Equitable Mechanism Design for Facility Location

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Abstract

We consider strategy proof mechanisms for facility location which maximize equitability between agents. As is common in the literature, we measure equitability with the Gini index. We first prove a simple but fundamental impossibility result that no strategy proof mechanism can bound the approximation ratio of the optimal Gini index of utilities for one or more facilities. We propose instead computing approximation ratios of the complemented Gini index of utilities, and consider how well both deterministic and randomized mechanisms approximate this. In addition, as Nash welfare is often put forwards as an equitable compromise between egalitarian and utilitarian outcomes, we consider how well mechanisms approximate the Nash welfare.

1 Introduction

Mechanism design is the problem of designing rules for a game to achieve a specific outcome, even though each participant may be self-interested. The aim is to design rules so that the participants are incentivized to behave as the designer intends. This typically includes achieving properties such as truthfulness, individual rationality, budget balance, and maximizing social welfare. Here we consider another desirable property that designers might look to achieve: equitability. How does a mechanism designer ensure that all participants are as equally happy with the outcome as is possible? Surprisingly, equitable mechanism design has received somewhat limited attention so far in the scientific literature.

Central to this question of equitable mechanism design is defining what it means for an outcome to be equitable. Consider the simple decision making problem of locating a facility along a line. This models a number of real world problems such as picking the room temperature for a classroom, or the deadline for a project. Agents are supposed to have single peaked preferences, prefering the facility to be nearer to their location. And we look to design mechanisms which locate the facility so that the distances which the different agents must travel are as small and as similar as is possible. In general, of course, the distances agents travel may have to be different. Consider locating a facility on $[0, 1]$, with three agents: one at 0, another at $1/2$ and the final at 1. The agent at $1/2$ inevitably has to be nearer the facility than at least one of the other two agents. Where then do we locate the facility to ensure the most equitable outcome? In this case, locating the facility at $1/2$ might seem best. The maximum distance any agent travels is the minimum it can be.

Our goal then is to design equitable mechanisms for facility location in which agents are incentivized to report sincerely. While our focus is on the facility location problem, there are some general conclusions that can be drawn from this study. First, designing mechanisms for equitability is **somewhat different** to designing mechanisms for objectives such as social welfare. For instance, equitability considers all agents, while egalitarian welfare considers just the worst off agent. We will show that mechanisms with approximate well the egalitarian welfare may not return equitable outcomes. For instance, the randomized LRM mechanism (described shortly) has good welfare properties but is less good at returning equitable solutions. Second, designing strategy proof mechanisms for equitability is possible if we sacrifice a little optimality. For example, we identify a **strategy proof mechanism for one facility with optimal equitability**. And by carefully considering how existing mechanisms perform poorly, we design a **new strategy proof mechanism for two facilities with close to optimal equitability**. And third, a key component in designing such new mechanisms with good performance is to **censor extreme outcomes**. We conjecture that mechanism design for equitability may offer promise in other domains such as fair division and ad auctions.

2 Facility location

The facility location problem is a classic problem in social choice and mulitagent decision making in which we need to decide where to locate a facility to serve a set of agents. The problem generalizes to locating two (or more) facilities, in which case we suppose agents are served by the nearest facility. We consider n agents located on the real line at x_1 to x_n and a facility at y. Without loss of generality, we suppose $x_1 \leq \ldots \leq x_n$. We let d_i be the distance of agent i to the facility: $d_i = |x_i - y|$. As in several previous studies, we assume agents and facilities are limited to the interval $[0, 1]$, and the utility of agent i is $u_i = 1 - d_i$. The interval could be $[a, b]$, in which case we normalise by $b - a$.

Supposing agents and facilities lie on an interval is interesting for both practical and theoretical reasons. In particular, agents and facilities are often limited to a finite interval and cannot be located outside those limits. For example, when setting a thermostat, we have a temperature range limited by the boiler. As a second example, when locating a water treatment plant on a river, the plant must be on the river itself. As a third example, when locating a shopping centre, the centre might have to be on the fixed (and finite) road network. There are thus many situations where agents are limited to a finite interval. Restricting agents to a finite interval also limits the extent to which agents can misreport their location to influence the outcome. Several other recent works have used a finite interval (e.g. [Cheng *et al.*, 2013; Feigenbaum and Sethuraman, 2015; Mei *et al.*, 2016; Aziz *et al.*, 2021; Aziz *et al.*, 2022]).

A deterministic mechanism f locates the facility at a location y. Formally, $f(x_1, \dots, x_n) = y$. A mechanism is *anonymous* iff any permutation of the agents returns the same facility location. Formally f is anonymous iff for any permutation σ, $f(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = f(x_1, \dots, x_n)$. A mechanism is *Pareto efficient* iff we cannot move the location of the facility and make one agent better off (nearer to the facility) and no agent worse off. Formally f is Pareto efficient iff for any $x_1, \dots, x_n$, there is no location z and agent i with $|x_i - z| < |x_i - f(x_1, \dots, x_n)|$ and $|x_j - z| \leq |x_j - f(x_1, \dots, x_n)|$ for all $j \in [1, n]$. We limit our attention to *unanimous* mechanisms that locate the facility where all agents agree. Formally f is unanimous iff for any x, $f(x, \dots, x) = x$. This is a weaker condition than Pareto efficiency. Unanimity rules out undesirable mechanisms such as the mechanism which always locates the facility at $1/2$. We consider how well mechanisms approximate some objective O like the (soon to be defined) Gini index. For a maximization (minimization) objective, the approximation ratio is the maximum ratio of O_{opt}/O_{approx} (O_{approx}/O_{opt}) where O_{opt} is the optimal value and O_{approx} is the approximately optimal value returned by the mechanism.

One of our major concerns is strategic manipulation. Agents are self-interested so we look for mechanisms where agents cannot improve their outcome by mis-reporting their location. A mechanism is *strategy proof* iff no agent can misreport their position and reduce their distance to the nearest facility. Formally f is strategy proof iff for any $x_1, \dots, x_n$, and agent i, it is not the case that there exists x_i' with $|x_i - f(x_1, \dots, x_i', \dots, x_n)| < |x_i - f(x_1, \dots, x_i, \dots, x_n)|$. Moulin [1980] has provided an elegant characterization of strategy-proof mechanisms: a mechanism is anonymous, strategy-proof and Pareto efficient iff it is the median rule (which locates the facility at the median agent) with at most $n-1$ phantoms (additional "agents" reporting fixed locations). We consider various strategy proof mechanisms from previous studies of facility location. The LEFTMOST mechanism locates the facility at x_1, the leftmost agent. This is equivalent to the median rule with $n-1$ phantoms at 0. The MEDIAN mechanism locates the facility at $x_{\lceil n/2 \rceil}$, the median agent. This is equivalent to the median rule with $\lfloor n/2 \rfloor$ phantoms at 0, and the rest of the phantoms at 1. The MIDORNEAREST mechanism locates the facility at x_n when $x_n < 1/2$, at $1/2$ when $x_1 \leq 1/2 \leq x_n$, and at x_1 when $x_1 > 1/2$. This is equivalent to the median rule with $n-1$ phantoms at $1/2$. When locating two facilities, the ENDPOINT mechanism locates one facility at x_1 and another at x_n. These mechanisms are all deterministic. We also consider randomized mechanisms which return a lottery over solution, and the expected distance of agents from the facility. For example, the strategy proof LRM mechanism (Left, Right, or Midpoint) locates the facility at x_1 with probability $1/4$, at $(x_1 + x_n)/2$ with probability $1/2$, and at x_n with the remaining probability $1/4$ [Procaccia and Tennenholtz, 2013].

While our results are, like many previous studies, focused on the 1-d setting, they are interesting more broadly. The 1-d facility location problem models several real world problems such as locating distribution centres along a highway. There are also non-geographical settings that are 1-d (e.g. setting a thermostat or tax rate). In addition, we can solve more complex problems in higher dimensions by decomposing them into 1-d problems. Finally, the 1-d problem is the starting point to consider more complex metrics such as trees and networks. For instance, lower bounds for 1-d can be inherited for 2-d and other metric spaces.

3 Minimizing the Gini index

The Gini index is one of the most widely used measures of equitability. Unsurprisingly it has been used in facility location problems. For example, Mulligan [1991] argues that simple equitability measures like maximum distance ignore the distribution of distances and recommends measures like the Gini index. The Gini index of distances is:

$$G_d = \frac{\sum_{i \leq n} \sum_{j \leq n} |d_i - d_j|}{2n \sum_{i \leq n} d_i}$$

This lies in $[0, 1]$, takes the value 0 for an equitable solution when $d_i = d_j$ for all i and j, and increases in value as distances become more unequal. If all agents are at the same location, then any facility location is an equitable solution since all agents travel the same distance. Therefore equitability alone is not sufficient to guarantee solutions are desirable. We might also demand additional properties like unanimity.

But does minimizing the Gini index of distances even guarantee an equitable and favourable outcome? Minimizing the Gini index of distances favours distances being large over distances being small. As an example, suppose we have n agents at $1/2$, and one agent at both 0 and 1, with $n > 2$. Locating the facility at 0 or 1, gives the minimum Gini index of distances of $\frac{2(n+1)}{(n+2)^2}$. In this solution, n of the agents have to travel a distance of $1/2$ and one agent has to travel the maximum distance of 1. A more equitable solution has the facility at $1/2$, with most of the agents travelling no distance at all, and just two agents travel a distance of $1/2$. Perversely this has a larger Gini index of distances of $\frac{n}{(n+2)}$.

We propose instead to **consider utilities rather than distances** by minimizing the Gini index of utilities:

$$G_u = \frac{\sum_{i \leq n} \sum_{j \leq n} |u_i - u_j|}{2n \sum_{i \leq n} u_i}$$

Minimizing the Gini index of utilities prefers most agents having a large utility over most agents having a large distance to travel. Returning to the previous example, minimizing the Gini index of utilities locates the facility at $1/2$ which is, as we argued, the most equitable facility location. Note that minimizing the Gini index is also compatible with Pareto efficiency. Whilst not all solutions that minimize the Gini index of utilities are Pareto efficient, there is always a solution minimizing the Gini index that is Pareto efficient.

Our first result is **an impossibility**. No strategy proof mechanism has a bounded approximation ratio.

Theorem 1. *No strategy proof mechanism for locating one or more facilities on* $[0, 1]$ *has a bounded approximation ratio for the Gini index of utilities.*

Proof. Suppose there exists a strategy proof mechanism for two agents with a bounded approximation ratio for locating a single facility. Consider $x_1 = 0$ and $x_2 = 1/2$. To have a bounded approximation ratio, the facility must be at the optimal location, $1/4$. Consider $x_1 = 0$ and $x_2 = 1$. To have a bounded approximation ratio, the facility must be at $1/2$. Hence, the agent at $1/2$ in the first scenario had an incentive to mis-report their location as 1. A similar construction can be made for more agents and facilities. □

Randomization does not help escape this impossibility. The proof works whether mechanisms are deterministic or randomized. The problem with the approximation ratio of the Gini index is that this focuses on equitable problems where the index is zero (and all distances/utilities are equal) A natural way around this problem is to consider the complemented Gini index (that is, $1 - G$). This is again in $[0, 1]$. It is 1 when utilities (or distances) are equal, and becomes smaller as problems become more inequitable. Our goal now is to *maximize* the complemented Gini index. Considering the approximation ratio of the complemented Gini index switches focus onto inequitable problems in which utilities (or distances) are necessarily imbalanced (such as the earlier example with agents at 0, $1/2$ and 1).

You might be concerned that by shifting to the complement of the Gini index, we are just replacing one problem (approximating equitable problems with Gini indices close to zero) with another (approximating inequitable problems with Gini indices close to 1, and complemented Gini indices close to zero). This is not the case. While Gini indices can indeed be close to zero (and hard to approximate within a constant factor), the *optimal* Gini index of utilities in facility location is never close to 1 and, as we will show, can be approximated well by strategy proof mechanisms.

4 One facility

We first deomstrate that there exist strategy proof mechanisms which approximate well the complemented Gini index. We start with one of the simplest possible mechanisms. The LEFTMOST mechanism is strategy proof and 2-approximates the maximum distance. This is optimal as no deterministic and strategy proof mechanism can do better[1]. However, the LEFTMOST mechanism does not return very equitable solutions.

Theorem 2. *For a facility location problem with n agents, the* LEFTMOST *mechanism n-approximates the complemented Gini index of utilities.*

Proof. If all agents are at the same location, then the LEFTMOST mechanism is optimal with respect to the complemented Gini index. Therefore we suppose agents are at two or more locations. The smallest possible complemented Gini index (which the LEFTMOST mechanism achieves) is when one agent is at 0 and the remaining $n - 1$ agents are at 1. The complemented Gini index in this case is just $1/n$. This compares to an optimal of 1 when the facility is at $1/2$. □

The MEDIAN mechanism does significantly better than the LEFTMOST mechanism at returning equitable solutions. This is unsurprising as the MEDIAN mechanism tends to be more balanced than the LEFTMOST mechanism which necessarily locates the facility at an extreme location. When considering the maximum distance agents must travel, we cannot distinguish between the LEFTMOST and MEDIAN mechanisms. Both mechanisms 2-approximate the maximum distance. Here we see that **the Gini index distinguishes them apart**, suggesting that MEDIAN is more equitable.

Theorem 3. *The* MEDIAN *mechanism* 2*-approximates the complemented Gini index of utilities.*

Proof. We prove that the MEDIAN mechanism always returns a facility location that gives a complemented Gini index of utilities of $1/2$ or greater. Hence the MEDIAN mechanism cannot be worse than a 2-approximation. Indeed, if one agent is at 0 and another at 1, it is at best a 2-approximation.

There are two cases. In the first, $n = 2k$. Note that the median agent is at x_k, which is the location of the facility. We assume $x_k \leq 1/2$ otherwise we reflect the position of any agent x onto $1-x$. We first prove that $\sum_i u_i \geq k$. In fact, the sum equals k when $x_1 = \ldots = x_k = 0$ and $x_{k+1} = \ldots = x_n = 1$. Suppose there is a smaller sum of utilities for some other x'_1 to x'_n. If we map x'_i onto 0 for $i \leq k$ then each u_i for $i \leq k$ increases less than each u_i for $i > k$ decreases. That is, the sum of utilities would decrease which is a contradiction. Hence, $x'_1 = \ldots = x'_k = 0$. Similarly, suppose $x'_{k+1} < 1$. Then mapping x_i onto 1 for $i > k$ would decrease the sum of utilities which is again a contradiction. Hence, the smallest sum of utilities occurs when $x_1 = \ldots = x_k = 0$ and $x_{k+1} = \ldots = x_n = 1$ and this sum is k.

We next prove that $\sum_i \sum_j |u_i - u_j| \leq 2k^2$. Again, the maximum double sum is when $x_1 = \ldots = x_k = 0$ and $x_{k+1} = \ldots = x_n = 1$. We consider different terms in the double sum. If we consider the pair of terms $|u_i - u_j| + |u_i - u_{n-j+1}|$ for $i < j \leq k$ then as x_i and x_j are at or to the left of x_k, and at or the right of x_k, it follows that the sum of these two differences equals or is less than 1. A similar argument applies to the sum of terms $|u_{n-i+1} - u_{n-j+1}| + |u_{n-i+1} - u_j|$. Hence the $4k^2$ terms have a maximum sum of $2k^2$. This again occurs when $x_1 = \ldots = x_k = 0$ and $x_{k+1} = \ldots = x_n = 1$. The complemented Gini index is $1 - \frac{\sum_i \sum_j |u_i - u_j|}{2n \sum_i u_i}$. The minimum value this takes is lower

[1]Procaccia and Tennenholtz [2013] demonstrate this for the real line. However, the result easily extends to any fixed interval.

bounded by the maximum value of $\sum_i \sum_j |u_i - u_j|$ divided by the minimum value of $\sum_i u_i$. That gives a lower bound of $1 - \frac{2k^2}{4k \cdot k}$ or $1/2$.

In the second case $n = 2k + 1$ is odd. We suppose $x_{k+1} \leq 1/2$. This is the median agent and therefore location of the facility. By a similar argument, $\sum_i u_i$ takes a minimum value of $k + 1$, and $\sum_i \sum_j |u_i - u_j|$ takes a maximum value of $2k(k + 1)$ when $x_1 = \ldots = x_{k+1} = 0$ and $x_{k+2} = \ldots = x_n = 1$ The complemented Gini index takes its minimum value of $1 - \frac{2k(k+1)}{2(2k+1)(k+1)}$ or $1 - \frac{k}{2k+1}$ which tends to $1/2$ from above as k goes to infinity. □

Can we do even better than this? Yes, we can. Consider the MIDORNEAREST mechanism. This is strategy proof, 2-approximates the maximum distance (recall that no strategy proof and deterministic mechanism can do better), and $3/2$-approximates the minimum utility (no strategy proof and deterministic mechanism can again do better [Walsh, 2024]). We now show that the MIDORNEAREST mechanism also approximates well the complemented Gini index of utilities.

Theorem 4. *The* MIDORNEAREST *mechanism* $\frac{(n^2+n)}{(n^2+1)}$*-approximates the optimal complemented Gini index of utilities for* n *agents (*$n \geq 1$*). The worst case is* $n = 2$ *or* $n = 3$ *when it provides a* $6/5$*-approximation.*

Proof. For $n = 1$, the MIDORNEAREST mechanism locates the facility at the single agent which is optimal. For $n \geq 2$, observe that the MIDORNEAREST mechanism guarantees that $u_i \geq 1/2$ for any i. The most inequitable outcome returned then is when $u_i = 1/2$ for $i < n$ and $u_n = 1$. This occurs, for example, when $x_i = 0$ for $i < n$ and $x_n = 1/2$, and the mechanism locates the facility at $1/2$. This gives a complemented Gini index of utilities of $\frac{(n^2+1)}{(n^2+n)}$. Coincidently, when $x_i = 0$ for $i < n$ and $x_n = 1/2$, there is an optimal and perfectly equitable solution which locates the facility at $1/4$, giving a complemented Gini index of 1. Hence, the most inequitable outcome for the MIDORNEAREST mechanism occurs when there is an optimal and perfectly equitable solution. The approximation ratio of the MIDORNEAREST mechansm is thus $\frac{(n^2+n)}{(n^2+1)}$. This ratio is maximized for $n = 2$ or $n = 3$ when it provides a $6/5$-approximation. □

In fact, **we cannot do better** than a $6/5$-approximation.

Theorem 5. *A deterministic mechanism that is anonymous, Pareto efficient and strategy proof at best* $6/5$*-approximates the complemented Gini index of utilities.*

Proof. Such a mechanism is a median mechanism with $n - 1$ phantoms. Consider $n = 2$, the phantom at a, and one agent at 0, and another at a. Suppose $a \leq 1/2$. The case of $a \geq 1/2$ is dual. The median mechanism locates the facility at a. The complemented Gini index of utilities is $\frac{(4-3a)}{(4-2a)}$. This compares to an optimal of 1. Therefore the approximation ratio is $\frac{(4-2a)}{(4-3a)}$ which is in the interval $[1, 6/5]$ for $0 \leq a \leq 1/2$. Now consider one agent at a, and another at 1. This median mechanism again locates the facility at a. The complemented Gini index of utilities is now $\frac{(1+3a)}{(2+2a)}$. This again compares to an optimal of 1. Therefore the approximation ratio is $\frac{(2+2a)}{(1+3a)}$ which is in the interval $[6/5, 2]$ for $0 \leq a \leq 1/2$. Over the two scenarios, the best ratio that can be achieved is $6/5$. □

5 Two facilities

With two facilities, the ENDPOINT mechanism is the only strategy proof and deterministic mechanism on the real line with a bounded approximation ratio of the optimal maximum distance [Fotakis and Tzamos, 2013]. With respect to the complemented Gini index of utilities, it provides a reasonable approximation ratio of the optimal. However, somewhat surprisingly, it **does not offer the best possible ratio** amongst strategy proof and deterministic mechanisms.

Theorem 6. *The* ENDPOINT *mechanism* $35/29$*-approximates the optimal complemented Gini index of utilities (*≈ 1.21*).*

Proof. With two or fewer agents, the ENDPOINT mechanism returns an optimal solution. Therefore we consider three or more agents. With the ENDPOINT mechanism, the leftmost and rightmost agents must have utility 1, while the other agents have utility between $1/2$ and 1. The minimum complemented Gini index of utilities is when these $n-2$ agents have utility $1/2$. This occurs when one agent is at 0, another is at 1, the final $n - 2$ are at $1/2$, and the facilities located at the two endpoints. The complemented Gini index of utilities is then $1 - \frac{2(n-2)}{n(n+2)}$. This is minimized for $n = 5$, when it is $29/35$. Coincidently, there is an optimal outcome in this case with facilities at $1/4$ and $3/4$, and an optimal complemented Gini index of 1. The approximation ratio is therefore $35/29$. □

We now define ENDPOINT$_\gamma$, a new mechanism which performs better by **truncating extreme locations** for the two facilities using a parameter $\gamma \in [0, 1/2]$. For two or fewer agents, this simply applies the ENDPOINT mechanism. For three or more agents, ENDPOINT$_\gamma$ locates the left facility at $\min(\max(x_1, \gamma), x_n)$ and the right facility at $\max(x_1, \min(1 - \gamma, x_n))$.

Theorem 7. *The* ENDPOINT$_\gamma$ *mechanism is strategy proof, and* α*-approximates the optimal complemented Gini index of utilities where* $\alpha \in [15/14, 35/29]$ *and* α *depends on* γ*. The ratio is minimized for* $\gamma = 1/4$ *when* $\alpha = 15/14$ *(*≈ 1.07*), and maximized for* $\gamma = 0$ *when* $\alpha = 35/29$ *(*≈ 1.21*).*

Proof. Truncating endpoints does not impact strategy proofness. With one or two agents, the ENDPOINT$_\gamma$ mechanism returns an optimal solution. Therefore we consider three or more agents. There are two cases. In the first case, $\gamma \leq 1/4$, and agents must have utility between $1/2 + \gamma$ and 1. Then the minimum complemented Gini index of utilities is when $n - 2$ agents have utility $1/2 + \gamma$, one agent has utility $1 - \gamma$, and the final agent has utility 1. This occurs when one agent is at 0, another is at $1 - \gamma$ and $n - 2$ are at $1/2$, and the ENDPOINT$_\gamma$ mechanism places facilities at γ and $1 - \gamma$. In this case, for fixed n, the complemented Gini index of utilities increases as γ increases towards $1/4$. For $\gamma = 0$, as shown in the previous theorem, the approximation ratio of the complemented Gini index of utilities is $35/29$. The approximation ratio decreases as γ increases towards $1/4$. For $\gamma = 1/4$, the complemented Gini index of utilities takes a minimum of $14/15$. Coincidently, when the agents are at 0, $1/2$ and $1-\gamma$, there is an optimal solution in which facilities are at $1/4 + \gamma/2$ and $3/4 - \gamma/2$, and the complemented Gini index of utilities is 1. The approximation ratio for $\gamma \leq 1/4$ is therefore in $[15/14, 35/29]$.

In the second case, $\gamma \geq 1/4$, and agents must have utility between $1-\gamma$ and 1. Then the minimum complemented Gini index of utilities is when $n-1$ agents have utility $1-\gamma$, and the final agent has utility 1. This occurs when one agent is at 0, another is at γ and $n-2$ are at 1, and the ENDPOINT_γ mechanism places facilities at γ and $1-\gamma$. In this case, for fixed n, the Gini index of utilities increases as γ increases from $1/4$. For $\gamma = 1/2$, the complemented Gini index takes a minimum value of $5/6$ when $n = 3$. Coincidently, when the agents are at 0, γ and 1 there is an optimal solution in which facilities are at $\gamma/2$ and $1 - \gamma/2$, and the complemented Gini index of utilities is 1. The approximation ratio for $\gamma \geq 1/4$ is therefore in $[15/14, 6/5]$ (and $6/5 < 35/29$). □

We now prove that no strategy proof and deterministic mechanism for two facilities can do better than $30/29$-approximate the optimal complemented Gini index (≈ 1.03). This leaves a small gap with the best approximation ratio of $15/14$ (≈ 1.07) achieved by the ENDPOINT_γ mechanism.

Theorem 8. *No strategy proof and deterministic mechanism for two facilities can do better than $30/29$-approximate the optimal complemented Gini index of utilities (≈ 1.03).*

Proof. Suppose a strategy proof and deterministic mechanism exists with an approximation ratio smaller that $30/29$. Consider three agents, one at 0, another at $1/2$ and the final agent at $3/4$. The optimal location of facilities that maximizes the complemented Gini index of utilities has one facility at $1/8$, and the other at $5/8$ giving a complemented Gini index of 1. The most left that the rightmost facility can be and the approximation ratio of the complemented Gini index be smaller than $30/29$ is to the right of $21/26$. If the rightmost facility is at $21/26$ then the minimal Gini index of utilities is when the leftmost facility is at $1/13$ and the Gini index is $1/30$. The complemented Gini index is then $29/30$, which corresponds to an approximation ratio of the optimal complemented Gini index of $30/29$. The agent at $3/4$ therefore travels a distance greater than $21/26 - 3/4$ (which is $3/52$).

Now suppose the agent at $3/4$ reports their location as 1. The optimal solution maximizing the complemented Gini index of utilities for the reported locations of the agents has one facility at $1/4$, and the other at $3/4$ giving a complemented Gini index of 1. The most right that the rightmost facility can be and the approximation ratio of the complemented Gini index be smaller than $30/29$ is to the left of $21/26$. If the rightmost facility is at $21/26$ then the minimal Gini index of utilities for the reported locations is when the leftmost facility is at $5/26$ and the Gini index is $1/30$. The complemented Gini index is then $29/30$, which corresponds to an approximation ratio of the optimal complemented Gini index of $30/29$. Note also that the rightmost facility cannot be to the left of $3/4 - 3/52$ as this gives an approximation ratio of the complemented Gini index greater than $30/29$. The agent at $3/4$ therefore travels a distance less than $21/26 - 3/4$ (which is $3/52$). Thus, by mis-reporting their location, the agent at $3/4$ reduces their distance from the facility from more than $3/52$ to less than $3/52$. □

6 Randomized mechanisms

Randomization is often a simple and attractive mechanism to achieve better performance in expectation. For example, the randomized LRM mechanism $3/2$-approximates the maximum distance any agent must travel in expectation [Procaccia and Tennenholtz, 2013]. This beats the 2-approximation lower bound that deterministic and strategy proof mechanisms can at best achieve. In addition, randomized and strategy proof mechanisms cannot do better than $3/2$-approximate the maximum distance[2]. The LRM mechanism is not quite as good at returning equitable solutions, **even being beaten** by a deterministic mechanism like MIDORNEAREST. The problem with the LRM mechanism is that, while it returns a solution that is close to the optimal facility location in expectation, the ex post solutions are at extreme locations half of the time.

Theorem 9. *The* LRM *mechanism α-approximates the optimal complemented Gini index of utilities in expectation with $\alpha = 4/3$ for $n \leq 3$, and $\alpha \in [4/3, 2]$ for $n \geq 4$.*

Proof. For $n = 2$, suppose one agent is at 0 and the other at a with $0 \leq a \leq 1$. With probability $1/2$, the facility is located at $a/2$ which gives an optimal complemented Gini index of utilities of 1. With the remaining probability, the facility is located at 0 or a which gives a sub-optimal complemented Gini index of utilities of $1 - \frac{a}{2(2-a)}$. This is minimized for $a = 1$ when the complemented Gini index of utilities is $1/2$. The LRM mechanism thus has an expected complemented Gini index of utilities that is at least $3/4$, compared to an optimal of 1. Hence the approximation ratio $\alpha = 4/3$.

For $n = 3$, we suppose one agent is at 0, another at a and the third at b with $0 \leq a \leq b \leq 1$. Without loss of generality, we suppose $2a \leq b$ (otherwise we reflect problem). The optimal Gini index of utilities has the facility at $b/2$ giving a complemented Gini index of utilities that is $1 - \frac{4a}{3(6-3b+2a)}$. With probability $1/2$, the LRM mechanism locates the facility at $b/2$ which gives an optimal complemented Gini index of utilities of $1 - \frac{4a}{3(6-3b+2a)}$. With probability $1/4$, the facility is located at 0 which gives a sub-optimal complemented Gini index of utilities of $1 - \frac{2b}{3(3-a-b)}$. With the remaining probability $1/4$, the facility is located at b which gives a sub-optimal complemented Gini index of utilities of $1 - \frac{2b}{3(3+a-2b)}$. The expected complemented Gini index of utilities is thus $1 - \frac{2a}{3(6-3b+2a)} - \frac{b}{6(3-a-b)} - \frac{b}{6(3+a-2b)}$. The ratio of this with the optimal is minimized by $a = 0$ and $b = 1$ when the expected value is $3/4$ compared to an optimal of 1. Hence the approximation ratio $\alpha = 4/3$.

For $n = 2k$, we consider k agents at 0 and k at 1. The optimal complemented Gini index is 1, but the expected complemented Gini index returned by LRM is $3/4$. Hence $\alpha \geq 4/3$. For $n = 2k+1$, we consider k agents at 0 and $k+1$ at

[2]Theorem 3.4 in [Procaccia and Tennenholtz, 2013] shows this for the real line. However, the proof works for any fixed interval.

1. The optimal complemented Gini index is 1, but the expected complemented Gini index returned by LRM is again $3/4$. Hence $\alpha \geq 4/3$.

If the facility is at the midpoint between agents, each agent gets at least an utility of $1/2$. The most inequitable outcome satisfying this constraint gives $n-1$ agents utility $1/2$ and one agent utility 1, with a complemented Gini index of $\frac{(n^2+1)}{(n^2+n)}$. On the other hand, if the facility is at one of the extreme agents, the most inequitable outcome has $n-1$ agents with utility 0 and one agent with utility 1, giving a complemented Gini index of at least $1/n$. Hence the expected complemented Gini index is at least $\frac{1}{2}\frac{1}{n} + \frac{1}{2}\frac{(n^2+1)}{(n^2+n)}$ or $\frac{(n^2+n+2)}{2(n^2+n)}$. Therefore $\alpha \leq \frac{2(n^2+n)}{(n^2+n+2)} \leq 2$. □

We also provide a lower bound on the best approximation ratio that randomized mechanisms can achieve.

Theorem 10. *Any randomized mechanism that is strategy proof can at best $8/7$-approximate the optimal complemented Gini index of utilities in expectation.*

Proof. Suppose a strategy proof mechanism exist with a better approximation ratio. Consider two agents at $1/3$ and $2/3$. We suppose the expected location of the facility is in $[0, 1/2]$. The case when it is in $(1/2, 1]$ is dual. Suppose the agent at $2/3$ mis-reports their location as 1. The optimal facility location is $2/3$, giving a Gini index of 0. If the facility is at $2/3 - x$ for $x \in [0, 1/3]$ then the Gini index is $3x/4$. And if the facility is at $2/3 + x$ for $x \in [0, 1/3]$ then the Gini index is $3x/4$. Hence, the expected Gini index is $3/4$ the expected distance of the facility from $2/3$. To achieve the required approximation ratio, the expected complemented Gini index must be less than $7/8$. This puts the expected location of facility in the interval $(1/2, 5/6)$. Hence, the agent at $2/3$ in the first setting has an incentive to misreport their location as 1. □

7 Nash welfare

The solution maximizing the Nash welfare is often considered to be an equitable compromise between the egalitarian and utilitarian solution. The Nash welfare is $\sqrt[n]{\prod_i u_i}$. We therefore also consider how well these strategy proof mechanisms approximate the Nash welfare.[3]

Theorem 11. *The* LEFTMOST *and* MEDIAN *mechanisms have an unbounded approximation ratio of the Nash welfare, while the* MIDORNEAREST *mechanism* 2*-approximates it.*

Proof. Consider an agent at 0 and 1. The LEFTMOST and MEDIAN mechanisms give one agent an utility of zero, thus giving a Nash welfare also of zero. However, the optimal Nash welfare is $1/2$ with the facility located at $1/2$.

The MIDORNEAREST mechanism ensures that every agent has utility of $1/2$ or greater. The Nash welfare is therefore $1/2$ or greater. The maximum Nash welfare is 1. Hence, the approximation ratio is at most 2. Consider $n-1$ agents at 0 and a final agent at $1/2$. The MIDORNEAREST mechanism locates the facility at $1/2$, giving a Nash welfare of $1/\sqrt[n]{2^{n-1}}$. This compares to an optimal of $1/\sqrt[n]{2}$ with the facility at 0. This gives an approximation ratio of $\sqrt[n]{2^{n-2}}$. This approaches 2 from below as n goes to infinity. Hence, the approximation ratio is at least 2. □

We next prove that any deterministic mechanism can at best approximate the optimal Nash welfare.

Theorem 12. *A deterministic mechanism that is anonymous, Pareto efficient and strategy proof at best $\frac{3}{2\sqrt{2}}$-approximates the Nash welfare (≈ 1.06).*

Proof. Such a mechanism is a median mechanism with $n-1$ phantoms. Consider $n = 2$ and a phantom at a. Suppose $a \leq 1/2$. The case with $a \geq 1/2$ is dual. Consider one agent at 0, and another at a. The median mechanism locates the facility at a giving a Nash welfare of $\sqrt{(1-a)}$. This compares to an optimal of $1 - a/2$. Therefore the approximation ratio is $\frac{(1-a/2)}{\sqrt{(1-a)}}$ which is in $[1, \frac{3}{2\sqrt{2}}]$ for $0 \leq a \leq 1/2$. Consider one agent at a, and another at 1. The median mechanism again locates the facility at a gving a Nash welfare of $\sqrt{a}$. This again compares to an optimal of $\frac{(a+1)}{2}$. Therefore the approximation ratio is $\frac{(a+1)}{2\sqrt{a}}$ which is in $[\frac{3}{2\sqrt{2}}, \infty)$ for $0 \leq a \leq 1/2$. Over the two scenarios, the best approximation ratio that can be achieved is $\frac{3}{2\sqrt{2}}$. □

It is an interesting question to close the gap between this lower bound and the 2-approximability provided by the MIDORNEAREST mechanism.

Can randomized mechanisms do better? Again the LRM mechanism is an obvious candidate given its optimality at minimizing the maximum distance. It is, however, less good at approximating the Nash welfare. Indeed, it is again beaten by deterministic mechanisms like the MIDORNEAREST mechanism.

Theorem 13. *The* LRM *mechanism α-approximates the Nash welfare in expectation with $\alpha = 2$ for $n = 2$, $\alpha = \frac{4}{\sqrt[3]{4}}$ (≈ 2.52) for $n = 3$ and $\alpha = 4$ for $n \geq 4$.*

Proof. For $n = 2$ and $n = 3$, we perform a similar case analysis to the proof of Theorem 9. For $n \geq 4$, consider the facility at $(x_1 + x_n)/2$. The utility of every agent is at least $1/2$. Hence the Nash welfare is also at least $1/2$. Since this occurs with probability $1/2$, this contributes at least $1/4$ to the expected Nash welfare. As the optimal Nash welfare is at most 1, the approximation ratio is at most 4. Consider $n-1$ agents at 0 ($n > 1$), and one final agent at 1. The expected Nash welfare of the probability distribution of solutions returned by the LRM mechanism is $1/4$. The optimal Nash welfare approaches 1 from below as n goes to infinity. Therefore the approximation ratio is at least 4. □

Finally, we give approximability results for two facilities.

Theorem 14. *The* ENDPOINT *mechanism* 2*-approximates the optimal Nash welfare.*

[3] We again see why it is better to use utilities rather than distances. The product of distances suffers from the drowning effect of any zero distance. On the other hand, in our facility location problem, the product of utilities never suffers such drowning as optimal utilities (unlike optimal distances) are never zero.

Proof. With two or more agents, the ENDPOINT mechanism, the leftmost and rightmost agents must have utility 1, while the other agents have utility between $1/2$ and 1. The minimum Nash welfare is when these $n-2$ agents have utility $1/2$ and the Nash welfare is $1/\sqrt[n]{2^{n-2}}$. This tends to $1/2$ for above as n goes to infinity. Hence the approximation ratio is at most 2. Suppose one agent is at 0, another is at 1, the final $n-2$ are at $1/2$, and the facilities located at the two endpoints. The Nash welfare of the solution returned by the mechanism is $1/\sqrt[n]{2^{n-2}}$. The optimal Nash welfare $\sqrt[n]{(1-1/n)^{n-2}/n^2}$ with the facility at $1/n$. This approaches 1 from below as n goes to infinity. Hence the approximation ratio is at least 2. □

Using the same examples as in the proof of Theorem 8, we can also show that any strategy proof and deterministic mechanism for two facilities cannot do better than $169/168$-approximate the Nash welfare (≈ 1.01).

8 Related Work

Several recent surveys summarize the considerable literature on mechanism design for facility location [Cheng and Zhou, 2015; Chan *et al.*, 2021]. Beginning with Procaccia and Tennholtz [2009], most studies of strategy proof mechanisms for facility location have focused on approximating the total and maximum distance (e.g. [Escoffier *et al.*, 2011; Fotakis and Tzamos, 2010; Fotakis and Tzamos, 2013; Goel and Hann-Caruthers, 2023; Lu *et al.*, 2010; Tang *et al.*, 2020; Zhang and Li, 2014]). Indeed, one of these recent surveys describes designing strategy proof mechanisms for facility location which approximate well the total or maximum distance that agents travel as the "classic setting" [Chan *et al.*, 2021].

One of the simplest measures of equitability in facility location is the maximum distance agents travel. Marsch and Schilling [1994] claim that this "is the earliest and most frequently used measure that has an equity component" in facility location problems, that "it has long been used as a more equitable alternative to the p-median problem which minimizes [total] travel distance", and that it "quantifies the popular Rawlsian criteria of equity which seeks to improve as much as possible those who are worst-off".

Another common measure of equitability is variance. For example, Maimon [1986] develops an algorithm to locate a facility on a tree network minimizing the variance in distances agents travel. Procaccia *et al.* [2018] study a different use of variance, exploring the tradeoff in randomized mechanisms between variance in the distribution of the location a facility and the approximation ratio of the optimal total or maximum distance agents travel. Other simple measures of equitability are the range and absolute deviation in distances agents travel [Marsh and Schilling, 1994]. For example, Berman and Kaplan [1990] argue that the absolute deviation is "a natural measure of the equity" of facility location problems and provide an efficient algorithm to compute the location of a facility on a general network to minimize this measure.

There are other indices of inequality besides the Gini index. For example, a common measure of income inequality is the Hoover index (also known as the Robin Hood or Schutz index), and this has been applied to facility location [Mulligan, 1991]. As a second example, the Atkinson index has been used in social choice settings such as resource allocation [Schneckenburger *et al.*, 2017]. It would be interesting to design mechanisms approximating such indices.

Lam *et al.* [2023] study mechanisms for facility location that maximize the product of agents' utilities. They give a polynomial time approximation algorithm to compute the facility location maximizing this product, and prove results suggesting that this achieves a good balance between fairness and efficiency. They also prove that no deterministic and strategy-proof mechanism provides a bounded approximation of the optimal product of utilities. Here we show that the optimal Nash welfare (the nth root of this product) can, on the other hand, be approximated well.

9 Conclusions

	1-G	**Nash**
1 facility, deterministic		
lower bound	$6/5$	$\frac{3}{2\sqrt{2}}$
LEFTMOST	n	∞
MEDIAN	2	∞
MIDORNEAREST, $n < 4$	$6/5$	2
MIDORNEAREST, $n \geq 4$	$\frac{n^2+n}{n^2+1}$	2
1 facility, randomized		
lower bound	$8/7$	–
LRM, $n = 2$	$4/3$	2
LRM, $n = 3$	$4/3$	$\frac{4}{\sqrt[3]{4}}$
LRM, $n \geq 4$	$[4/3, 2]$	4
2 facilities, deterministic		
lower bound	$30/29$	$169/168$
ENDPOINT	$35/29$	2
ENDPOINT$_\gamma$	$[15/14, 35/29]$	–
ENDPOINT$_\gamma$, $\gamma = 1/4$	$15/14$	–
ENDPOINT$_\gamma$, $\gamma = 1/2$	$6/5$	–

Table 1: Summary of approximation ratios of the complemented Gini index of utilities (**1-G**) and of the Nash welfare (**Nash**).

We have proposed designing mechanisms that approximate well equitability. We first proved an impossibility result that strategy proof mechanism cannot bound the approximation ratio of the optimal Gini index of utilities. We instead turn the problem on its head by considering approximation ratios of the complemented Gini index. This focuses on problems without perfect or near to perfect solutions where not all agents can be equi-distant from the facility. As Nash welfare is often considered to be an equitable compromise between egalitarian and utilitarian solutions, we also considered how well mechanisms optimize the Nash welfare. Our results are summarized in Table 1. They demonstrate that it is possible to design strategy proof mechanisms with near optimal equitable solutions. For a single facility, we identified a deterministic and strategy proof mechanism with an optimal approximation ratio. For two facilities, we designed a new strategy proof mechanism with a near optimal approximation ratio.

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