [y, x^2] = x$
            yx^2 = x^3y
366
367
      Step 1. Express y in terms of the basic commutator
368
      Put c = [x, y].
369
      Because [x, y^2] = [x, y][x, y]^y (= c c^y) we have
    8
370
         y = c c^y. (3)
    9
371
372
    11
      Multiply (3) on the right by (c^y)^{-1}.
373
      Since (c^y)^{-1} = y^{-1} c^{-1} y^2, (3) gives
    12
374
    13
375
         y (c^y)^{-1} = c \inf y y^{-1} c^{-1} y = c \inf c^{-1} y =
376
         \hookrightarrow c$,
377
```

<sup>9</sup>https://huggingface.co/deepseek-ai/DeepSeek-V3

```
378
    15 hence y = c^2. (4)
379
380
      So $y$ is the square of the commutator $c$ and therefore commutes
381
       \rightarrow with $c$.
382
383
      Step 2. Express $x$ in terms of the commutator
384
      Put d = [y, x]. From (2) we similarly get
385
386 22
        x = d d^x. (5)
387 23
388 24
      Because (d^x)^{-1} = x^{-1} d^{-1} x, the calculation used
       → above gives
389
390 25
        d^{-1} x = d \le x = d^2. (6)
391 26
392 27
      Step 3. Relate d to c
   28
393
      But \{[y, x] = [x, y]^{-1}\}, i.e. d = c^{-1}\}. (7)
394
395
      Combining (4), (6) and (7) we obtain
   31
396
   32
397
        y = c^2, x = (c^{-1})^2 = c^{-2}, (8)
    33
398
    34
399
      Step 4. xy = 1
    35
400
      Because powers of the same element commute,
    36
401
        xy = c^{-2} c^2 = 1, so y = x^{-1}. (9)
    37
402
403
      Step 5. Finish with either defining relation
404
      Insert y = x^{-1} in the first original relation:
    40
405
406
        x y^2 = y^3 x \in x^{-2} = x^{-3} \cdot x
    42
407
                             i.e. x^{-1} = x^{-2}$.
    43
408
409
      Multiplying by x^2$ gives x = 1$, and then (9) forces y = 1$.
   45
410 46
411 47
      Hence every generator of $G$ is the identity; therefore
412 48
413 49
        $G$ is the trivial group.
414
```

```
432
                 B.2 o3
433
434
                 Lines 36–37: The value k need not be an integer if n is infinity. (Critical)
435
                 Lines 36-41: The given map involves x and thus y need not be sent to an element of \langle y \rangle.
436
                 The model has made the assumption that the automorphism of G (conjugation by x) induces an
437
                 automorphism of the subgroup \langle y \rangle. (Critical)
438
439
                  Below is one way to see that the presentation
440
441
442
                  G=\label{eq:G-langle} G=\label{eq:G-langle
443
444
445
                  defines nothing more than the trivial group.
446
447
448
           10
449
                   ### 1.
                                         The abelianisation is trivial
           11
450
           12
451
                  452
                    \hookrightarrow on two generators.
453
                  The two relators become
454
455
                   \begin{tabular}{ll} $$ \begin{tabular}{ll} $x+2 \neq y=3 \begin{tabular}{ll} $x+2 \neq x, \neq x \end{tabular} \label{tabular}
456
          17
                   \frac{y+2}{ar} x=3\frac{x+\sqrt{y}}{ar} y
457
          19
458
           20
459
                  and both reduce to \ \bar x=\bar y=0\$.
          21
460
                  Hence G^{\left(G,G\right]=1}; in particular G=[G,G] is
461
                              *perfect*.
462
                   If we can show that $G$ is abelian, it will therefore be trivial.
           23
463
           24
464
           25
465
           26
466
                                         Conjugating the cyclic subgroups
                   ### 2.
           27
467
468
           29
                  Write $n=\lvert y\rvert\; (1\le n\le\infty)$.
469
                  From xy^{2}=y^{3}x we get
           30
470
           31
471
           32
472
                  x y^{2}x^{-1}=y^{3}. \text{ } tag\{2.1\}
           33
473
                  $$
           34
474
475
                  Because the subgroup $\langle y\rangle$ is cyclic, there is an
476

    integer

477
                  k\; (1\le k\le n)$ such that
          37
478
479
           39
480
                  x y x^{-1}=y^{k}. \tag{2.2}
         40
```

Squaring (2.2) and comparing with (2.1) yields the congruence

 $2k \neq 3 \pmod n$ ,  $\neq (k, n) = 1$ .  $tag\{2.3\}$ 

\$\$

481 41

482 42

483 43 484 44

```
486
    47 $$
487
    48
488
       Exactly the same argument with the second relator yx^{2}=x^{3}y
489

→ gives numbers

490
       $m=\lvert x\rvert$ and $\ell$ satisfying
491
492
       $$
493
       y \times y^{-1}=x^{\ell}, \quad 2\leq 1\leq 3\leq 3
494
       \rightarrow m,\qquad\gcd(\ell,m)=1.\tag{2.4}
495 54
       $$
496
   55
       *If n=\inf x then (2.3) would read 2k=3 in the integers,
497
498

→ impossible.

       Thus y=1 and, inserting this in the first relator, x=1.
499
       So from now on we may assume $n,m<\infty$; necessarily $n,m$ are
500
           **odd** (otherwise $2$ would not be invertible mod $n$ or
501
           $m$).
502
   59
503
    60
504
505
       ### 3. A useful commutator
    62
506
    63
507
       By (2.2) - (2.4),
    64
508
    65
509
510
       [x,y]=x y x^{-1}y^{-1}=y^{k-1}=x^{1-\ell}. \text{ } \{3.1\}
511
512
513
       Because of the congruences $2k\equiv3$ and $2\ell\equiv3$ we have
514
515
516
       \gcd(k-1, n) = \gcd(\ell-1, m) = 1. \tag{3.2}
517
       $$
    74
518 <sub>75</sub>
519 <sub>76</sub>
       (Indeed, if a divisor d\ satisfied d\ and d\ and d\
520

→ then $k\equiv1\pmod d$; reducing $2k\equiv3\pmod n$ modulo

521

    $d$ would give $2\equiv3\pmod d$, hence $d=1$.)
522 77
523 78
       From (3.1)-(3.2) the element
524
525 80
       c:=y^{k-1}=x^{1-\ell}
526 81
527 82
       $$
    83
528
       has order
    84
529
    85
530
    86
531
       \lvert c\rvert=n=\lvert y\rvert = m=\lvert x\rvert,
532
533
534
       so $\langle c\rangle=\langle y\rangle=\langle x\rangle$.
535
       Thus **both** generators are powers of the *same* element $c$;
536
           consequently $G$ is cyclic and therefore abelian.
537
538
    93
539
    94
```

```
540
    95 ### 4. Finish the argument
541
542
      We have shown
543
544
       * $G$ is abelian (Step 3),
545 100
       * $G$ is perfect (Step 1).
546 <sub>101</sub>
547 102
       The only perfect abelian group is the trivial one, so G=1.
548 103
       Equivalently x=1 and y=1.
549 104
550 105
       \boxed{\;G\text{ is the trivial group.}\;}
551 106
       $$
552 107
553 108
554 109
555 110
       ### Why the key step works
556 111
557 112
       The heart of the proof is the observation that the first relator
558 113
       → lets us
559
       *measure* how conjugation by $x$ acts on the cyclic subgroup
560
       → generated by $y$; it must multiply exponents by $k$ with
561
       \rightarrow $2k\equiv3$.
562
       Because k-1 is coprime to |y|, the commutator |x,y|=y^{k-1}
   115
563
           still has full order and ends up generating the **same**
564

→ cyclic subgroup as $y$.
565
       The second relator yields the mirror-image statement for $x$.
566
       That forces the whole group to be cyclic-hence, after the
   117
567
           abelianisation calculation, trivial.
568 118
569
570
```

#### B.3 O4 MINI HIGH

Lines 21-37: It is shown that  $\langle y^2 \rangle$  and  $\langle y^3 \rangle$  are conjugate in G. But the following comments require that they be conjugate in  $\langle y \rangle$ , which is not demonstrated. (Critical)

Lines 53-71: Again, conjugation is in G, not in the cyclic subgroup generated by y. Thus the stated conclusions are not supported. (Critical)

Lines 93-100: The given map involves x and may not even be a map from  $\langle y \rangle$  to itself, let alone an automorphism. (Critical)

```
604
       Here is a self-contained proof that
605
606
607
       G = \langle x, y \rangle = y^3x, \cdot; yx^2 = x^3y \rangle
608
       $$
609
610
       is the trivial group.
611
612
613
614
       ## 1. No element can have infinite order
615
    12
616
    13
       From
617
    14
       $$
618
    15
       xy^2 = y^3x
619
       \quad\Longrightarrow\quad
    17
620
       x \setminus y^2 \setminus x^{-1} = y^3
    18
621
       $$
    19
622
    20
623
       we see that the subgroups
    21
624
    22
625
    23
626
       24
627
       \quad\text{and}\quad
    25
628
       \langle y^3\rangle
    26
629
    27
       $$
630
631
       are conjugate in $G$. But in an infinite cyclic group $\langle
    29
632
           y\rightarrow y^{\ }, the index of \alpha y^{\ }
633
           Conjugate subgroups have the same index, so you would get
634
635
       $$
636
       637
       \qquad
638
       [\<y\>:\<y^3\>] \ \;=\;3,
639
    35
640
641 37
       a contradiction. Hence **$y$ must have finite order**. By an
642

→ entirely symmetric argument from

643 38
       $$
644
    39
       yx^2 = x^3y
645
   40
       \quad\Longrightarrow\quad
   41
646
       y \setminus x^2 \setminus y^{-1} = x^3,
    42
647
      $$
    43
```

```
648
649
       also **$x$ has finite order**.
650
651
652
653
       ## 2. Reduction to a cyclic, perfect group
654
655
       Let
    51
656
    52
657 53
658 54
       m = \mathbb{Q}  (y), \mathbb{Q}  (x).
       $$
659
   55
660
    56
       Because xy^2x^{-1}=y^3, conjugation by x carries
661
   57
       $\langle y^2\rangle$ onto $\langle y^3\rangle$. In a cyclic
662
            group of order $m$, the index of $\langle y^k\rangle$ is
663
664
       $$
    60
665
       \left| \left| \left| \left| \left| \left| \right| \right| \right| \right| \right| \right|
    61
666
       = frac\{m\} \{ \gcd(m, k) \}.
667
       $$
    63
668
669
       Equating the two indices gives
670
671
       $$
    67
672
       \frac{m}{\gcd(m,2)} \ ;=\ ; \frac{m}{\gcd(m,3)}
673
       \;\Longrightarrow\;
674
       \gcd(m, 2) = \gcd(m, 3).
675
       $$
676
677
       Since \c (m,2) \in \{1,2\}\ and \c (m,3) \in \{1,3\}\, the only
678
        → way they are equal is
679
680
    75
681
       \gcd(m, 2) = \gcd(m, 3) = 1,
    76
682
       $$
   77
683
       i.e.\ m is coprime to 6. In particular, **2 and 3 are
684
685
        → invertible in $\mathbb{Z}/m\mathbb{Z}$.**
686
       By exactly the same reasoning from the second relation one sees
687
        688
689
    83
690
       \gcd(n, 2) = \gcd(n, 3) = 1
691
       $$
    85
692
693
       and so **2 and 3 are invertible mod $n$.**
694
695
696
697
       ## 3. Linearising the conjugations
    91
698
699
       Because \d(2,m)=1, the map
700
    94
701
       $$
    95
```

```
702
       \varphi\colon\<y\>\to\<y\>,\quad
703
       \operatorname{varphi}(y^k) = x \setminus y^k \setminus x^{-1}
704
       $$
    98
705
706
       is an automorphism of the finite cyclic group of order $m$. From
   100
707
   101
708
   102
       $$
709
       \vert (y^2) = y^3
   103
710 104
       \quad\Longrightarrow\quad
711 105
       \operatorname{varphi}(y)^2 = y^3,
712 106
       $$
713 107
       we deduce that
714 108
715 109
716 110
       $$
717 111
       \vert varphi(y) = y^c,
       $$
   112
718
   113
719
       where $c$ is the unique integer mod $m$ satisfying
   114
720
721
   116
722
       2c\equiv3\pmod m.
   117
723
       $$
   118
724
   119
725
       Since \c (2,m)=1, we have
   120
726
727
   122
       $$
728
       c-1 \;\equiv\;2^{-1}\;\;(\bmod\,m),
   123
729
   124
730
   125
731
       126
732
733 <sub>128</sub>
734 129
       x \setminus y \setminus x^{-1} \setminus = \cdot; y^c.
735 130
736 131
737 132
       738
739 133
       $$
740 134
       y \, x \, y^{-1} \, = \; x^d,
741 135
742 136
   137
743
       where 2d\neq 0 and hence 3d - 1, n = 1.
   138
744
745
   140
746
   141
747
       \#\# 4. The commutator forces cyclicity {&} perfection
   142
748
   143
749
       Compute the two "commutators":
   144
750
751
   146
752
       [x,y] \ \ ;=\ ; \ x\,y\,x^{-1}\,y^{-1}
   147
753
       754
       $$
   149
755
    150
```

```
756
757
       [y,x] = y \setminus x \setminus y^{-1} \setminus x^{-1}
   153
       = x^d\setminus x^{-1} = x^{-1}.
759
       $$
760
   155
761
       But always \{[y,x]=[x,y]^{-1}\}, so
   156
762
763
   158
764 159
       x^{\{\,d-1\}\ \;=\;\ \bigl[y,x\bigr]}
765 160
       \;=\;\bigl[x,y\bigr]^{-1}
766 161
       \text{tag}\{*\}
767 162
       $$
768 163
769 164
770 165
       Now:
   166
771
       167
772
   168
773
   169
774
       u \setminus (c-1) \neq mod m.
   170
775
   171
776
   172
777
       Raising \{x,y\}=y^{c-1}\ to the \{u\}th power gives
   173
778
   174
779
       $$
   175
780
       [x,y]^u = y, (c-1) = y,
781
       $$
   177
782
   178
783
       i.e.\\
   179
784
   180
785
       $$
786
       y = (x, y)^u
787 <sub>183</sub>
       \;\in\; [G,G].
788 <sub>184</sub>
       \t\{1\}
789 <sub>185</sub>
       $$
790 186
791 187
       2. Since \c d(d-1,n)=1, there is an integer v with
792 188
793 189
       v\setminus, (d-1)\neq 0 n.
794
   190
795 191
   192
       Raising equation $(*)$ to the $v$th power gives
   193
797
   194
798
   195
799
       x^{(-1)} = y^{(-1)}
   196
800
   197
801
802
       but v(d-1) \neq 0 so x=y^{(1-c)}.
   199
803
   200
804
       $$
805
       x \;\in\;\langle y\rangle.
   202
806
       \text{tag}\{2\}
   203
807
       $$
   204
808
   205
809
```

```
810
    206 From \$(1)\$ and \$(2)\$, **both** generators \$x,y\$ lie in the
811

→ *derived subgroup* $[G,G]$, so
812 207
813 208
        $$
814 <sub>209</sub>
        [G,G] = G,
815 210
       $$
816 <sub>211</sub>
817 212
       i.e.\ $G$ is **perfect**. On the other hand $(2)$ shows $G$ is
818
        \rightarrow cyclic (generated by $y$), hence **abelian**. The only group
819
        → which is *both* abelian and perfect is the trivial group.
820 213
821 214
822 215
        ### Conclusion
823 216
824 217
825 218
       No non-trivial group can satisfy simultaneously
826 219
827 220
        $$
       xy^2=y^3x
828 221
829 <sup>222</sup>
        \quad\text{and}\quad
830 223
       yx^2=x^3y,
        $$
831 224
832 225
833 226
        so
    227
834
        $$
835
        \boxed{G\text{ is the trivial group.}}
    229
836
       $$
837 231
    230
838
```

## B.4 GPT-40

Line 118: The preceding lines don't demonstrate that conjugation of  $y^n$  by x increases the exponent by 1. (Incorrect but not used)

Line 348: It is not clear which equation x = yxy comes from. (Incorrect but not used)

Line 354: Having made no progress whatsoever, the model concludes the "only consistent solution" is the trivial group. (Critical)

```
872
873
       We are given a group $G$ with generators $x$ and $y$, and two
874
           relations:
875
876
877
           \quad xy^2 = y^3x, \quad text{and} \quad (2) \quad yx^2 =
       (1)
878
           x^3y.
879
       $$
880
881
       Our goal is to **prove that G is trivial**, i.e., x = y = 1.
882
883
884
885
       ### Step 1: Rewrite the Relations
    11
886
887
    13
       Let's manipulate each relation to bring like terms together.
888
    14
       #### From (1): xy^2 = y^3x
889
    16
890
       Multiply both sides on the right by x^{-1}:
    17
891
    18
892
    19
893
       xy^2x^{-1} = y^3.
    20
894
       $$
    21
895
    22
896
       So,
    23
897
    24
898
       $$
    25
899
       x y^2 x^{-1} = y^3. \quad \text{(1')}
900
    27
901
902
    29
       This says that conjugating $y^2$ by $x$ gives $y^3$, or
903
    30
904
    31
905
       x y^2 x^{-1} = y^3 \quad \text{quad } \text{Rightarrow} \quad y^2 x^{-1} y^{-3} =
906
           1.
907
       $$
    33
908
909
       This is equivalent to:
    35
910
911
    37
912
    38
       x y^2 x^{-1} y^{-3} = 1.
       $$
913
    39
914
    40
       #### From (2): yx^2 = x^3y$
915
    41
    42
916
       Multiply both sides on the right by y^{-1}:
    43
917
    44
```

```
918
919
    46
       yx^2 y^{-1} = x^3.
920
       $$
    47
921
922
       So,
    49
923
924
       $$
925
       y x^2 y^{-1} = x^3. \quad \text{(2')}
    52
926 53
927 54
928 55
       Similarly,
929
930
   57
       y x^2 y^{-1} x^{-3} = 1.
931 58
       $$
932
    60
933
    61
934
    62
935
       ### Step 2: Define Conjugation Maps
    63
936
937
       Let's define the following:
    65
938
939
       * From (1'): x y^2 x^{-1} = y^3 \times y^2 x^{-1} y^{-3}
940
941
       * From (2'): y x^2 y^{-1} = x^3 \Rightarrow y x^2 y^{-1} x^{-3}
942
       943
944
       These say that:
    70
945
946
    72
947
       x y^2 x^{-1} y^{-3} = 1 \quad \text{(quad \text{(text)} and)} \quad y^2 y^{-1}
948
       \Rightarrow x^{-3} = 1.
949
950
951
       These are conjugation identities showing how $x$ and $y$ act on
952

    each other.

953 77
954
       Let's now take (1') and apply the same conjugation again.
   78
955 79
956
957
   81
       ### Step 3: Conjugate (1') Again
958
   82
    83
959
       From (1'):
    84
960
961
    86
962
       x y^2 x^{-1} = y^3.
    87
963
       $$
964
965
       Now conjugate both sides by $x$:
966
967
       Left-hand side:
    92
968
    93
969
970
       x (x y^2 x^{-1}) x^{-1} = x^2 y^2 x^{-2}.
    95
971
       $$
```

```
972
973
        Right-hand side:
974
975 <sub>100</sub>
        $$
976 101
       x y^3 x^{-1}.
977 102
       $$
978 103
979 104
       So,
980 105
981 106
       x^2 y^2 x^{-2} = x y^3 x^{-1}.
982 107
       $$
983 108
984 109
       But from (1') again:
985 110
986 111
987 112
       x y^2 x^{-1} = y^3 \times x^{-1} = x (x y^2 x^{-1})
988 113
        \Rightarrow x^{-1} = x^2 y^2 x^{-2}.
989
990 114
991 115
992 116
        So this checks out.
    117
993
       But this also tells us that each time we conjugate y^n by x,
   118
994
        \rightarrow the exponent increases by one.
995
   119
996
       So maybe try to **write $x$ and $y$ in terms of each other**.
997
    121
998
    122
999
    123
1000 124
        ### Step 4: Solve for One Variable
1001 125
1002 126
        Go back to:
1003 127
1004 128
1005 129
       x y^2 = y^3 x.
1006 <sub>130</sub>
1007 131
1008 132 Let's try to **move all $x$'s to one side**.
1009 133
       Start with:
1010 134
1011 135
        $$
1012 136
        x y^2 = y^3 x
1013 137
1014 138
1015 139
1016 140
       Multiply both sides on the left by x^{-1}, then on the right by
        \Rightarrow $x^{-1}$:
1017
1018 141
1019 142
1020 143
        y^2 = x^{-1} y^3 x \Rightarrow x y^2 x^{-1} = y^3.
1021 144
       $$
1022 145
1022
1023
147
       As before.
1024 148
       Similarly, from (2):
1025 149
```

```
1026
1027
        yx^2 = x^3 y \setminus Rightarrow y x^2 y^{-1} = x^3.
1028 152
        $$
1029 153
1030 154
        Now, write these as:
1031 155
1032 <sub>156</sub>
        * $x y^2 x^{-1} = y^3
1033 <sub>157</sub>
        * y x^2 y^{-1} = x^3
1034 158
1035 159
        Let's now define a = x, b = y, to simplify notation.
1036 160
        So:
1037 161
1038 162
        * $a b^2 a^{-1} = b^3$
1039 163
                                     (1)
        * $b a^2 b^{-1} = a^3$ (2)
1040 164
1041 165
       Let's try to eliminate one of the variables using these
1042 166

→ relations.

1043
1044 167
1045^{\ 168}
1046 <sup>169</sup>
        ### Step 5: Try to Find Orders of $x$, $y$
1047 170
1048 <sup>171</sup>
1049 172
        Suppose x = 1, the identity.
1050 173
1050
1051
175
1052
176
        Then from (1):
1053
        1 \cdot y^2 = y^3 \Rightarrow y^2 = y^3 \Rightarrow y^{-1} = 1
1054
            \Rightarrow\ y = 1.
1055 178
        $$
1056 179
1057 180
        So $x = 1 \setminus Rightarrow y = 1$.
1058 181
1059 <sub>182</sub>
        Similarly, if y = 1:
1060 183
1061 184
        From (2):
1062 185
1063 186
        1 \cdot x^2 = x^3 \Rightarrow x^2 = x^3 \Rightarrow x^{-1} = 1
1064 187
        \rightarrow \Rightarrow x = 1.
1065
        $$
1066 188
1067 189
        So y = 1 \setminus Rightarrow x = 1.
1068 <sup>190</sup>
1069 <sup>191</sup>
1070 <sup>192</sup>
        So **if either generator is identity, the other must be too**.
1071 193
1072 <sup>194</sup>
        Now suppose neither is the identity, but we try to find an order.
1073 <sup>195</sup>
        Suppose $x$ and $y$ have finite order.
1074 196
1075
1076 198
        Let's say x^n = 1 and y^m = 1.
1076
1077
1077
200
        Try to find a contradiction.
1078 201
1079
```

```
^{1080}_{202} But a better approach is to **define a new variable** and work
1081

→ through simplification.

1082 203
1083 204
1084 205
1085 206
        ### Step 6: Try to Simplify with Substitutions
1086 207
1087 208
        From the conjugation relation:
1088 209
1089 210
        x y^2 x^{-1} = y^3 \setminus Rightarrow x y^2 = y^3 x.
1090 211
        $$
1091 212
1092 213
1093 214 Let's isolate $x$. Multiply both sides on the left by $y^{-3}$:
1094 215
1095 216
        $$
        y^{-3} = x \cdot y^{2} = x \cdot y^{3}
1096 217
        $$
1097 218
1098 219
1099 <sup>220</sup>
        So:
1100 <sup>221</sup>
        $$
1101 222
1102 223
        x = y^{-3} x y^{2}.
1103 224
        $$
1104 225
1104
1105
226
227
        Now cancel $x$ from both sides:
1106 228
        Multiply both sides on the left by x^{-1}:
1107 229
1108 230
1109 231
        1 = x^{-1} y^{-3} x y^{2}
1110 232
        $$
1111 233
1112 234
        Rewriting:
1113 235
1114 236
1115 237
        x^{-1} y^{-3} x y^{2} = 1.
1116 238
        $$
1117 239
        Similarly, from the second relation:
1118 240
1119 241
1120 242
        y x^2 = x^3 y \Rightarrow x^2 = y^{-1} x^3 y \Rightarrow 1 =
1121 243
        \rightarrow x^{-2} y^{-1} x^3 y.
1122
        $$
1123 <sup>244</sup>
1124 245
1125 <sup>246</sup>
        So:
1126 <sup>247</sup>
1127 248
1128 249
        x^{-2} y^{-1} x^3 y = 1.
1129 250
        $$
1130 251
1130
252
1131
253
        We now have:
1132_{254}^{253} \mid * \$x^{-1} y^{-3} x y^{2} = 1\$  (A)
1133_{255}^{255} | * $x^{-2} y^{-1} x^3 y = 1$ (B)
```

```
1135 257
        Let's work with (A):
1136 258
1137 259
        $$
1138 <sub>260</sub>
        x^{-1} y^{-3} x y^{2} = 1 \Rightarrow y^{-3} x y^{2} = x.
1139 261
1140 262
1141 263
        Let's multiply both sides by x^{-1} on the right:
1142 264
1143 265
1144 266
        y^{-3} \times y^{2} \times \{-1\} = 1.
        $$
1145 267
1146 268
        So:
1147 269
1148 270
1149 271
        $$
        y^{-3} x y^{2} x \{-1\} = 1.
1150 272
1151 <sup>273</sup>
1152 <sup>274</sup>
1153 <sup>275</sup>
        That is:
1154 276
1155 <sup>277</sup>
        $$
        x y^2 x^{-1} = y^3.
1156 <sup>278</sup>
1157 279
1158 280
1159 281
        We've gone in circles.
1160 282
1160 283
        Let's take a step back.
1161 284
1162 285
1163 286
1164 287
        ### Step 7: Try a Free Group Homomorphism
1165 <sub>288</sub>
1166 <sub>289</sub>
        Suppose $G$ is a group with these relations:
1167 290
1168 <sub>291</sub>
1169 292
        G = \langle x, y \rangle  wid x y^2 = y^3 x, y x^2 = x^3 y \rangle 
1170 293
        $$
1171 294
        Let's define a homomorphism \phi G \to \mathbb{Z} to
1172 295

→ abelianize the group.

1173
1174 296
        In the abelianization, xy = yx, so we treat all elements as
1175 297
        1176
1177 298
1178 <sup>299</sup>
        From the relations:
1179 <sup>300</sup>
1180 <sup>301</sup>
        1. x y^2 = y^3 x  Rightarrow x + 2y = 3y + x  Rightarrow 2y = 3y
        \rightarrow \Rightarrow y = 0$
1181
1182 302
        2. y x^2 = x^3 y  Rightarrow y + 2x = 3x + y  Rightarrow 2x = 3x
             \Rightarrow x = 0$
1183
1184 303
1185
        So in the abelianization, x = y = 0, so both are trivial in
        \Rightarrow $G^{ab}$.
1186
1187
```

```
1188 306
       Therefore, $G$ is a **perfect group** (equal to its commutator
1189

→ subgroup).

1190 307
1191 <sub>308</sub>
       Since the abelianization is trivial, we suspect $G$ is trivial or
1192

→ very small.

1193 309
1194 310
1195 311
1196 312
       ### Step 8: Try the Identity Directly
1197 313
1198 314
       From:
1199 315
1200 316
1201 317 | x y^2 = y^3 x
1202 318 \Rightarrow x y^2 x^{-1} = y^3
1203 319
       \Rightarrow x y^2 x^{-1} y^{-3} = 1.
       $$
1204 320
1205 321
       Let's define:
1206 <sup>322</sup>
1207 323
1208 324
       r = x y^2 x^{-1} y^{-3}, \quad s = y x^2 y^{-1} x^{-3}.
1209 <sup>325</sup>
1210 326
       $$
1211 327
1212 328
       Then G = \langle x, y \rangle = 1, s = 1 \rangle.
1213 329
1214 330
       Now define a **map from $G$ to the trivial group**, mapping $x
        → \mapsto 1, y \mapsto 1$. Clearly both relations are
1215
        \rightarrow satisfied. So the trivial group is a quotient of $G$.
1216 331
1217 332
       But we now show that **$x = y = 1$** in $G$.
1218 333
1219 334
       From:
1220 335
1221 336
1222 337
       x y^2 = y^3 x \Rightarrow x y^2 x^{-1} = y^3.
1223 338
1224 339
1225 340
       Let's suppose y \le 1, and see that the order of y must be
1226

    finite.

1227 341
       Let's try small orders.
1228 342
1229 343
       Suppose y^n = 1.
1230 344
1231 345
1232 346
       Try n = 1: y = 1 \Rightarrow x = 1.
1233 347
1234 348
       Try n = 2: then y^2 = 1 \cdot Rightarrow x = y x y \cdot Rightarrow x =
        \rightarrow x$, okay.
1235
1236 349
1237 350
       Try n = 3: y^3 = 1 \Rightarrow x y^2 = x \Rightarrow y^2 = 1
        \rightarrow \Rightarrow y = 1\$. Contradiction.
1238
    351
1239 352
       So we always get back to y = 1 \cdot Rightarrow x = 1.
1240 353
1241_{354} Therefore, the only consistent solution is:
```

```
1242
355
1243
356
1244 <sub>357</sub>
         x = y = 1.
1245 358
         $$
1246 <sub>359</sub>
1247 360
1248 <sub>361</sub>
1249 <sub>362</sub>
         ### Final Conclusion:
1250 <sub>363</sub>
1251 364
         $$
1252 365
         \boxed{G \text{ is the trivial group.}}
1253 366
         $$
1254 367
1255 368 All elements reduce to the identity under the given relations.
1256
1257
1258
1259
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1293
1294
1295
```

#### B.5 GEMINI 2.5 PRO

1296

1297

1299

Lines 39-40: a cannot be cancelled from the right, and even if this is a typo, the claimed identity  $a = b^3 a b^{-1}$  does not follow from  $bab = b^4 a b^{-1}$ . This is an algebra error. (Critical)

```
1300
1301
      Here is a proof that the group $G$ with generators $x$ and $y$
1302
          and relations xy^2 = y^3x and yx^2 = x^3y must be the
1303
          trivial group.
1304
1305
      ### **1. Rewriting the Relations**
1306
1307
      The given relations are:
1308
          xy^2 = y^3x
      1.
1309
          yx^2 = x^3y
      2.
1310
1311
      To simplify the manipulation of these relations, we can introduce
1312
       \rightarrow new generators. Let a = x and b = y^{-1}. Then y =
1313
          b^{-1}$. We can rewrite the original relations in terms of
1314
          $a$ and $b$.
1315 10
1316 11
      * **Relation 1:** xy^2 = y^3x becomes ab^{-2} = b^{-3}a.
1317 12
      * **Relation 2:** yx^2 = x^3y$ becomes b^{-1}a^2 = a^3b^{-1}$.
1318 13
1319 14
      Now, we can rearrange these new relations to get expressions for
1320
      * From ab^{-2} = b^{-3}a, if we multiply by b^3 on the left,
       \rightarrow we get b^3ab^{-2} = a. If we then multiply by b^2 on the
1322

→ right, we obtain:

1323
          b^3a = ab^2  (R1')
1324 16
      * From b^{-1}a^2 = a^3b^{-1}, if we multiply by b on the
       \rightarrow left, we get a^2 = ba^3b^{-1}. If we then multiply by $b$
1326
       \rightarrow on the right, we obtain:
1327
          a^2b = ba^3 (R2')
1328 18
1329 19
      So, the original relations are equivalent to the pair of
1330 20
       → relations:
1331
      * $ab^2 = b^3a
1332
      * $a^2b = ba^3$
1333
    23
1334
   24
1335 25
      ### **2. Deriving a Key Identity**
1336
1337 27
      Now, we will use these relations to show that $b$ must be the
1338
       1339 28
1340 <sub>29</sub>
      From ab^2 = b^3a, we can derive expressions by progressively
1341
      → multiplying by $b^{-1}$ on the right:
1342 30
      * $a = b^3ab^{-2}$
1343 31
      * $ab = b^3ab^{-1}$
1344 32
      * $ab^2 = b^3a$ (This is R1')
1345 33
      Let's focus on the identity ab = b^3ab^{-1}. We can use this to
1346 34
      → analyze the expression $bab$:
1347
      bab = b(ab)
1348 35
      Substitute the expression for $ab$:
1349 36
   | \$bab = b(b^3ab^{-1}) = b^4ab^{-1}
```

```
1350 38
1351
       So, we have found that bab = b^4ab^{-1}. By canceling b\ from
1352
       \rightarrow the left and $a$ from the right (if $a$ is not the identity),
1353
       → we get:
1354 40
       a = b^3ab^{-1}
1355 41
1356 <sub>42</sub>
       Now we have two expressions for $a$:
1357 <sub>43</sub>
       1. a = b^3ab^{-2} (from R1')
1358 44
       2. a = b^3ab^{-1} (derived above)
1359 45
1360 46
       Equating these two expressions gives:
       b^3ab^{-2} = b^3ab^{-1}$
1361 47
1362 48
       By canceling $b^3a$ from the left, we get:
1363 49
       b^{-2} = b^{-1}
1364 50
1365 51
       Multiplying by $b^2$ on the right gives:
1366 <sup>52</sup>
       e = b
1367 53
1368 54
1369 <sup>55</sup>
       ### **3. Proving Triviality**
1370 56
1371 57
       We have shown that b = e. Since we defined b = y^{-1}, this
1372 58
       \rightarrow means y^{-1}=e, which implies **y=e**.
1373
    59
1374
1375 60
       Now we can substitute $y=e$ back into one of the original
       → relations. Let's use the second relation:
1376 61
       yx^2 = x^3y
1377 62
       Substituting $y=e$:
1378 63
       ex^2 = x^3e
1379 64
       x^2 = x^3
1380 65
1381 66
       By canceling x^2 from the left, we find that **x=e**.
1382 67
1383 <sub>68</sub>
       Since both generators x and y are the identity element (p),
       → the group $G$ can only contain the identity element.
       \rightarrow Therefore, **$G$ is the trivial group**.
1385
1386 69
1387
1388
```

## B.6 DEEPSEEK R1

1404

1405 1406

1407

Lines 11–13: It is stated that  $xyx^{-1} = y^3$ . This is claimed to follow from  $gyg^{-1} = y^3$ . But this is not proved. The claimed proof only shows that  $(xyx^{-1})^2 = y^3$ . (Critical)

```
1408
1409
        The group \backslash (G\backslash) is defined by generators \backslash (x\backslash) and \backslash (y\backslash) and
1410
             relations (xy^2 = y^3x) and (yx^2 = x^3y). The goal is to
1411
            show that \backslash (G \backslash) is trivial, meaning \backslash (x = 1 \backslash) and \backslash (y = 1 \backslash),
1412
            where 1 is the identity element.
1413
1414 2
        From the given relations, it is possible to derive additional
1415
        → relations:
1416
        - From (xy^2 = y^3x), conjugation by (x) gives (x y^2 x^{-1})
1417
        \Rightarrow = y^3\). Since conjugation is an automorphism, \(x y^2 x^{-1}\)
1418
        \Rightarrow = (x y x^{-1})^2, so \((x y x^{-1})^2 = y^3\).
1419
        - Similarly, from (yx^2 = x^3y), conjugation by (y) gives (y)
1420
        \rightarrow x^2 y^{-1} = x^3\), so \((y x y^{-1})^2 = x^3\).
1421
1422
        Consider the element (g = x y). Conjugating (y) by (g):
1423
1424
        g y g^{-1} = (x y) y (x y)^{-1} = (x y) y (y^{-1} x^{-1}) = x y
            x^{-1}.
1426 <sub>10</sub>
        \1
1427 11
       From the earlier derivation, (x y x^{-1}) = k) where (k^2 = k)
1428
        \rightarrow y^3\), but also from the relation \(q y q^{-1} = y^3\) (since
1429
            (g y g^{-1}) = x y x^{-1}  and ((x y x^{-1})^2 = y^3)
1430
            implies that conjugating again or directly, it follows that
            (x y x^{-1}) = y^{3}). Thus:
1431
1432 12
        \[
       g y g^{-1} = x y x^{-1} = y^3.
1433 13
1434 14
        \]
       Therefore:
1435 15
1436 16
        \[
        x y x^{-1} = y^3. \quad (1)
1437 17
1438 <sup>18</sup>
        Similarly, for the other relation, consider (h = y x).
1439 19
        \hookrightarrow Conjugating \(x\) by \(h\):
1440
1441 20
        1 /
        h \times h^{-1} = (y \times x) \times (y \times x)^{-1} = (y \times x) \times (x^{-1}) y^{-1} = y \times x
1442 21
        \rightarrow y^{-1}.
1443
        \]
1444
        From the derivation, (y \times y^{-1}) = m) where (m^2 = x^3), and
1445
        \rightarrow similarly, it follows that:
1446 24
1447 25
       h \times h^{-1} = y \times y^{-1} = x^3.
1448 26
        \]
1449 27
       Thus:
1450 28
        1 /
1451 29
        y \times y^{-1} = x^3. \quad (2)
1452 <sub>30</sub>
1453 31
1454 32
       From equations (1) and (2):
       - Equation (1): \langle x y x^{-1} \rangle = y^3 \rangle implies \langle x y \rangle = y^3 x \rangle.
1456
            \qquad \qquad (3)
        - Equation (2): (y \times y^{-1}) = x^3 implies (y \times y^{-1}) = x^3.
1457 34
            \qquad \qquad \qquad (4)
```

```
1458
1459
        Now, use the original relations. From the original relation
        \rightarrow \(xy^2 = y^3x\) \quad (5), and from equation (3), multiply
1461
        \rightarrow both sides by \((y\):
1462 37
        1 /
1463 38
       x y^2 = y^3 x y.  \quad (6)
1464 39
1465 <sub>40</sub>
       Equating (5) and (6):
1466 41
        1 /
1467 <sub>42</sub>
       y^3 x = y^3 x y.
1468 43
       \]
       Since the group operation is cancellative, left-multiply both
1469 44
        \rightarrow sides by \(\(\(y^{-3}\\\)\):
1470
        \ [
1471 45
1472 46
       x = x y
1473 47
       \]
       which implies:
1474 48
1475 <sup>49</sup>
        \[
       x y x^{-1} = x x^{-1} \setminus y = 1. \quad (7)
1476 <sup>50</sup>
1477 51
1478 <sup>52</sup>
        Similarly, from the original relation (yx^2 = x^3y) \quad (8),
1479 <sup>53</sup>
        \rightarrow and from equation (4), multiply both sides by \(x\):
1480
1481 54
        1/
       y x^2 = x^3 y x.  \quad (9)
    55
1482
        \]
1483
       Equating (8) and (9):
1484
        1
1485
       x^3 y = x^3 y x.
1486 60
1487 61
       Left-multiply both sides by (x^{-3}):
1488 62
        1
1489 <sub>63</sub>
       y = y x,
1490 64
       \]
1491 65
        which implies:
1492 66
        1 /
1493 67
       y \times y^{-1} = y y^{-1} \setminus x = 1. \quad (10)
1494 68
1495 69
       From (7) and (10), (x = 1) and (y = 1). Therefore, the group
1496 70
        \rightarrow \(G\) is trivial.
1497
1498 71
1499 72
        \boxed{\text{The group } G \text{ is trivial.}}
1500
```

```
1512
       B.7 CLAUDE SONNET 4 (EXTENDED THINKING)
1513
       Line 39: The model promises to verify the assumption y^2 = e, but this is never independently
1515
       verified. (Critical)
1516
       Lines 51–54: The model assumes x^2 = e without justification. (Critical)
1517
1518
        Looking at this group presentation, I'll work systematically with
1519
        \rightarrow the given relations to show that both generators must equal
1520
        \hookrightarrow the identity.
1521
1522
        Given relations:
1523
       1) xy^2 = y^3x
1524
        2) yx^2 = x^3y
1525
1526
        **Step 1: Finding the orders of x and y**
1527
1528 9
       From relation (1): xy^2 = y^3x
1530 11
       Let me multiply both sides on the left by y^{-3} and on the
1531
        \rightarrow right by y^{-2}:
1532 <sub>12</sub> $$y^{-3}xy^2y^{-2} = y^{-3}y^3xy^{-2}$$
       $$y^{-3}x = xy^{-2}$
1533 <sub>13</sub>
1534 14
       Multiplying both sides on the right by $y^2$:
1535 15
       \$\$y^{-3}xy^{2} = x\$\$
1536 16
1537 17
1538 18
       But from our original relation, xy^2 = y^3x, so:
        \$\$y^{-3}(y^3x) = x\$
1539 19
1540 <sup>20</sup>
        $$x = x$$
1541 <sup>21</sup>
       Let me try a different approach. From xy^2 = y^3x, I'll
1542 <sup>22</sup>
        \rightarrow multiply on the left by y^{-3}:
1543
        \$\$y^{-3}xy^{2} = y^{-3}y^{3}x = x\$\$
1544 <sup>23</sup>
1545 <sup>24</sup>
1546 <sup>25</sup>
        So y^{-3}xy^2 = x, which means y^{-3}x = xy^{-2}.
1547 26
        Multiplying both sides on the right by $y^2$:
1548
        \$\$y^{-3}xy^{-2} = xy^{-2}y^{-2} = x\$\$
1549
1550
        From the original relation xy^2 = y^3x, we have y^{-3}xy^2 =
1551
        \Rightarrow y^{-3}y^{3}x = x^{3}.
1552 31
1553 <sub>32</sub>
       This confirms our manipulation. Now, from y^{-3}x = xy^{-2},
1554
        → multiplying on the right by $y^2$:
1555 33
       \$\$y^{-3}xy^{2} = x\$\$
1556 <sub>34</sub>
1557 35
       And since xy^2 = y^3x:
1558 <sub>36</sub>
       \$\$y^{-3}y^{3x} = x\$\$
1559 37
       $$x = x$$
1560 38
       But more importantly, from y^{-3}x = xy^{-2}, if we set y^2 =
1561 39

→ e$ (which we'll verify), then:
1562
        \$\$y^{-3}x = x\$\$
1563 40
1564 \ 41 \ \$\$y^{-}\{-3\} = e\$\$
       $$y^3 = e$$
1565 <sup>42</sup>
    43
```

```
1566 44
1567 45
       **Step 2: Using symmetry for x**
1568 <sub>46</sub>
        Similarly, from relation (2): yx^2 = x^3y
1569 <sub>47</sub>
1570 <sub>48</sub>
       Following the same process, multiply on the left by x^{-3}:
1571 <sub>49</sub>
       $x^{-3}yx^2 = x^{-3}x^3y = y$
1572 <sub>50</sub>
1573 <sub>51</sub>
        This gives us x^{-3}yx^2 = y, so x^{-3}y = yx^{-2}.
1574 52
1575 <sub>53</sub>
       Following similar logic, we get:
1576 54
       $$x^3 = e$$
1577 55
        **Step 3: Deriving stronger conditions**
1578 56
1579 57
       Now I have x^3 = e and y^3 = e.
1580 58
1581 <sup>59</sup>
       Substituting back into the original relations:
1582 <sup>60</sup>
1583 <sup>61</sup>
       From xy^2 = y^3x with y^3 = e:
1584 <sup>62</sup>
        $$xy^2 = ex = x$$
1585 <sup>63</sup>
        \$\$y^2 = e\$\$
1586 <sup>64</sup>
1587 65
       From yx^2 = x^3y with x^3 = e:
1588 66
1589 67
        \$yx^2 = ey = y\$
1590 68
        $x^2 = e$
1591
        **Step 4: Conclusion**
1592
1593 72 Now I have:
1594 73
        - $x^2 = e$ and $x^3 = e$
1595 74
        - \$y^2 = e\$  and \$y^3 = e\$
1596 <sub>75</sub>
1597 <sub>76</sub>
       Since \c (2,3) = 1, there exist integers a,b such that a+b
        \rightarrow 3b = 1$.
1599 77
1600 78
        For x: x = x^1 = x^{2a + 3b} = (x^2)^a(x^3)^b = e^a \cdot cdot e^b
1601
        \Rightarrow = e$
1602 79
       For y: y = y^1 = y^{2a + 3b} = (y^2)^a(y^3)^b = e^a \cdot d^b
1603 80
1604
1605 81
       Therefore, both generators equal the identity, which means $G$ is
1606 82

→ the trivial group.

1607
1608
1609
1610
1611
1612
1613
1614
1615
1616
1617
1618
```

#### B.8 CLAUDE OPUS 4 (EXTENDED THINKING)

1620

1621 1622

1623

Line 20: This is an unwarranted assumption. The conjugation map is a map on G and so the result of conjugating y need not be a power of y. (Critical)

```
1624
1625
      Looking at the given relations:
1626
      - $xy^2 = y^3x$
1627
      - \$yx^2 = x^3y
1628
1629
      I'll analyze what these relations tell us about the group
1630

    structure.

1631
1632
      From the first relation xy^2 = y^3x, I can rewrite this as:
1633
      \$xy^2x^{-1} = y^3\$
1634
1635
      This means that conjugating y^2 by x gives us y^3.
1636
1637
      Similarly, from the second relation yx^2 = x^3y:
    12
1638 13
      \$\$yx^2y^{-1} = x^3\$\$
1639 14
1640 15
      This means that conjugating x^2 by y gives us x^3.
1641 16
1642 17
      Let me define the conjugation map \phi = xgx^{-1}. Since
1643
       \rightarrow conjugation is a homomorphism, and \phi(y^2) = y^3, I
1644
       → have:
1645 18
      $\$\phi_x(y)^2 = y^3$
1646 19
      If I let \phi(y) = y^a for some integer a, then:
1647 20
      $$(y^a)^2 = y^3$$
1648 21
1649 22
      \$\$y^{2a} = y^3\$
1650 <sup>23</sup>
      This means y^{2a-3} = 1.
1651 24
1652 25
      For this to hold, either:
1653 <sup>26</sup>
      1. \$y\$ has infinite order, which would require \$2a = 3\$ in
1654 <sup>27</sup>
       \rightarrow $\mathbb{Z}$ (impossible for integer $a$)
1655
      2. y$ has finite order m$, and 2a \neq 3 \neq 0
1656 <sup>28</sup>
1657 <sup>29</sup>
       So $y$ must have finite order $m$. For the congruence $2a \equiv
1658 <sup>30</sup>
          1659
          Since 3 is odd, we need \c (2,m) = 1, which means m must
1660
         be odd.
1661
1662
      By the same reasoning applied to the second relation, $x$ must
1663
          have finite odd order $n$, and there exists $b$ such that
1664
          yxy^{-1} = x^b where $2b \equiv 3 \pmod{n}$.
1665 33
1666 34
      Now I'll use both relations together. From our conjugation
1667
       → equations:
1668 35
       - xyx^{-1} = y^a where $2a \equiv 3 \pmod{m}$
1669 <sub>36</sub>
      - yxy^{-1} = x^b where 2b \neq 3 \pmod{n}
1670 37
1671 38
      Let me compute xyxy^{-1}x^{-1} in two ways:
1672 39
      First way: x^{-1}x^{-1} = x(yxy^{-1})x^{-1} = xx^bx^{-1} = x^bx^{-1} = x^bx^{-1}
1673 40
          x^b$
```

```
1674
1675
       Second way: xyxy^{-1}x^{-1} = (xyx^{-1})(yy^{-1}) = y^a \cdot 1
1676
       \Rightarrow = v^a
1677 43
1678 44
       Therefore: x^b = y^a
1679 <sub>45</sub>
1680 <sub>46</sub>
       This is a key relationship. Now, since x^b = y^a:
1681 <sub>47</sub>
       - Applying \phi $\phi_y$: $y(x^b)y^{-1} = y(y^a)y^{-1}$, which gives
1682
       \Rightarrow $(x^b)^b = y^a$, so $x^{b^2} = y^a$
1683 48
       - But also x^{b^2} = (x^b)^b = (y^a)^b = y^{ab}
1684 49
       So y^a = y^{ab}, which means y^{a(b-1)} = 1.
1685 50
1686 51
       Since y$ has order m$, we need m \mid a(b-1)$.
1687 52
1688 53
       But we also know:
1689 54
       - From x^b = y^a, raising to the a-th power: x^a = th
1690 55
       \rightarrow y^{a^2}$
       - Applying \phi = y(y^{a^2})y^{-1} = y(y^{a^2})y^{-1}
1692 56
       - This gives (x^b)^{ab} = y^{a^2}, so x^{ab^2} = y^{a^2}
1693 <sup>57</sup>
       - But we already had x^{ab} = y^{a2}
1694 <sup>58</sup>
1695 59
1696 <sup>60</sup>
       So x^{ab} = x^{ab^2}, which means x^{ab(b-1)} = 1.
1697 <sup>61</sup>
1698 62
       Since x$ has order n$, we need n \mid ab(b-1)$.
1699
       Now, since $m$ and $n$ are odd, and $a$ and $b$ are odd (as they
1700
       \rightarrow satisfy $2a \equiv 3 \pmod{m}$ and $2b \equiv 3 \pmod{n}$
1701
       \rightarrow with odd $m,n$), we have:
1702 65
       - \ (since \ is odd and \ a,b\ are odd)
1703
       - Therefore m \mid (b-1) (since b-1 is even and m \mid a(b-1))
1704 67
1705 <sub>68</sub>
       But wait - this gives us m \mid (b-1) where b satisfies 2b
1706
       \rightarrow \equiv 3 \pmod{n}$.
1707 69
1708 70
       Similarly, we get n \mid (a-1) where a satisfies 2a \neq 3
1709
       \rightarrow \pmod\{m\}$.
1710 71
1711 72
       Let me check if any odd integer $m > 1$ can satisfy these
1712

→ conditions:
1713 73
       For m = 3: a \neq 0 \cdot 2^{-1} \neq 0 \cdot 2 \neq 0
1714 74
       \rightarrow \pmod{3}$
1715
       So a-1 \neq 0  \equiv -1 \equiv 2 \pmod{3}$
1716 <sup>75</sup>
       But we need 3 \mid (a-1), which means a-1 \neq 0 \pmod{3}.
1717 <sup>76</sup>
       → Contradiction!
1718
1719 77
       For m = 5: a \neq 3 \cdot 3 \cdot 4 \cdot 7
1720 <sup>78</sup>
       \Rightarrow $2^{-1} \equiv 3 \pmod{5}$)
1721
       So a-1 \neq 3 \neq 5
1722 79
1723 80
       But we need 5 \mid (a-1), which means a-1 \neq 0 \pmod{5}.
          Contradiction!
1724
1725
       By similar analysis, no odd $m > 1$ can satisfy the required
1726

→ conditions.

1727
```

```
1728 84 1729 85 1730 86 1731 87 1732 88 1733 89 1734 1735 Therefore $$ = 1$, which means $$ = 1$.  

Therefore $$ = 1$, which means $$ = 1$.  

Therefore $$ = 1$, which means $$ = 1$.  

Therefore $$ = 1$, which means $$ = 1$.  

Therefore $$ = 1$, which means $$ = 1$.
```

```
1782
       B.9 GROK 4 0709
1783
1784
       Note: Grok 4 repeatedly returned empty output in OpenRouter. Thus Grok 4 0709 was accessed
1785
       via LMArena. It is unclear if the two versions are identical.
1786
       Line 5: Nothing of the sort is established. (Critical)
1787
       Line 7: No reason is given for this. (Critical)
1788
1789
1790
        The group \backslash (G\backslash) is presented as \backslash (\large x, y \mid x y^2 = y^3)
            1791
1792 2
        Assume for contradiction that \((G\)) is non-trivial. Then at least
1793
            one of (x\setminus) or (y\setminus) is non-trivial.
1794
1795 4
        As established, the orders \(n) of \(y) and \(m) of \(x) must
1796

→ be finite and odd.

1797
1798
        The relations imply that conjugation by (x) induces an
1799
            automorphism on \(\langle y \rangle\) multiplying exponents
1800
            by \k equiv 3 \cdot 2^{-1} \pmod{n}\), with \k (\cdot (k, n) =
1801
        \hookrightarrow
            1\).
1802
1803
        Similarly, conjugation by (y) induces an automorphism on
1804
        \rightarrow \(\langle x \rangle\) multiplying exponents by \(l \equiv 3
1805
            \cdot 2^{-1} \pmod\{m\}\), with \cdot 2^{-1} \pmod\{m\}\).
1806 10
1807 11
        The order \ensuremath{(e\)} of \ensuremath{(k\)} modulo \ensuremath{(n\)} divides \ensuremath{(m\)} and \ensuremath{(1-
1808
        \hookrightarrow 1\).
1809 12
1810 13
        The order \(d) of \(l) modulo \(m) divides \(n) and \(k - m)
1811
1812 14
1813 15
        Now, (k - 1 \neq 3 \cdot 2^{-1} - 1 \neq n). Suppose an odd
        \rightarrow prime \(p\) divides both \(n\) and \(k - 1\). Then \(3 \)
        \rightarrow 2^{-1} \equiv 1 \pmod{p}\), so \(3 \equiv 2 \pmod{p}\), hence
1815
        \rightarrow \(p \mid 1\), impossible.
1816
1817 16
        Thus, \langle (\gcd(n, k-1) = 1 \rangle), so \langle (d = 1 \rangle).
1818 <sup>17</sup>
1819 18
        Then (1 \neq 1 \neq 1 \neq 1), so (2 \neq 1 \neq 3 \neq 1),
1820 <sup>19</sup>
        \rightarrow hence \((m \mid 1\), so \((m = 1\)) and \((x = 1\)).
1821
1822 <sup>20</sup>
        With (x = 1), the first relation becomes (y^2 = y^3), so (y
1823 <sup>21</sup>
        \Rightarrow = 1\), hence \((n = 1\)\).
1824
1825
1826 23
        This contradicts \(G\) non-trivial.
1827
        Therefore, \backslash (G \backslash) is trivial.
    25
1828
1829
        \boxed{G is trivial}
1830
1831
1832
1833
```

## В.10 Кімі К2

1837 1838

1839

1840 1841

1842 1843

1844

Note: The fact that at the end of the output generated by Kimi K2 a \$\$ is missing is not a typo. The model terminates its output without claiming to have found a solution. The error below occurs well before that time.

Line 184: The identity

$$y^3 = x^2 y^4 x^{-2}$$

does not follow by conjugating again by x, and in fact, does not follow from the previous line at all. (Critical)

```
1845
1846
        ### Understanding the Problem
1847
1848
       We have a group \$\$ G \$\$ defined by two generators \$\$ x \$\$ and \$\$
1849

    y $$ with two relations:

1850
       1. \$\$ xy^2 = y^3x \$\$
1851
       2. \$\$ vx^2 = x^3v \$\$
1852
1853
       Our goal is to prove that $$ G $$ is the trivial group, meaning
1854
            $$ x = 1 $$ and $$ y = 1 $$ in $$ G $$, so $$ G $$ has only
1855
            one element.
1856 <sub>8</sub>
1857 9
       ### Initial Approach
1858 10
       To show that $$ G $$ is trivial, we need to derive from the given
1859 11
            relations that \$\$ x = 1 \$\$ and \$\$ y = 1 \$\$. This means we
1860
            should manipulate the given equations to find simpler
1861
            relationships or equalities that force $$ x $$ and $$ y $$ to
1862
        \rightarrow be the identity.
1863
1864 12
        ### Manipulating the First Relation: $$ xy^2 = y^3x $$
1865 13
1866 <sup>14</sup>
       Let's start with the first relation:
1867 <sup>15</sup>
       $$ xy^2 = y^3x $$
1868 <sup>16</sup>
1869 <sup>17</sup>
       We can multiply both sides on the right by \$\$ x^{-1} \$ (the
    18
1870
        \rightarrow inverse of $$ x $$) to isolate terms:
1871
        $$ xy^2x^{-1} = y^3 $$
1872
1873
       This shows that \$\$ y^3 \$\$ is equal to \$\$ x y^2 x^{-1} \$\$, which
1874
        \hookrightarrow is the conjugation of $$ y^2 $$ by $$ x $$.
1875
1876
        ### Manipulating the Second Relation: $$ yx^2 = x^3y $$
1877 <sub>24</sub>
1878 <sub>25</sub>
       Similarly, take the second relation:
1879 <sub>26</sub>
       $$ yx^2 = x^3y $$
1880 <sub>27</sub>
1881 <sub>28</sub>
       Multiply both sides on the right by $$ v^{-1} $$:
1882 <sub>29</sub>
       $$ yx^2y^{-1} = x^3 $$
1883 30
1884 31
       Here, \$\$ x<sup>3</sup> \$\$ is the conjugation of \$\$ x<sup>2</sup> \$\$ by \$\$ y \$\$.
1885 32
       ### Exploring Conjugation and Powers
1886 33
1887 34
       From the first manipulation, we have:
1888 35
1889 36
       $$ x y^2 x^{-1} = y^3 $$
    37
```

```
This implies that \$\$ x \$\$ conjugates \$\$ y^2 \$\$ to \$\$ y^3 \$\$.
1891
        → Similarly, from the second manipulation:
1892 39
       $$ y x^2 y^{-1} = x^3 $$
1893 40
1894 41
       So \$\$ y \$\$ conjugates \$\$ x^2 \$\$ to \$\$ x^3 \$\$.
1895 42
1896 <sub>43</sub>
       ### Trying to Find Simpler Relations
1897 44
1898 45
       Let's see if we can find a relationship between $$ x $$ and $$ y
1899
       \Rightarrow $$ that leads to $$ x = 1 $$ or $$ y = 1 $$.
1900 46
       From \$\$ \times y^2 \times \{-1\} = y^3 \$\$, we can raise both sides to the
1901 47
       → power of $$ n $$ (for any integer $$ n $$):
1902
       $$ (x y^2 x^{-1})^n = y^{3n} $$
1903 48
       $ x y^{2n} x^{-1} = y^{3n} $$
1904 49
1905 50
       Similarly, from $$ y x^2 y^{-1} = x^3 $$:
1906 51
       $ (y x^2 y^{-1}) n = x^{3n} $$
1907 <sup>52</sup>
       $ y x^{2n} y^{-1} = x^{3n} $$
1908 53
1909 54
       These show how conjugation by \$\$ \times \$\$ and \$\$ \times \$\$ affects powers
1910 55
        \rightarrow of $$ y $$ and $$ x $$, respectively.
1911
1912 56
       ### Attempting to Find a Fixed Point
1913 <sup>57</sup>
1914 58
1915 59
       Let's consider the exponent of $$ y $$ in the first relation. The
       \rightarrow exponents on both sides of $$ x y^2 x^{-1} = y^3 $$ don't
1916
        → match directly, which is unusual. This suggests that the only
1917
           way this can hold is if \$\$ y = 1 \$\$, making both sides equal
1918
        \rightarrow to 1.
1919 60
1920 61
       Similarly, if \$\$ y = 1 \$\$, then plugging into the second
1921
       → relation:
       $$ 1 \cdot x^2 \cdot 1 = x^3 \cdot 1 $
1923 <sub>63</sub>
       $$ x^2 = x^3 $$
1924 64
       $$ x = 1 $$
1925 65
1926 66 By symmetry, if we assume \$\$ x = 1 \$\$, then:
1927 67 First relation:
       $$ 1 \cdot y^2 = y^3 \cdot 1 $
1928 68
       $$ y^2 = y^3 $$
1929 69
       $$ y = 1 $$
1930 70
1931 71
       This suggests that \$\$ x = y = 1 \$\$ is a solution that satisfies
1932 72
        \hookrightarrow both relations, making $$ G $$ trivial. But we need to ensure
1933
           that this is the only possible solution within the group $$ G
1934
           $$.
        \hookrightarrow
1935
1936 <sup>73</sup>
       ### Proving Uniqueness of Trivial Solution
1937 74
1938 75
       Let's formalize this without assuming \$\$ \times \$\$ or \$\$ y \$\$ is the
1939 76

    identity upfront.

1940
1941
1942
```

```
From \$\$ \times y^2 \times \{-1\} = y^3 \$\$, we can consider the exponent as a
1945
        \hookrightarrow kind of homomorphism. Specifically, the mapping that sends $$
1946
        \rightarrow y $$ to $$ y^3 $$ when conjugated by $$ x $$. Similarly, the
1947
        \rightarrow other relation gives a mapping sending $$ x $$ to $$ x^3 $$
1948
        → under conjugation by $$ y $$.
1949 79
1950 <sub>80</sub>
       This is reminiscent of the concept where in a group, if
1951
        → conjugation by an element raises powers to higher exponents,
1952
        → the group might be forced to be trivial.
1953 81
       Let's try to express everything in terms of a single element.
1954 82
       → Suppose we express one element in terms of the other.
1955
1956 83
       From \$\$ \times y^2 \times \{-1\} = y^3 \$\$, we can write:
1957 84
       $$ x y^2 = y^3 x $$
1958 85
1959 86
       Similarly, from $$ y x^2 y^{-1} = x^3 $$:
1960 87
       $$ y x^2 = x^3 y $$
1961 88
1962 <sup>89</sup>
       Now, let's try to find a word that must equal the identity.
1963 <sup>90</sup>
1964 <sup>91</sup>
       Compute \$\$ \times y \times^{-1} y^{-1} \$\$, but this might not directly
1965 <sup>92</sup>
        → help. Instead, let's try to find a power that collapses to
1966
        1967
    93
1968
       ### Using Commutators
1969
1970 96
       Notice that the relations resemble "expansive"
1971
        → behaviour-conjugation increases exponents. In some contexts,
1972
        → this implies the group must be trivial unless the elements
1973
        → commute in a certain way.
1974 97
1975 <sub>98</sub>
       Let's attempt to compute some commutators or express higher
1976
       → powers.
1977 99
1978 <sub>100</sub>
       From \$\$ \times y^2 \times \{-1\} = y^3 \$\$, we can apply this multiple times:
1979<sub>101</sub>
1980 102 First, raise both sides to the power of $$ k $$:
1981 103 \$ (x y^2 x^{-1}) k = y^{3k} $$
       $$ x y^{2k} x^{-1} = y^{3k} $$$
1982 104
1983 105
       Now, set \$\$ k = 0 \$\$, but that gives \$\$ 1 = 1 \$\$, which is
1984 106

    → trivial.

1985
1986 107
       Alternatively, express $$ y $$ in terms of itself:
1987 108
       $$ x y^2 x^{-1} = y^3 $$
1988 <sup>109</sup>
       $$ x y x^{-1} x y x^{-1} = y^3 $$
1989 110
1990 <sup>111</sup>
       $$ (x y x^{-1})^2 = y^3 $$
1991 112
1992 113
       Similarly, from $$ y x^2 y^{-1} = x^3 $$:
1993 114
       $$ (y x y^{-1})^2 = x^3 $$
1994 115
1995
       This seems to imply that the conjugates have certain power
       → relationships.
1996
1997 _{118}^{\dots} \mid ### Trying Specific Elements and Equalities
```

```
1998
1999 120
       Let's consider the order of $$ x $$ and $$ y $$. Suppose $$ x $$
       \rightarrow has finite order $$ n $$, then $$ x^n = 1 $$.
2001 121
2002 122
       From \$\$ y x^2 y\{-1\} = x^3 \$\$, we have:
2003 123
       $$ y x^2 y^{-1} = x^3 $$
2004 124
       \$\$ y x^4 y^{-1} = x^6 \$
2005 125
       $$ y x^{2k} y^{-1} = x^{3k} $$
2006 126
2007 127
       For \$\$ k = n \$\$, since \$\$ x^n = 1 \$\$, we have:
       $$ y 1 y^{-1} = 1 = x^{3n} $$
2008 128
       But \$\$ x^{3n} = (x^n)^3 = 1^3 = 1 \$, which doesn't give new
2009 129
       2010
2011 130
       Similarly, from \$\$ \times y^2 \times \{-1\} = y^3 \$\$, we have:
2012 131
2013 \, 132 \, \$ \, x \, y^{2k} \, x^{-1} = y^{3k} \, \$
2014 133
2015 134
       If \$\$ y \$\$ has finite order \$\$ m \$\$, then for \$\$ k = m \$\$:
       $$ x 1 x^{-1} = 1 = y^{3m} $$
2016 135
       $$ y^{3m} = 1 $$
2017 136
2018 137
       But since the order of $$ y $$ is $$ m $$, $$ m $$ divides $$ 3m
2019 138
        → $$, which is always true, so no contradiction arises directly
2020

    from this.

2021
2022 139
2023
       ### Introducing a Normal Subgroup Approach
2024
       Perhaps we can consider that these relations imply all elements
2025
        \rightarrow are trivial. Let's try to prove that $$ x $$ and $$ y $$
2026
        \hookrightarrow commute, and then see if this forces them to be trivial.
2027 143
2028 144
       Assume \$\$ xy = yx \$\$. Then from \$\$ xy<sup>2</sup> = y<sup>3</sup>x \$\$:
2029 145
       $$ xy^2 = y^3x $$
       $$ y^2x = y^3x $$ (since $$ x $$ and $$ y $$ commute)
2031 147
       Multiply both sides on the right by x^{-1} y^{-2} $$:
2032 148
       $$ 1 = y $$
2033 149
2034 150 | Similarly, from $$ yx^2 = x^3y $$:
2035 151 \$\$ yx^2 = x^3y \$\$
2036 152 \$\$ x^2y = x^3y \$\$
       $$ x^2 = x^3 $$
2037 153
       $$ x = 1 $$
2038 154
2039 155
       Thus, if \$\$ x \$\$ and \$\$ y \$\$ commute, both must be identity. But
2040 156
       → we don't know if they commute.
2041
2042 157
       ### Using the Composition of Conjugations
2043 158
2044 159
2045 160
       We have:
2046 161
       $$ \phi_x: y \mapsto x y x^{-1} $$
2047 162
       with \$ \phi_x(y^2) = y^3 \$, so \$ (\phi_x(y))^2 = y^3 \$,
           implying \$ \phi_x(y) \$ is a square root of \$ y^3 \$, i.e.,
2048
        \Rightarrow $$ x y x^{-1} = y^{3/2} $$, but this suggests fractional
2049

→ exponents, which are typically not in groups.

2050
2051 164
       This seems problematic unless \$\$ y = 1 \$\$.
```

```
2052 165
2053
        Let me rephrase: from \$\$ \times y^2 \times \{-1\} = y^3 \$\$, we can define \$\$
2054
        \Rightarrow z = x y x<sup>{-1}</sup> $$, then:
2055 167
        $$ z^2 = v^3 $$
2056 168
2057 169
       Similarly, from $$ y x^2 y^{-1} = x^3 $$, let $$ w = y x y^{-1}$
        \Rightarrow $$, then:
2059 170
        $$ w^2 = x^3 $$
2060 171
2061 172
       Now we have:
2062 173
       $$ x y x^{-1} = z $$ where $$ z^2 = y^3 $$
       $$ y x y^{-1} = w $$ where $$ w^2 = x^3 $$
2063 174
2064 175
       This seem to imply that $ z $ is some root of $ y ^3 $ and $
2065 176
        \rightarrow w $$ of $$ x^3 $$, but in groups, this often forces elements
2066

→ to be identity.

2067
2068 177
        ### Deriving Contradictions or Identities
2069 178
2070 179
       Let's try to express everything in terms of $$ x $$ and its
2071 180

→ conjugates.

2072
2073 181
2074 182
       From \$\$ \times y^2 \times \{-1\} = y^3 \$\$, we have:
2075 183
        $$ y^3 = x y^2 x^{-1} $$
2076 184
        \$\$ y^3 = x^2 y^4 x^{-2} \$ (conjugating by \$\$ x \$ again)
       $$ y^3 = x^2 y^4 x^{-2} $$
2076
2077
186
2078
2079 188
       But also:
       $$ y^3 = x y^2 x^{-1} $$
2080 189
        So:
2081 190
       $$ x y^2 x^{-1} = x^2 y^4 x^{-2} $$
2082 191
       $$ y^2 = x y^4 x^{-1} $$
2083 <sub>192</sub> But $$ x y^4 x^{-1} = (x y^2 x^{-1})^2 = (y^3)^2 = y^6 $$, so:
2084 <sub>193</sub> \$\$ y^2 = y^6 \$\$
2085 194
       $$ y^4 = 1 $$
2086 195
2087<sub>196</sub> | Similarly, from $$ y x^2 y^{-1} = x^3 $$, applying $$ y $$ twice:
2088 197 \$\$ \$ \$ \$ \$ \$ \$ \$ \$
2089 198 But also:
       $$ x^3 = y x^2 y^{-1} $$
2090 199
2091 200
       So:
       $$ y x^2 y^{-1} = y^2 x^4 y^{-2} $$
2092 201
       $$ x^2 = y x^4 y^{-1} $$
2093 202
        $$ x^2 = (x^3)^2 = x^6 $$
2094 203
       $$ x^4 = 1 $$
2095 204
2096 <sup>205</sup>
2097 <sup>206</sup>
       Now we have $$ x^4 = 1
2098 <sup>207</sup>
2099
2100
2101
2102
```

### B.11 QWEN3 235B A22B THINKING 2507

2106

2107 2108

Line 25: The model has assumed commutativity. (Critical)

```
2109
2110
2111 | To determine whether the group G = \langle x, y \rangle = 1
        \rightarrow y^3x,\ yx^2 = x^3y \rangle $ is trivial, we analyze the given
            relations and manipulate them using group-theoretic
2113

→ techniques.

2114
2115
2116
2117
       ### Step 1: Expressing Generators in Terms of a Single Element
2118
2119
       Define the **commutator** a = [x, y] = x y x^{-1} y^{-1} $.
2120
       → Then by definition:
2121
2122 9
       $$
2123 10
       x y = a y x.
2124 11
       $$
2125 12
2126 13 Now apply this to **Relation 1**:
2127 14 $$
2128 15
       xy^2 = y^3x.
2129 16
       $$
2130 17
2131 18 Left-hand side:
2132 19
       x y^2 = x y \cdot dot y = (a y x) \cdot dot y = a y x y.
2133 <sup>20</sup>
       $$
2134 21
2135 <sup>22</sup>
       Using x y = a y x again inside:
2136 23
2137 <sup>24</sup>
       a y x y = a y (a y x) = a^2 y^2 x.
2138 <sup>25</sup>
       $$
2139 <sup>26</sup>
2140 27
2141 28
       So:
2142
       $$
       x y^2 = a^2 y^2 x.
2143
       $$
    31
2144
    32
2145 33
       Equating with the right-hand side $ y^3 x $, we get:
2146 34
2147 35
       a^2 y^2 x = y^3 x \Rightarrow a^2 y^2 = y^3 \Rightarrow a^2 = y.
2148 36
2149 37
2150 38
       Thus, we have:
2151 39
       $$
2152 40
       y = a^2. \setminus tag\{1\}
2153 41
       $$
2154 42
2155 43
2156 44
       ### Step 2: Using the Second Relation
2157 45
2158 46
2159 47 | Now consider **Relation 2**:
    48 $$
```

```
yx^2 = x^3 y.
    49
2161
    50
2162 51
2163 52
       We define the inverse commutator b = [y, x] = y \times y^{-1} \times (-1)
2164
        \Rightarrow = a<sup>{-1}</sup> $, so:
2165 53
       $$
2166 54
       y x = b x y = a^{-1} x y.
2167 55
2168 56
2169 57
       Apply this to Relation 2:
2170 58
       $$
       y x^2 = y x \cdot (a^{-1} x y) \cdot (a x = a^{-1} x y x.
2171 59
2172 60
2173 61
2174 62
       Then:
2175 63
       $$
       y x^2 = a^{-1} x (a^{-1} x y) = a^{-2} x^2 y.
2176 64
2177 65
2178 66
       Equating with RHS $ x^3 y $, we get:
2179 67
2180 <sup>68</sup>
       a^{-2} x^2 y = x^3 y \neq a^{-2} x^2 = x^3 \neq x
2181 <sup>69</sup>
        \rightarrow a<sup>{-2}</sup>. \tag{2}
2182
2183 70
    71
2184
2185
2186
       ### Step 3: Substituting Back into Relations
2187 75
2188 76
       From (1) and (2), we now express both generators in terms of $ a
2189

⇒ $:

2190 77
       $$
2191 <sub>78</sub>
       x = a^{-2}, \quad y = a^{2}.
2192 79
       $$
2193 80
2194 81
       Substitute these into **Relation 1**:
2195 82
       $$
2196 83
       xy^2 = a^{-2}(a^2)^2 = a^{-2}a^4 = a^2
2197 84
       $$
2198 85
       $$
       y^3 x = (a^2)^3 a^{-2} = a^6 a^{-2} = a^4.
2199 86
       $$
2200 87
2201 88
2202 89
       So:
       $$
2203 90
2204 91
       a^2 = a^4 \setminus Rightarrow a^2 = e.
2205 92
2206 <sup>93</sup>
2207 94
       Therefore, $ a = e $, which implies:
2208 95
2209 96
       x = a^{-2} = e, \quad y = a^2 = e.
       $$
    97
2210
    98
2211
2212
    100
2213 101
       ### Step 4: Verifying Triviality
```

```
2214
2215
       With x = e  and y = e , all relations are trivially
2216
        → satisfied:
2217 104
       - xy^2 = e \cdot cdot e = e , and y^3x = e \cdot cdot e = e ,
2218 105
       - $ yx^2 = e \cdot e = e $, and $ x^3y = e \cdot e = e $.
2219 106
2220 <sub>107</sub>
       Hence, the only solution is the trivial group.
2221 108
2222 109
2223 110
2224 111
       ### Conclusion
2225 112
       By expressing the generators in terms of a single element derived
2226 113
       → from the commutator and analyzing the resulting constraints,
2227
        \rightarrow we find that the relations force both $ x $ and $ y $ to be
2228
        → the identity element. Therefore, the group defined by the
2229
        → presentation:
2230
2231 114
2232 115
       G = \langle x, y \rangle = y^3x, \langle yx^2 = x^3y \rangle
2233 116
       $$
2234 117
2235 118
2236 119
       is **trivial**.
2237 120
2238 121
       $$
2239 122
       \boxed{G \text{ is the trivial group}}
2240 124
2241
```

#### B.12 GLM-4.5

22422243

2244 2245

2246

2247

2248

Note: The generation began in the middle of the sentence, our reproduction below is not a typo.

Line 19: This identity is incorrect. The model has made an algebra error from the previous line. (Critical)

```
2249
      that the group G = \langle x, y \rangle = y^3x, yx^2 = x^3y
2250
          \rangle$ is the trivial group, I'll derive that both
2251
          generators $x$ and $y$ must be the identity element.
2252
2253
      Starting with the given relations:
2254
      1. $xy^2 = y^3x$
2255
      2. yx^2 = x^3y
2256
2257
      From relation (1), multiply both sides by y^{-2} on the left:
2258
      y^{-2}xy^2 = yx
                          (3)
2259
2260 10
      From relation (2), multiply both sides by x^{-2} on the left:
2261 11
      x^{-2}yx^2 = xy
                          (4)
2262 12
2263 13
      Let A = y^{-1}xy and B = x^{-1}yx. Then:
2264 14
      - From (3): y^{-1}Ay = yx$
      - From (4): x^{-1}Bx = xy
2265 15
2266 16
      Since A = y^{-1}xy, we have y^{-1}Ay = y^{-2}xy^2. From
2267 17
          equation (3), y^{-2}xy^2 = yx^3, so:
```

```
2268 18 \mid \$y^{-}\{-1\} Ay = yx\$
2269
       which implies A = y^2x.
2270 20
2271 21
       Similarly, since \$B = x^{-1}yx\$, we have \$x^{-1}Bx = x^{-2}yx^{2}\$.
2272
       \rightarrow From equation (4), x^{-2}yx^2 = xy, so:
2273 22
       x^{-1}Bx = xy
2274 <sub>23</sub>
       which implies B = x^2y.
2275 24
2276 <sub>25</sub>
       Now we have:
2277 <sub>26</sub>
       A = y^{-1}xy = y^{2x}
2278 27 \$B = x^{-1}yx = x^2y
2279 28
       From \$yA = xy\$ and \$A = y^2x\$, we get:
2280 29
       y(y^2x) = xy
2281 30
2282 31 | $y^3x = xy$
2283 32
       From the original relation (1), xy^2 = y^3x, so:
2284 33
       xy^2 = xy
2285 34
2286 35
       Now, multiply both sides of xy^2 = xy by x^{-1} on the left:
2287 36
       y^2 = y
2288 37
2289 38
       Multiply both sides of y^2 = y by y^{-1} on the right:
2290 39
2291 40
       v = e
2292 41
2293 42
       Since y is the identity element, we can substitute y = e into
       \rightarrow relation (1):
2294
       xe^2 = e^3x
2295 44
       x = x
2296 45
2297 46
       This is a tautology and doesn't give us new information. Let's
2298
       \rightarrow substitute $y = e$ into relation (2):
2299 47
       ex^2 = x^3e
2300 48
       x^2 = x^3
2301 49
2302 50
       Multiply both sides of x^2 = x^3 by x^{-2} on the right:
2303 51
       e = x
2304 52
2305 s3 Therefore, both $x$ and $y$ are the identity element, which means
2306
        \hookrightarrow $G$ is the trivial group.
2307
2308
2309
2310
2311
2312
2313
2314
2315
2316
2317
2318
2319
```

# B.13 GEMINI 2.5 DEEP THINK

2322

2323 2324

Line 18: The model claims  $yx^3 = (yx)x^3$ , which is an algebra error. (Critical)

```
2325
2326
      Yes, we can prove that G is the trivial group.
2327
2328
       Let G$ be a group with generators x$ and y$ and the relations:
2329
       1) xy^2 = y^3x
2330
       2) yx^2 = x^3y
2331
2332
       We want to show that x=e and y=e, where e is the identity
2333

→ element.

2334
2335
      Let's start by manipulating the second relation (2). We can
2336
       → rewrite it as:
2337 10
      x^3v = vx^2 (2')
2338 11
2339 12
       Now, we examine how the element x^3 interacts with the element
2340
       \rightarrow $yx$. We look at the product $x^3(yx)$:
2341 13
       x^3(yx) = (x^3y)x (by associativity)
2342 14
2343 15 We use the rearranged relation (2') to substitute $x^3y$ with
2344
       2345 16
      (x^3y)x = (yx^2)x = y(x^2x) = yx^3
2346 17
2347 18 So we have found that x^3(yx) = yx^3. Since yx^3 = (yx)x^3,
       \rightarrow this means that x^3 commutes with yx. In terms of
2348
       \rightarrow commutators, we have [x^3, yx] = e.
2349
2350 19
      Next, let's analyze the commutator \{x^3, yx\}. Recall that the
       \rightarrow commutator \{A, B\} is defined as ABA^{-1}B^{-1}.
2352
       [x^3, yx] = x^3 (yx) (x^3)^{-1} (yx)^{-1}
2353 <sup>21</sup>
       = x^3 y x x^{-3} (x^{-1}y^{-1})
2354 22
       = x^3 y (x^{1-3-1}) y^{-1}
2355 23
2356 24
       = x^3 y x^{-3} y^{-1}$
2357 25
2358 26
       This expression is exactly the definition of the commutator of
       \rightarrow $x^3$ and $y$, i.e., $[x^3, y]$.
2359
2360
       Since we established that \{x^3, yx\} = e and \{x^3, yx\} = [x^3, yx]
2361
       \rightarrow y]$, we must have:
2362 29
       [x^3, y] = e
2363 30
2364 31
      This means that x^3 commutes with y:
2365 _{32} \mid \$x^3y = yx^3\$.
2366 33
2367 34
      Now we have two equations involving $x^3y$:
2368 35
      From relation (2'): x^3y = yx^2
2369 36 From our derivation: x^3y = yx^3
2370 37
2371 38
      We can equate the right-hand sides:
2372 39
      yx^2 = yx^3
2373 40
2374 41 We multiply by y^{-1} on the left:
2375 42
      $x^2 = x^3$
    43
```

```
2376 <sub>44</sub>
2377 <sub>45</sub>
      We multiply by x^{-2} (on either side):
       e = x$.
2378 46
2379 47
       Now that we have shown x=e, we substitute this back into the
2380
        \hookrightarrow first relation (1):
2381 48
       xy^2 = y^3x
2382 49
       e^{y^2} = y^3 = y
2383 50
       y^2 = y^3
2384 51
2385 52
       We multiply by y^{-2}:
2386 53
       e = y.
2387 54
       Since both generators x and y are equal to the identity
2388 55
        \rightarrow element, the group $G$ is the trivial group.
2389
2390 56
2391 57
2392
2393
```

```
2430
      B.14 LLAMA 4 MAVERICK
2431
      Lines 4-8: A lot of effort is expended proving y^3x = y^3x.
2433
      Line 11: The model claims x(y^2)x^{-1}(y^2)^{-1} = e but never proves this. (Critical)
2434
2435
       ## Step 1: Understand the given problem and the relations
2436
       → provided for the group $G$.
2437
       The group G is generated by x and y with the relations
2438
       \Rightarrow $xy^2 = y^3x$ and $yx^2 = x^3y$. We need to determine if $G$
2439
        → is the trivial group, meaning it contains only the identity
2440
       \rightarrow element.
2441
2442
       ## Step 2: Manipulate the given relations to simplify or derive
2443

→ useful equations.

2444
       Starting with xy^2 = y^3x, we can rearrange this to xy^2x^{-1}
2445
       \rightarrow = y^3$, implying that $x$ conjugates $y^2$ to $y^3$.
2446
       \rightarrow Similarly, from yx^2 = x^3y^2, we get yxy^{-1} = x^3x^{-2} =
2447
       \rightarrow x$, showing that $y$ conjugates $x$ to $x$.
2448 6
2449 7
       ## Step 3: Further simplify the relations to check for any
2450
       → patterns or simplifications that could lead to proving $G$ is

    trivial.

       From xy^2 = y^3x, we can also derive x^{-1}y^3x = y^2. This
2452 8
       \rightarrow implies x^{-1}y^3x = y^2 and thus y^3 = xy^2x^{-1}.
2453
       \hookrightarrow Substituting y^2 = x^{-1}y^3x into the original equation
2454
       \rightarrow yields x(x^{-1}y^3x) = y^3x, simplifying to y^3x = y^3x,
2455
       → which is always true and doesn't provide new information
2456

→ directly.

2457
2458 <sup>9</sup>
       ## Step 4: Analyze the relation xy^2 = y^3x further to derive a
2459 <sup>10</sup>
       \hookrightarrow condition on $y$.
2460
       Rearranging xy^2 = y^3x gives xy^2x^{-1}y^{-3} = e, where e
2461 11
       \rightarrow is the identity. This can be written as
2462
       \Rightarrow $xy^2x^{-1}y^{-2}y^{-1} = e$, or
2463
       \Rightarrow $x(y^2)x^{-1}(y^2)^{-1}y^{-1} = e$, simplifying to $y^{-1} =
2464
       \rightarrow e$ because x(y^2)x^{-1}(y^2)^{-1} = e implies y^2 = e
2465
       \rightarrow after simplification, but directly we see $y = e$ from
2466
       \Rightarrow \$v^{-1} = e\$.
2467
2468 13
       ## Step 5: Examine the implication of y = e on the group G
2469
       \rightarrow and its generators.
2470 14
       If y = e, then substituting into one of the original relations,
2471
       \Rightarrow $xy^2 = y^3x$ becomes $x = x$, which is true for any $x$.
2472
       \rightarrow However, we need to check the other relation as well. yx^2 =
2473
       \rightarrow x^3y$ becomes $ex^2 = x^3e$, simplifying to $x^2 = x^3$. This
2474
       \rightarrow implies x^3x^{-2} = e^x, or x = e^x.
2475 15
2476 16
       ## Step 6: Verify the result by checking if both generators being

→ the identity implies $G$ is trivial.

       Since both x = e and y = e, and G is generated by x and
2478 17
       → $y$, $G$ must be the trivial group because it is generated by
2479
       \hookrightarrow the identity.
2480
2481 18
       The final answer is: $\boxed{Yes}$
2482 19
2483 <sup>20</sup>
```

### B.15 DEEPSEEK V3 0324

2484

2485 2486

2487

2488

2489

2490

2491

2492

2493

Line 46: The order is only a positive integer if it is finite. The model has ignored the infinite case. However, it recovers from this on line 62.

Line 82: The model worries that  $x^3 = x^2$  can only be simplified to x = e if x is cancellable. But this is an unfounded concern, as we are working in a group.

Line 194: The model makes no progress with the problem, other than showing |x| and |y| are coprime to 6, but then concludes the group is trivial. No part of the proof comes close to showing this. (Critical)

```
2494
       Alright, let's tackle this problem step by step. We have a group
2495
            $$ G $$ generated by two elements $$ x $$ and $$ y $$ with
2496
           the following relations:
2497
2498
       1. $$ xy^2 = y^3x $$
2499
       2. \$\$ vx^2 = x^3v \$\$
2500
2501
       Our goal is to show that $$ G $$ is the trivial group, meaning
2502
           that \ x \  and \  are both the identity element, and
2503
           thus $$ G = \{e\} $$.
2504 7
2505 8
       ### Understanding the Problem
2506 9
2507 10
       First, let's recall what these relations mean. In a group
           presentation, relations are equations that hold true among
2508
           the generators. Here, the two relations give us ways to
2509
           manipulate expressions involving $$ x $$ and $$ y $$.
2510
2511 11
       Our strategy will be to derive consequences from these relations,
2512 12
           hoping to find that $$ x $$ and $$ y $$ must be the identity.
2513
2514 13
2515 <sup>14</sup>
       ### Starting with the First Relation: $$ xy^2 = y^3x $$
2516 <sup>15</sup>
       Let's see what this tells us. If we multiply both sides on the
2517 <sup>16</sup>
           left by $$ x^{-1} $$, we get:
2518
    17
2519
       \$\$ x^{-1}xy^2 = x^{-1}y^3x \$\$
2520
2521
       Which simplifies to:
    20
2522
    21
2523
       $$ y^2 = x^{-1}y^3x $$
2524
2525 <sub>24</sub>
       This can be rewritten as:
2526 <sub>25</sub>
2527 <sub>26</sub>
       $ x y^2 x^{-1} = y^3 \tag{Equation 1}$$
2528 <sub>27</sub>
2529 <sub>28</sub>
       This suggests that conjugating $$ y^2 $$ by $$ x $$ gives $$ y^3
2530

⇒ $$.
2531 29
2532 30
       Similarly, from the second relation \$\$ yx^2 = x^3y \$\$, let's
        \hookrightarrow multiply both sides on the left by $$ y^{-1} $$:
2533
2534 31
       $ y^{-1}yx^2 = y^{-1}x^3y $
2535 32
2536 33
2537 34
       Simplifying:
    35
```

```
$$ x^2 = y^{-1}x^3y $$
2539
2540 38
       Which can be rewritten as:
2541 39
2542 40
       \$\$ y x^2 y^{-1} = x^3 \times \{Equation 2\}
2543 41
2544 42
       This suggests that conjugating x^2 x^2 \ by x^3 y \ gives x^3 y \
2545

⇒ $$.
2546 43
2547 44
        ### Exploring Equation 1: \$\$ \times y^2 \times \{-1\} = y^3 \$\$
2548 45
       This equation tells us that $$ y^2 $$ and $$ y^3 $$ are
2549 46
        → conjugates, implying that they have the same order in the
2550
        \hookrightarrow group. The order of an element $$ g $$, denoted $$ |g| $$, is
2551
        \rightarrow the smallest positive integer $$ n $$ such that $$ g^n = e
2552

⇒ $$.
2553
2554 47
       From $$ y^2 $$ and $$ y^3 $$ being conjugates:
2555 48
2556 49
       $$ |y^2| = |y^3| $$
2557 <sup>50</sup>
2558 <sup>51</sup>
       We know that in general, \$ |g^k| = \frac{|g|}{\gcd(|g|, k)} \$.
2559 <sup>52</sup>
        → So:
2560
2561 53
       frac{|y|}{\gcd(|y|, 2)} = \frac{|y|}{\gcd(|y|, 3)} 
2562 54
2563
       This implies:
2564
2565
       \$ \gcd(|y|, 2) = \gcd(|y|, 3) \$$
2566 59
2567
       The only way this can happen is if \$\$ \d(|y|, 6) = 1 \$\$,
2568
        \rightarrow meaning that $$ |y| $$ is coprime with 6. The simplest
2569
        \rightarrow possibility is $$ |y| = 1 $$, i.e., $$ y = e $$.
2570 61
2571 <sub>62</sub>
       But let's not jump to conclusions; maybe $$ |y| $$ is infinite or
2572
            another number coprime to 6. Let's explore further.
2573 63
       ### Exploring Equation 2: \$\$ y x^2 y^{-1} = x^3 \$\$
2574 64
2575 65
       Similarly, this tells us that $$ x^2 $$ and $$ x^3 $$ are
2576 66

→ conjugates, so:
2577
2578 67
2579 68
       $$ |x^2| = |x^3| $$
2580 69
       Again, using the order formula:
2581 <sup>70</sup>
2582 71
       frac{|x|}{\gcd(|x|, 2)} = \frac{|x|}{\gcd(|x|, 3)} 
2583 72
2584 <sup>73</sup>
2585 74
       Thus:
2586 <sup>75</sup>
2587 76
       \$ \gcd(|x|, 2) = \gcd(|x|, 3) \$$
    77
2588
       Similarly, this implies \$\$ \gcd(|x|, 6) = 1 \$\$, so \$\$ \|x\| \$\$ is
2589
        \rightarrow coprime with 6. Again, the simplest solution is $$ |x| = 1
2590
        \Rightarrow $$, i.e., $$ x = e $$.
2591
```

```
### Considering $$ x $$ and $$ y $$ are the Identity
2593
2594 82
        If \$\$ y = e \$\$, then the first relation \$\$ xy^2 = y^3x \$\$ becomes
2595
        \Rightarrow $$ x = x $$, which holds trivially. The second relation $$
2596
        \rightarrow yx^2 = x^3y $$ becomes $$ x^2 = x^3 $$, which implies $$ x =
2597
            e \$ (assuming \$ x \$ has finite order or is cancellable).
2598 <sub>83</sub>
2599 84
        Similarly, if \$\$ x = e \$\$, the second relation becomes \$\$ y = y
2600
        \Rightarrow $$, and the first relation becomes $$ y^2 = y^3 $$, implying
2601
        \Rightarrow $$ y = e $$.
2602 85
        This suggests that the only solution is x = y = e
2603 86
2604 87
        ### Trying to Find Non-Trivial Solutions
2605 88
2606 89
       But let's not assume the group is finite yet. Maybe the group is
2607 90
        \hookrightarrow infinite, and $$ x $$ and $$ y $$ have infinite order. Let's
2608
            see if we can find any non-trivial relations.
2609
2610 <sup>91</sup>
        From \$\$ \times y^2 \times \{-1\} = y^3 \$\$, we can think of this as an action
2611 <sup>92</sup>
        \rightarrow of $$ x $$ on $$ y $$. In general, this could suggest a
2612
        → semidirect product, but the relations seem too restrictive.
2613
2614 <sup>93</sup>
        Similarly, \$\$ y x^2 y^{-1} = x^3 \$\$ is an action of <math>\$\$ y \$\$ on \$\$
2615 <sup>94</sup>
        2616
2617
        Perhaps we can find a consistency condition by combining these.
2618
2619 98
        Let's see if we can express both $$ x $$ and $$ y $$ in terms of
2620
        \rightarrow one another.
2621 99
2622 <sub>100</sub>
        From \$\$ \times y^2 = y^3 \times \$\$, we can try to solve for \$\$ y \$\$ in
2623
        \rightarrow terms of $$ x $$, but it's not straightforward.
2624 101
2625 <sub>102</sub>
        Alternatively, let's see if we can find that \$\$ \times \$\$ and \$\$ \times \$
2626

→ commute.

2627 103
       Assume for a moment that $ x $ and $ y $ commute, i.e., $ xy
2628 104
2629
        \Rightarrow = yx $$. Then:
2630 105
        From \$\$ xy^2 = y^3x \$\$, with commutativity:
2631 106
2632 107
2633 108
        $$ y^2x = y^3x $$
2634 109
        Cancel \$\$ x \$\$ on the right (assuming \$\$ x \$\$ is cancellable,
2635 110

→ which it is in a group):
2636
2637 111
2638 <sup>112</sup>
        $$ y^2 = y^3 $$
2639 <sup>113</sup>
2640 114
        Thus \$\$ y = e \$\$.
2641 115
2642 116
        Similarly, from \$\$ yx<sup>2</sup> = x<sup>3</sup>y \$\$, with commutativity:
    117
2643
        $$ x^2y = x^3y $$
    118
2644 119
2645 120 | Cancel $$ y $$:
```

```
2646
2647
121
               $$ x^2 = x^3 $$
2648 123
2649 124
               Thus $$ x = e $$.
2650 125
2651 <sub>126</sub>
               So if \$\$ x \$\$ and \$\$ y \$\$ commute, they must both be the
                 → identity.
2653 127
2654 128
               But we haven't shown that $ x $ and $ y $ must commute. Maybe
2655
                \hookrightarrow the relations force them to commute.
2656 129
               Let's attempt to see if \$\$ xy = yx \$\$ follows from the given
2657 130
2658
                \rightarrow relations.
2659 131
               From \$\$ xy^2 = y^3x \$\$, let's try to express \$\$ xy \$\$:
2660 132
2661 133
               Multiply both sides by $$ y^{-1} $$ on the right:
2662 134
2663 135
               $$ xy = y^3x y^{-1} $$
2664 136
2665 <sup>137</sup>
               Similarly, from \$\$ yx<sup>2</sup> = x<sup>3</sup>y \$\$, multiply by \$\$ x<sup>{-1}</sup> \$\$ on
2666 <sup>138</sup>
                 → the right:
2667
2668 <sup>139</sup>
2669 <sup>140</sup>
               $$ yx = x^3y x^{-1} $$
2670 141
2671 <sup>142</sup>
               This seems circular. Maybe another approach is better.
2672
                ### Attempting to Find Orders
2673 .
145
2674 146
               Let me think about the orders of $ x $ and $ y $. From
2675
                \rightarrow earlier, $$ |y^2| = |y^3| $$, which implies $$ \gcd(|y|, 2) =
2676
                 \rightarrow \gcd(|y|, 3) \$\$.
2677 147
2678 <sub>148</sub>
               Possible orders for $$ y $$:
2679 149
2680 <sub>150</sub>
               If \$ |y| \$ is finite, then \$ \gcd(|y|, 2) = \gcd(|y|, 3) \$$.
2681
                → The numbers where this holds are those not divisible by 2 or
                 \rightarrow 3 (or both), i.e., numbers coprime to 6: 1, 5, 7, 11, etc.
2682
2683 151
               Similarly, \$\$ |x| \$\$ must be coprime to 6.
2684 152
2685 153
               The simplest case is \$\$ |x| = |y| = 1 \$\$, i.e., trivial group.
2686 154
2687 155
                If \$\$ |y| > 1 \$\$, say \$\$ |y| = 5 \$\$, then \$\$ y^2 \$\$ has order \$\$
2688 156
                 \rightarrow 5 / \qcd(5,2) = 5 \$, and \$\ \y^3 \$\ has order \$\ 5 /
2689
                        \gray \gra
2690

→ to a contradiction elsewhere?

2691
2692 <sup>157</sup>
               From \$\$ \times y^2 \times \{-1\} = y^3 \$\$, if \$\$ y \$\$ has order 5, then \$\$
2693 <sup>158</sup>
                 \Rightarrow y^3 = y^{-2} $$, so:
2694
2695 159
2696 160
               $$ x y^2 x^{-1} = y^{-2} $$
        161
2697
               But $$ y^2 $$ has order 5, and its inverse also has order 5, so
2698
                 \rightarrow this is possible, but doesn't immediately force $$ y = e $$.
2699 163
```

```
2700 164
        Similarly, for $$ x $$, if $$ |x| = 5 $$, from $$ y x^2 y^{-1} =
2701
        \rightarrow x<sup>3</sup> $$, and $$ x<sup>3</sup> = x<sup>{-2}</sup> $$, so:
2702 165
2703 166
        \$\$ y x^2 y^{-1} = x^{-2} \$\$
2704 167
2705 <sub>168</sub>
        Again, no immediate contradiction.
2706 169
2707 170
        However, the group seems too constrained to allow non-trivial
2708
        → elements, but I'm not seeing an immediate contradiction for
2709
        → higher orders coprimes to 6.
2710 171
        ### Trying Specific Cases
2711 172
2712 173
2713 174 Perhaps assume $$ y = e $$, then see what $$ x $$ must be.
2714 175
       If \$\$ y = e \$\$, then relations become:
2715 176
2716 177
        1. $$ x = x $$ (holds)
2717 178
        2. \$\$ x^2 = x^3 \in x = e \$\$
2718 179
2719 <sup>180</sup>
        So \$\$ x = e \$\$, \$\$ y = e \$\$. Trivial group.
2720 <sup>181</sup>
2721 182
2722 183
        Similarly, if \$\$ x = e \$\$, then relations become:
2723 <sup>184</sup>
2724 185
        1. \$\$ y^2 = y^3 \le y = e \$\$
2725 <sup>186</sup>
        2. \$\$ y = y \$\$  (holds)
2726
        Again, trivial group.
2727
2728
        This suggests that the only solution is the trivial group.
2729 191
2730 <sub>192</sub>
        ### Conclusion
2731 193
2732 194
        After exploring these relations, it appears that the only group
2733
        \rightarrow $$ G $$ satisfying both $$ xy^2 = y^3x $$ and $$ yx^2 = x^3y
2734
            \$\$ is the trivial group where both generators \$\$ x \$\$ and \$\$
2735
        → y $$ are the identity element. Any non-trivial assignment
        → leads to contradictions or forces the generators to be the
2736
        \rightarrow identity.
2737
2738 195
        ### Final Answer
2739 196
2740 197
        After carefully analyzing the given relations, we conclude that
2741 198
        \rightarrow the group $$ G $$ must be the trivial group. That is, $$ G =
2742
        \rightarrow \{e\} $$, where $$ e $$ is the identity element, and $$ x = y
2743
         \Rightarrow = e $$.
2744
2745 <sup>199</sup>
        ### Verification
2746 <sup>200</sup>
2747^{\ 201}
2748 <sup>202</sup>
        Let's verify:
2749 <sup>203</sup>
2750 204
        1. \$\$ xy^2 = e \cdot e^2 = e \$\$ and \$\$ y^3x = e^3 \cdot e = e \$\$,
        \rightarrow so $$ xy^2 = y^3x $$.
        2. \$\$ yx^2 = e \cdot cdot e^2 = e \$\$ and \$\$ x^3y = e^3 \cdot cdot e = e \$\$,
2752
        \rightarrow so $$ yx^2 = x^3y $$.
2753 <sub>206</sub>
```

```
2754
2755 <sup>207</sup>
        Both relations hold, and no smaller group satisfies them unless
         _{\rightarrow} both generators are the identity. Hence, $$ G $$ is indeed
2756
         \hookrightarrow the trivial group.
2757 <sub>208</sub>
2758
2759
2760
2761
2762
2763
2764
2765
2766
2767
2768
2769
2770
2771
2772
2773
2774
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2800
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2804
2805
2806
```

```
2808
       B.16 QwQ 32B
2809
2810
      Lines 7-9: There is an algebra error in deriving x^{-1}y^3xy^{-3} = e. (Critical)
2811
2812
      Line 13: Commutativity is incorrectly assumed. (Critical)
2813
       Line 25: Commutativity is incorrectly assumed. (Critical)
2814
2815
2816
       To prove that the group \ (G \ ) generated by \ (x \ ) and \ (y \ )
2817
        \rightarrow with relations \( xy^2 = y^3x \) and \( yx^2 = x^3y \) is
2818
        \hookrightarrow trivial, we start by analyzing the given relations.
2819
2820
       First, consider the relation (xy^2 = y^3x). Rearranging this,
2821

→ we have:

2822
       1/
2823
       x^{-1} y^{3} x = y^{2}
2824
       Multiplying both sides on the right by (y^{-3}), we get:
2826
2827 <sub>9</sub>
       x^{-1} y^3 x y^{-3} = e
2828 <sub>10</sub>
       \1
2829 11
       This simplifies to:
2830 12
       x^{-1} (y^3 y^{-3}) x = e \in x^{-1} y^0 x = e \in x^{-1}
2831 13
        \rightarrow x^{-1} e x = e
2832
       \1
2833 14
       However, we need to re-express this correctly. Starting again:
2834 15
2835 16
2836 17
       x y^2 x^{-1} = y^3
       \1
2837 18
       Multiply both sides by (y^{-3}):
2838 <sup>19</sup>
2839 <sup>20</sup>
       x y^2 x^{-1} y^{-3} = e
2840 <sup>21</sup>
2841 <sup>22</sup>
2842 23
       This can be rewritten as:
    24
       \[
2843
       x (y^{-1}) x^{-1} = e
2844
       \1
2845
       Multiplying both sides on the left by \ (x^{-1})\  and on the
2846
        \rightarrow right by \( x \), we get:
2847 28
        1/
2848 29
       y^{-1} = e \in y = e
2849 30
       \]
2850 31
       Thus, \setminus ( y \setminus) must be the identity element.
2851 32
2852 <sub>33</sub>
       Substituting \( y = e \) into the second relation \( yx^2 = x^3y
2853
           \backslash), we get:
2854 34
       \[
2855 35
       e \cdot x^2 = x^3 \cdot cdot = implies x^2 = x^3
2856 36
       \1
       This simplifies to:
2857 37
2858 38
       1/
       x^2 = x^3 \le x^2 = x^3 \le x^{-2} = x
2859 39
2860 40
       \]
2861 41
       42
```

```
2862 43
        Since both (x ) and (y ) are the identity element, the
2863
         \rightarrow group \( G \) is trivial.
2864 44
2865 45
         ] /
2866 46
         \boxed{1}
2867 <sub>47</sub>
         \]
2868 <sub>48</sub>
2869
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```

## B.17 GPT-OSS-120B

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2917 2918

2919

Line 24: The given conjugation relations only show that the subgroups generated by  $y^2$  and  $x^2$  are normal, not those generated by x and y. (Critical)

```
2920
2921
2922
       G=\langle x ,y\mid xy^{2}=y^{3}x ,\; yx^{2}=x^{3}y\rangle .
2923
2924
2925
2926
2927
       ### 1. Conjugation relations
2928
2929
       From the defining equalities we can rewrite
2930
2931
2932 11
       \begin{aligned}
2933 12
       xy^{2}_{e}y^{3}x \quad \
2934
       \rightarrow x\,y^{2}x^{-1}=y^{3},\$$2mm]
2935 13
       yx^{2}&=x^{3}y \quad duad\ y\ x^{2}y^{-1}=x^{3}.
2936 14
       \end{aligned}
2937 15
       \text{tag}\{1\}
2938 16
       $$
2939 17
       Thus
2940 18
2941 19
       *conjugation by $$x$$ sends $$y^{2}$$ to $$y^{3}$$;*
2942 20
2943 21
       *conjugation by $$y$$ sends $$x^{2}$$ to $$x^{3}$$.
2944 22
2945 23
       Hence the subgroups $$\langle y\rangle$$ and $$\langle x\rangle$$
2946 <sup>24</sup>
        \rightarrow are normal, and the
2947
       conjugation maps restrict to automorphisms of these cyclic
2948 <sup>25</sup>

→ subgroups.

2949
2950 <sup>26</sup>
    27
2951
2952
       ### 2. The automorphisms are given by a power map
2953
    29
2954
    30
       Let
2955
2956
2957 <sub>33</sub>
       x y x^{-1}=y^{\ell}, \quad y x y^{-1}=x^{k}
2958 34
       \text{tag}\{2\}
2959 <sub>35</sub>
       $$
2960 <sub>36</sub>
2961 37
       for some integers $$\ell ,k$$.
2962 38
       Because the restriction of conjugation to a cyclic group is an
2963

→ automorphism,

2964 39
       \ (if the orders are infinite
2965

    → the condition is

       $\ell ,k\neq0$$).
2966 40
2967 41
       Apply the first equality of (2) to the element $$\partial \gamma^{2}$$:
2968 42
2969 43
       $$
       x y^{2}x^{-1} = (x y x^{-1})^{2}=y^{2\ell}=y^{3}.
```

```
2971 46
       Hence
2973 48
2974 49
       2\left(y\right) \simeq 3\left(y\right) \qquad 1=3\left(y\right)
2975
        \rightarrow } | y | = \infty).
2976 <sub>50</sub>
       \text{tag}{3}
2977 51
       $$
2978 52
2979 53
       Similarly, applying the second equality of (2) to x^{2} gives
2980 54
2981 55
       y x^{2}y^{-1} = (y x y^{-1})^{2} = x^{2k} = x^{3},
2982 56
2983 57
2984 58
       so
2985 59
2986 60
       2k \neq 3 \pmod{|x|} \qquad \text{if } |x| = \inf y.
2987 61
       \text{tag}\{4\}
2988 <sup>62</sup>
2989 <sup>63</sup>
       $$
2990 64
       Because the congruences (3) and (4) have a solution, the orders
2991 65
        \rightarrow of $$x$$ and
2992
       $$v$$ cannot be even and cannot be multiples of $$3$$; in
2993 66

→ particular they are

2994
2995 67
       odd (or infinite). In any case the integers $$\ell ,k$$ are well

→ defined and

2996
       satisfy $$\ell\not\equiv 0$$ and $$k\not\equiv 0$$.
2997 69
2998 70
2999
3000 71
        \#\#\# 3. \$x\$\$ and \$\$y\$\$ lie in the same cyclic subgroup
3001 72
3002 73
       From (2) we compute the commutator:
3003 74
3004 75
       $$
3005 76
       \begin{aligned}
3006 77 | x y x^{-1} | y^{-1} |
3007 78
           \&= (x y x^{-1}) y^{-1}
           = y^{\{\{\}\}}y^{\{-1\}}
3008 79
            = y^{\{-1\}}, \$2mm]
3009 80
          \&= x (y x y^{-1}) x^{-1}
3010 81
3011 82
           = x x^{(k)}x^{(-1)}
            = x^{k}.
3012 83
       \end{aligned}
3013 84
3014 85
       $$
3015 86
3016 87
       Thus
3017 88
3018 89
3019 90
       x^{k}=y^{\ell-1}. \text{ } tag{5}
3020 91
3021
       Since \$k\not\equiv0\$ (mod \$\$|x|\$\$) the element \$\$x^{k} is a
3022

→ non-trivial power

3023
```

```
of $$x$$; (5) shows that this power is a power of $$y$$.
3025

→ Consequently

3026 95
3027 <sub>96</sub>
       $$
3028 <sub>97</sub>
       x\in\langle y\rangle .
3029 98
3030 99
3031 100
       A symmetric computation (or simply interchanging the roles of
3032
        \rightarrow $$x$$ and $$y$$)
3033 101
       gives
3034 102
3035 103
       y\in\langle x\rangle .
3036 104
3037 105
3038 106
3039 107 | Hence both generators belong to the same cyclic subgroup; the

→ whole group

3040
       $$G$$ is cyclic.
3041 108
3042 109
3043 110
3044
        ### 4. A cyclic group satisfying the relations is trivial
3045 111
3046 112
3047 113
       Write the cyclic group as $$\langle g\rangle$$ and set
3048
3049 115
       $$
3050 117
       x=g^{r}, \quad y=g^{s}.
       $$
3051
3052 119
       Insert these expressions in the defining relations:
3053 120
3054 121
       $$
3055 122
       \begin{aligned}
3056 123
       xy^{2} = y^{3} x
3057 124
           &\Longrightarrow q^{r+2s}=q^{3s+r},\\
3058 125
       yx^{2} = x^{3}y
3059 126
         &\Longrightarrow g^{s+2r}=g^{3r+s}.
3060 127
       \end{aligned}
3061 128
       $$
3062 129
       In a cyclic group equality of elements means equality of the
3063 130

→ exponents

       (modulo the order of $$q$$, or as integers if the order is
3065 131
        \hookrightarrow infinite). From the
        first equality we obtain
3067 132
3068 133
3069 <sup>134</sup>
3070 <sup>135</sup>
       2s\equiv 3s\pmod{|g|}\;\;\Longrightarrow\;\;s\equiv0,
3071 136
3072 137
3073 138
       and from the second equality
3074 139
3075 141
       2r\equiv 3r\pmod{|g|}\;\;\Longrightarrow\;\;r\equiv0.
3076 142
3077
```

```
3078
144
3079
145
       Thus $r=s=0; consequently x=y=e.
3080 146
3081
3082 147
        ### 5. Conclusion
3083 148
3084 149
        The group defined by
3085 150
3086 151
        $$
        \langle x ,y\mid xy^{2}=y^{3}x ,\; yx^{2}=x^{3}y\rangle
3087 152
3088 153
3089 154
        is cyclic, and the only cyclic group that satisfies the two
3090 155
        \hookrightarrow relations is the
       trivial group. Hence
3092 156
3093 157
3094 158
3095 159
        \boxed{G\text{ is the trivial group}.}
3096 <sup>160</sup>
3097
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```

## B.18 GPT-5 THINKING

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3133 3134

3135

3136

Lines 12, 20, 35: The definition of [s,t] used is  $sts^{-1}t^{-1}$ . The commutator identity given then holds only if  $s^t = tst^{-1}$  but the commutator definition is later taken to be  $s^t = t^{-1}st$ . (Critical)

```
3137
3138
       Yes. Write the relations in commutator form and use a short
3139
        → lemma.
3140
3141
       From
3142
3143
       $$
3144
       xy^2=y^3x\qquad yx^2=x^3y
3145
3146
3147
       we get
3148
3149 11
3150 12
       x y^2 x^{-1}=y^3 \ Longrightarrow \ [x,y^2]=x y^2 x^{-1}y^{-2}=y
3151 13
3152 14
3153 15
3154 16
       y x^2 y^{-1}=x^3 \setminus Longrightarrow = [y,x^2]=y x^2 y^{-1}x^{-2}=x.
3155 17
3156 18
       Let u=[x,y]. Using the standard commutator identity
       [a,bc]=[a,b]\,[a,c]^b, we have
3158 20
3159 21
3160 <sup>22</sup>
       [x,y^2] = [x,y] \setminus, [x,y]^y = u \setminus, u^y, qquad
       [y, x^2] = [y, x] \setminus [y, x]^x = u^{-1} (u^{-1})^x.
3162 24
3163 <sup>25</sup>
       $$
3164 <sup>26</sup>
3165 27
       Hence
3166 <sup>28</sup>
3167 <sup>29</sup>
       y=u\, u^y, qquad x=u^{-1}(u^{-1})^x. \tag{{}^x}
3168
3169
    32
3170
    33
       **Lemma.** If a=w\, w^a in a group, then a=w^2.
3171
3172
       *Proof.* a=w\,w^a\Rightarrow 1=w\,w^a\ a^{-1}=w\,a^{-1}=w\
3173
        \rightarrow a^{-1} w$, so a^{-1}=w^{-2}$ and thus a=w^{2}$. $\square$
3174 36
3175 37
       Apply the lemma to (a, w) = (y, u) and (a, w) = (x, u^{-1}) in
3176
        3177 38
3178 39
3179 40
       y=u^2, \qquad x=(u^{-1})^2=u^{-2}.
3180 41
       $$
3181 42
       Thus x=y^{-1}.
3182 43
3183 44
       Substitute x=y^{-1}\ into xy^2=y^3x:
3184 45
3185 46
       $$
    47
```

```
3186 48
      3187
       \rightarrow y=1,
3188 49
      $$
3189 50
3190 51
      whence x=1 as well. Therefore G is the trivial group.
3191
      3192
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