Compound Poisson Limits in Weighted Bernoulli Congestion Games: Theory Meets Experiments

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Abstract

Congestion games provide a key framework for modeling traffic in transportation networks. Cominetti et al. (2023) studied Bernoulli congestion games with homogeneous players and showed that Nash equilibria converge to Poisson random variables linked to the Wardrop equilibrium of the nonatomic game. In this work, we extend this model to weighted Bernoulli congestion games, allowing heterogeneous player weights. We have proven that arc loads and flows at equilibrium converge to compound Poisson random variables, thereby strengthening the bridge between atomic and nonatomic models. We provided numerical experiments that further demonstrate that even with a small number of players, the equilibrium closely approximates the limiting behavior, highlighting the model's ability to capture the natural stochasticity of traffic flows.

1 Introduction

Congestion games (CG) model the strategic use of network resources, such as transportation or data networks. We propose the weighted Bernoulli congestion game, where players are heterogeneous and participate with a certain probability. Under a regime of many players with low participation probabilities, we show that flows and loads in equilibrium converge in total variation to compound Poisson distributions. The parameters of these limiting distributions can be derived from the nonatomic Wardrop equilibrium with adjusted costs. This work extends Theorem 2 in Cominetti et al. [2023] to capture the random participation and the stochastic variability between the resources used by different players. Figures 1a and 1b show that Dublin's traffic density keeps its variability and does not concentrate around the average density. Moreover, the observed vehicle density (in blue) behaves similarly to a compound Poisson distribution (red), see Ross [1995]. Aside of the asymptotic theoretical result, we performed computational experiment to study if for a finite number of players the flow-load behaves according to our main result. Our analysis showed that, in the chose settings, even with that a moderate number of players the actual and limiting behavior are statistically indistinguishable. Furthermore, we analyze a variant of CG that does not satisfy our more

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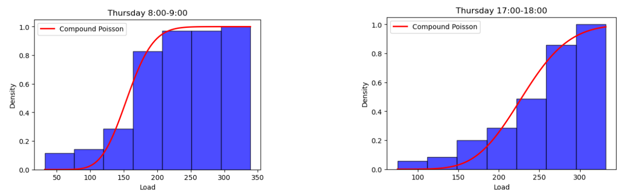
critical assumption and compare again the behavior for a fix number of players with the limiting flow and load.



(a) 8:00–9:00 a.m.   (b) 17:00–18:00 p.m.

Figure 1: Traffic count in Dublin (January–October 2024). Fuente: Dublin City Council (january–june) y (july–december).

This model enables us to go beyond the discrete flows from previous results; it allows us to characterize the random loads as continuous, discrete, or mixed random variables. The network's performance is linked to the probability of exceeding the resource's capacities. Therefore, to determine which resource capacities must increase, we need to describe the flows and load distributions accurately.

2 Methodology

We propose the model *Weighted Bernoulli Congestion Game* (WBCG) Mac Kenney [2025], which capture two previous extensions in the same setting: heterogeneity in player weights and random participation. More precisely, each player i has a random weight $W_i \sim \mathscr{W}$ and a Bernoulli participation variable $U_i \sim \text{Bernoulli}(u_i)$, which introduces uncertainty in both demand and congestion intensity. All these variables are mutually independent.

The analysis considers two approaches. In the atomic case, Monderer and Shapley [1996] states that the Nash equilibrium (NE) is obtained by the corresponding potential game and applying an iterative best response algorithm until reaching a point where no player can reduce their expected cost by unilaterally changing their strategy. In the non-atomic case, we construct a continuous potential function that defines a convex optimization problem; such that its solution determines the Wardrop equilibrium (WE), see Maillé and Tuffin [2014].

Finally, we investigate the asymptotic behavior as the number of players increases while their participation probabilities tend uniformly to 0. Under these conditions, we show that the flows and loads in NE converge in total variation to compound Poisson variables, with parameters given by the non-atomic WE with adjusted cost functions.

3 Theoretical results

3.1 Weighted Bernoulli Congestion Games Approach

The random variables $U_i \sim \text{Bernoulli}(u_i)$ with $i \in \mathscr{N}$ represent the random participation of player i and they are independent from each other. The distribution is commonly known to all players, although its realizations are unknown, so it is not certain who will actually participate in the game. Each player $i \in \mathscr{N}$ contributes a random weight W_i to the traffic, reflecting their impact on the network. Weights are distributed as $\mathscr{W}$ and they are iid among players. Let the average weight be $w := \mathbb{E}[W_i]$. The weight distribution is also common knowledge, but its realizations w_i are not.

We embedded the game in a probability space that incorporates three sources of uncertainty: each player's stochastic participation, their choice of strategy, and their random weight. Thus, the vector of participations, a profile of strategies, and a realization of weight describe the state of the system. The joint probability measure of these elements is the product of the probability measures for the random participations, the mixed strategy profile, and the randomly distributed weights.

Let $\mathscr{G} = (\mathscr{E}, (c_e)_{e\in\mathscr{E}}, \mathscr{T}, (\mathscr{S}_t)_{t\in\mathscr{T}})$ and considering these conditions, the game is defined as $\Gamma_{WB} = (\mathscr{G}, \mathscr{W}, (t_i, u_i)_{i\in\mathscr{N}})$. Thus, the random demand by type t is modeled as $D_t =$

$\sum_{i:t_i=t} U_i W_i$. If S_i is the random pure strategy of the aplyer i, the load on each resource and the flow in each strategy are defined as:

$$F_{t,s} = \sum_{i:t_i=t} U_i\, W_i\, \mathbb{1}_{\{S_i=s\}}, \quad \text{and} \quad L_e = \sum_{i\in\mathscr{N}} U_i W_i \mathbb{1}_{\{e\in S_i\}},$$

whose expectations are:

$$f_{t,s} := \mathbb{E}_{\boldsymbol{\sigma}}[F_{t,s}] = w \sum_{i:t_i=t} u_i\, \sigma_i(s), \quad \text{and} \quad l_e := \mathbb{E}_{\boldsymbol{\sigma}}[L_e] = w \sum_{i\in\mathscr{N}} u_i\, \sigma_{i,e}.$$

The pair $(\mathbf{f}, \mathbf{l})$ represents the vector of flows in the strategies and loads on the resources. A direct calculation shows that it must satisfies the following conditions:

$$\forall t \in \mathscr{T}, \quad d_t = \sum_{s\in\mathscr{S}_t} f_{t,s}, \quad \text{whit} \quad f_{t,s} \geq 0, \quad \text{and} \quad \forall e \in \mathscr{E}, \quad l_e = \sum_{t\in\mathscr{T}} \sum_{s\in\mathscr{S}_t} f_{t,s} \mathbb{1}_{\{e\in s\}}.$$

Therefore, any pair $(\mathbf{f}, \mathbf{l})$ that satisfies these conditions is considered feasible.

3.2 WBCG limit

We consider a sequence of WBCGs indexed by n to reflect the dependence on the number of players. This is defined as $\Gamma^n_{WB} = (\mathscr{G}, \mathscr{W}, (u^n_i, t^n_i)_{i\in\mathscr{N}^n})$. Without loss of generality, we will assume that:

$$|\mathscr{N}^n| \to \infty; \quad u^n := \max_{i\in\mathscr{N}^n} u^n_i \to 0; \quad \mathbb{E}_{\mathscr{W}}[W_i] = 1, \forall i \in \mathscr{N}; \quad d^n_t := \sum_{i:t^n_i=t} u^n_i \to d_t, \forall t \in \mathscr{T}. \tag{1}$$

3.3 Auxiliary Game

Let the nonatomic auxiliary game representing the limit of the WBCG, be denoted by $\check{\Gamma}^\infty = (\check{\mathscr{G}}, (d_t)_{t\in\mathscr{T}})$. This game is defined with demands d_t for each type $t \in \mathscr{T}$, and cost functions $\check{c}_e : \mathbb{R}_* \to \mathbb{R}_* \cup \{+\infty\}$ with $\mathbb{R}_* = \mathbb{R}_+ \cup \{0\}$ that are associated with each resource $e \in \mathscr{E}$. To obtain the auxiliary cost functions, we must first consider the function $c^*_e(k) := \mathbb{E}_{\mathscr{W}}[c_e(\sum_{j=1}^k \bar{W}_j)]$, where $\bar{W}_j$ are independent identical copies of $\mathscr{W}$. This function represents the expected cost of resource e given that k players are using it.

Given $m \in \mathbb{R}^+$, then let M be a Poisson random variable with parameter m. The auxiliary cost $\check{c}_e(m)$ is the expected cost faced by a player who plans to use the resource e. This player must consider both their own random weight and the load induced by the other participants, whose number is also random. We then define the auxiliary cost as $\check{c}_e(m) := \mathbb{E}_{\mathscr{W},M}[c^*_e(1 + M)]$. Here, we add 1 because the player must consider her own presence, while the Poisson variable M represents the random number of other players using the resource. The function c^*_e uses the total number of players in the resource as an argument and is structured in such a way that it captures the random load on the resource. This formulation reflects the fact that, in the limit regime, the total load is random and composed of multiple independent contributions with varying weights. To ensure that expected costs are well defined and uniformly bounded, the following mild condition is imposed:

$$\exists v \in \mathbb{R}, \text{ y } d_{\max} > d_{\text{tot}} \text{ with } \mathbb{E}[|\Delta^2 c^*_e(1 + V)|] \leq v, \ \forall e \in \mathscr{E} \text{ y } V \sim \text{Poisson}(d_{\max}), \tag{2}$$

These conditions reflect a large-scale regime, where individual participation is unlikely, but aggregation by type generates a non-trivial total load. This assumption is guarantied if the original costs functions are $\mathcal{C}^2$, but weaker condition may be also considered. With these preliminaries, we are ready to state the convergence of equilibrium for the WBCG, (the proof is given in B).

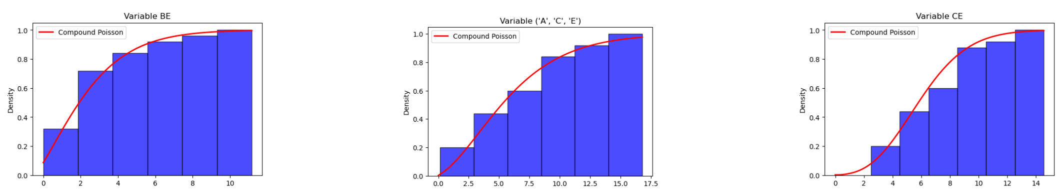
Theorem 3.1 *Let Γ_{WB}^n be a sequence of WBCGs satisfying* (1) *and* (2)*, and let $\hat{\boldsymbol{\sigma}}^n \in NE(\Gamma_{WB}^n)$ be a sequence of arbitrary Bayesian Nash equilibrium. Then, the corresponding sequence of expected flow-load pairs ($\hat{\mathbf{f}}^n$, $\hat{\mathbf{l}}^n$) is bounded, and any accumulation point ($\hat{\mathbf{f}}$, $\hat{\mathbf{l}}$) is a Wardrop equilibrium for the nonatomic congestion game $\check{\Gamma}^\infty$. Furthermore, throughout each of these convergent subsequences, the random flows $F_{t,s}^n$ and loads L_e^n converge in total variation to limiting random variables $F_{t,s} \sim$ compound Poisson($\hat{f}_{t,s}$, $\mathscr{W}$) and $L_e \sim$ compound Poisson($\hat{l}_e$, $\mathscr{W}$), where the variables $F_{t,s}$ are independent of each other.*

4 Experimental results

We recall that in Theorem 3.1, players choose their strategy without knowing their own weight W_i. This assumption is very restrictive, but without it the WBCG will no longer be a congestion game. As our techniques rely on this assumption, to remove it will required a different approach. Nevertheless, we conjecture that if the coefficient of variation is small enough, the limiting behavior of our main result should remain a valid approximation for the flows and loads. Therefore we design the experiments to study if the conjecture holds in the experiments setting.

We use the network and cost functions given in Appendix A with 100 players to validate Theorem 3.1. Also, we use two type of players, AD and AE, with $u_i = 0.16$ and $\mathbb{E}[W_i] = 1$ for all i. We have three configurations: for the first two, players choose their strategy unknowing their weights W_i (a) here weight are exponentially distributed, (b) here weight are gamma distributed, and for (c) players know only their own weight and they are gamma distributed. In all cases, the NE was calculated from the potential function and compared with the corresponding WE.

Figures 2a, 2b, and 2c show representative examples where the simulation histograms (blue) closely fit the predicted compound Poisson distributions (red). In all three experiments, the simulations show that the load and flow distributions in the NE conform to the compound Poisson distributions predicted by Theorem 3.1. Furthermore, Kolmogorov–Smirnov tests at the 95% confidence level did not reject the null hypothesis that the samples come from the theoretical distribution. This empirically confirms the validity of our model.



(a) Unknown exponential weights.

(b) Unknown gamma weights.

(c) Known gamma weights.

Figure 2: Simulated distributions (blue) vs. theoretical compound Poisson (red).

5 Conclusions and future work

This work introduces the *weighted Bernoulli congestion game*, extending previous models by incorporating simultaneously random weights and participation on players. Under a regime of many players with low participation probabilities, Nash equilibrium flows and loads converge in total variation to *compound Poisson* variables, whose parameters are derived from the nonatomic WE with adjusted cost functions.

We validate our theoretical result and conjecture empirically by performing three experiments: (a) unknown exponential weights, (b) unknown gamma weights, and (c) known gamma weights. In all cases, the simulated distributions closely matched the theory, and Kolmogorov–Smirnov tests at the 95% confidence level did not reject the null hypothesis that the samples come from a compound Poisson with parameters obtained from the WE.

As future work, we propose extending the our experimental analysis, in some settings it is natural that the player strategies depended on its own weight. Our experiment show that if the random weight has a small coefficient of variation, then 3.1 continues to approximate correctly. See Tables 1a and 1b for resource AC and BD respectively. Note that as the variance of the weights W_i increases, so does the difference between the two cases for the same 95th percentile. The same behavior should appear in other settings. Moreover, under suitable hypotheses, we can define a threshold for the coefficient of variation such that, below it, we can prove the conjecture.

(a) Table 1a: 95th percentiles for resource AC.

$\mathrm{Var}[W_i]$	0.1	1	10
Unknown weight	14.12	12.91	28.48
Known weight	15.22	18.74	37.23

(b) Table 1b: 95th percentiles for resource BD.

$\mathrm{Var}[W_i]$	0.1	1	10
Unknown weight	3.61	5.53	12.32
Known weight	3.2	2.17	0.61

These results are very useful at the moment of define networks capacities based on saturation probabilities, compare two or more networks in terms of efficiency and build future infrastructure designs.

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A Graph of the experiments

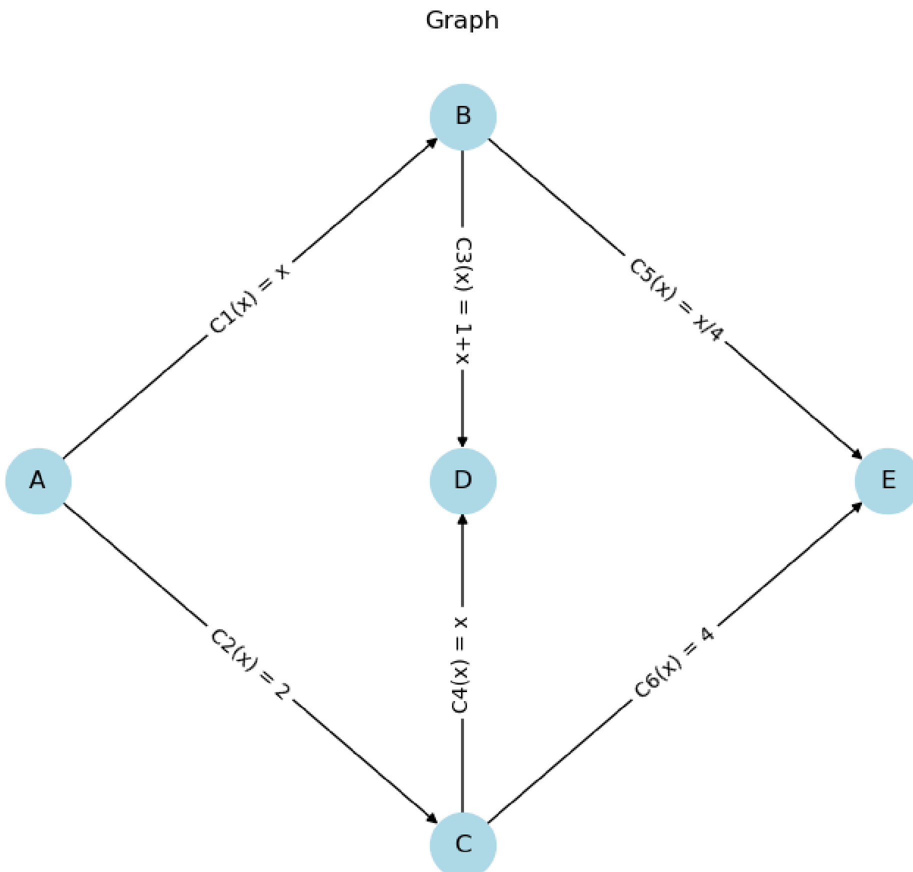

Figure 3: Graph of the experiments

The cost functions are:

- $c_{AB}(x) = x$.
- $c_{AC}(x) = 2$.
- $c_{BD}(x) = 1 + x$.
- $c_{BE}(x) = \frac{x}{4}$.
- $c_{CD}(x) = x$.
- $c_{CE}(x) = 4$.

B Related to the theorem

For any function f of class $\mathcal{C}^2$ and $h_1, h_2 > 0$ we have that,

$$\frac{1}{h_1}(f(x+h_1) - f(x)) = f(x) + \int_0^{h_1} (h_1 - t)\, f''(x+t)\, dt,$$
$$\frac{1}{h_2}(f(x) - f(x-h_2)) = f(x) - \int_0^{h_2} (h_2 - t)\, f''(x-t)\, dt.$$

Proof:

Let h_1, h_2 and h be real positive numbers. By the fundamental theorem of calculus,

$$f(x+h_1) - f(x) = \int_0^{h_1} f'(x+s)\, ds \quad \text{and} \quad f(x) - f(x-h_2) = \int_0^{h_2} f'(x-s)\, ds.$$

Using $f'(x+s) = f'(x) + \int_0^s f''(x+t)\, dt$ (valid almost always since f' is absolutely continuous) and Fubini/Tonelli,

$$\begin{aligned}
\frac{1}{h}\int_0^h f'(x+s)\, ds &= \frac{1}{h}\int_0^h \left(f'(x) + \int_0^s f''(x+t)\, dt\right) ds, \\
&= \frac{1}{h}\int_0^h f'(x)\, ds \; + \frac{1}{h}\int_0^h \int_0^s f''(x+t)\, dt\, ds, \\
&= f'(x) + \frac{1}{h}\int_0^h \int_t^h f''(x+t)\, ds\, dt, \\
&= f'(x) + \frac{1}{h}\int_0^h \Big(h\, f''(x+t) - t\, f''(x+t)\Big)\, dt, \\
&= f'(x) + \frac{1}{h}\int_0^h (h-t)\, f''(x+t)\, dt.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{1}{h}\int_0^h f'(x-s)\, ds &= \frac{1}{h}\int_0^h \left(f'(x) - \int_0^s f''(x-t)\, dt\right) ds, \\
&= \frac{1}{h}\int_0^h f'(x)\, ds \; - \frac{1}{h}\int_0^h \int_0^s f''(x-t)\, dt\, ds, \\
&= f'(x) - \frac{1}{h}\int_0^h \int_t^h f''(x-t)\, ds\, dt, \\
&= f'(x) - \frac{1}{h}\int_0^h \Big(h\, f''(x-t) - t\, f''(x-t)\Big)\, dt, \\
&= f'(x) - \frac{1}{h}\int_0^h (h-t)\, f''(x-t)\, dt.
\end{aligned}$$

Now, let be

$$I_f(x, h_1, h_2) = \frac{1}{h_1}(f(x+h_1) - f(x)) - \frac{1}{h_2}(f(x) - f(x-h_2)).$$

It follows that if $||f||_\infty = \sup\{f(y) : y \in \mathbb{R}\} = v$, then

$$|I_f(x, h_1, h_2)| \leq \frac{1}{2}(h_1 + h_2)\, v.$$

Corollary 1 (Supreme limit).

For any $h_1, h_2 > 0$, we have

$$
\begin{aligned}
I_f(x,h_1,h_2) &= \frac{1}{h_1}[f(x+h_1)-f(x)] - \frac{1}{h_2}[f(x)-f(x-h_2)], \\
&= \left[\int_0^{h_1} f'(x+s)\,ds\right] - \left[\int_0^{h_2} f'(x-s)\,ds\right], \\
&= \left[f'(x) + \frac{1}{h_1}\int_0^{h_1}(h_1-t)\,f''(x+t)\,dt\right] - \\
&\quad \left[f'(x) - \frac{1}{h_2}\int_0^{h_2}(h_2-t)\,f''(x-t)\,dt\right], \\
&= \frac{1}{h_1}\int_0^{h_1}(h_1-t)\,f''(x+t)\,dt + \\
&\quad \frac{1}{h_2}\int_0^{h_2}(h_2-t)\,f''(x-t)\,dt \\
&\le (\frac{h_1^2}{2h_1}v + \frac{h_2^2}{2h_2})\,v \\
&= \frac{h_1+h_2}{2}\,v
\end{aligned}
$$

which proves the statement.

Corollary 2.

Let V be a Poisson random variable with parameter $d_{\max}$ and let $(W_k)_{k\in\mathbb{N}}$ be random variables iid. We define:

$$S_x := \sum_{k=1}^{x} \bar{W}_k.$$

Let us consider f a function $\mathbb{R} \to \mathbb{R}, \mathcal{C}^2$ and $||f||_\infty = v$. We define $c^*(k) = \mathbb{E}(f(S_k))$. We have

$$\mathbb{E}_V\left[\Delta^2 c(V+1)\right] \le v.$$

Let's see

$$\Delta^2 c_e(V+1) = I_f(S_{V+2}, \bar{W}_{V+2}, \bar{W}_{V+3}) \le \frac{\bar{W}_{V+2}+\bar{W}_{V+1}}{2}\,||f||_\infty.$$

Taking expectation, we have that:

$$\mathbb{E}(\Delta^2 c_e(V+1)) \le \mathbb{E}(\bar{W}_1)||f||_\infty.$$