Banach distribution spaces for a Hilbert space

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***Abstract*—Associated with every separable Hilbert space $\mathcal{H}$ and given localized frame, there exists a natural test function Banach space $\mathcal{H}^1$ and a Banach distribution space $\mathcal{H}^\infty$ so that $\mathcal{H}^1 \subset \mathcal{H} \subset \mathcal{H}^\infty$. In this article we close some gaps in the literature and rigorously introduce the space $\mathcal{H}^\infty$ and its weighted variants $\mathcal{H}_w^\infty$ under minimal assumptions and discuss some of their properties. In particular, we compare several topologies associated with $\mathcal{H}_w^\infty$ and show that $(\mathcal{H}_w^\infty, \|\cdot\|_{\mathcal{H}_w^\infty})$ is a Banach space.**

***Index Terms*—Localized frame, co-orbit space, distribution space**

I. INTRODUCTION

The theory of distributions (or generalized functions) has become indispensable in modern mathematics, physics and engineering, and provides a suitable abstract framework for the analysis of various problems and the formalization of many phenomena. A classical example of a distribution space is the space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions, i.e. the (anti-linear) topological dual of the space $\mathcal{S}(\mathbb{R}^d)$ of Schwartz functions on $\mathbb{R}^d$. Since $\mathcal{S}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$ and since both $\mathcal{S}(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$ are invariant under the Fourier transform, the space of tempered distributions is well suited for extending Fourier theory from $L^2(\mathbb{R}^d)$ to the distributional setting, and provides a valuable framework for analyzing various kind of PDEs [9]. On the other hand, $\mathcal{S}(\mathbb{R}^d)$ is not a Banach space and $\mathcal{S}'(\mathbb{R}^d)$ not even a Fréchet space [2], which sometimes makes working with these objects a bit tedious. If one is less oriented towards derivatives, but more interested in time-frequency analysis and applications to quantum physics, the *Feichtinger algebra* $\mathbf{S}_0(\mathbb{R}^d)$ [5] and its dual space $\mathbf{S}_0'(\mathbb{R}^d)$, sometimes called the space of *mild distributions* [6], serve as an indisputably useful alternative to the latter, see also [4]. The space $\mathbf{S}_0(\mathbb{R}^d)$ is defined as the space of all elements in $L^2(\mathbb{R}^d)$, whose *short-time Fourier transform* [8] with respect to the Gaussian (or any other Schwartz function) is in $L^1(\mathbb{R}^{2d})$. In fact, $\mathbf{S}_0(\mathbb{R}^d)$ is not only a Banach space, but even a Banach algebra under both convolution and pointwise multiplication, and contains $\mathcal{S}(\mathbb{R}^d)$ as a norm-dense subspace. Thus $\mathbf{S}_0'(\mathbb{R}^d)$ is a Banach space of distributions contained in $\mathcal{S}'(\mathbb{R}^d)$. In particular, one can show that $\mathbf{S}_0(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \subset \mathbf{S}_0'(\mathbb{R}^d)$ (such a triple is called a *Banach-Gelfand triple*), and since $\mathbf{S}_0(\mathbb{R}^d)$ is invariant under both the Fourier transform and actions of time-frequency shifts, $(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$ is widely considered as *the* appropriate test function-/distribution space pair for time-frequency analysis and applications to quantum physics [4].

In this article we consider a generalization of the Banach distribution space $\mathbf{S}_0'(\mathbb{R}^d) \supset L^2(\mathbb{R}^d)$ to the abstract Hilbert space setting. In analogy to a characterization of $\mathbf{S}_0(\mathbb{R}^d)$ via Gabor frames [8], Fornasier and Gröchenig defined general co-orbit spaces $\mathcal{H}_w^p$ $(1 \leq p \leq \infty)$ associated with a localized frame in a given Hilbert space $\mathcal{H}$ [7]. In fact, for reasonable weights w, $\mathcal{H}_{1/w}^1 \subset \mathcal{H} \subset \mathcal{H}_w^\infty$ is a Banach-Gelfand triple, which (up to norm equivalence) coincides with the triple $\mathbf{S}_0(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \subset \mathbf{S}_0'(\mathbb{R}^d)$ in the case of certain Gabor frames [7]. While the spaces $\mathcal{H}_w^p$ $(1 \leq p < \infty)$ and their properties have been studied in detail in e.g. [7], a rigorous discussion of the space $\mathcal{H}_w^\infty$ in [7] was omitted by the authors *"to avoid tedious technicalities"*. In the book chapter [1], the authors gave a more detailed presentation of the space $\mathcal{H}_w^\infty$, −unfortunately containing some small errors. The purpose of this article is to close these gaps in the literature and give a detailed presentation of the space $\mathcal{H}_w^\infty$ under minimal assumptions.

II. PRELIMINARIES

Let $\mathcal{H}$ be a separable $\mathbb{C}$-Hilbert space with inner product $\langle \cdot, \cdot \rangle$ being conjugate linear in the second slot and let $\|\cdot\|$ be the induced norm. The set X is assumed to be countable and is used as the index set for the ℓ^p spaces. A weight w is defined to be a map from X to $\mathbb{R}_{>0}$. For a sequence $(\alpha_k)_{k\in X}$, the weighted ℓ^p-norm is given by

$$\|(\alpha_k)_{k\in X}\|_{\ell_w^p} := \|(\alpha_k w(k))_{k\in X}\|_{\ell^p} .$$

The space ℓ_w^p consists of all sequences with a finite ℓ_w^p-norm and is a Banach space. For the pointwise topology, we use the notation pw.

A. A certain locally convex topology

Let V be a $\mathbb{C}$-vector space with a countable Hamel basis. Let $\mathcal{T}$ be another $\mathbb{C}$-vector space and assume that $(\mathcal{T}, V)$ is a dual pair with associated nondegenerate sesquilinear form $\langle \cdot, \cdot \rangle_{\mathcal{T},V}$ (being conjugate-linear in the second slot). We equip $\mathcal{T}$ with the Hausdorff locally convex $\sigma(\mathcal{T}, V)$-topology generated by the seminorms $\{|\langle \cdot, v\rangle_{\mathcal{T},V}| : v \in V\}$. Equivalently, we can replace V by a Hamel basis, to get countably many seminorms generating the same topology, so that $\mathcal{T}$ is metrizable. Let $\overline{\mathcal{T}}$ denote the topological completion [10] of $\mathcal{T}$, which is again a metrizable Hausdorff locally convex space. For the sake of sanity, we make sure that the sesquilinear form $\langle\cdot,\cdot\rangle_{\mathcal{T},V}$ can be uniquely extended. For each $v\in V$, the linear functional

$$l_v:\mathcal{T}\to\mathbb{C},\quad l_v(x):=\langle x,v\rangle_{\mathcal{T},V}.$$

is continuous, hence there exists a unique continuous linear extension $\bar{l}_v:\overline{\mathcal{T}}\to\mathbb{C}$ [11, Theorem 5.1]. In particular, we may define the extended sesquilinear form $\langle\cdot,\cdot\rangle_{\overline{\mathcal{T}},V}$ on $\overline{\mathcal{T}}\times V$ by

$$\langle x,v\rangle_{\overline{\mathcal{T}},V}:=\bar{l}_v(x).$$

We easily check for conjugate linearity on the second slot. Let $(x_n)_{n=1}^\infty\subset\mathcal{T}$ be a sequence converging to some $x\in\overline{\mathcal{T}}$. Then

$$\begin{aligned}\langle x,\lambda u+v\rangle_{\overline{\mathcal{T}},V}&=\lim_{n\to\infty}\langle x_n,\lambda u+v\rangle_{\mathcal{T},V}\\&=\overline{\lambda}\lim_{n\to\infty}\langle x_n,u\rangle_{\mathcal{T},V}+\lim_{n\to\infty}\langle x_n,v\rangle_{\mathcal{T},V}\\&=\overline{\lambda}\langle x,u\rangle_{\overline{\mathcal{T}},V}+\langle x,v\rangle_{\overline{\mathcal{T}},V}.\end{aligned}$$

Finally, we make sure that our extended sesquilinear form is still nondegenerate. Suppose $\langle x,v\rangle_{\overline{\mathcal{T}},V}=0$ for all $v\in V$. Let $(x_n)_{n=1}^\infty\subseteq\mathcal{T}$ converge to $x\in\overline{\mathcal{T}}$. Then

$$0=\langle x,v\rangle_{\overline{\mathcal{T}},V}=\lim_{n\to\infty}\langle x_n,v\rangle_{\mathcal{T},V}\quad(\forall v\in V),$$

so $(x_n)_{n=1}^\infty$ converges to 0 with respect to $\sigma(\mathcal{T},V)$, implying $x=0$, since limits in Hausdorff spaces are unique.

B. Frames

A countable family $\psi=(\psi_k)_{k\in X}$ of vectors in $\mathcal{H}$ is called a *frame*, if

$$\sum_{k\in X}|\langle f,\psi_k\rangle|^2\asymp\|f\|^2\qquad(\forall f\in\mathcal{H}).$$

We refer to [3] for a comprehensive introduction to frame theory. Whenever ψ is frame, both the *coefficient operator* (or *analysis operator*), defined as

$$C_\psi:\mathcal{H}\to\ell^2,\quad C_\psi f=\big(\langle f,\psi_k\rangle\big)_{k\in X}$$

and the *synthesis operator*, defined as

$$D_\psi:\ell^2\to\mathcal{H},\quad D_\psi c=\sum_{k\in X}c_k\psi_k,$$

are bounded and adjoint to one another, where the latter series converges unconditionally in $\mathcal{H}$. Additionally, D_ψ is surjective. Their composition yields the *frame operator*

$$S_\psi=D_\psi C_\psi:\mathcal{H}\to\mathcal{H},\quad S_\psi f=\sum_{k\in X}\langle f,\psi_k\rangle\psi_k,$$

which is bounded, positive, self-adjoint and invertible. Composing the frame operator with its inverse (and vice versa) yields the *frame reconstruction* formulae

$$f=\sum_{k\in X}\langle f,\tilde{\psi}_k\rangle\psi_k=\sum_{k\in X}\langle f,\psi_k\rangle\tilde{\psi}_k\qquad(\forall f\in\mathcal{H}).\quad\text{(II.1)}$$

The family $\tilde{\psi}=\big(\tilde{\psi}_k\big)_{k\in X}:=\big(S_\psi^{-1}\psi_k\big)_{k\in X}$ is a frame as well and called the *canonical dual frame*. More generally, if $\psi^d=(\psi^d)_{k\in X}$ is another frame in $\mathcal{H}$ such that (II.1) holds after replacing each $S_\psi^{-1}\psi_k$ with ψ_k^d, then ψ^d is called a *dual frame* of ψ. Finally, the *cross Gram matrix* $G_{\tilde{\psi},\psi}$ associated with two frames ψ and $\tilde{\psi}$ is given by

$$G_{\tilde{\psi},\psi}=\big[\langle\psi_l,\tilde{\psi}_k\rangle\big]_{k,l\in X}.$$

In fact, $G_{\tilde{\psi},\psi}$ defines a bounded operator on ℓ^2 and

$$G_{\tilde{\psi},\psi}=C_{\tilde{\psi}}D_\psi.$$

III. Results

In order to introduce the co-orbit spaces $\mathcal{H}_w^\infty$ we fix the following assumptions for the remainder of this article:

(1) ψ is a frame for $\mathcal{H}$ and $\tilde{\psi}$ a dual frame.
(2) The cross Gram matrix $G_{\tilde{\psi},\psi}$ defines a bounded operator on ℓ_w^∞ for some fixed weight w.

We explicitly emphasize that conditions $(1-2)$ are met for any weight w whenever ψ is a Riesz basis and $\tilde{\psi}$ its canonical dual Riesz basis (and in particular, when $\psi=\tilde{\psi}$ is an orthonormal basis). Further examples of pairs of dual frames $(\tilde{\psi},\psi)$ and weights w satisfying $(1-2)$ are given by an intrinsically localized frame and its canonical dual frame, where w is a so-called *admissible weight*, see [7] and the references therein.

Now let $\mathcal{H}^{00}:=\operatorname{span}(\tilde{\psi}_k)_{k\in X}=D_{\tilde{\psi}}(\ell^{00})$. Since ℓ^{00} a dense subspace of ℓ^2 and $D_{\tilde{\psi}}:\ell^2\to\mathcal{H}$ bounded and onto, we have that $\mathcal{H}^{00}$ is a dense subspace of $\mathcal{H}$. Consequently, $(\mathcal{H},\mathcal{H}^{00})$ is a dual pair with associated nondegenerate sesquilinear form $\langle\cdot,\cdot\rangle$ restricted to $\mathcal{H}\times\mathcal{H}^{00}$. Next, let $\overline{\mathcal{H}}$ be the completion of $\mathcal{H}$ with respect to the $\sigma(\mathcal{H},\mathcal{H}^{00})$-topology, with induced sesquilinear form $\langle\cdot,\cdot\rangle_{\overline{\mathcal{H}},\mathcal{H}^{00}}$. Then, we define $\mathcal{H}_w^\infty$ as the subspace of all $f\in\overline{\mathcal{H}}$, for which there exists a sequence $(f_n)_{n=1}^\infty\subseteq\mathcal{H}$ satisfying

$$\overset{\sigma(\overline{\mathcal{H}},\mathcal{H}^{00})}{\lim_{n\to\infty}}f_n=f\ \text{ and }\ \big|\langle f_n,\tilde{\psi}_k\rangle\big|w(k)\lesssim 1\ (\forall k\in X,n\in\mathbb{N}).$$

Let $\langle\cdot,\cdot\rangle_{\mathcal{H}_w^\infty,\mathcal{H}^{00}}$ be the restriction of $\langle\cdot,\cdot\rangle_{\overline{\mathcal{H}},\mathcal{H}^{00}}$ to $\mathcal{H}_w^\infty\times\mathcal{H}^{00}$.

A. The coefficient operator

Having defined our space, we can easily define the coefficient operator with respect to $\tilde{\psi}$.

Definition III.1. *We define the coefficient operator as*

$$C_{\tilde{\psi}}:\mathcal{H}_w^\infty\to\ell_w^\infty,\quad f\mapsto\big(\langle f,\tilde{\psi}_k\rangle_{\mathcal{H}_w^\infty,\mathcal{H}^{00}}\big)_{k\in X}.$$

By the definition of $\mathcal{H}_w^\infty$ this operator is well-defined and easily seen to be linear.

Proposition III.2. *The coefficient operator is injective.*

Proof. Suppose $C_{\tilde{\psi}}f=0$ for some $f\in\mathcal{H}_w^\infty$. Then $\langle f,\tilde{\psi}_k\rangle_{\mathcal{H}_w^\infty,\mathcal{H}^{00}}=0$ for all $k\in X$. Since the frame elements $\tilde{\psi}_k$ span $\mathcal{H}^{00}$, we must have $f=0$, since $\langle\cdot,\cdot\rangle_{\mathcal{H}_w^\infty,\mathcal{H}^{00}}$ is nondegenerate with respect to the first slot. □

Since $\|\cdot\|_{\ell_w^\infty}$ is a norm and the coefficient operator, as defined above, is injective, we immediately obtain the following.

Corollary III.3. *The map*

$$\|\cdot\|_{\mathcal{H}_w^\infty}:\mathcal{H}_w^\infty\to\mathbb{R},\quad \|f\|_{\mathcal{H}_w^\infty}:=\left\|C_{\tilde\psi}f\right\|_{\ell_w^\infty}$$

defines a norm on $\mathcal{H}_w^\infty$. In particular, $C_{\tilde\psi}$ is an isometry.

Proposition III.4. *For every $f\in\mathcal{H}_w^\infty$ there exists a $\|\cdot\|_{\mathcal{H}_w^\infty}$-bounded sequence $(f_n)_{n=1}^\infty\subseteq\mathcal{H}_w^\infty\cap\mathcal{H}$ converging to f with respect to $\sigma(\mathcal{H}_w^\infty,\mathcal{H}^{00})$.*

Proof. By definition, there exists a sequence $(f_n)_{n=1}^\infty\subset\mathcal{H}$ converging to f with respect to $\sigma(\overline{\mathcal{H}},\mathcal{H}^{00})$ satisfying

$$\left|\langle f_n,\tilde\psi_k\rangle\right|w(k)\lesssim 1\ (\forall k\in X,n\in\mathbb{N}).$$

For each $n\in\mathbb{N}$, we choose the constant sequence $(f_n)_{m=1}^\infty$ to verify that $f_n\in\mathcal{H}_w^\infty$. Indeed,

$$\|f_n\|_{\mathcal{H}_w^\infty}=\sup_{k\in X}\left|\langle f_n,\tilde\psi_k\rangle\right|w(k)\lesssim 1$$

for each $n\in\mathbb{N}$, as was to be shown. □

Next we show that the norm topology is stronger than the $\sigma(\mathcal{H}_w^\infty,\mathcal{H}^{00})$ topology.

Theorem III.5. *If a sequence $(f_n)_{n=1}^\infty\subset\mathcal{H}_w^\infty$ converges in norm, then it converges with respect to $\sigma(\mathcal{H}_w^\infty,\mathcal{H}^{00})$.*

Proof. Without loss of generality, assume that $(f_n)_{n=1}^\infty$ converges to 0. This means that

$$\lim_{n\to\infty}\left\|C_{\tilde\psi}f_n\right\|_{\ell_w^\infty}=0.$$

This implies for each $k\in X$ that

$$\lim_{n\to\infty}\left|\langle f_n,\tilde\psi_k\rangle\right|=\lim_{n\to\infty}|(C_{\tilde\psi}f_n)_k|=0,$$

as was to be shown. □

Remark III.6. *The converse of the above statement is not true. Indeed, let $X=\mathbb{N}$, $w=1$ and $(e_n)_{n\in\mathbb{N}}=(\psi_n)_{n\in\mathbb{N}}=(\tilde\psi_n)_{n\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$. Then the sequence $(f_n)_{n=1}^\infty=(\psi_n)_{n=1}^\infty$ converges to 0 with respect to $\sigma(\mathcal{H}_w^\infty,\mathcal{H}^{00})$ since*

$$\lim_{n\to\infty}\langle e_n,e_k\rangle_{\mathcal{H}_w^\infty,\mathcal{H}^{00}}=\lim_{n\to\infty}\langle e_n,e_k\rangle=0.$$

However, it does not converge in norm since

$$\lim_{n\to\infty}\|e_n-0\|_{\mathcal{H}_w^\infty}=\lim_{n\to\infty}\sup_{k\in\mathbb{N}}|\langle e_n,e_k\rangle_{\mathcal{H}_w^\infty,\mathcal{H}^{00}}|=1.$$

B. Completeness

The main goal of this article is to show that $(\mathcal{H}_w^\infty,\|\cdot\|_{\mathcal{H}_w^\infty})$ is a Banach space. Before we are able to do so, we need a bit of work.

Lemma III.7. *For each $l\in X$, $\psi_l\in\mathcal{H}_w^\infty$.*

Proof. It suffices to show that $\left(\langle\psi_l,\tilde\psi_k\rangle\right)_{k\in X}\in\ell_w^\infty$ for each $l\in X$. This is indeed the case, since

$$\begin{aligned}\sup_{k\in X}|\langle\psi_l,\tilde\psi_k\rangle|w(k)&=w(l)\sup_{k\in X}|\langle\psi_l,\tilde\psi_k\rangle|w(k)w(l)^{-1}\\&\le w(l)\sup_{k\in X}\sum_{m\in X}|\langle\psi_m,\tilde\psi_k\rangle|w(k)w(m)^{-1}\\&=w(l)\left\|G_{\tilde\psi,\psi}(w(m)^{-1})_{m\in X}\right\|_{\ell_w^\infty}\\&=w(l)\left\|G_{\tilde\psi,\psi}\right\|_{\mathcal{B}(\ell_w^\infty)}<\infty,\end{aligned}$$

where we used that $\|(w(m)^{-1})_{m\in X}\|_{\ell_w^\infty}=1$. □

Note that the latter shows that $\langle\psi_l,\tilde\psi_k\rangle$ and $\langle\psi_l,\tilde\psi_k\rangle_{\mathcal{H}_w^\infty,\mathcal{H}^{00}}$ are the same for all $k,l\in X$.

Lemma III.8. *Let $(f_n)_{n=1}^\infty\subseteq\mathcal{H}\cap\mathcal{H}_w^\infty$ be a $\mathcal{H}^\infty$-norm bounded sequence and fix $k\in X$. Then the sequence*

$$\left(\langle f_n,\tilde\psi_l\rangle\langle\psi_l,\tilde\psi_k\rangle\right)_{l\in X}$$

is dominated by an ℓ^1 sequence.

Proof. Recall from the definition of $\mathcal{H}_w^\infty$ that

$$\left|\langle f_n,\tilde\psi_l\rangle\right|w(l)\le\left\|C_{\tilde\psi}f_n\right\|_{\ell_w^\infty}=\|f_n\|_{\mathcal{H}_w^\infty}\lesssim 1\quad\forall n\in\mathbb{N}.$$

This implies that

$$\begin{aligned}\left|\langle f_n,\tilde\psi_l\rangle\right|\left|\langle\psi_l,\tilde\psi_k\rangle\right|&=\left|\langle f_n,\tilde\psi_l\rangle\right|w(l)\left|\langle\psi_l,\tilde\psi_k\rangle\right|w(l)^{-1}\\&\lesssim\left|\langle\psi_l,\tilde\psi_k\rangle\right|w(l)^{-1}.\end{aligned}$$

To show that the latter is in ℓ^1 (with respect to l), we estimate similarly as in the proof of Lemma III.7:

$$\begin{aligned}&\sum_{l\in X}\left|\langle\psi_l,\tilde\psi_k\rangle\right|w(l)^{-1}\\&\le w(k)^{-1}\sup_{m\in X}\sum_{l\in X}\left|\langle\psi_l,\tilde\psi_m\rangle\right|w(m)w(l)^{-1}\\&\le w(k)^{-1}\left\|G_{\tilde\psi,\psi}\right\|_{\mathcal{B}(\ell_w^\infty)}.\end{aligned}$$

□

Now we are able to show that the Gramian matrix is the identity on the range of $C_{\tilde\psi}:\mathcal{H}_w^\infty\to\ell_w^\infty$.

Theorem III.9. *It holds $G_{\tilde\psi,\psi}|_{R(C_{\tilde\psi})}=id_{R(C_{\tilde\psi})}$.*

Proof. Let $f\in\mathcal{H}_w^\infty$. By Proposition III.4, there exists a $\mathcal{H}_w^\infty$-norm bounded sequence $(f_n)_{n=1}^\infty\subseteq\mathcal{H}\cap\mathcal{H}_w^\infty$ such that

$$\overset{\sigma(\mathcal{H}^\infty,\mathcal{H}^{00})}{\lim_{n\to\infty}}f_n=f.$$

Our goal is to show that

$$(G_{\tilde\psi,\psi}C_{\tilde\psi}f)_k=(C_{\tilde\psi}f)_k\quad\forall k\in X.$$

The idea is to replace f by the limit of the sequence $(f_n)_{n=1}^\infty$, then apply dominated convergence to swap the limit with the sum. Finally, we apply the frame reconstruction formula (II.1) to f_n. To make the computation simpler, we start from the inside and move step by step to the outside. First, note that

$$(C_{\tilde\psi}f)_l=\langle f,\tilde\psi_l\rangle_{\mathcal{H}_w^\infty,\mathcal{H}^{00}}=\lim_{n\to\infty}\langle f_n,\tilde\psi_l\rangle.$$

Since $\psi_l \in \mathcal{H}_w^\infty$ by Lemma III.7, this yields

$$(C_{\tilde{\psi}} f)_l \psi_l = \left(\lim_{n\to\infty} \langle f_n, \tilde{\psi}_l \rangle \right) \psi_l = \overset{\mathcal{H}_w^\infty}{\lim_{n\to\infty}} \left(\langle f_n, \tilde{\psi}_l \rangle \psi_l \right).$$

Then

$$\begin{aligned} \langle (C_{\tilde{\psi}} f)_l \psi_l, \tilde{\psi}_k \rangle_{\mathcal{H}_w^\infty, \mathcal{H}^{00}} &= \left\langle \overset{\mathcal{H}_w^\infty}{\lim_{n\to\infty}} \left(\langle f_n, \tilde{\psi}_l \rangle \psi_l \right), \tilde{\psi}_k \right\rangle_{\mathcal{H}_w^\infty, \mathcal{H}^{00}} \\ &= \lim_{n\to\infty} \langle \langle f_n, \tilde{\psi}_l \rangle \psi_l, \tilde{\psi}_k \rangle \\ &= \lim_{n\to\infty} \langle f_n, \tilde{\psi}_l \rangle \langle \psi_l, \tilde{\psi}_k \rangle. \end{aligned}$$

Now, observe that

$$\begin{aligned} (G_{\tilde{\psi},\psi} C_{\tilde{\psi}} f)_k &= \sum_{l\in X} \langle \psi_l, \tilde{\psi}_k \rangle (C_{\tilde{\psi}} f)_l \\ &= \sum_{l\in X} \langle (C_{\tilde{\psi}} f)_l \psi_l, \tilde{\psi}_k \rangle_{\mathcal{H}_w^\infty, \mathcal{H}^{00}} \\ &= \sum_{l\in X} \lim_{n\to\infty} \langle f_n, \tilde{\psi}_l \rangle \langle \psi_l, \tilde{\psi}_k \rangle = (*). \end{aligned}$$

By Lemma III.8, $\left(\langle f_n, \tilde{\psi}_l \rangle \langle \psi_l, \tilde{\psi}_k \rangle \right)_{l\in X}$ is dominated by an ℓ^1-sequence for each $n \in \mathbb{N}$, so we can interchange the sum and the limit using dominated convergence and obtain

$$\begin{aligned} (*) &= \lim_{n\to\infty} \sum_{l\in X} \langle f_n, \tilde{\psi}_l \rangle \langle \psi_l, \tilde{\psi}_k \rangle \\ &= \lim_{n\to\infty} \Bigg\langle \underbrace{\sum_{l\in X} \langle f_n, \tilde{\psi}_l \rangle \psi_l}_{= f_n \text{ (frame reconstruction)}}, \tilde{\psi}_k \Bigg\rangle \\ &= \lim_{n\to\infty} \langle f_n, \tilde{\psi}_k \rangle = \langle f, \tilde{\psi}_k \rangle_{\mathcal{H}_w^\infty, \mathcal{H}^{00}} = (C_{\tilde{\psi}} f)_k. \end{aligned}$$

This shows that $G_{\tilde{\psi},\psi} C_{\tilde{\psi}} f = C_{\tilde{\psi}} f$ for all $f \in \mathcal{H}_w^\infty$. ☐

Next, we show that the range of the coefficient operator is the only set on which the Gramian matrix acts as the identity.

Lemma III.10. *Let $\alpha \in \ell_w^\infty$ such that $G_{\tilde{\psi},\psi}\alpha = \alpha$. Then $\alpha \in R(C_{\tilde{\psi}})$.*

Proof. Choose a nested sequence $(F_n)_{n\in\mathbb{N}}$ of finite subsets of X such that $\bigcup_{n\in\mathbb{N}} F_n = X$. Let

$$f_n := \sum_{l\in F_n} \alpha_l \psi_l \in \mathcal{H} \cap \mathcal{H}_w^\infty.$$

Observe that for each $k \in X$

$$\begin{aligned} \left| \langle f_n, \tilde{\psi}_k \rangle \right| w(k) &= \left| \left\langle \sum_{l\in F_n} \alpha_l \psi_l, \tilde{\psi}_k \right\rangle \right| w(k) \\ &\le \sum_{l\in F_n} |\alpha_l| \left| \langle \psi_l, \tilde{\psi}_k \rangle \right| w(k) \\ &\le \sum_{l\in X} |\alpha_l| w(l) w(l)^{-1} \left| \langle \psi_l, \tilde{\psi}_k \rangle \right| w(k) \\ &\le \|\alpha\|_{\ell_w^\infty} \sum_{l\in X} w(l)^{-1} \left| \langle \psi_l, \tilde{\psi}_k \rangle \right| w(k) \\ &\le \|\alpha\|_{\ell_w^\infty} \left\| G_{\tilde{\psi},\psi} \right\|_{\mathcal{B}(\ell_w^\infty)}. \end{aligned}$$

This implies that $(f_n)_{n=1}^\infty$ is $\mathcal{H}_w^\infty$-bounded. Similarly, we see that this sequence is a Cauchy sequence with respect to $\sigma(\mathcal{H}_w^\infty, \mathcal{H}^{00})$, since for $m \ge n$

$$\begin{aligned} \left| \langle f_m - f_n, \tilde{\psi}_k \rangle \right| w(k) &\lesssim \sum_{l\in F_m \setminus F_n} w(l)^{-1} \left| \langle \psi_l, \tilde{\psi}_k \rangle \right| w(k) \\ &\le \sum_{l\in X\setminus F_n} w(l)^{-1} \left| \langle \psi_l, \tilde{\psi}_k \rangle \right| w(k) \\ &\overset{n\to\infty}{\to} 0. \end{aligned}$$

Also,

$$\begin{aligned} \alpha_k = (G_{\tilde{\psi},\psi}\alpha)_k &= \left\langle \overset{\sigma(\mathcal{H}_w^\infty, \mathcal{H}^{00})}{\sum_{l\in X}} \alpha_l \psi_l, \tilde{\psi}_k \right\rangle_{\mathcal{H}_w^\infty, \mathcal{H}^{00}} \\ &= \lim_{n\to\infty} \left\langle \sum_{l\in F_n} \alpha_l \psi_l, \tilde{\psi}_k \right\rangle \\ &= \lim_{n\to\infty} \langle f_n, \tilde{\psi}_k \rangle. \end{aligned}$$

Now let

$$f := \overset{\sigma(\mathcal{H}_w^\infty, \mathcal{H}^{00})}{\lim_{n\to\infty}} f_n \in \mathcal{H}_w^\infty.$$

Then, we see that

$$\begin{aligned} C_{\tilde{\psi}} f = \left(\langle f, \tilde{\psi}_k \rangle \right)_{k\in X} &= \left(\lim_{n\to\infty} \langle f_n, \tilde{\psi}_k \rangle \right)_{k\in X} \\ &= (\alpha_k)_{k\in X} = \alpha. \end{aligned}$$

Thus we conclude that $\alpha \in R(C_{\tilde{\psi}})$. ☐

From the last two results, we obtain the following theorem:

Theorem III.11. *Let $V := \{\alpha \in \ell_w^\infty : \alpha = G_{\tilde{\psi},\psi}\alpha\}$. Then $C_{\tilde{\psi}} : \mathcal{H}_w^\infty \to V$ is an isometric isomorphism.*

Finally, we can show the completeness of $\mathcal{H}_w^\infty$.

Theorem III.12. *V is a closed subspace of ℓ_w^∞. Consequently, $\mathcal{H}_w^\infty$ is a Banach space.*

Proof. Let $(\alpha^n)_{n=1}^\infty \subseteq V$ be a sequence converging to some $\alpha \in \ell_w^\infty$. Since $G_{\tilde{\psi},\psi}$ is bounded on ℓ_w^∞, we get

$$G_{\tilde{\psi},\psi}\alpha = G_{\tilde{\psi},\psi} \overset{\ell_w^\infty}{\lim_{n\to\infty}} \alpha^n = \overset{\ell_w^\infty}{\lim_{n\to\infty}} G_{\tilde{\psi},\psi}\alpha^n = \overset{\ell_w^\infty}{\lim_{n\to\infty}} \alpha^n = \alpha.$$

This implies that V is a closed subspace of ℓ_w^∞. Since, by Theorem III.11, V is isometrically isomorphic to $\mathcal{H}_w^\infty$, the latter is complete as well. ☐

Acknowledgment

This research was funded in whole or in part by the Austrian Science Fund (FWF) 10.55776/P34624. For open access purposes, the author has applied a CC BY public copyright license to any author accepted manuscript version arising from this submission.

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